Existence of Homogeneous Polynomial Solutions for Parameter-Dependent Linear Matrix Inequalities with Parameters in the Simplex

P.-A. Bliman, R. C. L. F. Oliveira, V. F. Montagner, and P. L. D. Peres

***Abstract*— This paper presents some general results concerning the existence of homogeneous polynomial solutions to parameter-dependent linear matrix inequalities whose coefficients are continuous functions of parameters lying in the unit simplex. These results are useful in the context of robust analysis and synthesis of parameter-dependent feedback gains (gain-scheduling) for uncertain linear systems in polytopic domains. A result showing the generality of the class of static gains with homogeneous polynomial dependence and a result dealing with the solutions of parameter-dependent linear matrix inequalities with slowly time-varying parameters are also given.**

I. INTRODUCTION

Robust analysis and control design of linear systems depending on uncertain parameters are well-rooted problems within robust control theory and have attracted remarkable research efforts in the last decades. Frequently, problems like these can be formulated in terms of parameter-dependent linear matrix inequalities (LMIs) [1]. In general, the feasibility of a parameter-dependent LMI must be checked in the whole space of parameters, thus yielding an infinite-dimensional problem [2]. It is very natural for robustness problems in linear systems investigated by means of Lyapunov theory to be expressed in terms of parameter-dependent LMIs [3].

Within the many descriptions for linear systems with parametric uncertainties, the polytopic representation has been receiving great attention in the last years, mainly due to its simple expression, given by a convex combination of precisely known vertices. The study of this class of systems through the Lyapunov theory naturally yields optimization problems involving parameter-dependent LMIs, whose solution is written in terms of a matrix $P(\alpha)$ that depends arbitrarily on the vector of parameters α which lies in the unit simplex. This matrix, also known as the Lyapunov matrix, is associated, in most cases, to a quadratic Lyapunov function $v(x) = x^T P(\alpha)x$.

Assuming a general dependence of $P(\alpha)$ in terms of the parameters, the resulting optimization problems are numerically intractable (infinite-dimensional), since the whole space of parameters must be tested. On the other hand, imposing a particular structure to the matrix $P(\alpha)$ produces optimization problems of finite dimension, formulated in terms of standard (parameter-independent) LMIs. Despite of a possible introduction of conservativeness be inherent to the solution of the problem under this strategy, many important problems like robust analysis and control synthesis could find a convex solution using $P(\alpha) = P$ (quadratic stability) [3]. The results show that this choice is very conservative, being more suitable to handle time-varying systems [4]. Lyapunov functions with affine dependence on the parameters emerged as the next candidate to the structure of $P(\alpha)$ with the aim of improve the results of quadratic stability [5, **?**–11]. In the context of time-varying systems, the use of affine parameter-dependent Lyapunov functions [5, 12, 13] allows to take into account bounds on the time derivative of the parameters in the LMI conditions.

Quadratic [8] and polynomial structures for $P(\alpha)$ rose naturally in order to obtain a complete characterization of parameter-dependent LMIs [14–20]. In this context, it is worth mentioning [21], where a result is given ensuring that solutions to LMIs depending continuously on parameters in a compact set either do not exist for some values of the parameters, or include some solutions which are polynomial in the parameters.

The contribution of this paper is to provide some extensions of the results of [21] for the case of parameter-dependent LMIs emerging from robust analysis and control design of linear uncertain systems with polytopic representation. The first result establishes that parameter-dependent LMIs with parameters in the unit simplex can be completely characterized by means of homogeneous polynomial solutions, without loss of generality. This result is important from the numerical point of view, since the complexity associated with the numerical tests can be drastically reduced. Applications to the synthesis of parameter-dependent controllers and to slowly time-varying systems are also presented. Numerical examples are given to illustrate the proposed approach.

II. MAIN RESULTS

Consider the following general parameter-dependent LMI:

$$\forall\alpha \in \Delta_m, \exists x \in \mathbb{R}^p,\ G(x,\alpha) \triangleq G_0(\alpha) + x_1 G_1(\alpha) + \dots + x_p G_p(\alpha) > 0_n \qquad (1)$$

where Δ_m is the unit simplex, given by

$$\Delta_m \triangleq \left\{ \alpha \in \mathbb{R}^m \ :\ \sum_{i=1}^{m} \alpha_i = 1,\ \alpha_i \geq 0,\ i = 1,\dots m \right\}$$

Supported by the Brazilian agencies CNPq and FAPESP.

