Easy-to-use realistic dry friction models for automatic control

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Abstract

We present two state variable dry friction models with some desired properties for control purposes, which permit a unified description of kinetic and static friction, Dahl and Stribeck effects and stick slip. We give basic theoretical properties, as well as an open-loop identification procedure, the relation with Coulomb model, sufficient conditions on PID parameters for quenching of limit cycles for a one-degree-of-freedom system. Proofs are sketched in the Appendix.

1 Introduction - Classical modeling of dry friction

In a first approach, friction phenomena subdivide in viscous and dry friction, including Coulomb friction, stiction and Dahl effects ...

Classical Coulomb and stiction models are of the form[1]

$$F_{dry} = \begin{cases} f_k & \text{for} \quad \dot{u} > 0 \\ [-f_s, f_s] & \text{for} \quad \dot{u} = 0 \\ -f_k & \text{for} \quad \dot{u} < 0 \end{cases} \qquad (1)$$

where $\dot{u}$ is the relative speed between two pieces in contact: F_{dry} takes values between $-f_s$ and f_s for $\dot{u} = 0$ and is equal to the constant value f_k or $-f_k$ if $\dot{u} > 0$ or $\dot{u} < 0$ respectively. Model (1) includes stiction effects for $f_s > f_k$ and reduces to pure Coulomb friction for $f_s = f_k$. f_s is the *static friction value* and f_k the *kinetic friction value*. Model (1) has some severe drawbacks:

- It provides few informations on the behavior of friction during velocity sign reversals: only bounds are given. However, when u is a position error, these reversals occur frequently, and the transient behavior of friction is fundamental. It leads to limit cycles that restrict the precision $|u|$. When $f_s > f_k$ these limit cycles are of *stick-slip* [1] type.

- Moreover, model (1) is multivalued: when coupled with an equation of motion it yields to a differential inclusion [16, 3]. Unfortunately, uniqueness does not hold when $f_s > f_k$ (see 3.2). Some approximation

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[1] The friction force is $-F_{dry}$.

schemes are then necessary to recover a classical well-posed differential equation.

A precise analysis [1, 11, 18] of the transient phases shows that dry friction, starting from rest, exhibits an *elastic behavior* until a relative microdisplacement s_e, where a maximum value, the *static friction* value f_s is attained. Beyond this value, the friction decreases until it approaches f_k for displacements greater than a value s_p, which are no more "microdisplacements". The behavior is then of *plastic* type (Coulomb friction behavior). When the velocity switches, an analogous transient curve begins from the reached position. This behavior is irreversible, and so gives rise to hysteresis cycles, i.e. cycles with rate independent shapes (see Fig. 1).
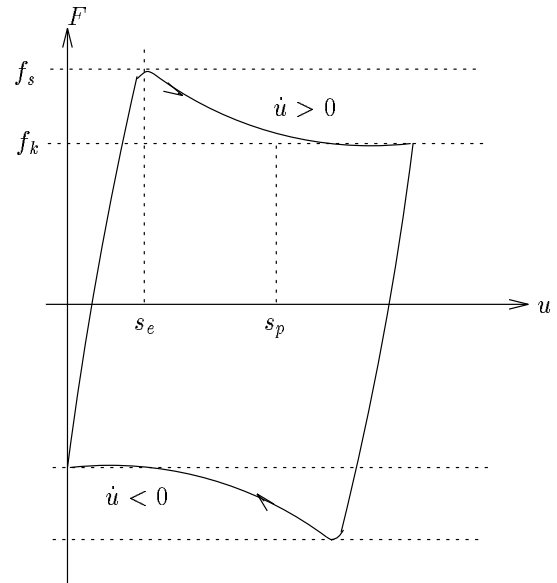

Figure 1

2 State space modeling of dry friction: LSI models

Modeling of dry friction in view of control-law synthesis has been the subject of various recent contributions (see [1] for a survey and [4, 9, 11, 19]), dedicated e.g. to line of sight stabilization for pointing devices or path following for robots. In particular, [10] provides a related analysis.

2.1 Desired properties of dry friction models

Friction models must agree with the previous experimental observations and must be mathematically sound:

1. First, the transient behavior seems to be independent from the velocity $|\dot{u}|$, but depends upon the covered distance $\int |\dot{u}| \cdot dt$ (upon the position when sgn$\dot{u}$ remains constant) [18]. This is the *rate independence* or *hysteresis property*.

