Multi-Parameter Dependent Lyapunov Functions for the Stability Analysis of Parameter-Dependent LTI Systems

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***Abstract*— In this paper it is shown that robust stability of multi-parameter affinely-dependent LTI systems is equivalent to the existence of a multi-parameter polynomially-dependent quadratic Lyapunov function of known, bounded degree in terms of the system parameters. Testing the stability of multi-parameter dependent LTI systems over a compact, connected set can be cast in terms of two linear matrix inequalities (LMI's).**

I. INTRODUCTION

The objective of this paper is to find computable, non-conservative conditions for checking the asymptotic stability of parameter-dependent LTI (PDLTI) systems that depend affinely on a parameter vector $\rho := (\rho_1, \ldots, \rho_m)^\mathsf{T} \in \Omega \subset \mathbb{R}^m$, of the form

$$\dot{x} = A(\rho)x, \quad A(\rho) := \sum_{i=0}^{m} \rho_i A_i, \quad \rho_0 = 1, \tag{1}$$

where $A_i \in \mathbb{R}^{n\times n}, (i = 0, 1, 2, \ldots, m)$. It is assumed that Ω is a compact and connected subset of $\mathbb{R}^m$. Stability criteria for LTI parameter-dependent systems can be derived through the search for a suitable Lyapunov function whose time derivative remains negative along the trajectories of (1). In general, a quadratic Lyapunov function $x^\mathsf{T} P x$ is postulated. The Lyapunov matrix P is either constant or parameter dependent. If P is constant, quadratic stability for (1) is ensured. Quadratic stability is a rather strong form of robust stability [1]. It leads to conservative results, especially if it is known that ρ is constant. Stability conditions closer to being necessary can be derived by using a parameter-dependent Lyapunov matrix instead, i.e., $P = P(\rho)$; see, for example [2], [3], [4], [5], [6]. Nonetheless, sufficient *and* necessary conditions result only if the correct parameter dependence of the Lyapunov matrix is used. Till recently, this "correct" form of the Lyapunov matrix $P(\rho)$ leading to non-conservative conditions have eluded the researchers in the field, except some very special cases [7], [8]. In fact, it is well known that robust stability analysis of multi-parameter dependent LTI systems is, in general, an NP-hard problem [9], [10], [11].

A very important recent result by Bliman [12] states that parameter-dependent LMI's admit polynomial solutions, provided they are feasible for each parameter value. This result paved the way to *necessary* Lyapunov-based conditions for parameter-varying and PDLTI systems. In [13], in particular, the author was able to show that the stability of (1) can be analyzed *without conservatism* via the use of *polynomial* parameter-dependent quadratic (PPQD) Lyapunov functions. However, the degree of the polynomial dependence in [13] is not known a priori. A similar result was obtained independently around the same time in [14] using a different approach. For the single-parameter case ($m = 1$), in particular, Refs. [14], [15] provided an upper bound for the polynomial dependence of the matrix $P(\rho)$. When the parameter is known to belong inside a compact interval, the stability of the matrix $A(\rho) = A_0 + \rho A_1$ can be tested through a pair of linear matrix inequalities (LMI's) without conservatism. Moreover, in general, the LMI conditions in [15] are computationally more efficient than the ones given in [13].

The problem of robust stability of PDLTI systems has also been addressed in the recent work of Chesi et al [16]. Therein the authors show that homogeneous polynomially parameter-dependent quadratic Lyapunov functions can be used to characterize robust stability of polytopic linear models. For systems affected by *polynomial* time-invariant uncertainty Chesi [16] has also provided computationally attractive LMI conditions, albeit these conditions are not Lyapunov based.

This paper contains two contributions. First, we extend the results of [15] to the case of multi-parameter dependent LTI systems of the form (1). Specifically, we show that the polynomial parameter-dependent Lyapunov function matrix of degree M

$$P(\rho) = \sum_{|k|=0}^{M} \rho_1^{k_1} \cdots \rho_m^{k_m} P_{k_1,\ldots,k_m}, \tag{2}$$

where $M \leq \frac{1}{2}n(n+1) - 1$ can be used to derive a non-conservative (i.e., necessary and sufficient) stability condition for (1). Second, we provide convex conditions for stability when the parameter vector belongs to a compact and connected set. Finally, the results of this paper can be considered as an improvement over the results of [13] by providing upper bounds on the polynomial degree of the Lyapunov function that *characterizes* stability of LTI multi-parameter dependent systems.

II. MATHEMATICAL PRELIMINARIES

Let a multi-variable polynomial $\pi(\rho)$, given by

$$\pi(\rho) = \sum_{|k|=0}^{N} \rho_1^{k_1} \cdots \rho_m^{k_m} p_{k_1,\ldots,k_m},$$

where $\rho := (\rho_1, \rho_2, \ldots, \rho_m)^\mathsf{T} \in \mathbb{R}^m$ and $|k| := k_1 + k_2 + \cdots + k_m$ with $k_i \in \mathbb{N}_0$.

Definition 1: The *multi-parameter* degree of $\pi(\rho)$ with respect to the vector ρ is defined as $\deg \pi(\rho) := \max\{k_1 + \ldots + k_m,\ p_{k_1,\cdots,k_m} \neq 0\}$. Similarly, the *single-parameter* degree

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of $\pi(\rho)$ with respect to the scalar parameter ρ_i is defined as $\deg_i \pi(\rho) := \max\{k_i,\ p_{\ldots,k_i,\ldots} \neq 0\}$. $\square$

It is clear from Definition 1 that $\deg_i \pi(\rho) \leq \deg \pi(\rho)$ for all $i = 1, 2, \ldots, m$. In addition, $\deg \pi(\rho) \leq \sum_{i=1}^m \deg_i \pi(\rho)$.

For a matrix $\Pi(\rho) \in \mathbb{R}^{n\times n}$ with polynomial elements we define similarly the multi-parameter degree and the single-parameter degree of $\Pi(\rho)$ as $\deg \Pi(\rho) := \max_{1\leq q,\ell\leq n} \deg \Pi_{(q,\ell)}(\rho)$ and $\deg_i \Pi(\rho) := \max_{1\leq q,\ell\leq n} \deg_i \Pi_{(q,\ell)}(\rho)$, respectively.

Note that in the definition of the single-parameter degree of the multi-variable polynomial $\pi(\rho)$ with respect to the parameter ρ_i, all other parameters ρ_j, $j \neq i$, are treated as constants. The following result, adapted from the single-parameter case [15], is therefore immediate. Note that in the following $\mathcal{N}(A)$ denotes the null space of the matrix A.

Lemma 2.1 ([14]): Given matrices $A_i \in \mathbb{R}^{n\times n}$, ($i = 0, 1, 2, \ldots, m$) and $\rho = (\rho_1, \rho_2, \ldots, \rho_m)^\intercal \in \mathbb{R}^m$, let $\pi(\rho) := \det \sum_{i=0}^m \rho_i A_i$. Then $\deg \pi(\rho) \leq n$. If $\dim \mathcal{N}(A_i) = r_i$ then $\deg_i \pi(\rho) \leq n - r_i$. Moreover, if $\dim[\bigcap_{i=1}^m \mathcal{N}(A_=i)] = r$, then $\deg \pi(\rho) \leq n - r$. $\square$

Proof: Since in the calculation of $\deg_i \pi(\rho)$ the parameters ρ_j, $j \neq i$ are assumed to be constant, the fact that $\deg_i \left(\det \sum_{j=0}^m \rho_j A_j\right) \leq n - r_i$ where $\dim \mathcal{N}(A_i) = r_i$ follows directly from the single-parameter case [15].