R. C. L. F. Oliveira, V. F. Montagner and P. L. D. Peres are with the School of Electrical and Computer Engineering, University of Campinas, CP 6101, 13081-970, Campinas, SP, Brazil {ricfow,montagne,peres}@dt.fee.unicamp.br.

P.-A. Bliman is with the INRIA, Bat. 12, Rocquencourt BP 105, 78153 Le Chesnay Cedex, France pierre-alexandre.bliman@inria.fr.

and $G_0(\cdot), G_1(\cdot), \ldots, G_p(\cdot)$ are functions defined in Δ_m, with values in the set of symmetric matrices of dimension $\mathbb{R}^{n\times n}$. $G(x,\alpha) > 0_n$ means that the matrix $G(x,\alpha)$ is positive definite. For a given value of the parameter α, the existence of x in $\mathbb{R}^p$ verifying the inequality in (1) is a standard LMI.

Theorem 1: Assume that $G_0(\cdot), G_1(\cdot), \ldots, G_p(\cdot)$ are continuous. If for all $\alpha \in \Delta_m$ there exists $x(\alpha) \in \mathbb{R}^p$ such that $G(x(\alpha),\alpha) > 0_n$, then there exists a homogeneous polynomial function $x^* : \Delta_m \to \mathbb{R}^p$ such that, for all $\alpha \in \Delta_m$, $G(x^*(\alpha),\alpha) > 0_n$.

Moreover, if there exists a polynomial function $x^{**}(\alpha)$ of degree d such that, for all $\alpha \in \Delta_m$, $G(x^{**}(\alpha),\alpha) > 0_n$, then there exists a homogeneous polynomial function of degree d with the same property. $\square$

Proof: It has been shown in [21, Theorem 1] that, under the hypothesis of the theorem, if there exists a solution $x(\alpha) \in \mathbb{R}^p$ then there exists, without loss of generality, a polynomial solution $x^{**}(\alpha)$ such that, for all $\alpha \in \Delta_m$, $G(x^{**}(\alpha),\alpha) > 0_n$.

Taking d as the higher degree present in the monomials of $x^{**}(\alpha)$, the following decomposition can be obtained

$$x^{**}(\alpha) = \sum_{0\le\beta_1+\cdots+\beta_m\le d,\ \beta_i\ge 0} c_{\beta_1,\ldots,\beta_m}\alpha_1^{\beta_1}\ldots\alpha_m^{\beta_m}$$

Consider now the following homogeneous polynomial $x^*(\alpha)$ of degree d:

$$x^*(\alpha) = \sum_{0\le\beta_1+\cdots+\beta_m\le d,\ \beta_i\ge 0} c_{\beta_1,\ldots,\beta_m}\alpha_1^{\beta_1}\ldots\alpha_m^{\beta_m}\Big(\sum_{i=1}^{m}\alpha_i\Big)^{d-\sum_i\beta_i}$$

Clearly, $x^*(\alpha)$ coincides with $x^{**}(\alpha)$ in Δ_m. As a conclusion, the homogeneous polynomial $x^*(\alpha)$ is a solution to the parameter-dependent LMI given in (1) for any value of $\alpha \in \Delta_m$. $\blacksquare$

Remark 1: Note that Theorem 1 does not restrict the coefficients $G_0(\cdot), \ldots, G_p(\cdot)$ to be homogeneous neither polynomials in the parameters; continuity is enough.

Remark 2: Unfortunately, there are no available results concerning the existence of a bound to the degree d of the polynomial solution of the general parameter-dependent LMI given by (1), which seems to be a very complicated problem. Some bounds could be indirectly established for the particular case of LMI conditions certifying the robust stability of a linear uncertain system, since in this case equivalent parameter-dependent Lyapunov equations can be obtained. Following these ideas, the maximum degrees of the polynomial parameter-dependent Lyapunov matrices needed to assess the robust stability of an uncertain continuous-time system with n states belonging to a polytope with N vertices was estimated as $n(n+1)/2$ in [17] or $2nN$ in [22].

Some direct consequences of Theorem 1 are now stated (Corollaries 2 to 4).

