PERIODIC SOLUTIONS OF LINEAR SYSTEMS COUPLED WITH RELAY*

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Key words: Linear systems, relay, switching systems, periodic solutions
AMS classification: 34K35, 34C25, 93B12, 34C15

1. THE PROBLEM

We study the existence of periodic solutions of autonomous linear time-invariant differential systems with non-ideal relay, namely

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathcal{R}_{\alpha,\beta}\,(C\mathbf{x}) \tag{1.1}$$

where A is a stable $n \times n$ matrix, B and $C^T \in \mathbb{R}^n$ and $\alpha < \beta$ are the thresholds values of relay nonlinearity $\mathcal{R}_{\alpha,\beta}$ (figure 1). The system therefore behaves at each time according to one of the two modes $\dot{\mathbf{x}} = A\mathbf{x} + B$, $\dot{\mathbf{x}} = A\mathbf{x} - B$. The exact definition of the relay operator can be found e.g. in [6].

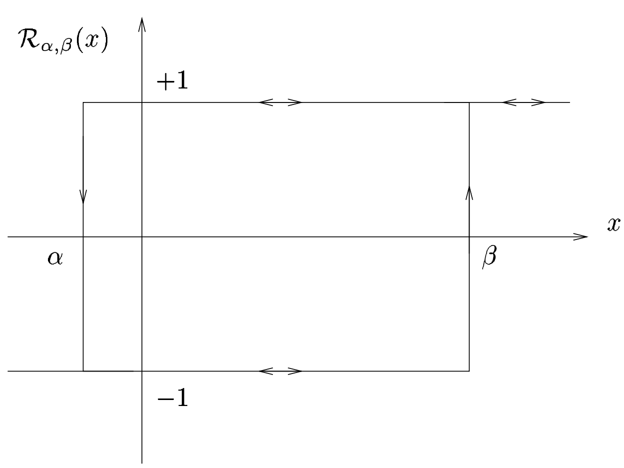

Fig. 1. Relay nonlinearity

We suppose that

$$-\,CA^{-1}B < \alpha < \beta < CA^{-1}B \tag{1.2}$$

*This work, partly done during the visit of A.M. Krasnosel'skii to I.N.R.I.A.– Rocquencourt, was partially supported by Grant # 93-01-00884 of Russian Foundation of Fundamental Researchs. The authors want to thank Dr. Rauf Izmailov from C&C Research Labs, NEC USA, Inc. for useful advices.

This implies that, before the unique equilibrium $\mathbf{x} = -A^{-1}B$ (resp. $\mathbf{x} = A^{-1}B$) of the 1st (resp. 2nd) mode is attained, switching occurs at the crossing of the hyperplane $C\mathbf{x} = \beta$ (resp. $C\mathbf{x} = \alpha$), see figure 2. This situation is described for more general cases in [8].

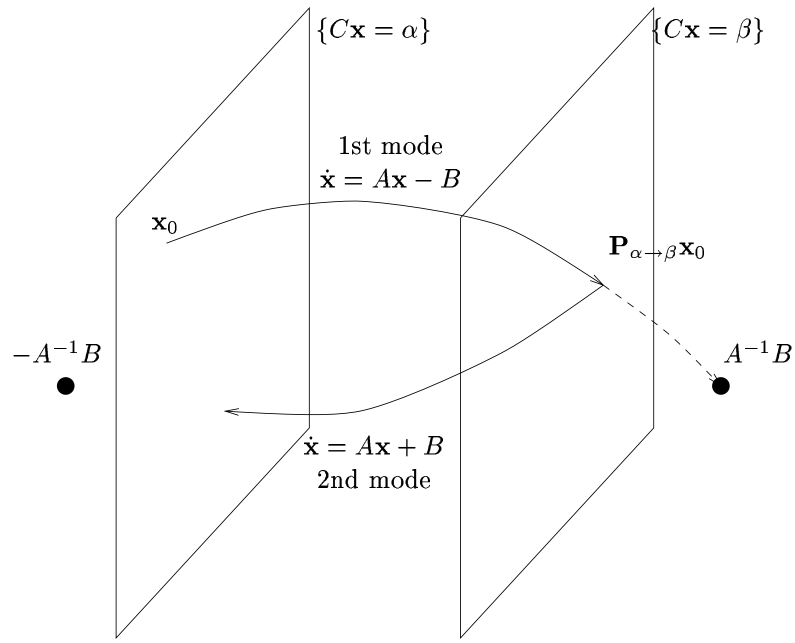

Fig. 2. Switching hyperplanes in the phase space

For $\mathbf{x}_0 \in \{C\mathbf{x} = \alpha\}$ (resp. $\mathbf{x}_0 \in \{C\mathbf{x} = \beta\}$), denote by $\mathbf{P}_{\alpha\to\beta}\mathbf{x}_0$ (resp. $\mathbf{P}_{\beta\to\alpha}\mathbf{x}_0$) the first point of the trajectory $\mathbf{x}(t)$, $t > 0$, with initial value $\mathbf{x}_0$ for the state and initial value -1 (resp. $+1$) for the relay, which lies on $\{C\mathbf{x} = \beta\}$ (resp. $\{C\mathbf{x} = \alpha\}$).