2. Another noticeable property is that friction dissipates energy. We hence consider *dissipative hysteresis models*.

3. Identification of the models must be easy, when knowing the main experimental results (f_k, f_s, s_e, s_p).

4. The friction model, together with the equation of motion (including control feedback) must constitute well-posed set of equations.

5. It must agree with (1) when $s_p \to 0$

6. The models must also be simple enough in order to be used in real-time algorithms.

The available friction models, as far as known by the authors, do not meet all the previous requirements. It is the reason why in [4, 5], new friction models are proposed. They are refinements of Coulomb model simpler than those obtained in the framework of continuous media mechanics [17]. Compared to these ones, they are "black box" models, fitted for use in real-time, and directly expressing macroscopic characteristics, rather than microscopic and distributed. Their scope is different. Compared to models commonly used in Control, the main advantages of the proposed models are the following:

- They are able to represent friction behaviors at any speed and in particular at low speed. In fact, due to the rate independence property, they are insensitive to time scale.

- They no more involve the multivalued operator $u \to \text{sgn}\dot{u}$: they are described by ordinary differential equations, no more by differential inclusions. They are ready to use with standard ode solvers.

2.2 The proposed models

They are ordinary differential equations defining the friction operator $u \mapsto F(u)$.

- A first order model satisfies all the previous requirements, with a restriction for transient phases: it reproduces only the elastic behavior (Dahl effect). It is already a very simple and reliable regularization of Coulomb model. It is a particular case of Dahl models [11]:

$$\begin{cases} \varepsilon_f \dot{x} = -|\dot{u}|x + f_1\dot{u}, & x(0) = 0 \\ F(u)(t) = x(t) \end{cases} \tag{2}$$

- The following second order model satisfies all the enumerated requirements:

$$\begin{cases} \varepsilon_f \dot{x} = |\dot{u}| \begin{pmatrix} -\dfrac{1}{\eta} & 0 \\ 0 & -1 \end{pmatrix} x + \dot{u} \begin{pmatrix} \dfrac{f_1}{\eta} \\ -f_2 \end{pmatrix} \\ x(0) = 0, \quad x \in \mathbb{R}^2 \\ F(u)(t) = (\, 1 \quad 1\,)x(t) \end{cases} \tag{3}$$

where $f_1 > 0$, $f_2 \geq 0$ are forces, $\varepsilon_f > 0$ is a distance and $\eta > 0$ is dimensionless. Both models are of the form

$$\begin{cases} \dot{x} = |\dot{u}| \cdot Ax + B\dot{u}, & x(0) = 0 \\ F(u)(t) = Cx(t) \end{cases} \tag{4}$$

with

$$\begin{cases} A = -\dfrac{1}{\varepsilon_f}, \quad B = \dfrac{f_1}{\varepsilon_f}, \quad C = 1 \quad \text{for (2)} \\ A = -\dfrac{1}{\varepsilon_f}\begin{pmatrix} \frac{1}{\eta} & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \dfrac{1}{\varepsilon_f}\begin{pmatrix} \frac{f_1}{\eta} \\ -f_2 \end{pmatrix} \\ \text{and } C = (\, 1 \quad 1\,) \quad \text{for (3)} \end{cases}$$

Through a change of time-variable $ds = |\dot{u}|dt$, model (4) is related to the Linear "Space Invariant" system:

$$\begin{cases} \dfrac{dx_S}{ds} = Ax_S + Bu_S, & x(0) = 0 \\ y_S = Cx_S \end{cases} \tag{5}$$

so we call (4) a *LSI friction model*. This relation is made precise in theorem 1 and in 2.4. These models are state space models, x being the state variable. First appearance of hysteresis models described by integro-differential equations seems to come back to Bouc [8].

In order to identify the parameters in (2) or (3), we need some precise definitions for the physically significant values:

- The *kinetic friction value* is defined as the *asymptotic value of friction when $\dot{u} > 0$*:

$$f_k \stackrel{\Delta}{=} \lim_{\substack{\dot{u}>0,\ u(t)\to +\infty, \\ t \to +\infty}} F(u)(t)$$

and s_p is the *distance above which $F(u)(t)$ is within 5% of f_k*[2].

