LMI Characterisation of Robust Stability for Time-Delay Systems: Singular Perturbation Approach

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Abstract— **We study here robust stability of linear systems with several uncertain incommensurate delays, more precisely delay-dependent stability. The main result of this paper consists in establishing that this property is *equivalent* to the feasibility of some Linear Matrix Inequality (LMI), a convex optimization problem.**

The method is based on two main ideas:

- **use of *Padé approximation* to transform the system into some singularly perturbed finite-dimensional system, for which robust dichotomy has to be checked;**
- **recursive applications of *Generalized Kalman-Yakubovich-Popov (KYP) lemma* to characterize by an LMI the previous property.**

I. INTRODUCTION

The analysis of linear time-delay systems has attracted much interest, and much work has been done on that subject, see e.g. [9], [17], [7]. However, surprisingly simple questions have not been totally solved so far (that is, by methods both precise and numerically tractable), including ones related to stability analysis, as testified by the large number of papers on the subject published monthly in the journals.

Many delay-independent and delay-dependent stability conditions have been formulated by frequency domain techniques. The latter include polynomial criteria [14], [15], matrix pencil techniques [4], [16], integral quadratic constraints [8], [13] and other, see references in [18]. Analysis in the time domain uses in general Lyapunov-Krasovskii functionals or Lyapunov-Razumikhin functions, see [17], [7]. The latter results are usually expressed as solvability condition of some LMI problems, a class of convex problems solvable by efficient numerical methods. To date, a class of LMIs has been proved to characterize delay-independent stability [1], [2], based on the search for common Lyapunov-Krasovskii functional.

To the best of our knowledge, no LMI characterisation (that is both necessary and sufficient condition) of delay-dependent stability has been given until now. This issue is the subject of the present contribution. We treat here the fully general multi-delay case. The results given here are primarily intended to provide effective method of testing delay-dependent stability for linear systems with several delays. They also prepare for future systematic LMI treatment of more difficult issues, namely performance analysis (synthesis seems to need much more work). Last, from an abstract point of view, they enlarge the domain of application of the LMIs and show the generality of this class of problems, and specifically the powerfulness of the Generalized KYP lemma [11], [12].

The paper is organised as follows. We first put some notations and technical preliminaries (including adequate statement of GKYP lemma) in Section II. We then state and demonstrate in Section III an original, exact, LMI characterisation of robust stability for singularly perturbed systems (Theorem 1). This result is then used in Section IV, together with an idea of Padé's approximation borrowed from [18], to obtain a necessary and sufficient condition for delay-dependent stability (Theorem 4). The latter applies to the linear systems with several incommensurate delays. Section V concludes the paper.

II. NOTATIONS AND TECHNICAL PRELIMINARIES

We gather in this section the notations used throughout the paper, together with needed technical tools.

A. Matrices and Representation of Matrix-Valued Polynomials

The sets of positive integers, resp. real numbers, resp. complex numbers, are denoted as usual $\mathbb{N}$, resp. $\mathbb{R}$, resp. $\mathbb{C}$. Let $\mathscr{H}^n$ be the set of $n\times n$ Hermitian matrices. For square matrices N, we let $\mathrm{He}\{N\} \doteq N+N^*$. Kronecker product is denoted $\otimes$, and: $M^{0\otimes}=1$, $M^{p\otimes} \doteq M^{(p-1)\otimes}\otimes M$.

We now fix some definitions, related to the representation of matrix-valued polynomials. Let us define, for all positive $k\in\mathbb{N}$, the matrices $\hat{J}_k, \check{J}_k \in \mathbb{R}^{k\times(k+1)}$ by:

$$\hat{J}_k \doteq \begin{pmatrix} I_k & 0_{k\times 1}\end{pmatrix}, \quad \check{J}_k \doteq \begin{pmatrix} 0_{k\times 1} & I_k\end{pmatrix} \ . \tag{1}$$

We also define, for all positive integers k,l,

$$\hat{J}_{l,k} \doteq \hat{J}_l\hat{J}_{l+1}\dots\hat{J}_{l+k-1} = \begin{pmatrix} I_l & 0_{l\times k}\end{pmatrix}, \tag{2a}$$

$$\check{J}_{l,k} \doteq \check{J}_l\check{J}_{l+1}\dots\check{J}_{l+k-1} = \begin{pmatrix} 0_{l\times k} & I_l\end{pmatrix} \ . \tag{2b}$$

By convention, we let $\hat{J}_{l,0}=\check{J}_{l,0}\doteq I_l$. With these definitions, $\hat{J}_{l,k},\check{J}_{l,k}\in\mathbb{R}^{l\times(l+k)}$. Notice that

$$\hat{J}_{l,k}\otimes I_n = \hat{J}_{ln,kn}, \quad \check{J}_{l,k}\otimes I_n = \check{J}_{ln,kn} \ .$$

In particular, $\hat{J}_k\otimes I_n = \hat{J}_{k,1}\otimes I_n = \hat{J}_{kn,n}$ and $\check{J}_k\otimes I_n=\check{J}_{kn,n}$.