Recall next that the determinant of a matrix $F \in \mathbb{R}^{n\times n}$ can be computed from [17]

$$\det F = \sum_{a\in \mathbf{A}} \operatorname{sign}(a) \prod_{i=1}^n F_{i,a_i}, \tag{3}$$

where $a := (a_1, a_2, \ldots, a_n)$, $\mathbf{A}$ is the set of permutations of $\{1, 2, \ldots, n\}$, and $\operatorname{sign}(a)$ is the signature of the permutation a taking the values of either $+1$ or -1. The determinant of $A(\rho) = A_0 + \rho_1 A_1 + \cdots + \rho_m A_m$ is thus a sum of $n!$ terms, each term being the product of n elements. Moreover, each of these elements is chosen from a different row and column of the matrix $A(\rho)$. Therefore, for every possible permutation $(a_1, a_2, \ldots, a_n) \in \mathbf{A}$,

$$\deg \prod_{i=1}^n F_{i,a_i} \leq n,$$

which together with (3), yields that

$$\deg \pi(\rho) = \deg \left(\det \sum_{i=0}^m \rho_i A_i\right) \leq n \tag{4}$$

Assume now that $\dim[\bigcap_{i=1}^m \mathcal{N}(A_i)] = r$. Then there exist r of linearly independent constant vectors $v_1, v_2, \ldots, v_r \in \mathbb{R}^n$ such that

$$A_i v_j = 0, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, r.$$

Choose now $n - r$ linearly independent constant vectors $u_1, u_2, \ldots, u_{n-r} \in \mathbb{R}^n$ such that the matrix

$$T = [u_1, u_2, \ldots, u_{n-r}, v_1, v_2, \ldots, v_r] \tag{5}$$

is invertible. Furthermore,

$$\begin{aligned} \det \sum_{i=0}^m \rho_i A_i &= \det \left(T^{-1}\Big(\sum_{i=0}^m \rho_i A_i\Big)T\right) \\ &= \det T^{-1} \det \left(\sum_{i=0}^m \rho_i A_i T\right) \\ &= \det T^{-1} \det [\bar{u}_1, \cdots, \bar{u}_{n-r}, \bar{v}_1, \cdots, \bar{v}_r] \end{aligned}$$

where $\bar{u}_i = \sum_{j=0}^m \rho_j A_j u_i$, $i = 1, 2, \ldots, n - r$ and $\bar{v}_i = A_0 v_i$, $i = 1, 2, \ldots, r$. Since $\bar{v}_i$ is constant, together with the determinant formula (3), one has

$$\begin{aligned} &\det [\bar{u}_1, \ldots, \bar{u}_{n-r}, \bar{v}_1, \ldots, \bar{v}_r] \\ &= \sum_{a_1\neq a_2\neq \ldots \neq a_n} \pm(\bar{u}_{1,a_1}\bar{u}_{2,a_2}\ldots \bar{u}_{(n-r),a_{(n-r)}}\bar{v}_{1,a_{(n-r+1)}}\ldots \bar{v}_{r,a_n}) \end{aligned}$$

For every possible permutation $(a_1, a_2, \ldots, a_n) \in \mathbf{A}$, we have that $\deg\ \bar{u}_{1,a_1}\bar{u}_{2,a_2}\cdots \bar{u}_{(n-r)a_{(n-r)}}\bar{v}_{1,a_{(n-r+1)}}\cdots \bar{v}_{r,a_n} = \deg\ \bar{u}_{1,a_1}\bar{u}_{2,a_2}\cdots \bar{u}_{(n-r)a_{(n-r)}} \leq n - r$. It follows that $\deg\big(\det [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-r}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_r]\big) \leq n - r$ and hence

$$\deg \left(\det \sum_{i=0}^m \rho_i A_i\right) \leq n - r,$$

thus completing the proof. $\blacksquare$

The next result deals with the multi- and single-parameter degrees of the adjoint of the matrix $A(\rho) = A_0 + \rho_1 A_1 + \cdots + \rho_m A_m$.

Corollary 2.1: Let matrices $A_i \in \mathbb{R}^{n\times n}$, ($i = 0, 1, 2, \ldots, m$) and $\rho = (\rho_1, \rho_2, \ldots, \rho_m)^\intercal \in \mathbb{R}^m$. Assume that $\dim \mathcal{N}(A_i) = r_i$ and $\dim[\bigcap_{i=1}^m \mathcal{N}(A_i)] = r$. Then

$$\deg \operatorname{Adj}\Big(\sum_{j=0}^m \rho_j A_j\Big) \leq \min\{n - 1,\ n - r\} \tag{6}$$

and

$$\deg_i \operatorname{Adj}\Big(\sum_{j=0}^m \rho_j A_j\Big) \leq \min\{n - 1,\ n - r_i\}, \tag{7}$$

for all $i = 1, 2, \ldots, m$. $\square$

Equation (6) implies, in particular, that

$$\operatorname{Adj}\Big(\sum_{j=0}^m \rho_j A_j\Big) = \sum_{|k|=0}^{\min\{n-1,\,n-r\}} \rho_1^{k_1} \ldots \rho_m^{k_m} N_{k_1,\cdots,k_m}$$

for some constant matrices $N_{k_1,\cdots,k_m} \in \mathbb{R}^{n\times n}$.

Example 1: Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ where

$$A_0 = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 3 \\ 0 & -2 \end{bmatrix}.$$

Here $n = 2$ and $\dim \mathcal{N}(A_1) = \dim \mathcal{N}(A_2) = 1$. Therefore, $r_1 = r_2 = 1$. Furthermore, $\dim\big[\mathcal{N}(A_1) \cap \mathcal{N}(A_2)\big] = \dim\big(\operatorname{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\}\big) = 1$, thus $r = 1$. One can compute the adjoint of the matrix $A(\rho)$ directly as follows

$$\operatorname{Adj}\big(A(\rho)\big) = \begin{bmatrix} 1 + 3\rho_1 - 2\rho_2 & -3 - \rho_1 - 3\rho_2 \\ -2 & 1 \end{bmatrix}.$$

For this example one verifies that $\deg \operatorname{Adj}\big(A(\rho)\big) = \deg_1 \operatorname{Adj}\big(A(\rho)\big) = \deg_2 \operatorname{Adj}\big(A(\rho)\big) = 1$.

Given a square matrix $P \in \mathbb{R}^{n\times n}$, let $\mathrm{vec}\,(P) \in \mathbb{R}^{n^2}$ denote the usual operation that stacks the columns of the matrix P on top of each other [18]. In case the matrix P is symmetric, $\mathrm{vec}\,(P)$ contains repeated entries. In order to remove any repetitions we introduce the following alternative.

Definition 2 ([19]): Given a symmetric matrix $P = P^{\intercal} \in \mathbb{R}^{n\times n}$, let $\mathrm{svec}(P) := (P_{11}, \cdots, P_{n1}, P_{22}, \cdots, P_{n2}, \cdots, P_{nn})^{\intercal} \in \mathbb{R}^{\frac{1}{2}n(n+1)}$.