- The mechanical system with friction is said to be *at rest* when

$$\dot{u} = 0 \text{ and } x = -A^{-1}B\sigma, \quad \sigma = \pm 1$$

The last condition means that before rest, a motion with constant direction (sgn$\dot{u} = \sigma$)

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[2]s_p is a rise distance, analogous to the 0-to-95% rise time.

and for example a vanishing speed takes place from $t = -\infty$, so that friction asymptotic values are reached for finite t. Taking $\sigma = 1$, the *static friction* f_s is the *maximum value of the friction reached starting from rest with* $\dot{u} > 0$:

$$f_s \triangleq \max_{\substack{t \geq 0, \ \dot{u} > 0 \\ x(0) = \pm A^{-1}B}} F(u)(t)$$

It is reached for the displacement value $s = s_e$, which is the *breakaway distance from rest*. It can be shown, that for the models we consider, it is also the constraint-free maximum value:

$$f_s = \sup_{\substack{u, \ t \geq 0 \\ x(0) = 0}} F(u)(t)$$

- The minimal and maximal slopes k_F^+ and k_F^- of the curves u *versus* F (hysteresis cycles) are also physically significant, they are the extremal values of the positive or negative stiffness of the friction:

$$k_F^\pm \triangleq \sup_{\substack{u, t \geq 0, \ \dot{u}(t) \neq 0 \\ x(0) = 0}} \pm \frac{\dot{F}(u)(t)}{\dot{u}(t)},$$

Remark that the value k_F^- corresponds to Stribeck effect.

2.3   Main results for LSI friction models

Theorem 1 (Well-posedness and hysteresis property of $F(u)$)

(i) For any absolutely continuous input function u, i.e. u and $\dot{u}$ locally integrable on $\mathbb{R}^+$, or in short $u \in W^{1,1}_{loc}(0,\infty)$, $F(u)$ is given by

$$F(u)(t) = \int_0^t C e^{A \int_{t'}^t |\dot{u}(\tau)| \cdot d\tau} B \dot{u}(t') \cdot dt' \tag{6}$$

Denote F_S the input-output operator associated to (5):

$$F_S(u_S)(s) = \int_0^s C e^{A(s-s')} B u_S(s') \cdot ds' \tag{7}$$

and define $(\Sigma(u), S(u))$ as follows:

$$S(u)(t) \triangleq \int_0^t |\dot{u}(\tau)| \cdot d\tau, \ \ \Sigma(u) \triangleq u \circ S(u)^{-1} \tag{8}$$

i.e. $\Sigma(u)(s) = u(t)|_{S(u)(t) = s}$. Then the pair $(\Sigma(u), S(u))$ is characterized by:

$$\begin{cases} u = \Sigma(u) \circ S(u), \ \ \left| \dfrac{d\Sigma(u)}{ds} \right| = 1, \\ \dfrac{dS(u)}{dt} \geq 0, \ \ S(u)(0) = 0 \end{cases} \tag{9}$$

and we have

$$F(u)(t) = F_S \left(\frac{d\Sigma(u)}{ds} \right) \circ S(u)(t) \tag{10}$$

(ii) $F : u \mapsto F(u)$ has the following hysteresis property:

$$\begin{array}{c} F(u \circ \varphi) = F(u) \circ \varphi \ for \ any \\ increasing \ diffeomorphism \ \varphi \ on \ \mathbb{R}^+ \end{array} \tag{11}$$

(iii) F is a locally Lipschitz operator from the space $W^{1,1}_{loc}(0,\infty)$ into itself.

Theorem 2 (Dissipativity condition)

(i) For any $t \geq 0$,

$$\int_0^t F(u)(\tau) \cdot \dot{u}(\tau) \cdot d\tau = \tag{12}$$

$$= \int_0^{S(u)(t)} F_S \left(\frac{d\Sigma(u)}{ds} \right)(s) \cdot \frac{d\Sigma(u)}{ds}(s) \cdot ds$$

(ii) If there exists P such that

$$P = P^T \geq 0, -A^T P - PA \geq 0, C^T = PB \tag{13}$$

then the operator $\dot{u} \mapsto F(u)$ is dissipative, i.e. for all positive t

$$2 \int_0^t F(u) \cdot \dot{u} \cdot d\tau \geq x(t)^T P x(t) - x(0)^T P x(0) \tag{14}$$

More precisely, (2) is dissipative for every $\varepsilon_f > 0$, and (3) is dissipative as soon as

$$\begin{cases} f_1 > f_2 \geq 0 \\ \varepsilon_f > 0 \ \ and \ \ 0 < \eta < 1 \end{cases} \tag{15}$$

Theorem 3 (Parameter identification)

(i) The values of the parameters for model (2) are related to the measurements by:

$$f_k = f_1, \ \ s_p = 3\varepsilon_f, \ \ k_F^- = 0, k_F^+ = 2\frac{f_1}{\varepsilon_f}$$