For any positive integer k and any $s\in\mathbb{C}$, we define the vector $s^{[k]}\in\mathbb{R}^k$ of *monomials* in s as follows:

$$s^{[k]} \doteq \begin{pmatrix} 1 \\ s \\ \vdots \\ s^{k-1}\end{pmatrix} \ .$$

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The matrices $\hat{J}_k, \check{J}_k$ defined in (1) are such that

$$s^{[l]} = \hat{J}_{l,k} s^{[l+k]}, \quad s^k s^{[l]} = \check{J}_{l,k} s^{[l+k]} \ .$$

One shows directly that, for any matrix $M \in \mathbb{C}^{p\times q}$, for any $u \in \mathbb{C}$,

$$(u^{[k]} \otimes I_p) M = (I_k \otimes M)(u^{[k]} \otimes I_q) \ . \tag{3}$$

Last, we have to define maps $\Phi_{k,n} : \mathbb{R}^{n\times kn} \to \mathbb{R}^{n\times kn}$ such that, for any $M \in \mathbb{R}^{n\times kn}$ and any $h \in \mathbb{R}$, the following equality on matrix-valued polynomials holds:

$$\mathrm{He}\{M(h^{[k]} \otimes I_n)\} \equiv \Phi_{k,n}(M)(h^{[k]} \otimes I_n) \ .$$

The matrix $\Phi_{k,n}(M)$ is just obtained from M by block by block symmetrisation (symmetrisation of the coefficients). In other words, for any $i = 1, \ldots, k$, it is given analytically by:

$$\Phi_{k,n}(M) \begin{pmatrix} 0_{(i-1)n\times n} \\ I_n \\ 0_{(k-i)n\times n} \end{pmatrix} \doteq \mathrm{He}\left\{ M \begin{pmatrix} 0_{(i-1)n\times n} \\ I_n \\ 0_{(k-i)n\times n} \end{pmatrix} \right\} \ . \tag{4}$$

In particular, $\Phi_{1,n}(M) = \mathrm{He}\{M\}$ for $M \in \mathbb{R}^{n\times n}$. A useful identity for the sequel is as follows: for any $k, l, n \in \mathbb{N}$, for any $M \in \mathbb{R}^{n\times kln}$, for any $h \in \mathbb{R}$,

$$\Phi_{k,n}(M(h^{[l]} \otimes I_{kn})) = \Phi_{kl,n}(M)(h^{[l]} \otimes I_{kn}) \ . \tag{5}$$

B. Generalized KYP Lemma and Consequences

We now recall the generalized KYP lemma, as found in Theorem 2 of [11]. For $G \in \mathbb{C}^{n\times m}$ and $\Pi \in \mathscr{H}^{n+m}$, a function $\sigma : \mathbb{C}^{n\times m} \times \mathscr{H}^{n+m} \to \mathscr{H}^m$ is defined by

$$\sigma(G, \Pi) \doteq \begin{pmatrix} G \\ I_m \end{pmatrix}^* \Pi \begin{pmatrix} G \\ I_m \end{pmatrix} \ . \tag{6}$$

For given matrices $\Phi, \Psi \in \mathscr{H}^2$, define

$$\Lambda \doteq \{ \ \lambda \in \mathbb{C} \mid \sigma(\lambda, \Phi) = 0, \ \sigma(\lambda, \Psi) \geq 0 \ \}. \tag{7}$$

We assume $\infty \in \Lambda$ if Λ is unbounded. Here, σ has to be taken as in the definition above, with $m = n = 1$. By an appropriate choice of Φ and Ψ, the set Λ can be made to represent a certain curve on the complex plane. Let

$$\Gamma_\lambda \doteq \begin{cases} \begin{pmatrix} I_m & -\lambda I_m \end{pmatrix} & \text{if} \quad \lambda \in \mathbb{C}, \\ \begin{pmatrix} 0_m & -I_m \end{pmatrix} & \text{if} \quad \lambda = \infty. \end{cases} \tag{8}$$

Lemma 1 (Generalized KYP): Let matrices $\Phi, \Psi \in \mathscr{H}^2$, $F \in \mathbb{C}^{2m\times(m+n)}$ and $\Pi \in \mathscr{H}^{m+n}$ be given and define Λ by (7). Suppose Λ represents curves on the complex plane. Denote by N_λ the null space of $\Gamma_\lambda F$ where Γ_λ is defined in (8). The following statements are equivalent.

(i) $N_\lambda^* \Pi N_\lambda > 0 \quad \forall \ \lambda \in \Lambda$.

(ii) There exist $P, Q \in \mathscr{H}^m$ such that $Q > 0$ and $F^*(\Phi \otimes P + \Psi \otimes Q)F < \Pi$. □

In the sequel of the present section, we let $\overline{\rho}$ be a nonzero real number (with no prescribed sign). The following result is a polynomial version of the GKYP lemma obtained by specializing Lemma 1.

Corollary 1 (Polynomial version of GKYP): Let $\Pi \in \mathscr{H}^{kn}$ and $m \doteq (k-1)n$. Then, $(\rho^{[k]} \otimes I_n)^\intercal \Pi (\rho^{[k]} \otimes I_n) > 0_n$ for all $\rho \in \mathbb{R}$ such that $\rho(\overline{\rho} - \rho) \geq 0$ *if and only if* there exist $P \in \mathbb{C}^{m\times m}$, $P + P^* = 0_m$, $Q \in \mathscr{H}^m$, $Q > 0_m$, such that

$$\begin{pmatrix} \check{J}_{m,n} \\ \hat{J}_{m,n} \end{pmatrix}^\intercal \begin{pmatrix} -Q & P + \frac{\overline{\rho}}{2} Q \\ P^* + \frac{\overline{\rho}}{2} Q & 0_m \end{pmatrix} \begin{pmatrix} \check{J}_{m,n} \\ \hat{J}_{m,n} \end{pmatrix} < \Pi \ .$$