If the map $\mathbf{P} \stackrel{\Delta}{=} \mathbf{P}_{\alpha\to\beta} \circ \mathbf{P}_{\beta\to\alpha}$ has a fixed point, then equation (1.1) has a periodic solution with two switches. Indeed, the question of existence of periodic solutions for this system reduces to the search of fixed points for the map $\mathbf{P}$ and its iterates.

It is possible to choose a convex compact set $\mathbf{K} \subset \mathbb{R}^n$ invariant for both equations $\dot{\mathbf{x}} = A\mathbf{x} \pm B$ and containing any periodic trajectory (see figure 3). Indeed, one may choose:

$$\mathbf{K} \stackrel{\Delta}{=} \{\mathbf{x} \in \mathbb{R}^n : \forall M^T \in \mathbb{R}^n, M\mathbf{x} \leq \int_0^{+\infty} |Me^{As}B| \cdot ds\} \tag{1.3}$$

The set $\mathbf{K}_\alpha \stackrel{\Delta}{=} \mathbf{K} \cap \{C\mathbf{x} = \alpha\}$ is a convex compact subset of $\{C\mathbf{x} = \alpha\}$, invariant for $\mathbf{P}$. If $\mathbf{P}$ is *continuous on* $\mathbf{K}_\alpha$, then it has at least one fixed point (Brouwer fixed point theorem).

To find sufficient conditions for continuity for $\mathbf{P}_{\alpha\to\beta}$ on $\mathbf{K}_\alpha$ and $\mathbf{P}_{\beta\to\alpha}$ on $\mathbf{K}_\beta \stackrel{\Delta}{=} \mathbf{K} \cap \{C\mathbf{x} = \beta\}$ is the main difficulty in this approach. In this goal, the notion of *anomalous point* is useful:

DEFINITION 1.1 *A point* $\mathbf{x}_0 \in \{C\mathbf{x} = \alpha\}$ *(resp.* $\{C\mathbf{x} = \beta\}$*) is called an* **anomalous point** *if* $C(A\mathbf{x}_0 + B) = 0$ *and* $CA(A\mathbf{x}_0 + B) \geq 0$ *(resp.* $C(A\mathbf{x}_0 - B) = 0$ *and* $CA(A\mathbf{x}_0 - B) \leq 0$*). We denote* $\mathbf{A}_\alpha$ *(resp.* $\mathbf{A}_\beta$*) the set of anomalous points.*

Any point of the switching hyperplanes on which the incoming trajectory is tangent without crossing it, is an anomalous point (see figure 4). Those points are of interest because they are the points of discontinuity of $\mathbf{P}_{\alpha\to\beta}$ and $\mathbf{P}_{\beta\to\alpha}$. Note that if $n > 3$ and C, CA, CA^2 are linearly independent (a natural observability assumption on the realization of the transfer $C(sI - A)^{-1}B$), the sets $\mathbf{A}_\alpha$ and $\mathbf{A}_\beta$ contain at least a closed half-hyperplane of codimension $n - 2$. In particular, if $n \geq 3$, operator $\mathbf{P}$ is never continuous on the whole hyperplane $\{C\mathbf{x} = \alpha\}$.

Notice that an anomalous point may be a point of continuity of $\mathbf{P}_{\alpha\to\beta}$ (resp. $\mathbf{P}_{\beta\to\alpha}$), unlike the definition of [8, 9].