(ii) The values of the parameters for model (3) are related to the measurements by:

$$\begin{cases} f_k = f_1 - f_2 \\ f_s = f_k + 2f_2 \left(\dfrac{\eta f_2}{f_1} \right)^{\frac{\eta}{1-\eta}} (1 - \eta) \\ s_e = \dfrac{\varepsilon_f \eta}{1-\eta} \log \dfrac{f_1}{\eta f_2} \\ s_p = 3\varepsilon_f \\ k_F^- = \dfrac{2f_2}{\varepsilon_f} \left(\dfrac{\eta^2 f_2}{f_1} \right)^{\frac{\eta}{1-\eta}} (1 - \eta) \\ k_F^+ = \dfrac{2}{\eta \varepsilon_f}(f_1 - \eta f_2) - k_F^- \end{cases} \tag{16}$$

(iii) It is convenient for the applications to express parameters as functions of measurements. This is straightforward for model (2). For (3), define

$$m_1 = \frac{f_s - f_k}{f_k} \ \ and \ \ m_2 = e^{3s_e/s_p}$$

We suppose $f_k \geq 0$ (which is natural for a friction force) and $s_e < +\infty$. Then, the parameters are given by

$$\begin{cases} f_1 = \frac{(m_1 m_2 + 2)p}{2(p-1)} f_k \\ f_2 = \frac{m_1 m_2 p + 2}{2(p-1)} f_k \\ \varepsilon_f = \frac{s_p}{3} \\ \eta = \frac{m_1 m_2 + 2}{m_1 m_2 p + 2} \end{cases} \tag{17}$$

where p is the solution of

$$\frac{m_1 m_2 + 2}{m_1 m_2} \ln p = (p-1)\ln m_2 \quad and \quad p > 1$$

This solution exists and is unique if and only if $\ln m_2 < \frac{m_1 m_2 + 2}{m_1 m_2}$, for example when $3s_e < s_p$. Dissipativity is guaranteed under these conditions.

2.4 Comments

Hysteresis property (11) (adapted from [20]) means that curves $u \mapsto F(u)$ do not depend upon the velocity with which they are covered, as:

$$H(u \circ \varphi) \circ (u \circ \varphi)^{-1} = H(u) \circ u^{-1}$$

precisely represents this graph. As an example, consider $u \mapsto \text{sgn}\dot{u}$.

$S(u)$ is the curvilinear abscissa of the curve $t \mapsto u(t)$ and $s \mapsto \Sigma(u)(s)$ the parametrization of this curve by its curvilinear abscissa.

As $S(u \circ \varphi) = S(u) \circ \varphi$ (S is itself a hysteresis operator) and $\Sigma(u \circ \varphi) = \Sigma(u)$, it is natural (and in fact *necessary* when H is continuous-function-valued [5]) to express hysteresis operators as

$$F(u) = F_S(\Sigma(u)) \circ S(u)$$

F_S being any causal operator with continuous function values: this is the meaning of (10).

As F_S is linear and stationary, we call these models Linear Space Invariant (LSI) friction models, by analogy with LTI models. This property allows to use Linear System Theory for these (nonlinear) operators F. In particular, the notions of state, minimal realization, stability, static gain ($-CA^{-1}B = f_k$), accessible set under bounded input, are useful to prove properties of the proposed friction models [5].

Due to (9), $\dfrac{d\Sigma(u)}{ds}$ is the unitary tangent to contact trajectory, oriented in the sense of motion: models (2), (3) appear as Linear Space Invariant (resp. 1st- and 2nd-order) filters applied to Coulomb friction. In this framework, Dahl models [11] are 1st order nonlinear SI filters.

3 Properties of basic mixed LSI/LTI models

Let us consider the following equation of motion for a one-degree-of-freedom mechanical system with friction and a PID position controller:

$$\ddot{u} + k_D \dot{u} + k_P u + k_I \int_0^t u = -F(u) + w(t) \quad \text{a.e.}$$
$$(u(0), \dot{u}(0)) = (u_0, u_1) \text{ fixed in } \mathbb{R}^2 \tag{18}$$

w represents all the other external forces.