□

Proof: The result follows from Lemma 1 by replacing λ by ρ, choosing

$$F \doteq \begin{pmatrix} \check{J}_{m,n} \\ \hat{J}_{m,n} \end{pmatrix}, \quad \Phi \doteq \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix}, \quad \Psi \doteq \begin{pmatrix} -1 & \frac{\overline{\rho}}{2} \\ \frac{\overline{\rho}}{2} & 0 \end{pmatrix},$$

and noting that $N_\rho \doteq \rho^{[k]} \otimes I_n$, $\Gamma_\rho F N_\rho = (\check{J}_{m,n} - \rho \hat{J}_{m,n}) N_\rho = 0$. ■

We use in the sequel the following variant of Corollary 1.

Corollary 2: Let $\Pi \in \mathbb{R}^{n\times kn}$. Then, $\mathrm{He}\{\Pi(\rho^{[k]} \otimes I_n)\} > 0_n$ for all $\rho \in \mathbb{R}$ such that $\rho(\overline{\rho} - \rho) \geq 0$ *if and only if* there exist $M \in \mathbb{R}^{(k-1)n\times(k-1)n}$ such that the two LMIs (9) (see next page) are fulfilled. □

Proof: Apply Corollary 1 with $Q = M + M^*$, $P = \frac{\overline{\rho}}{2}(M - M^*)$, and replacing $\Pi \in \mathscr{H}^{kn}$ by $\mathrm{He}\left\{ \begin{pmatrix} \Pi \\ 0_{(k-1)n\times kn} \end{pmatrix} \right\}$. ■

III. SINGULARLY PERTURBED SYSTEMS

Consider now, for $h = (h_1, \ldots, h_m) \in \mathbb{R}^m$, the singularly perturbed system

$$E_h \dot{x} = Ax, \quad x \doteq \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_m \end{bmatrix}, \ E_h \doteq \mathrm{diag}\{I_{n_0}; h_1 I_{n_1}; \ldots h_m I_{n_m}\} \ . \tag{10}$$

Here, A and E_h are square matrices of size $n \times n$, where $n \doteq n_0 + n_1 + \cdots + n_m$, for nonnegative integers n_i. Our goal in this section is to characterize robust dichotomy of (10), for any $(h_1, \ldots, h_m) \in [0, \hbar_1] \times \cdots \times [0, \hbar_m]$, where the $\hbar_i$ are fixed nonnegative real numbers. Recall that, by definition, dichotomy is the absence of purely imaginary roots to the characteristic equation.

A. Lyapunov Dichotomy Condition

The following result converts dichotomy condition for singularly perturbed system to a Lyapunov inequality, see also [5], [10].

Lemma 2: Let $A, E \in \mathbb{R}^{n\times n}$ be given and define $r \doteq \mathrm{rank}(E)$. Statements (11) and (12) are equivalent.

$$\lim_{\lambda\to\infty} \det(A - \lambda E)/\lambda^r \neq 0, \tag{11a}$$

$$\det(A - \lambda E) \neq 0, \quad \forall \ \lambda \in j\mathbb{R}. \tag{11b}$$

$$\exists \ S \in \mathbb{R}^{n\times n}, \quad ES = (ES)^\intercal, \quad AS + (AS)^\intercal > 0. \tag{12}$$

□

Proof: If $r = n$, then the result follows from a slight modification of the standard Lyapunov theory, so consider the case $r < n$. Let $U, V \in \mathbb{R}^{n\times n}$ be nonsingular matrices

$$\Psi^{0,0}_{k,n}(M,\Pi,\overline{\rho}) \doteq \mathrm{He}\{M\} > 0_{(k-1)n} \ , \tag{9a}$$

$$\Psi^{1,0}_{k,n}(M,\Pi,\overline{\rho}) \doteq \mathrm{He}\left\{ \check{J}^{\intercal}_{(k-1)n,n} M \left(\check{J}_{(k-1)n,n} - \overline{\rho}\hat{J}_{(k-1)n,n} \right) + \begin{pmatrix} \Pi \\ 0_{(k-1)n\times kn} \end{pmatrix} \right\} > 0_{kn} \ . \tag{9b}$$

$$\Psi^{0,i}_{k,n}(M,\Pi,\hbar) \doteq \Phi_{k^i,(k-1)n}(M) \ , \tag{13a}$$

$$\Psi^{1,i}_{k,n}(M,\Pi,\hbar) \doteq \Phi_{k^i,kn}\left(\check{J}^{\intercal}_{(k-1)n,n} M \left(I_{k^i} \otimes (\check{J}_{(k-1)n,n} - \hbar\hat{J}_{(k-1)n,n}) \right) + \begin{pmatrix} \Pi \\ 0_{(k-1)n\times k^{i+1}n} \end{pmatrix} \right) \ . \tag{13b}$$

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that transform E into a special diagonal form and let A_{ij} be defined accordingly:

$$UEV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad UAV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