Corollary 2: Assume that $G_0(\cdot), G_1(\cdot), \ldots, G_p(\cdot)$ are continuous. Let $E(\alpha)$ be a continuous function defined in Δ_m and assuming values in the set of positive definite symmetric matrices of dimension $\mathbb{R}^{n\times n}$. If for all $\alpha \in \Delta_m$ there exists $x(\alpha) \in \mathbb{R}^p$ such that $G(x(\alpha),\alpha) \ge 0_n$, then there exists a homogeneous polynomial function $x^* : \Delta_m \to \mathbb{R}^p$ such that, for all $\alpha \in \Delta_m$, $G(x^*(\alpha),\alpha) > -E(\alpha)$.

Moreover, if there exists a polynomial function $x^{**}(\alpha)$ of degree d such that, for all $\alpha \in \Delta_m$, $G(x^{**}(\alpha),\alpha) > -E(\alpha)$, then there exists a homogeneous polynomial function of degree d with the same property. $\square$

Another immediate consequence of Theorem is the following result, which provides information about the existence of solutions in the level sets that are defined by $G(x,\alpha)$.

Corollary 3: Assume that $G_0(\cdot), G_1(\cdot), \ldots, G_p(\cdot)$ are continuous. Let $\underline{E}(\alpha), \overline{E}(\alpha)$ be continuous functions defined in Δ_m, assuming values in the set of symmetric matrices of dimension $\mathbb{R}^{n\times n}$. If for all $\alpha \in \Delta_m$ there exists $x(\alpha) \in \mathbb{R}^p$ such that $\overline{E}(\alpha) > G(x(\alpha),\alpha) > \underline{E}(\alpha)$, then there exists a homogeneous polynomial function $x^* : \Delta_m \to \mathbb{R}^p$ such that, for all $\alpha \in \Delta_m$, $\overline{E}(\alpha) > G(x^*(\alpha),\alpha) > \underline{E}(\alpha)$.

Moreover, if there exists a polynomial function $x^{**}(\alpha)$ of degree d such that, for all $\alpha \in \Delta_m$, $\overline{E}(\alpha) > G(x^{**}(\alpha),\alpha) > \underline{E}(\alpha)$, then there exists a homogeneous polynomial function of degree d with the same property. $\square$

Consider the functions $c_0(\cdot), c_1(\cdot), \ldots, c_p(\cdot)$ defined in Δ_m with real values, and consider the question of determining the worst case value, with respect to α, of the affine objective function given by

$$c(x,\alpha) \triangleq c_0(\alpha) + x_1c_1(\alpha) + \cdots + x_pc_p(\alpha)$$

under the LMI constraint $G(x,\alpha) > 0_n$. For that, the following constants are defined:

$$\gamma_\infty \triangleq \sup_{\alpha\in\Delta_m} \inf\{c(x,\alpha) \ : \ x \in \mathbb{R}^p,\ G(x,\alpha) > 0_n\}$$

$$\gamma_d \triangleq \sup_{\alpha\in\Delta_m} \inf\{c(x^*(\alpha),\alpha) \ : \ x^* \text{ is a polynomial of degree less or equal to } d \text{ and } \forall\ \alpha \in \Delta_m, G(x^*(\alpha),\alpha) > 0_n\}$$

$$\tilde{\gamma}_d \triangleq \sup_{\alpha\in\Delta_m} \inf\{c(x^*(\alpha),\alpha) \ : \ x^* \text{ is a homogeneous polynomial of degree } d, \text{ and } \forall\ \alpha \in \Delta_m, G(x^*(\alpha),\alpha) > 0_n\}$$

The constants γ_d and $\tilde{\gamma}_d$ are defined for all nonnegative integer d and obviously satisfy the following inequalities for all d:

$$\gamma_\infty \le \gamma_{d+1} \le \gamma_d \quad \text{ and } \quad \gamma_d \le \tilde{\gamma}_d$$

Indeed, more can be said on these constants, as stated in the sequel.

Corollary 4: Assume that $G_0(\cdot)$, $G_1(\cdot)$, $\ldots$, $G_p(\cdot)$ and $c_0(\cdot)$, $c_1(\cdot)$, $\ldots$, $c_p(\cdot)$ are continuous. Then the sequence $\tilde{\gamma}_d$ is decreasing, with limit $\tilde{\gamma}_d$ when $d \to +\infty$.

Moreover, the sequences $\tilde{\gamma}_d$ and γ_d are equal. $\square$

The proofs of Corollaries 2, 3 and 4 are not presented since, as the proof of Theorem 1 is based on the results of [21], which assures the existence of polynomial solutions, these results can be extended to the homogeneous case using the same arguments.