3.1 Well-posedness of Cauchy problem

Theorem 4 *If $w \in L^1_{loc}(0,\infty)$, the systems (2)-(18) and (3)-(18) admit unique solutions $u \in W^{2,1}_{loc}(0,\infty)$, continuous wrt the data.*

3.2 Singular perturbation of LSI models and sgn regularization

Theorem 5 *Let $w \in L^2_{loc}(0,\infty)$. The solution of (2)-(18) tends when $s_p \to 0$ in $W^{2,2}_{loc}(0,\infty)$ towards the unique solution of*

$$\ddot{u} + k_D \dot{u} + k_P u + k_I \int_0^t u \in -f_k \text{sgn}\dot{u} + w \quad \text{a.e.}$$
$$(u(0), \dot{u}(0)) = (u_0, u_1) \tag{19}$$

The solution of (3)-(18) admits when $s_p \to 0$ cluster point in $W^{2,2}_{loc}(0,\infty)$ verifying

$$\ddot{u} + k_D \dot{u} + k_P u + k_I \int_0^t u \in -f_k \text{sgn}_{f_s/f_k}\dot{u} + w \quad \text{a.e.}$$
$$(u(0), \dot{u}(0)) = (u_0, u_1) \tag{20}$$

where we define, for $\lambda \geq 1$, $\text{sgn}_\lambda z = \text{sgn}z$ if $z \neq 0$, $\text{sgn}_\lambda 0 = [-\lambda, +\lambda]$.

(2) is hence a regularization of Coulomb model. It gives an alternative to Hille-Yosida regularization of Coulomb model (1) with $f_s = f_k$ (given by $F = f_k\text{sgn}\dot{u}$ for $\dot{u} > \rho$ and $f_k/\rho\dot{u}$ for $|\dot{u}| \leq \rho$ where $\rho > 0$ is the (small) regularization parameter). This latter cannot be used in control, as it assignes to friction a viscous transient behavior, non consistent with experiment.

(20) is the classical way of modeling stiction, using (1) with $f_s > f_k$, but it is ill-posed: as $\text{sgn}z \subset \text{sgn}_\lambda z$ (graph inclusion for multivalued maps), the solution of (19) is also a solution of (20) and is not the expected one (indeed, a continuum of solutions exists in general [5]).

The desired coherency requirement between the classical and the proposed dry friction models is fulfilled. Furthermore, (3) is more reliable and expressive than (1): it leads to well-posed motion equations describing the qualitative behavior usually associated with (1).

3.3 Stability criterion for PID position controller in presence of dry friction

The quenching of autonomous limit-cycles by adequate tuning of PID parameters is an already known technique [2, 13], although known
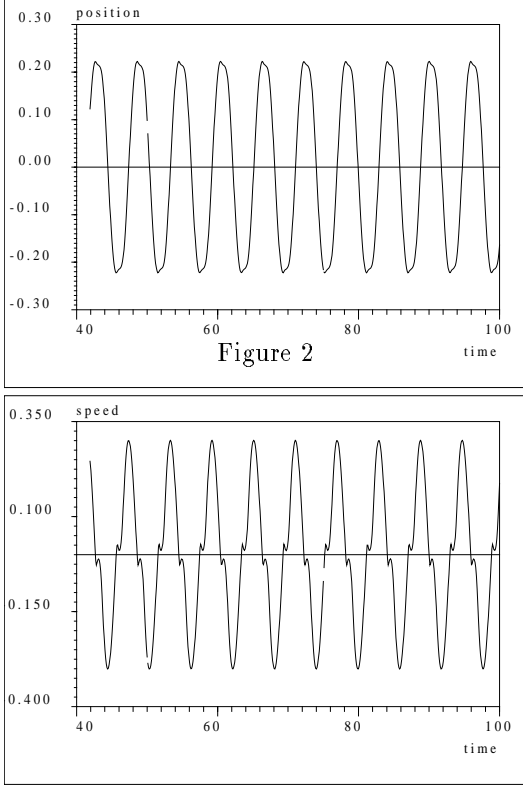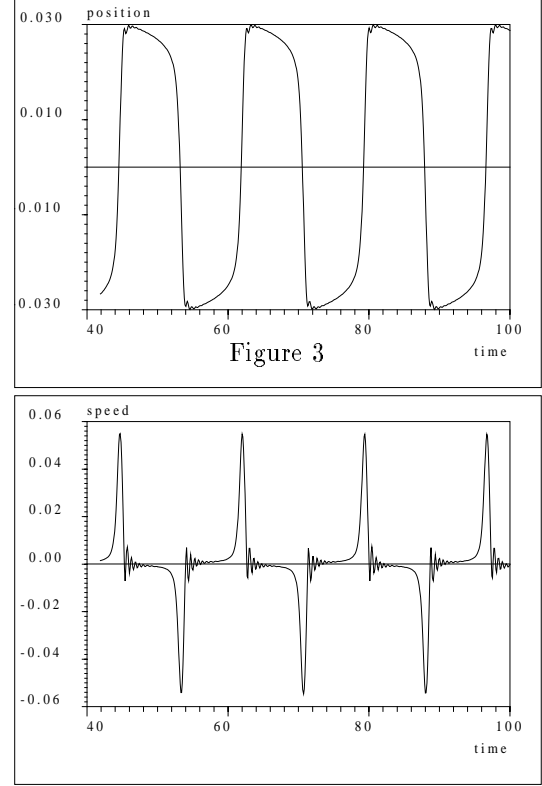
Figure 2