Then, condition (11a) holds if and only if $\det(A_{22}) \neq 0$. To see this, note that

$$\lim_{\lambda\to\infty} \frac{\det(A-\lambda E)}{\lambda^r} = \lim_{\lambda\to\infty} \frac{1}{\lambda^r \det(UV)} \ \det \begin{bmatrix} A_{11}-\lambda I_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

which is also equal to

$$\lim_{\lambda\to\infty} \frac{\det(A_{11}-\lambda I_r)}{\lambda^r} \cdot \frac{\det(A_{22}-A_{21}(A_{11}-\lambda I_r)^{-1}A_{12})}{\det(UV)} \ ,$$

that is $(-1)^r \det(A_{22})/\det(UV)$. Hence, in the transformed coordinates, the two statements can be rewritten respectively as (14) and (15) below:

$$\det(A_{22}) \neq 0, \tag{14a}$$

$$\det(A_o - \lambda I_r) \neq 0, \ \forall\ \lambda \in j\mathbb{R}, \quad A_o \doteq A_{11} - A_{12}A_{22}^{-1}A_{21}. \tag{14b}$$

$$\exists\ S_{11}, S_{21}, S_{22},\ S_{11} = S_{11}^{\intercal},\ \mathrm{He}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} > 0. \tag{15}$$

Now, suppose (14) holds. Then there exists $S_o \in \mathbb{R}^{r\times r}$ satisfying

$$S_o = S_o^{\intercal}, \quad A_oS_o + S_oA_o^{\intercal} + A_{12}A_{12}^{\intercal} > 0.$$

It is straightforward to verify that the choice $S_{11} \doteq S_o$, $S_{21} \doteq -A_{22}^{-1}A_{21}S_o$, $S_{22} \doteq -A_{22}^{\intercal}$ satisfies the condition in (15). Thus (14) $\Rightarrow$ (15). To show the converse, suppose (15) holds. First note that the $(2,2)$ block of the inequality in (15) reads

$$A_{22}S_{22} + (A_{22}S_{22})^{\intercal} > 0,$$

implying that $\det(A_{22}) \neq 0$. Noting the symmetry of S_{11}, we see that

$$\mathrm{He}\begin{bmatrix} A_{11}-\lambda I_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix} > 0, \quad \forall\ \lambda \in j\mathbb{R}.$$

This implies

$$\det\begin{bmatrix} A_{11}-\lambda I_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \neq 0, \quad \forall\ \lambda \in j\mathbb{R}$$

which in turn is equivalent to (14b). ■

B. LMI Condition

For $i = 0,\ldots,m$, we let the matrices $K_{k,i} \in \mathbb{R}^{k^m n\times(k+1)^m n}$ be defined by (powers of Kronecker products are used below):

$$K_{k,0} \doteq \hat{J}_k^{\otimes n} \otimes I_n, \tag{16a}$$

$$K_{k,i} \doteq \hat{J}_k^{\otimes(n_0+\cdots+n_{i-1})} \otimes \check{J}_k \otimes \hat{J}_k^{\otimes(n_{i+1}+\cdots+n_m)} \otimes I_n,\ i > 0\ . \tag{16b}$$

In the previous formulas, the indication of $n = n_0 + \cdots + n_m$ is voluntarily omitted, for simplicity. We also let, for any $i = 1,\ldots,m-1$ and $\varepsilon \in \{0,1\}$, the functions

$$\Psi^{\varepsilon,i}_{k,n} : \mathbb{R}^{(k-1)n\times(k-1)k^i n} \times \mathbb{R}^{n\times k^{i+1}n} \times \mathbb{R} \to \mathbb{R}^{(k+\varepsilon-1)n\times(k+\varepsilon-1)k^i n}$$

be defined in (13), where the functions Φ have been defined in (4). Notice that $\Psi^{\varepsilon,i}_{k,n}(M,\Pi,\hbar)$ is *affine* in (M,Π), and that (13) extends the definition of $\Psi^{\varepsilon,0}_{k,n}$ given in (9).

Theorem 1: For all $h \in [0,\hbar_1]\times\cdots\times[0,\hbar_m]$ system (10) is dichotomic, that is there exists S_h such that

$$E_hS_h = (E_hS_h)^{\intercal},\ \mathrm{He}\{AS_h\} > 0_n \ , \tag{17}$$

if and only if there exist a positive integer k, a matrix $S \in \mathbb{R}^{n\times k^m n}$ and $2^m - 1$ matrices indexed by $\varepsilon_i \in \{0,1\}$, $i = 1,\ldots,m-1$:

$$M_\emptyset \in \mathbb{R}^{(k-1)n\times(k-1)k^{m-1}n},$$
$$M_{\varepsilon_1} \in \mathbb{R}^{(k-1)(k+\varepsilon_1-1)n\times(k-1)(k+\varepsilon_1-1)k^{m-2}n},\ldots,$$
$$M_{\varepsilon_{m-1}\ldots\varepsilon_1} \in \mathbb{R}^{(k-1)(k+\varepsilon_1-1)\ldots(k+\varepsilon_{m-1}-1)n\times(k-1)(k+\varepsilon_1-1)\ldots(k+\varepsilon_{m-1}-1)n} \ ,$$

such that, for any $r = 1,\ldots,(k+1)^m$,

$$\left(\sum_{i=0}^{m} \mathrm{diag}\{0_{n_0+\cdots+n_{i-1}}; I_{n_i}; 0_{n_{i+1}+\cdots+n_m}\} S K_{k,i}\right)\begin{pmatrix} 0_{(r-1)n\times n} \\ I_n \\ 0_{((k+1)^m-r)n\times n} \end{pmatrix}$$
$$\text{is symmetric in } \mathbb{R}^{q\times q} \tag{18a}$$