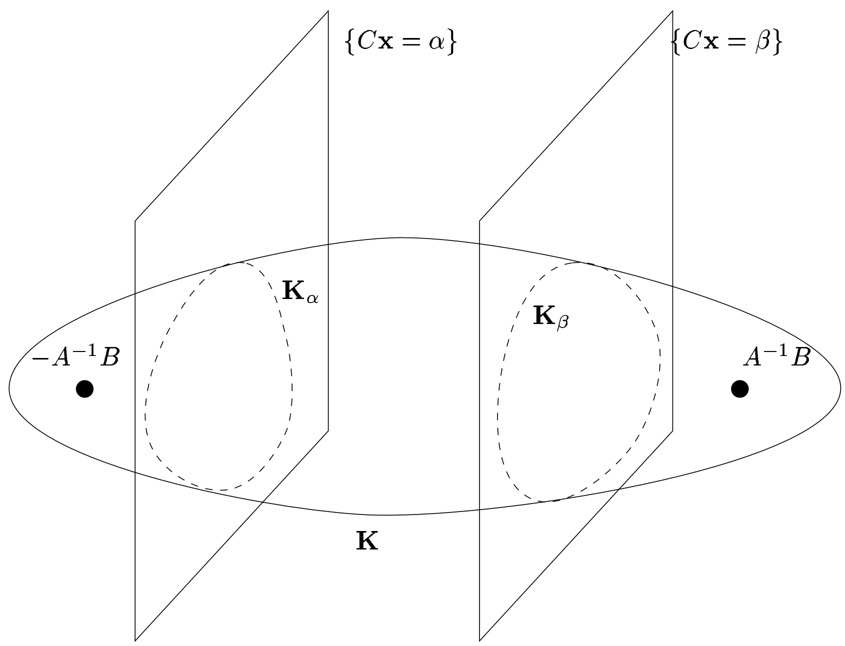

Fig. 3. The invariant set K

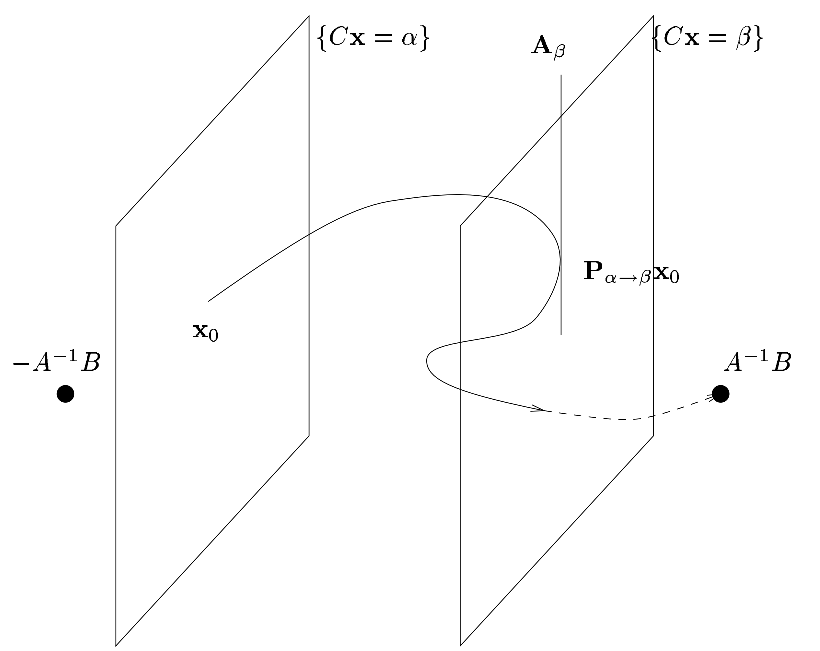

Fig. 4. A trajectory crossing a switching hyperplane in an anomalous point

From the preceding considerations, the following result is easily deduced:

THEOREM 1.0 *If* $\mathbf{P}_{\alpha\to\beta}\mathbf{K}_\alpha \cap \mathbf{A}_\beta = \mathbf{P}_{\beta\to\alpha}\mathbf{K}_\beta \cap \mathbf{A}_\alpha = \emptyset$*, then* $\mathbf{P}_{\alpha\to\beta}$ *(resp.* $\mathbf{P}_{\beta\to\alpha}$*) is continuous on* $\mathbf{K}_\alpha$ *(resp.* $\mathbf{K}_\beta$*), and* $\mathbf{P}$ *is continuous on* $\mathbf{K}_\alpha$*.*

Theorem 1.0 applies e.g. if $\mathbf{K}_\alpha \cap \mathbf{A}_\alpha = \mathbf{K}_\beta \cap \mathbf{A}_\beta = \emptyset$ (figure 5), but this is a strong hypothesis.

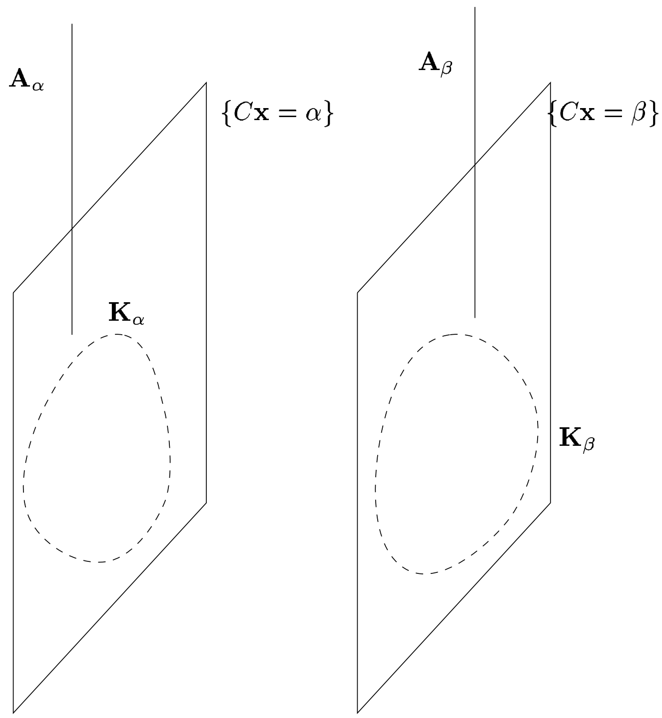

Fig. 5. Condition $\mathbf{K}_\alpha \cap \mathbf{A}_\alpha = \mathbf{K}_\beta \cap \mathbf{A}_\beta = \emptyset$

Up to this point, the construction is not new, see [3, 4, 8, 9].

The papers by Friedman and al. [3] and Gripenberg [4] are concerned with systems whose evolution is governed by heat equation. Reference [8] by Seidman provides sufficient condition (expressed in terms of impulse-response of the possibly infinite-dimensional linear system) for existence of periodic solutions. Paper [5] by Kolesov gives results of the same nature, expressed in terms of the (rational) transfer function. All these references are concerned with periodic solutions with two switchings per period, a case in which the value of the period may be computed exactly by Tsypkin method, see e.g. [2].