Figure 3

results seem to apply to the viscous friction case. We study this problem here, that is to say the asymptotic behavior of (18) with w constant. Results on existence of periodic oscillations (i.e. w periodic) for systems (4)-(18) are under study [6]. See also [12] for related properties for (19).

We denote Ω_1 (resp. Ω_2) the sets of equilibria for systems (2)-(18) (resp. (3)-(18)): $\Omega_i = \{(\dot{u}, u, \int u, F_i) : \dot{u} = u = k_I \int u + F_i = 0\}$, $i = 1, 2$, with F_1, F_2 given by (2), (3) respectively. Recall that the stability of the linear part of (18) is equivalent to

$$k_P > \frac{k_I}{k_D}, k_D > 0, k_I > 0 \qquad (21)$$

Theorem 6 (Sufficient condition for asymptotic stability) *When (21) holds:*

(i) Ω_1 *is globally asymptotically stable.*

(ii) Ω_2 *is locally asymptotically stable if*

$$k_P > k_F^- + \frac{k_I}{k_D} \qquad (22)$$

(iii) Ω_2 *is globally asymptotically stable if*

$$\begin{cases} k_P > k_F^- + \frac{(k_F^+ + k_F^-)^2}{4k_D^2} \\ \frac{k_I}{k_D} < k_D \sqrt{k_P - k_F^-} \cdot \left(\sqrt{k_P - k_F^-} - \frac{k_F^+ + k_F^-}{2k_D}\right) \end{cases} \qquad (23)$$

or if

$$\max_{\omega \in \mathbb{R}} \left| \frac{j\omega}{j\omega(k_P - \omega^2) + k_I - k_D\omega^2} + \frac{1}{2}\left(\frac{1}{k_F^+} - \frac{1}{k_F^-}\right) \right|$$

$$< \frac{1}{2}\left(\frac{1}{k_F^+} + \frac{1}{k_F^-}\right) \qquad (24)$$

(i) means that pure Dahl friction has no destabilizing effect. *(ii)* is the useful rule for PID tuning in presence of Stribeck effect. *(iii)* shows that global stability is probably hard to reach, due to the high stiffness involved.

Cycles may be observed for example with $f_1 = 1, f_2 = 0.9, \varepsilon_f = 0.01, \eta = 0.5$, then $f_k = 0.1$, $f_s = 0.505$, $k_F^- = 20.25$, $k_F^+ = 199.75$. Fig. 2 (resp. 3) shows the limit cycles obtained with $k_I = 3, k_D = 2, k_P = 2$, resp. $k_P = 10$, whose periods are approximately $T = 5.9$, resp. $T = 17.3$. Numerically, the cycles seem to disappear for $k_P \geq 12$, where the theoretical bounds are 21.75 with (22) and ... 9 995 with (23).

3.4 Stability criterion for PD controller in presence of dry friction

We examine the case $k_I = 0$ corresponding to the speed control problem. Denoting v the constant target speed and $\tilde{u} \triangleq u - vt$, we have

$$\ddot{\tilde{u}} + k_D\dot{\tilde{u}} + k_P\tilde{u} + F(\tilde{u} + vt) = 0 \qquad (25)$$

where F is given by (2) or (3). We denote Ω_1 (resp. Ω_2) the sets of equilibria for systems (2)-(25) (resp. (3)-(25)): $\Omega_i = \{(\dot{\tilde{u}}, \tilde{u}, F_i) : \dot{\tilde{u}} = 0, k_P\tilde{u} = -F_i = -f_k \text{sgn} v\}$, $i = 1, 2$, with F_1, F_2

given by (2), (3) respectively. The stability of the linear part of (25) is equivalent to

$$k_P > 0, k_D > 0 \qquad (26)$$

Theorem 7 (Sufficient condition for a-symptotic stability) *When* (26) *holds:*
(i) Ω_1 is globally asymptotically stable.