and, $\forall \varepsilon_i \in \{0,1\}$, $i = 1,\ldots,m$,

$$\Pi_{\varepsilon_m\ldots\varepsilon_1} > 0_{(k+\varepsilon_1-1)\ldots(k+\varepsilon_m-1)n} \ , \tag{18b}$$

where $\Pi_{\varepsilon_m\ldots\varepsilon_1}$ is recursively defined by:

$$\begin{aligned} \Pi_\emptyset &\doteq AS, \\ \Pi_{\varepsilon_i\ldots\varepsilon_1} &\doteq \Psi^{\varepsilon_i,m-i}_{k,(k+\varepsilon_1-1)\ldots(k+\varepsilon_{i-1}-1)n}\left(M_{\varepsilon_{i-1}\ldots\varepsilon_1}, \Pi_{\varepsilon_{i-1}\ldots\varepsilon_1}, \hbar_i\right), \\ & \quad i = 1,\ldots,m \ . \end{aligned} \tag{18c}$$

Moreover,

- if (18) is solvable for an integer k, it is also fulfilled for any larger integer;
- if (18) is solvable, then (17) is fulfilled for

$$S_h \doteq S(h_m^{[k]} \otimes \cdots \otimes h_1^{[k]} \otimes I_n) \ . \tag{19}$$

□

For any positive integer k, the system (18) is a system of linear matrix inequalities in the 2^m unknowns. For given k, system (18) is sufficient for robust stability of system (10). The precision of each of these tests increases with k, as well as the complexity of the associated semidefinite program. Last, the inaccuracy vanishes asymptotically, when k goes to infinity: this is the "only if" part of the result.

How to choose the integer k is a natural but difficult question. Indeed, it may be shown that, if the parameter-dependent LMI (17) is fulfilled, then one may replace the right-hand side of the inequality therein by a sum of squares of matrices depending polynomially upon h. It turns out that the integer k is linked to the degree and number of the terms of this sum.

Proof of Theorem 1: Define S_h as in (19). Letting for simplicity $n_{*i} \doteq \sum_{i=0}^{i-1} n_i$, $n_i^* \doteq \sum_{i=i+1}^{m} n_i$, one may write:

$$E_h = \mathrm{diag}\{I_{n_{*m}}; h_m I_{n_m}; I_{n_m^*}\} \times \cdots \times \mathrm{diag}\{I_{n_{*1}}; h_1 I_{n_1}; I_{n_1^*}\} \ .$$

Now, for S_h defined by (19), $E_h S_h$ is equal to

$$\sum_{i=0}^{m} \mathrm{diag}\{0_{n_{*i}}; I_{n_i}; 0_{n_i^*}\} S \left(h_m^{[k]} \otimes \cdots h_{i+1}^{[k]} \otimes h_i h_i^{[k]} \otimes h_{i-1}^{[k]} \otimes \cdots \otimes h_1^{[k]} \otimes I_n \right) \ ,$$

or again:

$$\left(\sum_{i=0}^{m} \mathrm{diag}\{0_{n_{*i}}; I_{n_i}; 0_{n_i^*}\} S K_{k,i} \right) \left(h_m^{[k+1]} \otimes \cdots \otimes h_1^{[k+1]} \otimes I_n \right) \ .$$

It is then clear that (18a) expresses the symmetry of all the coefficients of $E_h S_h$, that is the symmetry of $E_h S_h$ itself.

We now prove that (18b)-(18c) express the inequality condition in (17). Writing

$$S_h = S(h_m^{[k]} \otimes \cdots \otimes h_1^{[k]} \otimes I_n) = S(h_m^{[k]} \otimes \cdots \otimes h_2^{[k]} \otimes I_{kn})(h_1^{[k]} \otimes I_n)$$

and applying Corollary 2 for any value of $(h_2, \ldots, h_m)$ yields: $\forall h \in [0, \hbar_1] \times \cdots \times [0, \hbar_m]$, $\mathrm{He}\{AS_h\} > 0_n$ if and only if, for any $(h_2, \ldots, h_m) \in [0, \hbar_2] \times \cdots \times [0, \hbar_m]$, there exist $M_{h2,\ldots,h_m} \in \mathbb{R}^{(k-1)n \times (k-1)n}$ such that, $\forall (h_2, \ldots, h_m) \in [0, \hbar_2] \times \cdots \times [0, \hbar_m]$,

$$\Psi_{k,n}^{\varepsilon_1, 0} \left(M_{h2,\ldots,h_m}, AS(h_m^{[k]} \otimes \cdots \otimes h_2^{[k]} \otimes I_{kn}), \hbar_1 \right) > 0_{(k+\varepsilon_1-1)n}, \ \varepsilon_1 \in \{0,1\} \ .$$

Now, invoking [3], one may assume without loss of generality that $M_{h2,\ldots,h_m}$ is polynomial with respect to the parameters $(h_2, \ldots, h_m)$ in $[0, \hbar_2] \times \cdots \times [0, \hbar_m]$. Up to an increase of k (which amounts to represent polynomials as degenerate polynomials of higher degree), one may even assume that the degree of the latter is equal to k. In other words, there should exist $M_\emptyset \in \mathbb{R}^{(k-1)n \times k^{m-1}(k-1)n}$ such that, for any $(h_2, \ldots, h_m)$ in $[0, \hbar_2] \times \cdots \times [0, \hbar_m]$,

$$M_{h2,\ldots,h_m} \equiv M_\emptyset (h_m^{[k]} \otimes \cdots \otimes h_2^{[k]} \otimes I_{kn}) \ .$$