But things are not always so simple: Szczechla [9] proved that for any positive integer n, there exists systems like (1.1) which do not possess periodic solution with less than $2n$ switchings per period. To our knowledge, the only method leading to sufficient conditions for existence of solution with an arbitrary number of switchings, is the one by Macki and al. [7], based on harmonic balance and topological degree theory.

Remark that Alexeev achieved more than fourty years ago some analogic computations which show the rich behavior already present in a 2nd-order model with delay [1].

Here, we present two original approaches to obtain continuity of the map $\mathbf{P}$ on $\mathbf{K}_\alpha$ in the case where $\mathbf{K}_\alpha \cap \mathbf{A}_\alpha$ or $\mathbf{K}_\beta \cap \mathbf{A}_\beta$ is non-empty.

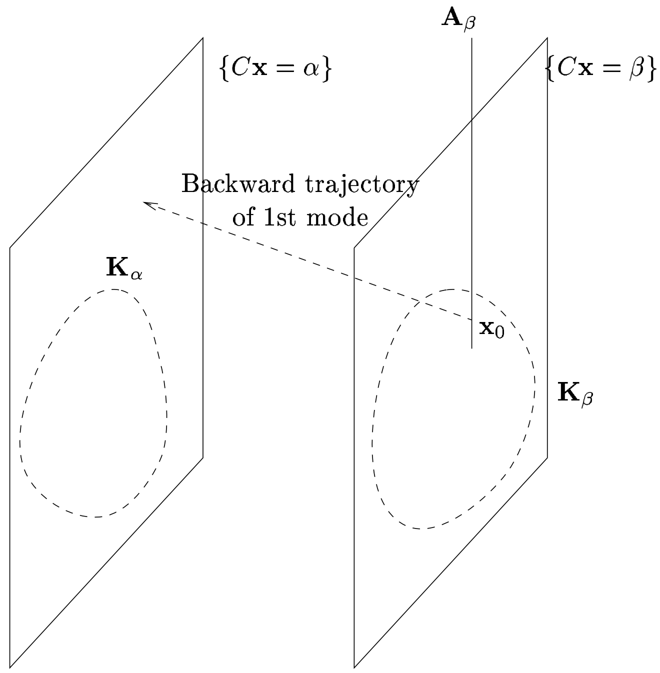

Fig. 6. Hypothesis of Theorem 1.1

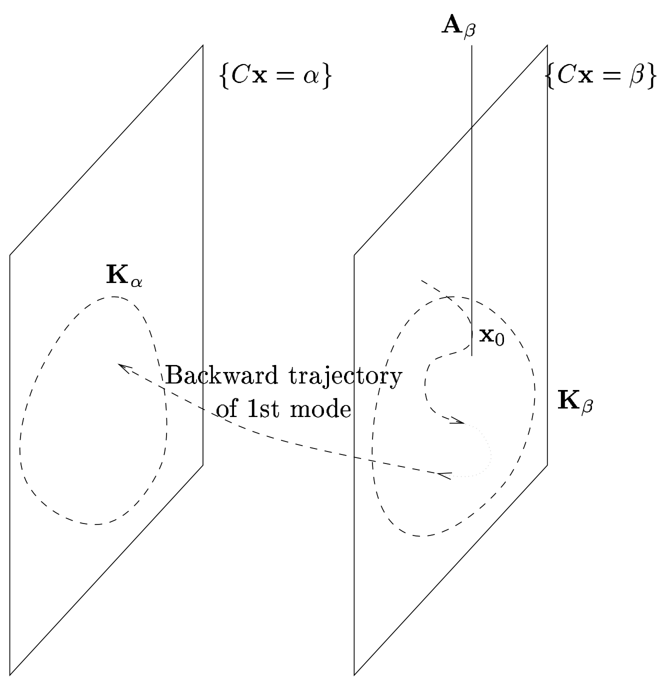

Fig. 7. Hypothesis of Theorem 1.2

THEOREM 1.1 *If the backward trajectory departing from any point* $x_0 \in \mathbf{K}_\beta \cap \mathbf{A}_\beta$ *(resp.* $\mathbf{K}_\alpha \cap \mathbf{A}_\alpha$*) attaines* $\{C\mathbf{x} = \alpha\}$ *(resp.* $\{C\mathbf{x} = \beta\}$*) outside* $\mathbf{K}_\alpha$ *(resp.* $\mathbf{K}_\beta$*), then* $\mathbf{P}_{\alpha\to\beta}\mathbf{K}_\alpha \cap \mathbf{A}_\beta = \mathbf{P}_{\beta\to\alpha}\mathbf{K}_\beta \cap \mathbf{A}_\alpha = \emptyset$ *and* $\mathbf{P}$ *is continuous on* $\mathbf{K}_\alpha$*.*

THEOREM 1.2 *If the backward trajectory departing from any point* $x_0 \in \mathbf{K}_\beta \cap \mathbf{A}_\beta$ *(resp.* $\mathbf{K}_\alpha \cap \mathbf{A}_\alpha$*) crosses* $\{C\mathbf{x} = \beta\}$ *(resp.* $\{C\mathbf{x} = \alpha\}$*) before attaining* $\{C\mathbf{x} = \alpha\}$ *(resp.* $\{C\mathbf{x} = \beta\}$*), then* $\mathbf{P}_{\alpha\to\beta}\mathbf{K}_\alpha \cap \mathbf{A}_\beta = \mathbf{P}_{\beta\to\alpha}\mathbf{K}_\beta \cap \mathbf{A}_\alpha = \emptyset$ *and* $\mathbf{P}$ *is continuous on* $\mathbf{K}_\alpha$.