(ii) Ω_2 is locally asymptotically stable if
$$k_P > k_F^- \qquad (27)$$

(iii) Ω_2 is globally asymptotically stable if
$$|v| > f_s \int_0^\infty \left| \begin{pmatrix} 0 & 1 \end{pmatrix} e^{Mt} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right| \cdot dt \qquad (28)$$

with $M \triangleq \begin{pmatrix} 0 & 1 \\ -k_P & -k_D \end{pmatrix}$ *or if*

$$k_P > k_F^- + \frac{(k_F^+ + k_F^-)^2}{4k_D^2} \qquad (29)$$

or if

$$\max_{\omega \in \mathbb{R}} \left| \frac{1}{k_P - \omega^2 + jk_D\omega} + \frac{1}{2}\left(\frac{1}{k_F^+} - \frac{1}{k_F^-} \right) \right|$$
$$< \frac{1}{2}\left(\frac{1}{k_F^+} + \frac{1}{k_F^-} \right) \qquad (30)$$

Again we see that Dahl friction has no destabilizing effect. (28) means that for high enough speed, no stability problems occur. For low speed, it is necessary to counteract Stribeck effect using (27). The price of global stability at low speed is again, by (29), high stiffness.

4 Conclusion

The simple differential form of the proposed models permits easy computer simulations and their underlying linear structure offers possibilities to synthesize on a rigorous basis control laws to compensate for the friction effects. Here we have shown how to tune PID controllers. More sophisticated control laws can also be studied, see [4] for an application in Robotics. [7] presents an application to tyre/road contact modeling.

5 Appendix: Proofs

We omit the proofs of existence/uniqueness and just sketch those of results of practical interest.

Proof of theorem 2 (13) $\Rightarrow$ (14) is an application of the Kalman-Yakubovitch-Popov lemma [14, 21] to the linear operator F_S in the right-hand side of (12).

Proof of theorem 3 is based on the identities:
$$f_s = \int_0^\infty |Ce^{As}B| \cdot ds$$
$$k_F^\pm = \int_0^\infty \pm|CAe^{As}B| \cdot ds + CB$$
In particular, $k_F^- \le k_F^+$.

Proof of theorem 5 As $w \in L^2_{loc}(0, \infty)$, u is bounded in $W^{2,2}_{loc}(0, \infty)$ independently of $s_p > 0$: there exists a subsequence u^{s_p} converging weakly in $W^{2,2}_{loc}(0, \infty)$ when $s_p \to 0$. The corresponding friction states x^{s_p}, being bounded in $L^\infty(0, \infty)$, may be supposed to converge weakly in $L^2_{loc}(0, \infty)$. It may be deduced [5], using the stability of matrix A, that the difference of the power $F^{s_p}(u^{s_p})\dot{u}^{s_p} - f_k|\dot{u}^{s_p}|$ tends strongly to 0 in $L^2_{loc}(0, \infty)$. On the other hand, as $\|F^{s_p}(u^{s_p})\dot{u}^{s_p}\|_{L^\infty} \le f_s$, it is shown that the cluster point verifies (20). For (2), $f_s = f_k$, which ensures that $\text{sgn}_{\frac{}{f_k}} = \text{sgn}$ is maximal monotone: convergence follows from uniqueness of solution for (19) [3].

Proof of theorem 6 We first show that $\dot{u} \in L^2(0, \infty)$. By derivation of (18), multiplication by $\ddot{u} + \alpha\dot{u}$ and integration, we get for any $t \ge 0$:

$$(k_D - \alpha)\|\ddot{u}\|_2^2 + (\alpha k_P - k_I)\|\dot{u}\|_2^2 + \qquad (31)$$
$$+\alpha \int_0^t \dot{F}\dot{u} + \int_0^t \dot{F}\ddot{u} + [C(t')]_0^t = 0$$