Note that $\mathrm{svec}(P)$ consists of all the elements of $\mathrm{vec}(P)$ without repetitions.

For every *symmetric* matrix $P = P^{\intercal} \in \mathbb{R}^{n\times n}$, there exists a unique full column rank matrix $D_n \in \mathbb{R}^{n^2\times\frac{1}{2}n(n+1)}$ called *the duplication matrix,* such that $\mathrm{vec}(P) = D_n\mathrm{svec}(P)$. Moreover, the pseudo-inverse of D_n satisfies $\mathrm{svec}(P) = D_n^+\mathrm{vec}(P)$, $D_n^+D_n = I_{\frac{1}{2}n(n+1)}$ and $\mathrm{rank}\,D_n = \mathrm{rank}\,D_n^+ = \frac{1}{2}n(n+1)$; see, for instance, [20], [19]. Note, in particular, that D_n has always full column rank. Subsequently, $D_n^+ = (D_n^{\intercal}D_n)^{-1}D_n^{\intercal}$.

Definition 3 ([19]): Given $A \in \mathbb{R}^{n\times n}$, let $\widehat{A}$ denote the matrix of dimension $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$, defined by

$$\widehat{A} := D_n^+(A \oplus A)D_n, \tag{8}$$

where $A \oplus A = A \otimes I_n + I_n \otimes A$ is the Kronecker sum of a matrix with itself. □

Lemma 2.2 ([20], [19]): Let $A \in \mathbb{R}^{n\times n}$ and $\widehat{A}$ as in Definition 3. The eigenvalues of $\widehat{A}$ are the $\frac{1}{2}n(n+1)$ numbers $\lambda_i+\lambda_j$, $(1 \le j \le i \le n)$ where λ_i, λ_j are the eigenvalues of A. □

The following is immediate from Lemma 2.2.

Corollary 2.2: Let a matrix $A \in \mathbb{R}^{n\times n}$ be Hurwitz. Then the following hold:

(i) $\widehat{A}$ is Hurwitz.
(ii) $\det \widehat{A} \neq 0$. □

It is clear that if $A(\rho) = \sum_{i=0}^m \rho_i A_i$ then $\widehat{A}(\rho) = \sum_{i=0}^m \rho_i \widehat{A}_i$. Moreover, the following is true.

Lemma 2.3 ([15]): Let $A \in \mathbb{R}^{n\times n}$ with $\dim \mathcal{N}(A) = r$. Then $\hat{r} := \dim \mathcal{N}(\widehat{A}) \ge \frac{1}{2}r(r+1)$. □

III. MAIN RESULTS

A. A Class of Polynomial Parameter-Dependent Lyapunov Matrices

In the following, Ω denotes any subset of $\mathbb{R}^m$. The following theorem shows that the existence of a parameter-dependent, polynomial Lyapunov matrix of a given degree is equivalent to the fact that $A(\rho)$ is Hurwitz for each $\rho \in \Omega$.

Theorem 3.1: Let matrices $A_i \in \mathbb{R}^{n\times n}$, $(i = 0, 1, 2, \dots, m)$, and $\rho = (\rho_1, \rho_2, \dots, \rho_m)^{\intercal} \in \Omega \subset \mathbb{R}^m$. Assume that $\dim[\bigcap_{i=1}^m \mathcal{N}(\widehat{A}_i)] = \hat{r}$. Then the following two statements are equivalent:

(i) $A(\rho) = \sum_{i=0}^m \rho_i A_i$ is Hurwitz for all $\rho \in \Omega$.
(ii) There exist matrices $P_{k_1,k_2,\dots,k_m} \in \mathbb{R}^{n\times n}$, where $k_i \in \mathbb{N}_0$, $(i = 1, \dots, m)$, such that for all $\rho \in \Omega$,

$$A(\rho)P(\rho) + P(\rho)A^{\intercal}(\rho) < 0, \quad P(\rho) > 0, \tag{9}$$

where,

$$P(\rho) = \sigma(\rho) \sum_{|k|=0}^{M} \rho_1^{k_1} \dots \rho_m^{k_m} P_{k_1,\dots,k_m} \tag{10}$$

and where $M = \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}n(n+1) - \hat{r}\}$ and $\sigma(\rho) := -\mathrm{sign}(\det \widehat{A}(\rho))$. Equivalently,

$$\deg P(\rho) \le \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}n(n+1) - \hat{r}\}.$$

Moreover, if $\dim \mathcal{N}(A_i) = r_i$ then

$$\deg_i P(\rho) \le \frac{1}{2}(n - r_i)(n + r_i + 1). \qquad \square$$

Proof: (ii) ⇒ (i): It is obvious.
(i) ⇒ (ii): Since $A(\rho)$ is Hurwitz for all $\rho \in \Omega$, it follows from Corollary 2.2 that $\det \widehat{A}(\rho) \neq 0$ for all $\rho \in \Omega$. Consider the parameter-dependent, positive definite matrix

$$Q(\rho) := |\det \widehat{A}(\rho)|\, I_n > 0. \tag{11}$$

Since $A(\rho)$ is Hurwitz for all $\rho \in \Omega$, the following Lyapunov equation has a unique, positive definite solution $P(\rho) > 0$ for each $\rho \in \Omega$ [21]

$$A(\rho)P(\rho) + P(\rho)A^{\intercal}(\rho) + |\det \widehat{A}(\rho)|\, I_n = 0. \tag{12}$$

The solution to this Lyapunov equation yields

$$\begin{aligned} \big(A(\rho) \oplus A(\rho)\big)\mathrm{vec}(P) &= -|\det \widehat{A}(\rho)|\,\mathrm{vec}(I_n) \\ D_n^+\big(A(\rho) \oplus A(\rho)\big)D_n\mathrm{svec}(P) &= -|\det \widehat{A}(\rho)|\,\mathrm{svec}(I_n) \\ \mathrm{svec}(P) &= -|\det \widehat{A}(\rho)|\widehat{A}^{-1}(\rho)\,\mathrm{svec}(I_n) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{svec}(P) &= -|\det \widehat{A}(\rho)|\frac{1}{\det \widehat{A}(\rho)}\mathrm{Adj}\Big(\widehat{A}(\rho)\Big)\,\mathrm{svec}(I_n) \\ &= \sigma(\rho)\mathrm{Adj}\big(\sum_{i=0}^m \rho_i\widehat{A}_i\big)\,\mathrm{svec}(I_n) \end{aligned} \tag{13}$$

From Corollary 2.1 one obtains that

$$\deg \mathrm{Adj}\big(\sum_{j=0}^m \rho_j\widehat{A}_j\big) \le \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}n(n+1) - \hat{r}\}$$

If $\dim \mathcal{N}(A_i) = r_i$, from Lemma 2.3 one has that $\dim \mathcal{N}(\widehat{A}_i) = \hat{r}_i \ge \frac{1}{2}r_i(r_i + 1)$. Therefore, $\frac{1}{2}n(n+1) - \hat{r}_i \le \frac{1}{2}(n - r_i)(n + r_i + 1)$. It follows from Corollary 2.1 that

$$\deg_i \mathrm{Adj}\big(\sum_{j=0}^m \rho_j\widehat{A}_j\big) \le \min\{\frac{1}{2}n(n+1)-1, \frac{1}{2}(n-r_i)(n+r_i+1)\}$$

The result follows from the fact that the map $\mathrm{svec}(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{R}^{\frac{1}{2}n(n+1)}$ is one-to-one and thus invertible over the space of symmetric matrices. ■

Remark 1: Note that when Ω is connected then $\sigma(\rho)$ has the same sign for all $\rho \in \Omega$ and (10) reduces to

$$P(\rho) = \sum_{|k|=0}^{M} \rho_1^{k_1} \dots \rho_m^{k_m} P_{k_1,\dots,k_m}, \quad \forall \rho \in \Omega. \tag{14}$$

Example 2: This example is taken from [22]. Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$\begin{aligned} A_0 &= \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ A_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned} \tag{15}$$

The exact robust stability region for this problem is $(-\infty, 1.75) \times (-\infty, 3)$ (see [22]). Reference [23] suggests an algorithm to calculate this stability domain exactly. Here we compute the stability domain of (15) using Theorem 3.1.