Now, one may check easily that the definition (13) of the functions Ψ is such that, for any $M \in \mathbb{R}^{(k-1)n \times (k-1)k^{i+1}n}$, $\Pi \in \mathbb{R}^{n \times k^{i+2}n}$, $\hbar \in \mathbb{R}$, the following fundamental identity holds, whose proof is left to the reader (Hint: use (5) and (3)):

$$\Psi_{k,n}^{\varepsilon,i} \left(M(h^{[k]} \otimes I_{(k-1)k^i n}), \Pi(h^{[k]} \otimes I_{k^{i+1}n}), \hbar \right) = \Psi_{k,n}^{\varepsilon,i+1}(M, \Pi, \hbar)(h^{[k]} \otimes I_{(k+\varepsilon-1)k^i n}) \ . \tag{20}$$

Thus, concentrating on the dependence on h_2 uniquely,

$$\Psi_{k,n}^{\varepsilon_1,0} \left(M_\emptyset (h_m^{[k]} \otimes \cdots \otimes h_2^{[k]} \otimes I_{kn}), AS(h_m^{[k]} \otimes \cdots \otimes h_2^{[k]} \otimes I_{kn}), \hbar_1 \right)$$

is equal, for any $\varepsilon_1 \in \{0,1\}$, to

$$\Psi_{k,n}^{\varepsilon_1,1} \left(M_\emptyset (h_m^{[k]} \otimes \cdots \otimes h_3^{[k]} \otimes I_{k^2 n}), AS(h_m^{[k]} \otimes \cdots \otimes h_3^{[k]} \otimes I_{k^2 n}), \hbar_1 \right) (h_2^{[k]} \otimes I_{(k+\varepsilon_1-1)n}) \ .$$

Eliminating now $h_2 \in [0, \hbar_2]$ shows that the inequality part in (17) is equivalent to the existence of k such that, for any $\varepsilon_1 \in \{0,1\}$, for any $(h_3, \ldots, h_m) \in [0, \hbar_3] \times \cdots \times [0, \hbar_m]$, there exists $M_{\varepsilon_1, h_3, \ldots, h_m}$ such that, for any $\varepsilon_2 \in \{0,1\}$, (21) holds (see next page).

By the argument previously cited, $M_{\varepsilon_1, h_3, \ldots, h_m}$ may be in turn written as a polynomial in $(h_3, \ldots, h_m)$. Proceeding, one eliminates the variables h_i one by one, to finally end up with (18b)-(18c). Thus, the existence, for any $h \in [0, \hbar_1] \times \ldots [0, \hbar_m]$, of S_h fulfilling (17) is equivalent to the existence of a positive integer k such that LMI (18) holds. Incidentally, notice that the number of the unknowns of type M is indeed $2^m - 1$, being the result of $1 + 2 + \cdots + 2^{m-1}$.

The proof of the two remarks at the end of the statement is straightforward. The solvability of (18) for larger integers is obtained by directly constructing new solution from the basic one, corresponding to addition to zero higher degree terms in the underlying polynomials. ■

IV. APPLICATION TO TIME-DELAY SYSTEMS ANALYSIS

A. Problem Formulation

The system studied in the sequel is:

$$\dot{x}(t) = \mathscr{A}_0 x(t) + \sum_{i=1}^{m} \mathscr{A}_i x(t - h_i), \tag{22}$$

for fixed matrices $\mathscr{A}_i \in \mathbb{R}^{n \times n}$, $i = 0, \ldots, m$. Notice that more involved configurations, where multiples or sums of delays are present, may be treated by the same considerations. We would like to determine whether the system is stable for all $h_i \in [0, \hbar_i]$, $\hbar_i \geq 0$ fixed. We assume that the system is stable when $h = 0_m$ i.e.:

Assumption 1: The matrix $\sum_{i=0}^{m} \mathscr{A}_i$ is Hurwitz. □

The following result, borrowed from [18] reduces the delay-dependent stability problem to dichotomy.

$$\Psi^{\varepsilon_2,0}_{k,(k+\varepsilon_1-1)n}\left(M_{\varepsilon_1,h_3,\dots,h_m},\Psi^{\varepsilon_1,1}_{k,n}\left(M_\emptyset(h_m^{[k]}\otimes\cdots\otimes h_3^{[k]}\otimes I_{k^2n}),AS(h_m^{[k]}\otimes\cdots\otimes h_3^{[k]}\otimes I_{k^2n}),\hbar_1\right),\hbar_2\right)>I_{(k+\varepsilon_1-1)(k+\varepsilon_2-1)n}\ . \quad (21)$$

$$A\doteq\begin{pmatrix}\mathscr{A}_0+\sum_{i=1}^m\mathscr{A}_i(I_n\otimes D_q) & \mathscr{A}_1(I_n\otimes C_q) & \mathscr{A}_2(I_n\otimes C_q) & \dots & \mathscr{A}_m(I_n\otimes C_q)\\ (I_n\otimes B_q) & (I_n\otimes A_q) & 0 & \dots & 0\\ (I_n\otimes B_q) & 0 & (I_n\otimes A_q) & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ (I_n\otimes B_q) & 0 & 0 & \dots & (I_n\otimes A_q)\end{pmatrix}, \quad (23)$$