where $C(t) \triangleq \frac{1}{2}((\ddot{u} + \alpha\dot{u})^2 + \alpha k_I(u + \frac{\dot{u}}{\alpha})^2 + (k_P + \alpha k_D - \alpha^2 - \frac{k_I}{\alpha})\dot{u}^2)$ and $\|.\|_p$ denotes $L^p(0, t)$-norm. Using the identity $\int_0^t \dot{F}\ddot{u} = \int_0^t CAX|\dot{u}|\ddot{u} + CB \int_0^t \dot{u}\ddot{u} = -\int_0^t CA\dot{X}\frac{\dot{u}|\dot{u}|}{2} + [CAX\frac{\dot{u}|\dot{u}|}{2} + CB\frac{\dot{u}^2}{2}]_0^t = \frac{1}{2}[\dot{F}\dot{u}]_0^t - \frac{1}{2}CA \int_0^t \dot{x}\dot{u}|\dot{u}|$, and taking $\alpha = k_D$, we deduce $((k_P - k_F^-)k_D - k_I - \frac{k_F'}{2}\|\dot{u}\|_\infty)\|\dot{u}\|_2^2 \le C(0) + \frac{1}{2}[\dot{F}\dot{u}]_0^t \le C(0) + \frac{1}{2}k_F^+\dot{u}^2(0) - \frac{1}{2}k_F^-\dot{u}^2(t)$, where $k_F' \triangleq \sup -CA\frac{\dot{x}}{\dot{u}}$ is finite and $C(0)$ is positive. For 1st order model, $k_F^- = k_F' = 0$, so $\|\dot{u}\|_2$ is bounded independently of t. For 2nd order model, $(k_P - k_F^- - \frac{k_I}{k_D})\dot{u}^2(t) \le C(t) + \frac{1}{2}\dot{F}\dot{u}(t) \le C(0) + \frac{1}{2}\dot{F}\dot{u}(0)$ if $\|\dot{u}\|_\infty \le 2\frac{k_P - k_F^- - \frac{k_I}{k_D}}{k_F'}$, so the same result is true if (22) holds and $C(0) + \frac{1}{2}\dot{F}\dot{u}(0) \le 2\frac{((k_P - k_F^-)k_D - k_I)^3}{k_D k_F'^2}$. To prove (23), we remark that the centered nonlinearity $F(u) - \frac{k_F^+ - k_F^-}{2}u$ is such that $|\dot{F}(u) - \frac{k_F^+ - k_F^-}{2}\dot{u}| \le \frac{k_F^+ + k_F^-}{2}|\dot{u}|$ a.e., we bound $\alpha \int_0^t \dot{F}\dot{u} + \int_0^t \dot{F}\ddot{u}$ in (31) from below by $-\|\dot{u}\|_2(\alpha k_F^-\|\dot{u}\|_2 + \frac{k_F^+ + k_F^-}{2}\|\ddot{u}\|_2) + [\frac{k_F^+ - k_F^-}{4}\dot{u}^2]_0^t$, as $k_F^+ \ge k_F^-$, and get a positive quadratic form in $\|\dot{u}\|_2$, $\|\ddot{u}\|_2$ *iff* there exists α s.t. $4(k_D - \alpha)(\alpha(k_P - k_F^-) - k_I) \ge \frac{(k_F^+ + k_F^-)^2}{4}$. This leads to (23), and again an upper bound on $\|\dot{u}\|_2$. (6) is obtained from circle criterion [22] applied to the derived version of (18).

Now it is possible to prove that $\ddot{u} \in L^2(0, \infty)$ too, taking $\alpha = \frac{k_I}{k_P}$ in (31). The use of u as

a multiplier then shows that $\|u\|_2$ is bounded too (and hence $\|k_I \int u + F\|_2$). These are standard techniques. We then deduce that $\dot{u}$, u and $k_I \int u + F$ tends to 0, as e.g. $\dot{u}^2 = 2\int_0^t \dot{u}\ddot{u} + \dot{u}^2(0) \leq 2\|\dot{u}\|_2 \cdot \|\ddot{u}\|_2 + \dot{u}^2(0)$. It may be shown as well that both F and $\int u$ admit limits.

Proof of theorem 7 *(i)*, *(ii)* and the second part and third parts of *(iii)* are proved with the same techniques than for theorem 6. Let us prove the first part of *(iii)*. As $F \leq f_s$, we have $\limsup |\dot{\tilde{u}}| \leq f_s \int_0^\infty |(\ 0\ \ 1\ )e^{Mt}(\ 0\ \ 1\ )^T| \cdot dt$. Hypothesis (28) implies that $\liminf |\dot{u}|$ is bounded from below by a strictly positive number. Hence $F \to f_k \mathrm{sgn}v$, because $\int_0^{+\infty} |\dot{u}| = +\infty$, and $\dot{\tilde{u}}, k_P \tilde{u} + F$ tend to zero.

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