The elements of the parameter-dependent Lyapunov matrix $P(\rho) \in \mathbb{R}^{3\times 3}$ can be computed as follows

$$\begin{aligned}
P_{11} =& -16\rho_2^2\rho_1^3 - 8592\rho_1 - 6720\rho_2 - 96\rho_2^3 + 56\rho_2^3\rho_1 - 8\rho_2^3\rho_1^2 \\
&+ 9792 - 1456\rho_2\rho_1^2 + 128\rho_2\rho_1^3 + 2496\rho_1^2 \\
&+ 5456\rho_2\rho_1 - 240\rho_1^3 + 1440\rho_2^2 - 1032\rho_2^2\rho_1 + 232\rho_2^2\rho_1^2, \\
P_{21} =& -16\rho_2^2\rho_1^3 - 1008\rho_1 - 936\rho_2 - 24\rho_2^3 + 32\rho_2^3\rho_1 - 8\rho_2^3\rho_1^2 \\
&+ 648 - 568\rho_2\rho_1^2 + 64\rho_2\rho_1^3 + 408\rho_1^2 + 1440\rho_2\rho_1 \\
&- 48\rho_1^3 + 312\rho_2^2 - 464\rho_2^2\rho_1 + 168\rho_2^2\rho_1^2 \\
P_{22} =& -16\rho_2^2\rho_1^3 - 7152\rho_1 - 2592\rho_2 - 48\rho_2^3 + 40\rho_2^3\rho_1 - 8\rho_2^3\rho_1^2 \\
&- 936\rho_2\rho_1^2 + 96\rho_2\rho_1^3 + 2728\rho_1^2 + 2808\rho_2\rho_1 - 336\rho_1^3 \\
&+ 528\rho_2^2 - 536\rho_2^2\rho_1 + 168\rho_2^2\rho_1^2 + 5976, \\
P_{31} =& 16\rho_2^2\rho_1^3 + 3984\rho_1 + 1680\rho_2 + 24\rho_2^3 - 32\rho_2^3\rho_1 + 8\rho_2^3\rho_1^2 \\
&- 2448 + 1072\rho_2\rho_1^2 - 128\rho_2\rho_1^3 - 1776\rho_1^2 - 2624\rho_2\rho_1 \\
&+ 240\rho_1^3 - 360\rho_2^2 + 528\rho_2^2\rho_1 - 184\rho_2^2\rho_1^2 \\
P_{32} =& 16\rho_2^2\rho_1^3 + 912\rho_1 + 1200\rho_2 + 48\rho_2^3 - 40\rho_2^3\rho_1 + 8\rho_2^3\rho_1^2 \\
&- 792 + 504\rho_2\rho_1^2 - 64\rho_2\rho_1^3 - 360\rho_1^2 - 1336\rho_2\rho_1 \\
&+ 48\rho_1^3 - 456\rho_2^2 + 464\rho_2^2\rho_1 - 152\rho_2^2\rho_1^2, \\
P_{33} =& -16\rho_2^2\rho_1^3 - 5520\rho_1 - 3360\rho_2 - 48\rho_2^3 + 40\rho_2^3\rho_1 - 8\rho_2^3\rho_1^2 \\
&+ 4896 - 1200\rho_2\rho_1^2 + 128\rho_2\rho_1^3 + 2016\rho_1^2 + 3568\rho_2\rho_1 \\
&- 240\rho_1^3 + 720\rho_2^2 - 696\rho_2^2\rho_1 + 200\rho_2^2\rho_1^2.
\end{aligned}$$

The matrix inequalities (9) were checked over a fine grid in the

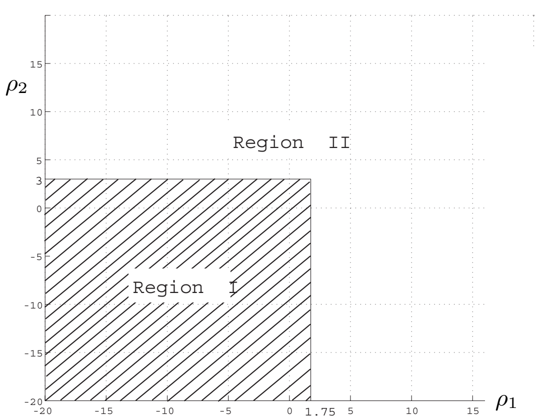

Fig. 1. Stability region for Example 2.

parameter space ρ_1-ρ_2. The results are shown in Fig. 1. Region I denotes the domain where both inequalities were satisfied. On the other hand, in Region II at least one of the matrix inequalities in (9) was violated. The results coincide with those of [22] and [23].

In the next section we present LMI conditions for checking the matrix inequalities (9). With the use of these LMI's gridding of the parameter space can be avoided.

IV. LMI CONDITIONS

A. An Alternative Expression for the Parameter-Dependent Lyapunov Matrix

Theorem 3.1 provides Lyapunov-based conditions for checking the stability of (1). Condition (9) represents a parameterized family of matrix inequalities. It is of interest to develop finite-dimensional counterparts to (9). In this section, and in preparation for the derivation of such conditions, we present a more convenient form for expressing a polynomially-dependent matrix. The notation and results below closely follow those of [13].

To this end, let the vector $z^{[q]} := (1, z, z^2, \cdots, z^{q-1})^\intercal \in \mathbb{R}^q$. In the sequel we will often use the shorthand notation $\otimes_{i=m}^{1} z_i^{[k_i]}$ to denote the product $z_m^{[k_m]} \otimes \cdots \otimes z_1^{[k_1]}$. Next, notice that the matrix $P(\rho)$ in (10) can be written as follows,

$$P(\rho) = \sum_{|k|=0}^{k_1=\bar{k}_1,\ldots,k_m=\bar{k}_m} \rho_1^{k_1} \ldots \rho_m^{k_m} P_{k_1,\ldots,k_m}, \qquad (16)$$

where $\deg_i P(\rho) = \bar{k}_i \le \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}(n - r_i)(n + r_i + 1)\}$ and where $r_i = \dim \mathcal{N}(A_i)$. Let $\bar{\alpha}_i := \lceil \frac{\bar{k}_i}{2} \rceil + 1$. Then the parameter-dependent matrix in (16) can be expressed as

$$P(\rho) = \Big(\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i]} \otimes I_n \Big)^\intercal P_\Sigma \Big(\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i]} \otimes I_n \Big) \qquad (17)$$

where P_Σ is a symmetric matrix of dimension $(\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_m n) \times (\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_m n)$. The matrix P_Σ can be divided into $(\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_m)\times(\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_m)$ blocks, each of dimension $n\times n$. In fact, the nonzero blocks of the matrix P_Σ are just the matrix coefficients of the polynomial matrix $P(\rho)$; see also Example 3 below.