The proof of these results follows directly from Theorem 1.0. The situation described by the hypotheses is illustrated in figures 6 and 7.

This approach may be generalized. For example, instead of $\mathbf{P}_{\beta\to\alpha}(\mathbf{K}_\beta) \cap \mathbf{A}_\alpha = \emptyset$, it is sufficient to assess $\mathbf{P}_{\beta\to\alpha}(\mathbf{P}_{\alpha\to\beta}\mathbf{K}_\alpha) \cap \mathbf{A}_\alpha = \emptyset$, etc ... Also, it may potentially be used for more general switching systems, that is systems whose evolution (in each given mode) is not described by a linear ODE.

2. A 3rd ORDER EXAMPLE

We apply the previous ideas to the equation

$$L\left(\frac{d}{dt}\right) x = -\mathcal{R}_{\alpha,\beta}\, x \tag{2.1}$$

where the real polynomial $L(s)$ writes:

$$L(s) \equiv (s+\lambda)(s+u+iv)(s+u-iv) \quad \text{with} \quad \lambda > 0,\ u > 0,\ v > 0$$

This equation can be rewritten in the form (1.1) using the state variable $\mathbf{x} = (x, x', x'')^T$ with the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda(u^2+v^2) & -(u^2+v^2) - 2\lambda u & -\lambda - 2u \end{pmatrix},\quad B = (0,0,-1)^T,\quad C = (1,0,0) \tag{2.2}$$

Remark that $CB = CAB = 0$, which implies that $\mathbf{K}_\alpha \cap \mathbf{A}_\alpha$ and $\mathbf{K}_\beta \cap \mathbf{A}_\beta$ are not empty (consider e.g. $x = \frac{\alpha}{CA^{-1}B} A^{-1}B$ and $x = \frac{\beta}{CA^{-1}B} A^{-1}B$). Throughout the paper, we will use the following notations

$$k_\alpha \triangleq \alpha + \frac{1}{\lambda(u^2+v^2)},\quad k_\beta \triangleq -\beta + \frac{1}{\lambda(u^2+v^2)}$$

$$\sigma \triangleq (u-\lambda)^2 + v^2,\quad w \triangleq \frac{u-\lambda}{v},\quad e^{i\xi} \triangleq \frac{u+iv}{\sqrt{u^2+v^2}},\quad 0 < \xi < \pi/2$$

Remark that $\sigma = v^2(1+w^2)$. Assumption (1.2) is equivalent to $k_\alpha,\ k_\beta > 0$.

3. PRELIMINARIES

- Denote $x(t)$ the solution of equation $L\left(\frac{d}{dt}\right) x = 1$ such that $x(0) = \beta,\ x'(0) = 0,\ x''(0) = y \le 0$: the solution meets the set $\mathbf{A}_\beta$ at time $t = 0$. One may write:

$$x(t) = \beta + \frac{1}{\sigma}(R(t)k_\beta + Q(t)y) \tag{3.1}$$

$$Q(t) \triangleq e^{-\lambda t} - e^{-ut}\cos vt + e^{-ut}\frac{\lambda - u}{v}\sin vt$$

$$R(t) \triangleq \sigma - e^{-\lambda t}(u^2+v^2) + e^{-ut}(-\lambda^2 + 2\lambda u)\cos vt + \frac{e^{-ut}}{v}(\lambda(u^2-v^2) - \lambda^2 u)\sin vt$$

$R'(t) = \lambda(u^2+v^2)Q(t),\ R(0) = Q(0) = Q'(0) = 0,\ Q''(0) = \sigma.$

Analogously, a solution $x(t)$ of $L\left(\frac{d}{dt}\right)x = -1$ attaining $\mathbf{A}_\alpha$ at $t=0$ with $x''(0) = y$ writes as

$$x(t) = \alpha + \frac{1}{\sigma}(-R(t)k_\alpha + Q(t)y)$$

• Consider the equation (in the unknown τ)

$$Q(\tau/v) = 0, \quad \text{that is } e^{w\tau} = \cos\tau + w\sin\tau \tag{3.2}$$

If $w > 0$, this equation has a root in $\tau = 0$ (of 2nd order) and negative roots. We define:

$$\text{Let } \tau^* < \tau_* < 0 \text{ be the two largest negative roots of (3.2)} \tag{3.3}$$

It is clear from the graph of Q that $Q(t) < 0$ (resp. $>$, resp. $=$) if $\tau^* < vt < \tau_*$ (resp. $\tau_* < vt < 0$, resp. $vt = \tau^*, \tau_*, 0$). The values of τ^* and τ_* are functions of $w > 0$. For $w \to \infty$ one has $\tau^* \to -2\pi,\ \tau_* \to -\pi$. Figure 8 shows an example of curves $\tau^*(w)$ and $\tau_*(w)$.