The second part of Theorem 1 and of Corollaries 2, 3 and 4 shows that the conservativeness of the linear matrix inequality is not reduced when general polynomials are considered instead of homogeneous polynomials. This remark is especially important in the applications, since it allows to considerably reduce the number of scalar variables, and consequently, the computational burden.

Suppose that the optimal homogeneous polynomial function of degree d was determined, yielding a cost with value $\tilde{\gamma}_d$. Then the homogeneous polynomial of degree $(d+1)$ given by $x_{d+1}(\alpha) \triangleq (\alpha_1 + \cdots + \alpha_m) x_d(\alpha)$ provides a cost equal to $\tilde{\gamma}_d$. This new function can be used for initialization during the evaluation of $\tilde{\gamma}_{d+1}$, saving computational time as illustrated in Example 1, Section IV.

III. Applications

Before presenting some applications, it is worth to mention that the main rationale for using homogeneous polynomials instead of general polynomials relies on the fact that homogeneous polynomials, as well as general ones, are dense in the Banach space of continuous functions defined in the compact set Δ_m. Similar argument could be used in situations where more appropriate decomposition bases exist, e.g. trigonometric functions for periodic problems.

A. Static State Feedback Stabilization

Consider the linear time invariant uncertain system

$$\dot{x} = A(\alpha)x + B(\alpha)u \tag{2}$$

with matrices $A(\alpha) \in \mathbb{R}^{n\times n}$ and $B(\alpha) \in \mathbb{R}^{n\times p}$ depending continuously on the parameter $\alpha \in \Delta_m$. One is interested here in the study of robust stabilizability of (2), i.e. the existence, for any $\alpha \in \Delta_m$, of a gain $L(\alpha)$ such that $A(\alpha) + B(\alpha)L(\alpha)$ is Hurwitz.

Theorem 5: The system (2) is stabilizable by static state feedback $\forall\ \alpha \in \Delta_m$ if and only if there exists a homogeneous polynomial stabilizing gain $L(\alpha) \in \mathbb{R}^{p\times n}$, to be determined, such that $A(\alpha) + B(\alpha)L(\alpha)$ is stable $\forall\ \alpha \in \Delta_m$. □

The homogeneous form used in Theorem 5 is not the best one for effective determination of a stabilizing gain, but it is notably simple. This theorem establishes that the class of homogeneous polynomial gains is absolutely general for stabilization, and this is the main interest of this formulation.

Proof: [Proof of Theorem 5] The stability of system (2) is equivalent to the following condition:

$$\begin{aligned}\forall \alpha \in \Delta_m, \exists\ L(\alpha) \in \mathbb{R}^{p\times n}, \exists P(\alpha) = P(\alpha)^T > 0 :\\ (A(\alpha) + B(\alpha)L(\alpha))P(\alpha)\\ + P(\alpha)(A(\alpha) + B(\alpha)L(\alpha))^T < 0\end{aligned} \tag{3}$$

Inequality (3) is nonlinear in the variables $L(\alpha)$ and $P(\alpha)$. Using the usual linearizing method [23], (3) can be written as

$$\begin{aligned}\forall \alpha \in \Delta_m, \exists M(\alpha) \in \mathbb{R}^{p\times n}, \exists P(\alpha) = P(\alpha)^T > 0 :\\ A(\alpha)P(\alpha) + B(\alpha)M(\alpha)\\ + P(\alpha)A(\alpha)^T + M(\alpha)^T B(\alpha)^T < 0\end{aligned} \tag{4}$$

For all values of $\alpha \in \Delta_m$, problem (4) presents a parameter-dependent LMI in $M(\alpha)$ and $P(\alpha)$, with continuous coefficients in terms of the vector of parameters α. Thus, Theorem 1 can be used to deduce that (4) is fulfilled if and only if there exist homogeneous polynomials $M^*(\alpha)$ and $P^*(\alpha)$ such that

$$\begin{aligned}\forall \alpha \in \Delta_m,\ P^*(\alpha) = P^*(\alpha)^T > 0 \quad \text{and}\\ A(\alpha)^T P^*(\alpha) + B(\alpha)M^*(\alpha)\\ + P^*(\alpha)A(\alpha) + M^*(\alpha)^T B(\alpha)^T < 0\end{aligned} \tag{5}$$

Notice that the corresponding gain $L^*(\alpha) \triangleq M^*(\alpha)P^*(\alpha)^{-1}$ is not polynomial in α.