Similarly, $\rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n$ is a matrix composed of $\bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_m$ blocks, each block having dimension $n \times n$.

The matrix P_Σ in (17) is not unique. One method for constructing a possible P_Σ is as follows.

Definition 4: Given $\bar{\alpha} := (\bar{\alpha}_1\,\bar{\alpha}_2\,\ldots\bar{\alpha}_m)^\intercal \in \mathbb{N}^m$ the *index function* $f_{\bar{\alpha}}$ is defined as

$$\begin{aligned}
f_{\bar{\alpha}}(\alpha_1, \alpha_2, \ldots, \alpha_m) :=\ & \bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{m-1}\alpha_m \\
&+ \bar{\alpha}_1\bar{\alpha}_2\cdots\bar{\alpha}_{m-2}\alpha_{m-1} \\
&+ \cdots + \bar{\alpha}_1\alpha_2 + \alpha_1 + 1.
\end{aligned}$$

Let the square matrix $\bar{P}_\Sigma$ having the same dimension as P_Σ and let $\bar{P}_{\Sigma,(i,j)}$ stand for the (i, j) block of $\bar{P}_\Sigma$. For every non-zero term $\rho_1^{k_1}\rho_2^{k_2}\ldots\rho_m^{k_m}P_{k_1,k_2,\ldots,k_m}$ in (16), let $\alpha_i = \lceil \frac{k_i}{2} \rceil$ and $\beta_i = \lfloor \frac{k_i}{2} \rfloor$ for $i = 1, 2, \ldots, m$ and let

$$\bar{P}_{\Sigma,(f_1,f_2)} = P_{k_1,k_2,\ldots,k_m},$$

where the indices f_1 and f_2 are given by $f_1 = f_{\bar{\alpha}}(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $f_2 = f_{\bar{\alpha}}(\beta_1, \beta_2, \ldots, \beta_m)$. Finally, let

$$P_\Sigma = \frac{1}{2}(\bar{P}_\Sigma + \bar{P}_\Sigma^\intercal).$$

Example 3: Consider the parameter-dependent matrix

$$\begin{aligned}
P(\rho) &= \sum_{k_1+k_2=0}^{k_1+k_2=2} \rho_1^{k_1}\rho_2^{k_2}P_{k_1,k_2} \\
&= P_{0,0} + \rho_1P_{1,0} + \rho_2P_{0,1} + \rho_1^2P_{2,0} + \rho_1\rho_2P_{1,1} + \rho_2^2P_{0,2}
\end{aligned}$$

The matrix $P(\rho)$ can be written as

$$P(\rho) = \Big(\rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I\Big)^\intercal P_\Sigma \Big(\rho_2^{[2]} \otimes \rho_1^{[2]} \otimes I\Big)$$

for some matrix P_Σ. Using the previous approach one obtains for example,

$$P_\Sigma = \frac{1}{2}\begin{bmatrix} 2P_{0,0} & P_{1,0} & P_{0,1} & P_{1,1} \\ P_{1,0} & 2P_{2,0} & 0 & 0 \\ P_{0,1} & 0 & 2P_{0,2} & 0 \\ P_{1,1} & 0 & 0 & 0 \end{bmatrix}.$$

Nonetheless, it can be easily verified that the following matrix will also work

$$P_\Sigma = \frac{1}{2}\begin{bmatrix} 2P_{0,0} & P_{1,0} & P_{0,1} & 0 \\ P_{1,0} & 2P_{2,0} & P_{1,1} & 0 \\ P_{0,1} & P_{1,1} & 2P_{0,2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We now turn our attention to finding a similarly convenient form for the polynomial matrix

$$R(\rho) := A(\rho)P(\rho) + P(\rho)A^{\mathsf{T}}(\rho). \tag{18}$$

It is desirable to express $R(\rho)$ in a form similar to that of (17). In order to achieve this we use the following results [13].

Let $\hat{J}_k := [I_k \; 0_{k\times 1}]$ and $\check{J}_k := [0_{k\times 1} \; I_k]$. It is clear that $\hat{J}_k z^{[k+1]} = z^{[k]}$ and $\check{J}_k z^{[k+1]} = z z^{[k]}$. Moreover, for any matrices $A \in \mathbb{R}^{n\times n}$ and $M \in \mathbb{R}^{p\times q}$, the following properties can be easily verified [13]:

(i) $\big(z^{[k]} \otimes I_p\big)M = \big(I_k \otimes M\big)\big(z^{[k]} \otimes I_q\big)$

(ii) $\big(z_m^{[k_m]} \otimes \cdots \otimes z_1^{[k_1]} \otimes I_n\big)A = z_m^{[k_m]} \otimes \cdots \otimes z_1^{[k_1]} \otimes A = \big(I_{k_1 k_2 \cdots k_m} \otimes A\big)\big(z_m^{[k_m]} \otimes \cdots \otimes z_1^{[k_1]} \otimes I_n\big)$.

(iii) $z_m^{[k_m]} \otimes \cdots \otimes z_1^{[k_1]} \otimes I_n = \big(\hat{J}_{k_m} \otimes \cdots \otimes \hat{J}_{k_1} \otimes I_n\big)\big(z_m^{[k_m+1]} \otimes \cdots \otimes z_1^{[k_1+1]} \otimes I_n\big)$.

(iv) $z_m^{[k_m]} \otimes \cdots \otimes z_j z^{[k_j]} \otimes \cdots \otimes z_1^{[k_1]} \otimes I_n = \big(\hat{J}_{k_m} \otimes \cdots \otimes \check{J}_{k_j} \otimes \ldots \otimes \hat{J}_{k_1} \otimes I_n\big)\big(z_m^{[k_m+1]} \otimes \cdots \otimes z_1^{[k_1+1]} \otimes I_n\big)$.

Lemma 4.1 ([13]): Given a matrix $A(\rho) \in \mathbb{R}^{n\times n}$ as in (1) and a symmetric matrix $P(\rho) \in \mathbb{R}^{n\times n}$ as in (17), let $R(\rho) = A^{\mathsf{T}}(\rho)P(\rho) + P(\rho)A(\rho)$. Then

$$R(\rho) = \Big(\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i+1]} \otimes I_n\Big)^{\mathsf{T}} R_\Sigma \Big(\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i+1]} \otimes I_n\Big) \tag{19}$$

and

$$R_\Sigma := H_\Sigma^{\mathsf{T}} P_\Sigma F_\Sigma + F_\Sigma^{\mathsf{T}} P_\Sigma H_\Sigma \tag{20}$$

where,

$$H_\Sigma := \hat{J}_{\bar{\alpha}_m} \otimes \ldots \otimes \hat{J}_{\bar{\alpha}_1} \otimes I_n,$$

and

$$F_\Sigma := \hat{J}_{\bar{\alpha}_m} \otimes \ldots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_0 + \sum_{i=1}^{m} \hat{J}_{\bar{\alpha}_m} \otimes \ldots \hat{J}_{\bar{\alpha}_{i+1}} \otimes \check{J}_{\bar{\alpha}_i} \otimes \hat{J}_{\bar{\alpha}_{i-1}} \otimes \ldots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_i. \quad \square$$

The important observation here is that, for given A_i, $i = 1, 2, \ldots, m$, the matrix R_Σ depends linearly on P_Σ.