$$A\doteq\begin{pmatrix}\mathscr{A}_0+\sum_{i=1}^m\mathscr{A}_i(I_n\otimes D_q) & \mathscr{A}_1(I_n\otimes C_q) & \mathscr{A}_2(I_n\otimes C_q) & \dots & \mathscr{A}_m(I_n\otimes C_q)\\ \frac{1}{\delta_q}(I_n\otimes B_q) & \frac{1}{\delta_q}(I_n\otimes A_q) & 0 & \dots & 0\\ \frac{1}{\delta_q}(I_n\otimes B_q) & 0 & \frac{1}{\delta_q}(I_n\otimes A_q) & \dots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ \frac{1}{\delta_q}(I_n\otimes B_q) & 0 & 0 & \dots & \frac{1}{\delta_q}(I_n\otimes A_q)\end{pmatrix}. \quad (24)$$

Theorem 2 (see [18, Lemma 2]): Under Assumption 1, system (22) is asymptotically stable for any $(h_1,\dots,h_m)\in[0,\hbar_1]\times\cdots\times[0,\hbar_m]$ *if and only if* $\forall\omega\in\mathbb{R}$, $\forall(h_1,\dots,h_m)\in[0,\hbar_1]\times\cdots\times[0,\hbar_m]$,

$$\det\left(j\omega I_n-\mathscr{A}_0-\sum_{i=1}^m\mathscr{A}_ie^{-j\omega h_i}\right)\neq 0\ .$$

□

Under frequency domain form, the system can be written

$$sx=\mathscr{A}_0x+\sum_{i=1}^m\mathscr{A}_iu_i,\quad u_i=e^{-sh_i}x.$$

B. Padé Approximation of $e^{-j\omega h}$

The method displayed in [18] consists in replacing the delay transfer function e^{-sh} by a parametrized Padé approximation $p_q(sh)$ of order q (which can be any nonnegative integer). The scalar transfer function p_q is rational, proper and stable, and constructed in such a way that, for a certain $\delta_q>1$ and for all real ω and $\hbar\geq 0$,

$$\{\ p_q(j\omega h):\ 0\leq h\leq\hbar\ \}\subset\{\ e^{-j\omega h}:\ 0\leq h\leq\hbar\ \}\subset\{\ p_q(j\delta_q\omega h):\ 0\leq h\leq\hbar\ \}\ . \quad (25)$$

Inclusions (25) express that, for any real ω, the arc of the unit circle defined by $\{\ e^{-j\omega h}:\ 0\leq h\leq\hbar\ \}$ is jammed between an inner and an outer approximation, two other arcs that are defined by the Padé approximation p_q. Note in particular that $p_q(0)=1$. The parameter δ_q in (25) is chosen in such a way as to be minimal, that is [18]:

$$\delta_q\doteq\frac{1}{2\pi}\min\{\omega>0\ :\ p_q(j\omega)=1\}\ . \quad (26)$$

We now reproduce a result given in [18].

Lemma 3: A possible choice for a stable transfer function p_q fulfilling (25) is

$$p_q(s)\doteq\frac{d_q(-s)}{d_q(s)}\ ,$$

where d_q is a polynomial of degree q, given by:

$$d_q(s)\doteq\sum_{j=0}^q\frac{(2q-j)!q!}{(2q)!j!(q-j)!}s^j\ .$$

Moreover, the corresponding sequence $(\delta_q)_{q\geq 1}$, defined by (26), is decreasing and tends towards 1 when $q\to+\infty$. □

Using the approximation defined in Lemma 3, the gap between inner and outer approximations can be made arbitrarily small by choosing high enough order q. Numerically, $\delta_3\simeq 1.2329$, $\delta_4\simeq 1.0315$, $\delta_5\simeq 1.00363$.

We call *inner system*, resp. *outer system*, the systems obtained from (22) by replacing the delays with Padé approximation $p_q(sh_i)$, resp. $p_q(\delta_qsh_i)$. The interest of this specific choice for approximation lies in the following result.

Theorem 3 ([18, Theorems 1 to 3]): Robust stability of the outer system implies delay-dependent stability of (22), which in turn implies robust stability of the inner system.

Moreover, if $(\hbar_1,\dots,\hbar_m)$ is *maximal* in the subset of $\mathbb{R}^{+m}$ for which the outer system is robustly stable on $[0,\hbar_1]\times\cdots\times[0,\hbar_m]$ (meaning that robust stability fails on any $[0,\delta\hbar_1]\times\cdots\times[0,\delta\hbar_m]$, for $\delta>1$), then system (22) is *not* delay-dependently stable on

$$[0,\delta_q\hbar_1]\times\cdots\times[0,\delta_q\hbar_m]\ .$$

□

As an example, for $q=5$, the robustness margin associated to the outer system provides a lower estimate of the delay margin for the system of interest (22), with a relative error smaller than 0.4%.