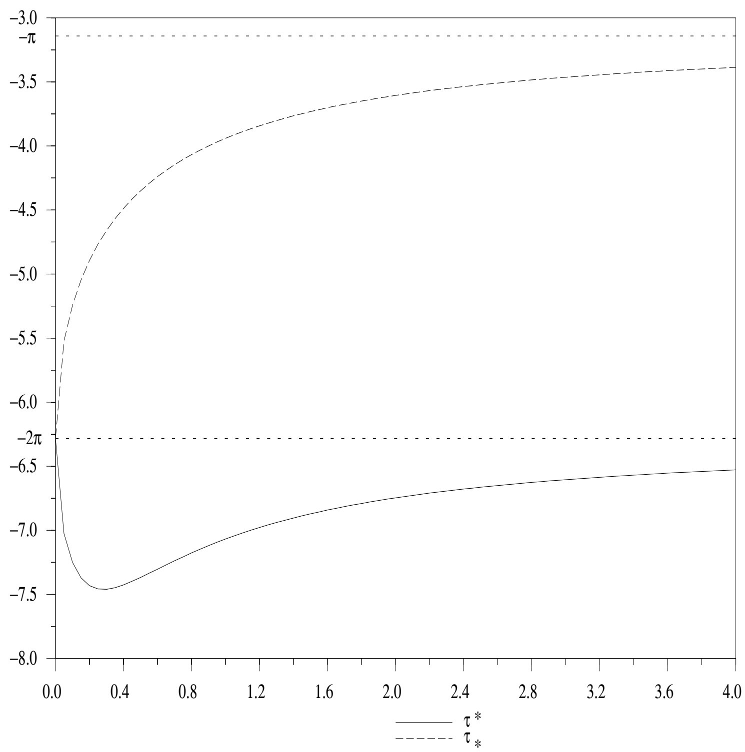

Fig. 8. Functions $\tau^*(w)$ and $\tau_*(w)$ for $\lambda = 0.01$, $u = 0.4$, $v = 1$

• In the statement of the results and in their proof, we will use the following definitions of the functions η and N:

$$\eta(\xi) \triangleq \frac{2e^{(2\xi-\pi)\tan\xi}}{1 - e^{-\pi\tan\xi}}, \quad 0 < \xi < \pi/2 \tag{3.4}$$

and, denoting $\rho e^{i\psi} \stackrel{\Delta}{=} (\lambda u^2 - uv^2 - u^3 - \lambda v^2) + iv(v^2 + u^2 - 2\lambda u)$, $\rho \geq 0, 0 \leq \psi < 2\pi$ (which implies $\rho = \sqrt{\sigma}(u^2 + v^2)$):

$$N(u,v,\lambda) \stackrel{\Delta}{=} \frac{2}{\sqrt{\sigma}} \cdot \frac{e^{(\psi-\pi)u/v}}{1-e^{-\pi u/v}} \text{ if } 0 \leq \psi < \pi, \quad \frac{2\lambda}{\sigma} + \frac{2}{\sqrt{\sigma}} \cdot \frac{e^{(\psi-2\pi)u/v}}{1-e^{-\pi u/v}} \text{ if } \pi \leq \psi < 2\pi \tag{3.5}$$

N is continuous (as $\psi = \pi$ implies $u^2 + v^2 = 2\lambda u$, so $\sigma = \lambda^2$).

4. RESULTS

THEOREM 4.1 *For any choice of the constants $\alpha < \beta$, $\lambda > 0$, $\xi = \tan\frac{u}{v} > 0$ such that*

$$(\beta - \alpha)\lambda^3 > (3 + \eta(\xi)) \tag{4.1}$$

if $u^2 + v^2$ is sufficiently small, then Theorem 1.1 *holds and equation* (2.1) *has at least one periodic solution with two switches.*

Recall that η is defined by formula (3.4).

Theorem 4.1 holds e.g. in cases where "the real root is large, the modulus of the complex roots is small".

THEOREM 4.2 *For any choice of the positive constants u, v, λ such that $u > \lambda$,*

$$\sigma + \lambda e^{-u\tau^*/v}\sqrt{\sigma - v^2 e^{2(u-\lambda)\tau^*/v}} \geq e^{-\lambda\tau^*/v}(u^2 + v^2 - \lambda u) \tag{4.2}$$

$$\frac{2v\sigma}{\lambda(u^2+v^2)\left(ve^{-\lambda\tau_*/v} + \sqrt{\sigma}e^{-u\tau_*/v}\right)} > N(u,v,\lambda) \tag{4.3}$$

if $k_\alpha, k_\beta > 0$ are sufficiently small, then Theorem 1.2 *holds and equation* (2.1) *has at least one periodic solution with two switches.*

Recall that $\tau_\star$ and $\tau^\star$ (resp. N) are defined by equation (3.3) (resp. (3.5)).