B. Stability Conditions in Terms of LMI's

In this section, we express the infinite-dimensional matrix inequalities $P(\rho) > 0$ and $R(\rho) < 0$ into a pair of finite-dimensional, linear matrix inequalities. The following lemma will be helpful in this context.

Lemma 4.2 ([5]): Let matrices $Q = Q^{\mathsf{T}}$, F, and a compact subset of real matrices $\mathcal{H}$ be given. The following statements are equivalent:

(i) For each $H \in \mathcal{H}$

$$\xi^{\mathsf{T}} Q \xi < 0, \quad \forall\, \xi \neq 0 \text{ such that } HF\xi = 0. \tag{21}$$

(ii) There exist $\Theta = \Theta^{\mathsf{T}}$ such that

$$Q + F^{\mathsf{T}}\Theta F < 0, \quad N_H^{\mathsf{T}}\Theta N_H \geq 0, \quad \forall\, H \in \mathcal{H},$$

where N_H denotes a matrix whose columns form a basis for the null space of H. $\square$

Given an integer $n \geq 2$ and the vector $K = (k_1, k_2, \ldots, k_m)^{\mathsf{T}} \in \mathbb{N}^m$, let the matrices

$$C_{K,i} := \check{\hat{J}}_m \otimes \check{\hat{J}}_{m-1} \otimes \cdots \otimes \check{\hat{J}}_{i+1} \otimes \hat{J}_{k_i-1} \otimes \check{\hat{J}}_{i-1} \otimes \cdots \otimes \check{\hat{J}}_1 \otimes I_n \tag{22}$$

$$J_{K,i} := \check{\hat{J}}_m \otimes \check{\hat{J}}_{m-1} \otimes \cdots \otimes \check{\hat{J}}_{i+1} \otimes \check{J}_{k_i-1} \otimes \check{\hat{J}}_{i-1} \otimes \cdots \otimes \check{\hat{J}}_1 \otimes I_n \tag{23}$$

where $\check{\hat{J}}_i$, $(i = 1, \ldots, m)$ stands for either $\hat{J}_{k_i-1}$ or $\check{J}_{k_i-1}$. Each matrix $C_{K,i}$ and $J_{K,i}$ is of dimension $q \times \ell$ where $q = (k_1 - 1)(k_2 - 1)\ldots(k_m - 1)n$ and $\ell = k_1 k_2 \ldots k_m n$. Moreover, since $\check{\hat{J}}_i$ can be either $\hat{J}_{k_i-1}$ or $\check{J}_{k_i-1}$, there are 2^{m-1} combinations for $C_{K,i}$ and $J_{K,i}$ for each i. Enumerating these matrices by $C_{K,i}^j$ and $J_{K,i}^j$, $(j = 1, 2, \ldots, 2^{m-1})$ we introduce the matrices $\mathcal{C}_K$, $\mathcal{J}_K$ and $\Delta_K(\rho)$ as follows

$$\mathcal{C}_K := \begin{bmatrix} C_{K,1}^1 \\ C_{K,1}^2 \\ \vdots \\ C_{K,1}^{2^{m-1}} \\ C_{K,2}^1 \\ \vdots \\ C_{K,i}^j \\ \vdots \\ C_{K,m}^{2^{m-1}} \end{bmatrix}, \quad \mathcal{J}_K := \begin{bmatrix} J_{K,1}^1 \\ J_{K,1}^2 \\ \vdots \\ J_{K,1}^{2^{m-1}} \\ J_{K,2}^1 \\ \vdots \\ J_{K,i}^j \\ \vdots \\ J_{K,m}^{2^{m-1}} \end{bmatrix}, \tag{24}$$

and

$$\Delta_K(\rho) := \operatorname{diag}\left[\rho_1 I_{q2^{m-1}}, \ldots, \rho_m I_{q2^{m-1}}\right] \tag{25}$$

It can be readily verified that the matrices $\mathcal{C}_K$ and $\mathcal{J}_K$ have dimension $2^{m-1}mq \times \ell$ and $\Delta_K(\rho)$ has dimension $2^{m-1}mq \times 2^{m-1}mq$, respectively.

Example 4: Let $K = (2,\, 2)^{\mathsf{T}}$ and $n = 2$. Then

$$\mathcal{C}_K = \begin{bmatrix} C_{K,1}^1 \\ C_{K,1}^2 \\ C_{K,2}^1 \\ C_{K,2}^2 \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix},$$

$$\mathcal{J}_K = \begin{bmatrix} J_{K,1}^1 \\ J_{K,1}^2 \\ J_{K,2}^1 \\ J_{K,2}^2 \end{bmatrix} = \begin{bmatrix} \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix},$$

and

$$\Delta_K(\rho) = \begin{bmatrix} \rho_1 I_4 & \\ & \rho_2 I_4 \end{bmatrix}.$$

Lemma 4.3: Let $K = (k_1, k_2, \ldots, k_m)^{\mathsf{T}} \in \mathbb{N}^m$ and define the matrices $\mathcal{J}_K$, $\mathcal{C}_K$ and $\Delta_K(\rho)$ as in (24) and (25). Let the sets

$$\mathcal{S}_1 := \{\, \xi \in \mathbb{R}^\ell : \big(\mathcal{J}_K - \Delta_K(\rho)\mathcal{C}_K\big)\xi = 0,\ \xi \neq 0,\ \text{some } \rho \in \mathbb{R}^m \,\}$$

and

$$\mathcal{S}_2 := \{\, \xi \in \mathbb{R}^\ell : \xi = \big(\otimes_{i=m}^{1} \rho_i^{[k_i]} \otimes I_n\big)x,\ \xi \neq 0,\ \text{some } \rho \in \mathbb{R}^m,\ \text{and } x \in \mathbb{R}^n \,\}.$$

Then $\mathcal{S}_1 = \mathcal{S}_2$. $\square$

Proof: Assume that $\xi \in \mathcal{S}_2$. Then there exists $\rho \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ such that $\xi = \big(\rho_m^{[k_m]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n\big)x$. Use this particular ρ and construct the matrices $\mathcal{J}_K$, $\mathcal{C}_K$ and $\Delta_K(\rho)$ as in (24) and

(25). It can be easily verified that $\big(\mathcal{J}_K - \Delta_K(\rho)\mathcal{C}_K\big)\big(\rho_m^{[k_m]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes I_n\big) = 0$. It follows that $\xi \in \mathcal{S}_1$. Hence $\mathcal{S}_2 \subseteq \mathcal{S}_1$. Assume now that $\xi \in \mathcal{S}_1$ and $\xi \neq 0$. Rewrite $\xi \in \mathbb{R}^{k_m \cdots k_1 n}$ in the form $\xi = \xi_m \otimes \xi_{m-1} \otimes \cdots \otimes \xi_1 \otimes x$ where $\xi_i \in \mathbb{R}^{k_i}$ and $x \in \mathbb{R}^n$. It is easy to show that such vectors $\xi_1, \xi_2, \ldots, \xi_m$ and x always exist. From hypothesis, we have that

$$(\mathcal{J}_K - \Delta_K(\rho)\mathcal{C}_K)\,(\xi_m \otimes \xi_{m-1} \otimes \cdots \otimes \xi_1 \otimes x) = 0, \quad \rho \in \mathbb{R}^m \tag{26}$$