Consider now the condition:

$$\begin{aligned}\forall \alpha \in \Delta_m, \exists L(\alpha) \in \mathbb{R}^{p\times n} : (A(\alpha) + B(\alpha)L(\alpha))P(\alpha)^*\\ + P(\alpha)^*(A(\alpha) + B(\alpha)L(\alpha))^T < 0\end{aligned} \tag{6}$$

where $P(\alpha)^*$ is obtained from the solution of (5) which has already been found. For all values of α, problem (6) is a parameter-dependent LMI (in the variable $L(\alpha)$), and the coefficients of the inequality are continuous in terms of α, since $A(\alpha)$, $B(\alpha)$ and the polynomial $P^*(\alpha)$ are also continuous. In this case, the problem given in (6) has a solution if and only if there exists a homogeneous polynomial solution $(M^*(\alpha), P^*(\alpha))$ for the problem given in (5).

As a consequence, from Theorem 1 one has that the initial property (stabilizability of system (2)) is equivalent to the existence of a homogeneous polynomial $L^*(\alpha)$, such that

$$\begin{aligned}(A(\alpha) + B(\alpha)L^*(\alpha))P(\alpha)^*\\ + P(\alpha)^*(A(\alpha) + B(\alpha)L^*(\alpha))^T < 0\end{aligned}$$

for all $\alpha \in \Delta_m$. The gain $L^*(\alpha)$ provides the desired answer, concluding the proof of Theorem 5. ■

Notice that there do exist cases where system (2) is robustly stabilizable and where robust stabilizing gains (i.e. degree zero) or affine parameter-dependent gains (degree one) do not exist: polynomially parameter-dependent gains with larger degree are mandatory in such situations.

B. LMIs with Slowly Time-Varying Parameters

The existence of solutions to LMIs with slowly time-varying parameters is an important issue in the context of control of linear systems with time-varying uncertain parameters. This problem was tackled in the references [24–26] in a particular context.

When the evolution of $\alpha(t)$ in the set Δ_m is sufficiently regular in terms of t, its time derivative $\dot{\alpha}(t)$ evolves in the set Δ'_m defined by

$$\Delta'_m \triangleq \left\{ \alpha' \in \mathbb{R}^m\ :\ \sum_{i=1}^{m} \alpha'_i = 0 \right\}$$

For all values of the constant $\varepsilon \geq 0$ or $\varepsilon = \infty$, the following function set is introduced:

$$W_\varepsilon \triangleq \big\{ \alpha(\cdot) \in W^{1,\infty}(\mathbb{R},\mathbb{R}^m) \ : \ \forall t \in \mathbb{R},\ \alpha(t) \in \Delta_m \text{ and } \dot{\alpha}(t) \in \Delta'_m \cap B(0_m,\varepsilon) \ t\text{-almost surely} \ \big\} \quad (7)$$

where $W^{1,\infty}$ represents the usual Sobolev space and $B(0_m,\varepsilon)$ represents the ball of $\mathbb{R}^m$ of diameter 2ε centered in the origin.

Consider the following function:

$$H(x,\alpha,\alpha') \triangleq H_0(\alpha,\alpha') + x_1 H_1(\alpha,\alpha') + \cdots + x_p H_p(\alpha,\alpha')$$

where $H_0(\cdot), H_1(\cdot), \ldots, H_p(\cdot)$ are functions defined in $\Delta_m \times \Delta'_m$, with values in the set of symmetric matrices of dimension $\mathbb{R}^{n\times n}$.

A result of existence of solutions is presented in the following theorem.

Theorem 6: Assume that $H_0(\cdot), H_1(\cdot), \ldots, H_p(\cdot)$ are continuous. If for all value of $\alpha \in \Delta_m$ there exists $x(\alpha) \in \mathbb{R}^p$ such that $H(x(\alpha),\alpha,0_p) > 0_n$, then there exists a constant $\varepsilon > 0$ and a homogeneous polynomial function $x^*(\alpha) : \Delta_m \to \mathbb{R}^p$ such that, for all function $\alpha(\cdot) \in W_\varepsilon$, the inequality $H(x^*(\alpha(t)),\alpha(t),\dot{\alpha}(t)) > 0_n$ is t-almost surely verified in $\mathbb{R}$. □