C. Reduction to Singular Perturbation Problem

Let a state space realization of the (scalar) Padé approximation p_q of Lemma 3 be given by the system

$$h\dot{\xi}=A_q\xi+B_qx,\quad u=C_q\xi+D_qx\ ,$$

where $A_q \in \mathbb{R}^{q\times q}$, $B_q \in \mathbb{R}^{q\times 1}$, $C_q \in \mathbb{R}^{1\times q}$, $D_q \in \mathbb{R}$. Then the inner, resp. outer, comparison system

$$sx = \mathscr{A}_0 x + \sum_{i=1}^{m} \mathscr{A}_i u_i,\ u_i = (I_n \otimes p_q(sh_i))x\ ,$$

resp.

$$sx = \mathscr{A}_0 x + \sum_{i=1}^{m} \mathscr{A}_i u_i,\ u_i = (I_n \otimes p_q(\delta_q sh_i))x\ ,$$

can be realised by (10), where

$$n_0 \doteq n,\ n_i \doteq qn,\ i = 1,\dots,m\ , \tag{27}$$

and the matrix $A \in \mathbb{R}^{(mq+1)n\times(mq+1)n}$ is given in (23), resp. (24).

One is now in position to deduce the main result of the paper.

Theorem 4: Suppose Assumption 1 holds. System (22) is delay-dependently stable *if and only if* there exist positive integers q and k such that LMI (18) is fulfilled, with (27) and A given by (24). □

Recall that (A_q, B_q, C_q, D_q) is a realisation of the transfer $p_q(s)$ defined in Lemma 3. The family of LMIs constructed from the statement, which is indexed by the integers q (corresponding to the degree of Padé approximation) and k (see the comments after Theorem 1), provides more and more precise LMI conditions which are sufficient for delay-dependent stability of (22). They are also asymptotically necessary, in the sense that they are fulfilled for large enough q and k when the system is delay-dependently stable. In consequence, Theorem 4 provides a semi-decidable necessary and sufficient condition for delay-dependent stability.

The choice of q, the degree of Padé approximation, is not a big deal, as the assorted precision may be estimated quantitatively by δ_q (see Section IV-B). In practice, $q = 5$ is enough.

V. Conclusions

A family of more and more precise sufficient conditions has been proposed in this paper, to check delay-dependent stability of linear systems with multiple incommensurate delays. It is shown that these conditions are also asymptotically necessary, meaning that delay-dependent stability implies their solvability from a certain rank and beyond. Moreover, they may be checked exactly, being expressed as feasibility problem for some LMIs.

References

[1] P.-A. Bliman, Lyapunov equation for the stability of linear delay systems of retarded and neutral type, *IEEE Trans. Automat. Control* **47** no 2, 327–335 (2002)

[2] P.-A. Bliman, From Lyapunov-Krasovskii Functionals for Delay-Independent Stability to LMI Conditions for μ-Analysis, *in Advances in Time-Delay Systems*, S.-I. Niculescu, K. Gu (Eds.), Lecture Notes in Computational Science and Engineering, Vol. **38**, Springer, 75–88 (2004)

[3] P.-A. Bliman, An existence result for polynomial solutions of parameter-dependent LMIs, *Systems and Control Letters* **51** no 3-4, 165–169 (2004)

[4] J. Chen, G. Gu, C.N. Nett, A new method for computing delay margins for stability of linear delay systems, *Systems and Control Letters* **26**, 101–117 (1995)

[5] L. Dai, *Singular control systems*, Lecture Notes in Control and Information Sciences, Vol. **118**, Springer Verlag, Berlin (1989)

[6] J. Dieudonné, *Foundations of Modern Analysis*, Enlarged and corrected printing. Pure and Applied Mathematics Vol. 10-I, Academic Press, New-York London (1969)

[7] K. Gu, V. Kharitonov, J. Chen, *Stability of time-delay systems*, Birkhäuser, Boston (2003)

[8] R. Gunawan, E.L. Russell, R.D. Braatz, Robustness analysis of multivariable systems with time delays, *Proc. of the European Control Conf.*, Porto, Portugal, 1882–1887 (2001)

[9] J.K. Hale, *Theory of functional differential equations*, Applied Mathematical Sciences 3, Springer Verlag, New York (1977)

[10] J.-Y. Ishihara, M.-H. Terra, On the Lyapunov theorem for singular systems, *IEEE Transactions on Automatic Control* **47** no 11, 1926–1930 (2002)

[11] T. Iwasaki, S. Hara, Generalized KYP lemma: unified frequency domain inequalities with design applications, *IEEE Transactions on Automatic Control* **50** no 1, 41–59 (2005)

[12] T. Iwasaki, S. Hara, A.L. Fradkov, Time domain interpretations of frequency domain inequalities on (semi)finite ranges, *Systems and Control Letters* **54** no 7, 681–691 (2005)

[13] M. Jun, M.G. Safonov, IQC robustness analysis for time-delay systems, *Int. J. Robust Nonlinear Control* **11**, 1455–1468 (2001)

[14] E.W. Kamen, On the relationship between zero criteria for two-variable polynomials and asymptotic stability of delay differential equations, *IEEE Trans. Automat. Control* **25** no 5 (1980) 983–984

[15] E.W. Kamen, Linear systems with commensurate time delays: stability and stabilization independent of delay, *IEEE Trans. Automat. Control* **27** no 2 (1982) 367–375

[16] S.-I. Niculescu, Stability and hyperbolicity of linear systems with delayed state: a matrix pencil approach, *IMA J. Math. Contr. Information* **15**, 331–347 (1998)

[17] S.-I. Niculescu, *Delay Effects on Stability. A Robust Control Approach*, Lecture Notes in Control and Information Sciences, Vol. **269**, Springer (2001)

[18] J. Zhang, C.R. Knospe, P. Tsiotras, New results for the analysis of linear systems with time-invariant delays, *International Journal of Robust Nonlinear Control* **13** no 12, 1149–1175 (2003)