The previous equation implies, in particular, that

$$\Big(\check{\hat{J}}_m \otimes \check{\hat{J}}_{m-1} \otimes \cdots \otimes \check{\hat{J}}_2 \otimes (\check{J}_{k_1-1} - \rho_1 \hat{J}_{k_1-1}) \otimes I_n\Big) \cdot \big(\xi_m \otimes \xi_{m-1} \otimes \cdots \otimes \xi_2 \otimes (\xi_1 \otimes x)\big) = 0,$$

or that

$$\Big((\check{\hat{J}}_m \otimes \cdots \otimes \check{\hat{J}}_2)(\xi_m \otimes \cdots \otimes \xi_2)\Big) \otimes \Big((\check{J}_{k_1-1} - \rho_1 \hat{J}_{k_1-1})\xi_1\Big) \otimes \big(I_n \otimes x\big) = 0. \tag{27}$$

Assume now that $(\check{\hat{J}}_m \otimes \cdots \otimes \check{\hat{J}}_2)(\xi_m \otimes \cdots \otimes \xi_2) = \otimes_{i=m}^{2} \check{\hat{J}}_i \xi_i = 0$. If $\check{\hat{J}}_j \xi_j = 0$ for some $j \in \{2, \ldots, m\}$ then $\hat{J}_{k-j}\xi_j = \check{J}_{k-j}\xi_j = 0$. It follows that $\xi_j = 0$ and thus $\xi = 0$, leading to a contradiction. Similarly, if $I_n \otimes x = 0$ we have that $x = 0$ and hence $\xi = 0$, also a contradiction. Therefore, from (27) we have necessarily,

$$(\check{J}_{k_1-1} - \rho_1 \hat{J}_{k_1-1})\xi_1 = 0.$$

Using now Lemma 4.3 of [15] the last equation implies that $\xi_1 = \rho_1^{[k_1]}$ for some $\rho_1 \in \mathbb{R}$. Therefore, $\xi = \xi_m \otimes \cdots \otimes \xi_2 \otimes \rho_1^{[k_1]} \otimes x$. Repeating this process, one can show that (26) also implies that

$$\Big(\check{\hat{J}}_m \otimes \cdots \otimes \check{\hat{J}}_3 \otimes (\check{J}_{k_2-1} - \rho_2 \hat{J}_{k_2-1}) \otimes \check{\hat{J}}_1 \otimes I_n\Big) \cdot \Big(\xi_m \otimes \cdots \otimes \xi_2 \otimes \rho_1^{[k_1]} \otimes x\Big) = 0,$$

or that,

$$\Big((\check{\hat{J}}_m \otimes \cdots \otimes \check{\hat{J}}_3)(\xi_m \otimes \cdots \otimes \xi_3)\Big) \otimes \Big((\check{J}_{k_2-1} - \rho_2 \hat{J}_{k_2-1})\xi_2\Big) \otimes \Big((\check{\hat{J}}_1 \otimes I_n)(\rho_1^{[k_1]} \otimes x)\Big) = 0, \tag{28}$$

from which we have that, necessarily,

$$\Big(\check{J}_{k_2-1} - \rho_2 \hat{J}_{k_2-1}\Big)\,\xi_2 = 0,$$

implying that $\xi_2 = \rho_2^{[k_2]}$. It follows that ξ can be written as $\xi = \xi_m \otimes \cdots \otimes \rho_2^{[k_2]} \otimes \rho_1^{[k_1]} \otimes x$. Repeating this process, one obtains that ξ can be written as $\xi = \rho_m^{[k_m]} \otimes \cdots \otimes \rho_1^{[k_1]} \otimes x = \big(\otimes_{i=m}^{1} \rho_i^{[k_i]} \otimes I_n\big)x$, for some $\rho_i \in \mathbb{R}$ $(i = 1, \cdots, m)$ and $x \in \mathbb{R}^n$. Therefore $\xi \in \mathcal{S}_2$ and thus $\mathcal{S}_1 \subseteq \mathcal{S}_2$. This completes the proof that $\mathcal{S}_1 = \mathcal{S}_2$. ∎

Returning now to Theorem 3.1, recall that the condition $P(\rho) > 0$ is equivalent to the fact that for any $x \in \mathbb{R}^n$, $x^\intercal P(\rho) x > 0$, which via (17) can be written as

$$x^\intercal (\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i]} \otimes I_n)^\intercal P_\Sigma (\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i]} \otimes I_n) x > 0.$$

Let $\xi = (\rho_m^{[\bar{\alpha}_m]} \otimes \cdots \otimes \rho_1^{[\bar{\alpha}_1]} \otimes I_n)x$. Using Lemma 4.3 the condition $P(\rho) > 0$ is equivalent to the condition

$$\xi^\intercal P_\Sigma \xi > 0 \text{ such that } (\mathcal{J}_K - \Delta_K(\rho)\mathcal{C}_K)\xi = 0, \quad \forall \xi \neq 0.$$

For the following result we assume that Ω is a compact and connected subsect of $\mathbb{R}^m$. Without loss of generality we take $\Omega = [-1, +1]^m$.

Theorem 4.1: Assume that there exist positive definite matrices $D_1, D_2, \ldots, D_m \in \mathbb{R}^{q2^{m-1} \times q2^{m-1}}$ where $q = n(\bar{\alpha}_1 - 1)(\bar{\alpha}_2 - 1)\ldots(\bar{\alpha}_m - 1)$ and skew-symmetric matrices $G_1, G_2, \ldots, G_n \in \mathbb{R}^{q2^{m-1} \times q2^{m-1}}$ such that

$$-P_\Sigma + \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix}^\intercal \begin{bmatrix} -D & G \\ G^\intercal & D \end{bmatrix} \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix} < 0, \tag{29}$$

where $D := \mathrm{diag}[D_1, \ldots, D_m]$, $G := \mathrm{diag}[G_1, \ldots, G_m]$, $\mathcal{J}_K$ and $\mathcal{C}_K$ as in (24) with $K = (\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_m)$. Then

$$\big(\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i]} \otimes I_n\big)^\intercal P_\Sigma \big(\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i]} \otimes I_n\big) > 0$$

for all $\rho = (\rho_1, \rho_2, \cdots, \rho_m)^\intercal \in [-1, +1]^m$. □

Proof: The matrix inequality $P(\rho) > 0$ for all $\rho \in [-1, +1]^m$ is equivalent to the scalar inequality $x^\intercal P(\rho) x > 0$ for all $x \in \mathbb{R}^n$ and for all $\rho \in [-1, +1]^m$. Let $\xi = \big(\otimes_{i=m}^{1} \rho_i^{[\bar{\alpha}_i]} \otimes I_n\big)x$. It follows from Lemma 4.3 that $(\mathcal{J}_K - \Delta_K(\rho)\mathcal{C}_K)\xi = 0$, where $\mathcal{J}_K$, $\Delta_K(\rho)$ and $\mathcal{C}_K$ as in (24). Therefore, the condition $P(\rho) > 0$ for all $\rho \in [-1, +1]^m$ is equivalent to the condition