Proof: From Theorem 1, there exists a homogeneous polynomial function $x^*(\alpha) : \Delta_m \to \mathbb{R}^p$ such that, for all $\alpha \in \Delta_m$, $H(x^*(\alpha),\alpha,0_p) > 0_n$. By continuity of H, the function $(\alpha,\alpha') \mapsto H(x^*(\alpha),\alpha,\alpha')$ is uniformly continuous in any compact set of the form $\Delta_m \times (\Delta'_m \cap B(0_m,\varepsilon))$. Therefore, there exists an $\varepsilon > 0$ such that, for all $\alpha \in \Delta_m$ and for all $\alpha' \in \Delta'_m \cap B(0_m,\varepsilon)$, $H(x^*(\alpha),\alpha,\alpha') > 0_n$. This value of ε is such that, when $\alpha(\cdot) \in W_\varepsilon$, one has $H(x^*(\alpha(t)),\alpha(t),\dot{\alpha}(t)) > 0_n$ t-almost surely in $\mathbb{R}$. ■

Notice that it is possible to choose the norm in $\mathbb{R}^m$ such that the obtained convex set $\Delta'_m \cap B(0_m,\varepsilon)$ be a polytope.

In the context of control of uncertain linear systems with time-varying parameters, the functions $H_0(\cdot),\ldots,H_p(\cdot)$ are, typically, polynomials of degree one in α', coming from affine expressions in the time derivative $\dot{\alpha}(t)$. It is important to remark that, if there exists a polynomial solution to the previous problem, it cannot be independent from the value of ε, as stated now.

Theorem 7: Assume that the functions $H_0(\cdot), \ldots, H_p(\cdot)$ are polynomials of degree one in terms of α' and continuous in terms of α. If there exists a polynomial function $x^*(\alpha) : \Delta_m \to \mathbb{R}^p$ such that, for all function $\alpha(\cdot) \in W_\infty$, the inequality $H(x^*(\alpha(t)),\alpha(t),\dot{\alpha}(t)) > 0_n$ is t-almost surely verified in $\mathbb{R}$, then, for all integer $i = 0,\ldots,p$, there exist a continuous function $\tilde{H}_i$, such that:

$$\forall (\alpha,\alpha') \in \Delta_m \times \Delta'_m,\ H_i(\alpha,\alpha') = \tilde{H}_i(\alpha) \ .$$

□

Proof: Considering larger values for the entries of α', it can be deduced from the affine dependence of H in the variable α' that the diagonal coefficients, and also the other coefficients, are indeed independent of α' in Δ'_m. ■

Therefore, when the hypotheses of Theorem 7 are satisfied, either the homogeneous polynomial solution x^* in Theorem 6 exists for all positive ε, or the solution exists only for small ε. This statement is illustrated by means of examples, presented in the next section (Examples 2 and 3).

IV. Examples

Example 1

Consider the linear time invariant uncertain system described by the following state space representation

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + B(\alpha)w(t) \\ y(t) &= C(\alpha)x(t) + D(\alpha)w(t) \end{aligned} \quad (8)$$

with $x \in \mathbb{R}^n$, $w \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times p}$.

Suppose that $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ and $D(\alpha)$ are not precisely known, but belong to a polytopic domain $\mathcal{P}$. In this case, any quadruple $(A(\alpha),B(\alpha),C(\alpha),D(\alpha))$ inside the uncertain domain $\mathcal{P}$ can be written as a convex combination of the vertices of the polytope (A_i,B_i,C_i,D_i), $i = 1,\ldots,m$, that is, $(A,B,C,D)(\alpha) \in \mathcal{P}$ with

$$\mathcal{P} \triangleq \Big\{ (A,B,C,D)(\alpha) \ : \ (A,B,C,D)(\alpha) = \sum_{i=1}^{m} \alpha_i (A_i,B_i,C_i,D_i); \alpha \in \Delta_m \Big\} \quad (9)$$

For a fixed α, the transfer matrix from the input w to the output y is given by

$$T_\alpha(s) = C(\alpha)\left(s\mathbf{I} - A(\alpha)\right)^{-1} B(\alpha) + D(\alpha) \quad (10)$$