Condition (4.3) is valid for λ small enough and (4.2) may be rewritten for $w = (u-\lambda)/v \simeq u/v$ as $e^{-2w\tau^*} - 1 + 2w\tau^* - (w^2+1)\tau^{*2} > 0$, where we recall that τ^* is a function of w which tends to -2π in $+\infty$: for $w \geq 0.271$, conditions (4.2) and (4.3) are true for small enough $\lambda > 0$. Hence, Theorem 4.2 holds e.g. in cases where "the real root is small, the real part of the complex roots is large, the thresholds are close from the static gains".

Remark that the "sufficiently small" conditions of these statements may be stated precisely with the help of the proofs. On the other hand, the impulse-response of the system we consider is $\frac{1}{\sigma}Q(t)$, and it is clear that Theorem 4.2 may provide results in cases where it is not ultimately decreasing, unlike the result for linear systems given in [8, Theorem 3.1].

5. ESTIMATE OF x''

Using realization (2.2) and formula (1.3), we deduce that any periodic solution x of (2.1) verifies:

$$\forall t \in \mathbb{R}, \quad |x''(t)| \leq \int_0^\infty |De^{As}B| \cdot ds, D = (0,0,1)$$

On the other hand, the decomposition (3.1) of x shows that $\frac{1}{\sigma}Q(t) = Ce^{At}(0,0,1)^T = Ce^{At}B$. Hence, $\frac{1}{\sigma}Q''(t) = CA^2e^{At}B = De^{At}B$, so $\forall t \in \mathbb{R}$, $|x''(t)| \leq \frac{1}{\sigma}\int_0^\infty |Q''(t)| \cdot dt \stackrel{\Delta}{=} M$.

LEMMA 5.1 *$M \leq N(u,v,\lambda)$, where N is defined by* (3.5).

The proof of this Lemma consists in computing Q'', writing

$$M \leq \frac{\lambda^2}{\sigma}\int_0^\infty e^{-\lambda t}\,dt + \frac{\rho}{\sigma v}\int_0^\infty e^{-ut}|\sin(\psi+vt)|\,dt = \frac{\lambda}{\sigma} + \frac{\rho e^{\psi u/v}}{\sigma v^2}\int_\psi^\infty e^{-u\tau/v}|\sin\tau|\,d\tau$$

and evaluating the former expression on the intervals $[k\pi,(k+1)\pi]$.

6. PROOF OF THEOREM 4.1

• Consider x defined in (3.1). For a finite $t_0<0$, its trajectory reaches the plane $\{C\mathbf{x}=\alpha\}$ in some point $\mathbf{x}(t_0)$. We shall first estimate t_0, and then show that if the hypotheses of Theorem 4.1 are fulfilled, then $\mathbf{x}(t_0)\notin\mathbf{K}_\alpha$, as $x''(t_0)>M$ defined in the previous section.
• For sufficiently small u^2+v^2, we have $Q(t)\geq 0$ for $t<0$. Since $y\leq 0$, the (negative) number t_0 may be bounded from below by the time at which $\beta+R(t)k_\beta/\sigma$ reaches $\{C\mathbf{x}=\alpha\}$. The value of $k_\beta=\frac{1}{\lambda(u^2+v^2)}-\beta$ is positive for small enough u^2+v^2, and the inequality $\beta+R(t)k_\beta/\sigma\leq\alpha$ is equivalent to

$$R(t)\leq 0 \text{ and } \left(\frac{1}{\lambda(u^2+v^2)}-\beta\right)|R(t)|\geq\sigma(\beta-\alpha)$$

As $R'=Q$ and e.g. for $u<\lambda/2$, $Q(t)\geq e^{-\lambda t}-e^{-\lambda t/2}+\frac{1}{2}e^{-\lambda t/2}\lambda t=e^{-\lambda t/2}(e^{-\lambda t/2}-1+\frac{1}{2}\lambda t)\geq 0$, one gets for small enough values of u^2+v^2 an estimate of t_0 *independent of* u^2+v^2.
• Now for u^2+v^2 sufficiently small we have, (the sign $\simeq$ will denote equality up to terms in $O(\sqrt{u^2+v^2})$; *uniform* estimate of t_0 is necessary): $\sigma\simeq\lambda^2$, $Q(t_0)\simeq e^{-\lambda t_0}-1+\lambda t_0$, $R(t_0)k_\beta\simeq -\frac{1}{\lambda}(e^{-\lambda t_0}-1+\lambda t_0-\frac{\lambda^2 t_0^2}{2})$. Hence, $x(t_0)=\alpha$ implies $(\beta-\alpha)\lambda^2\simeq|y|(e^{-\lambda t_0}-1+\lambda t_0)+\frac{1}{\lambda}(e^{-\lambda t_0}-1+\lambda t_0-\frac{\lambda^2t_0^2}{2})$, and

$$|y|\simeq\frac{(\beta-\alpha)\lambda^2}{e^{-\lambda t_0}-1+\lambda t_0}-\frac{1}{\lambda}\frac{e^{-\lambda t_0}-1+\lambda t_0-\frac{\lambda^2t_0^2}{2}}{e^{-\lambda t_0}-1+\lambda t_0}>\frac{(\beta-\alpha)\lambda^2}{e^{-\lambda t_0}}-\frac{1}{\lambda}$$