The $\mathcal{H}_\infty$ norm of (10) can be evaluated through the *bounded real lemma* [27], i.e. $\|T_\alpha(s)\|_\infty < \gamma$ if and only if there exists $P(\alpha) = P(\alpha)^T > 0$ such that ($\star$ denotes symmetric blocks)

$$\begin{bmatrix} A(\alpha)^T P(\alpha) + P(\alpha)A(\alpha) + \gamma^{-2} C(\alpha)^T C(\alpha) & P(\alpha)B(\alpha) + \gamma^{-2} C(\alpha)^T D(\alpha) \\ \star & \gamma^{-2} D(\alpha)^T D(\alpha) - \mathbf{I} \end{bmatrix} < 0 \quad (11)$$

The aim is to determine an upper bound to the $\mathcal{H}_\infty$ norm for any convex combination of the vector of parameters α, i.e. a guaranteed cost γ such that

$$\gamma \geq \|T_\alpha(s)\|_\infty, \ \forall \ (A,B,C,D)(\alpha) \in \mathcal{P} \quad (12)$$

The optimal $\mathcal{H}_\infty$ guaranteed cost is given by

$$\gamma_\infty = \min_{(12)\text{ holds}} \gamma = \max_{\alpha \in \Delta_m} \|T_\alpha(s)\|_\infty \quad (13)$$

As the constraints $P(\alpha) > 0$ and (11) are parameter-dependent LMIs whose parameters lie in the unit simplex, Corollary 4 shows that the structure for the desired solution $P(\alpha)$ can be constrained to the class of homogeneous polynomials solutions of arbitrary degree on the parameters. Suppose that, initially, an affine parameter-dependent Lyapunov matrix $P(\alpha)$ is used, that is, $P(\alpha) = P_1(\alpha)$, and a guaranteed cost $\tilde{\gamma}_1$ is obtained. If a homogeneous polynomially parameter-dependent Lyapunov matrix of degree two is used in the sequel, from Corollary 4 one has that the new evaluated guaranteed cost $\tilde{\gamma}_2$ is not larger than $\tilde{\gamma}_1$, tending asymptotically monotonically to the optimal guaranteed cost γ_∞, that is, the $\mathcal{H}_\infty$ worst case norm, as the degree of the Lyapunov matrix grows. From the numerical point of view, matrix $(\alpha_1+\cdots+\alpha_m)P_1(\alpha)$ can be used to initialize the search of $P_2(\alpha)$ with initial guaranteed cost $\tilde{\gamma}_1$, saving computational burden. The same idea can be used for larger degrees.

A numerical experiment has been performed in a Pentium IV 2.6 GHz, 512 MB RAM, using the LMI routines from Robust Control Toolbox of Matlab [28]. Consider the following continuous-time uncertain system with $m=3$ vertices given by

$$A_1 = \begin{bmatrix} -1.1 & -0.6 & 0.1 & 0.9 \\ 0.2 & -0.2 & -0.5 & -0.2 \\ -0.4 & 0.2 & -1.2 & 0.4 \\ -0.4 & 0.9 & 0.2 & -0.2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.7 & -0.4 & -0.4 & 0.8 \\ -0.5 & -1.5 & 0.8 & 0.7 \\ -0.8 & -0.4 & -0.9 & 0.0 \\ -0.7 & -0.6 & 0.6 & 0.1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1.0 & -0.9 & -0.1 & 0.4 \\ -0.6 & -0.8 & -0.7 & -0.8 \\ 0.7 & 0.5 & -1.0 & 0.5 \\ -0.5 & 0.2 & 0.3 & -0.8 \end{bmatrix}$$

$$B_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \quad C_i = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}; \quad D_i = 0; \quad i = 1,\ldots,3$$

whose worst-case $\mathcal{H}_\infty$ norm, computed through exhaustive gridding, is $\gamma_\infty = 2.306$.

The aim here is to compute $\mathcal{H}_\infty$ guaranteed costs as close as possible to γ_∞ through the solution of the parameter-dependent LMI given by (11). Using the method proposed in [29, Theorem 1], where the solution $P(\alpha)$ is obtained by means of LMI relaxations providing homogeneous polynomially parameter-dependent Lyapunov matrices $P_d(\alpha)$ of arbitrary degree d, two different strategies are compared: the first one uses the LMI solver without providing a feasible initial condition while the second one uses the initial condition constructed from the previous step (except for $d=1$), as discussed in the comments of Corollary 4.

Table I shows the elapsed time required by the two above mentioned strategies to obtain a homogeneous polynomially parameter-dependent Lyapunov matrix $P_d(\alpha)$ solving (11) for $d=1,\ldots,9$ (no feasible solution has been found for $d=0$).

It can be noted that the use of a feasible initial condition can greatly reduce the computational burden. Using the initial condition, the total time required to obtain a homogeneous polynomially parameter-dependent Lyapunov solution $P_d(\alpha)$ of degree $d=9$ (less than 0.1% close to the worst case $\mathcal{H}_\infty$ norm) has been reduced by 72% (from 35.7 to 10.08). Even if weaker precision is required, for instance, less than 1%, the computational time to construct $P_4(\alpha)$ has been reduced by 23% (from 1.86 to 1.43). Note that, if a strategy of incrementing the degree of the polynomial one-by-one was chosen, the advantage of using the initial conditions would be even more apparent. The use of a feasible initial condition could also improve the performance of other similar methods to obtain homogeneous polynomially parameter-dependent solutions to parameter-dependent LMIs.

TABLE I

$\mathcal{H}_\infty$ GUARANTEED COST ESTIMATES $\tilde{\gamma}_d$ USING [29, THEOREM 1] FOR $d=1,\ldots,9$ WITH (LABELED IC) AND WITHOUT INITIAL CONDITION (NO FEASIBLE SOLUTION HAS BEEN FOUND FOR $d=0$). THE COMPUTATIONAL TIMES ARE GIVEN IN SECONDS. THE $\mathcal{H}_\infty$ WORST CASE NORM IS $\gamma_\infty = 2.306$.

d	$\tilde{\gamma}_d$	Time	Time (IC)	Cumulated time (IC)
0	–	–	–	–
1	2.619	0.13	0.13	0.13
2	2.409	0.36	0.30	0.43
3	2.336	0.78	0.39	0.82
4	2.320	1.86	0.61	1.43
5	2.315	3.61	0.70	2.13
6	2.312	6.03	1.05	3.18
7	2.311	9.98	1.41	4.59
8	2.309	19.8	2.16	6.75
9	2.308	35.7	3.33	10.08

Example 2

An example of LMI with time-varying parameters that satisfy the assumptions from Theorem 7 and has a polynomial solution for all positive ε is presented here. Consider matrix

$$H(x,\alpha,\alpha') \triangleq \begin{pmatrix} x+\alpha_1' & 2\alpha_2+\alpha_1^2 \\ 2\alpha_2+\alpha_1^2 & 1 \end{pmatrix} .$$

Then, $H(x,\alpha,\alpha') > 0_2 \Leftrightarrow x > (2\alpha_2+\alpha_1^2)^2 - \alpha_1'$ and, for all $\varepsilon \geq 0$,

$$\forall(\alpha,\alpha') \in \Delta_2 \times (\Delta_2' \cap B(0_2,\varepsilon)),\ H(x^*(\alpha),\alpha,\alpha') > 0_2\ , \quad (14)$$

where x^* is a homogeneous polynomial of degree 4, given by:

$$x^*(\alpha) \triangleq (2\alpha_2+\alpha_1^2)^2+\varepsilon+1 = \left(2\alpha_2(\alpha_1+\alpha_2)+\alpha_1^2\right)^2+(\varepsilon+1)(\alpha_1+\alpha_2)^4$$

Example 3

In contrast with Example 2, this third example presents an LMI with time-varying parameters satisfying the assumptions from Theorem 7, which however admits a polynomial solution only for small values of ε. Consider matrix

$$H(x,\alpha,\alpha') \triangleq \begin{pmatrix} x+\alpha_1'+3 & 2\alpha_2+\alpha_1^2 \\ 2\alpha_2+\alpha_1^2 & -x-\alpha_1'+3 \end{pmatrix}$$

In this case,

$$H(x,\alpha,\alpha') > 0_2 \Leftrightarrow |x+\alpha_1'| < \sqrt{9-(2\alpha_2+\alpha_1^2)^2}$$

As $\max\{(2\alpha_2+\alpha_1^2)^2\ : \alpha \in \Delta_2\} = 4$, property (14) cannot be verified if $\varepsilon \geq \sqrt{5}$.

V. CONCLUSION

General solutions of parameter-dependent LMIs whose parameters lie in the unit simplex have been characterized by means of homogeneous polynomial solutions. This result has important applications from the numerical and theoretical point of views in many control problems like synthesis of parameter-dependent controllers (gain-scheduling) and analysis of linear uncertain systems with slowly time-varying parameters.

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