For small enough u^2+v^2, we then have

$$e^{-\lambda t_0}\left(|y|+\frac{1}{\lambda}\right)>(\beta-\alpha)\lambda^2 \tag{6.1}$$

On the other hand, $|x''(t_0)|\simeq|y|e^{-\lambda t_0}+\frac{1}{\lambda}(e^{-\lambda t_0}-1)$. Inequality (4.1) guarantees that $\xi=u/v$ is bounded from below by a positive number (independent of u^2+v^2). For u^2+v^2 small, $\tan\psi\simeq\tan 2\xi$, and $\psi\simeq 2\xi+\pi$, as $0<\xi<\frac{\pi}{2}$ and $\sin\psi<0$. From Lemma 1, we get $M\leq\frac{2}{\lambda}+\frac{1}{\lambda}\eta(\xi)+O\left(\sqrt{u^2+v^2}\right)$.

Suppose now that $|x''(t_0)|\leq M$, then $e^{-\lambda t_0}\left(|y|+\frac{1}{\lambda}\right)\leq 3\frac{1}{\lambda}+\frac{1}{\lambda}\eta(\xi)$. This, together with (6.1), implies $(\beta-\alpha)\lambda^2\leq\frac{1}{\lambda}(3+\eta(\xi))$ which contradicts assumption (4.1).

Hence, $|x''(t_0)|>M$ and $\mathbf{K}_\alpha\cap\mathbf{P}^{-1}_{\alpha\to\beta}(\mathbf{A}_\alpha)=\emptyset$. Proof of $\mathbf{K}_\beta\cap\mathbf{P}^{-1}_{\beta\to\alpha}(\mathbf{A}_\beta)=\emptyset$ is analogous.

7. PROOF OF THEOREM 4.2

To prove Theorem 4.2, we show that, for any $\mathbf{x}(0)\in\mathbf{A}_\beta$, $t_0\overset{\Delta}{\equiv}\tau^*/v<0$ is such that $x(t_0)\geq\beta$ and $x(t)>\alpha$ for $t\in(t_0,0)$. Since $Q(t_0)=0$, then $x(t_0)=\beta+R(t_0)k_\beta/\sigma$. Defining $e^{i\varphi}\overset{\Delta}{\equiv}\frac{1}{\sqrt{\sigma}}(u-\lambda+iv)$ with $\varphi\in]0,\frac{\pi}{2}[$, one has $\sin(vt_0+\varphi)=\sin\varphi\, e^{(\lambda-u)t_0}$ and, as $vt_0\in]-\frac{\pi}{2},0[$, $\cos(vt_0+$

$\varphi) = \sqrt{1 - e^{2(u-\lambda)t_0} \sin^2 \varphi}$. We deduce $\cos vt_0 = \frac{u-\lambda}{\sigma}\sqrt{\sigma - v^2 e^{2(u-\lambda)t_0}} + \frac{v^2}{\sigma} e^{(u-\lambda)t_0}$ and $\sin vt_0 = \frac{v}{\sigma}[(u-\lambda)e^{(u-\lambda)t_0} - \sqrt{\sigma - v^2 e^{2(u-\lambda)t_0}}]$. Finally, we compute $R(t_0) = \sigma + \lambda e^{-ut_0}\sqrt{\sigma - v^2 e^{2(u-\lambda)t_0}} - e^{-\lambda t_0}(u^2 + v^2 - \lambda u)$, which is non negative due to (4.2).

Now we need to check that $t \in (t_0, 0)$ implies $x(t) > \alpha$, i.e.

$$yQ(t) > (\alpha - \beta)\sigma - k_\beta R(t), \quad t_0 < t < 0 \tag{7.1}$$

For $t_0 < t \leq \tau_*/v$, inequality (7.1) is true for sufficiently small $k_\beta > 0$, as $\alpha < \beta$ and $yQ(t) \geq 0$. For $\tau_*/v < t < 0$, function $Q(t)$ is positive and inequality (7.1) is equivalent to $y > [(\alpha-\beta)\sigma - k_\beta R(t)]/Q(t)$, that is ($y \leq 0$, $\alpha = k_\alpha + 1/\lambda(u^2+v^2)$, $\beta = -k_\beta - 1/\lambda(u^2+v^2)$)

$$|y| < \frac{(\beta-\alpha)\sigma + k_\beta R(t)}{Q(t)} = \frac{2\sigma + O(k_\alpha, k_\beta)}{\lambda(u^2+v^2)Q(t)}$$

Due to (4.3), the last inequality is true for small enough $k_\alpha, k_\beta > 0$, as $|y| \leq M \leq N(u, v, \lambda)$, and $|Q(t)| \leq e^{-\lambda t} + e^{-ut}\sqrt{\sigma}/v$. Therefore, operator $\mathbf{P}_{\alpha\to\beta}$ is continuous. Proof of continuity for $\mathbf{P}_{\beta\to\alpha}$ is similar.

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