$$\xi^\intercal P_\Sigma \xi > 0, \quad \text{such that} \quad [I,\ -\Delta_K(\rho)] \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix} \xi = 0, \quad \forall \xi \neq 0. \tag{30}$$

and for all $\rho \in [-1, +1]^m$. According to Lemma 4.2, inequality (30) is equivalent to the existence of a matrix $\Theta = \Theta^\intercal$, such that

$$-P_\Sigma + \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix}^\intercal \Theta \begin{bmatrix} \mathcal{J}_K \\ \mathcal{C}_K \end{bmatrix} < 0, \tag{31}$$

and

$$\begin{bmatrix} \Delta_K(\rho) \\ I \end{bmatrix}^\intercal \Theta \begin{bmatrix} \Delta_K(\rho) \\ I \end{bmatrix} \geq 0, \quad \forall \rho \in [-1, +1]^m. \tag{32}$$

Now, the condition $\rho_i \in [-1, +1]$ for all $i = 1, 2, \ldots, m$ is equivalent to the existence of positive definite matrices $D_1, D_2, \ldots, D_m \in \mathbb{R}^{q2^{m-1} \times q2^{m-1}}$ and skew-symmetric matrices $G_1, G_2, \ldots, G_m \in \mathbb{R}^{q2^{m-1} \times q2^{m-1}}$ such that

$$\begin{bmatrix} (1-\rho_1^2)D_1 & & & \\ & (1-\rho_2^2)D_2 & & \\ & & \ddots & \\ & & & (1-\rho_m^2)D_m \end{bmatrix} + \begin{bmatrix} \rho_1(G_1 + G_1^\intercal) & & & \\ & \rho_2(G_2 + G_2^\intercal) & & \\ & & \ddots & \\ & & & \rho_m(G_m + G_m^\intercal) \end{bmatrix} \geq 0,$$

Equivalently,

$$\begin{bmatrix} \Delta_K(\rho) \\ I \end{bmatrix}^\intercal \begin{bmatrix} -D & G \\ G^\intercal & D \end{bmatrix} \begin{bmatrix} \Delta_K(\rho) \\ I \end{bmatrix} \geq 0, \tag{33}$$

where $D = \mathrm{diag}[D_1, \ldots, D_m]$ and $G = \mathrm{diag}[G_1, \ldots, G_m]$.

In summary, the condition $P(\rho) > 0$ is equivalent to the feasibility of the inequalities (31) and (32), the latter being implied by the feasibility of inequality (33). ∎

Using the previous result we are now ready to use the main result from Theorem 3.1 to cast the stability question of (1) over the domain $\Omega = [-1, +1]^m$ as a convex feasibility problem in terms of LMI's.

Theorem 4.2: Let $A_i \in \mathbb{R}^{n \times n}$, $(i = 0, 1, 2, \ldots, m)$ and $\rho = (\rho_1, \rho_2, \ldots, \rho_m)^\intercal \in [-1, +1]^m$ and assume that $\dim \mathcal{N}(A_i) = r_i$. Let $\bar{k}_i = \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}(n -$

$r_i)(n+r_i+1)\}$, $\bar{\alpha}_i = \lceil \frac{\bar{k}_i}{2} \rceil + 1$, $q_1 = n\prod_{i=1}^m(\bar{\alpha}_i - 1)$, $q_2 = n\prod_{i=1}^m \bar{\alpha}_i$, $K_1 = (\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_m)^\intercal$ and $K_2 = ((\bar{\alpha}_1+1), (\bar{\alpha}_2+1), \ldots, (\bar{\alpha}_m+1))^\intercal$. Assume that There exist symmetric matrix $P_\Sigma \in \mathbb{R}^{q_2 \times q_2}$, positive definite matrices $D_1, D_2, \ldots, D_m \in \mathbb{R}^{q_1 2^{m-1} \times q_1 2^{m-1}}$, skew-symmetric matrices $G_1, G_2, \ldots, G_m \in \mathbb{R}^{q_1 2^{m-1} \times q_1 2^{m-1}}$ positive definite matrices $E_1, E_2, \ldots, E_m \in \mathbb{R}^{q_2 2^{m-1} \times q_2 2^{m-1}}$ and skew-symmetric matrices $\Gamma_1, \Gamma_2, \ldots, \Gamma_m \in \mathbb{R}^{q_2 2^{m-1} \times q_2 2^{m-1}}$ such that

$$-P_\Sigma + \begin{bmatrix} \mathcal{J}_{K_1} \\ \mathcal{C}_{K_1} \end{bmatrix}^\intercal \begin{bmatrix} -D & G \\ G^\intercal & D \end{bmatrix} \begin{bmatrix} \mathcal{J}_{K_1} \\ \mathcal{C}_{K_1} \end{bmatrix} < 0,$$

$$H_\Sigma^\intercal P_\Sigma F_\Sigma + F_\Sigma^\intercal P_\Sigma H_\Sigma + \begin{bmatrix} \mathcal{J}_{K_2} \\ \mathcal{C}_{K_2} \end{bmatrix}^\intercal \begin{bmatrix} -E & \Gamma \\ \Gamma^\intercal & E \end{bmatrix} \begin{bmatrix} \mathcal{J}_{K_2} \\ \mathcal{C}_{K_2} \end{bmatrix} < 0,$$

where,

$$\begin{aligned} H_\Sigma &= \hat{J}_{\bar{\alpha}_m} \otimes \ldots \otimes \hat{J}_{\bar{\alpha}_1} \otimes I_n, \\ F_\Sigma &= \hat{J}_{\bar{\alpha}_m} \otimes \ldots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_0 \\ &+ \sum_{i=1}^m \hat{J}_{\bar{\alpha}_m} \otimes \ldots \hat{J}_{\bar{\alpha}_{i+1}} \otimes \check{J}_{\bar{\alpha}_i} \otimes \hat{J}_{\bar{\alpha}_{i-1}} \otimes \ldots \otimes \hat{J}_{\bar{\alpha}_1} \otimes A_i \end{aligned}$$

and where $D = \operatorname{diag}\left[D_1, \ldots, D_m\right]$, $G = \operatorname{diag}\left[G_1, \ldots, G_m\right]$, $E = \operatorname{diag}\left[E_1, \ldots, E_m\right]$ and $\Gamma = \operatorname{diag}\left[\Gamma_1, \ldots, \Gamma_m\right]$. Then the matrix $A(\rho) = \sum_{i=0}^m \rho_i A_i$ is Hurwitz for all $\rho \in [-1, +1]^m$.□

V. Numerical Example

A numerical example is given below to demonstrate the application of Theorem 4.2.

Example 5: Consider the two-parameter dependent matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$ where

$$A_0 = \begin{bmatrix} -2 & 7 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{1}{2} & 3 \\ 0 & -\frac{1}{3} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -4 \\ 0 & \frac{1}{2} \end{bmatrix}$$

The eigenvalues of $A(\rho)$ are given by $\lambda_1 = -2 + \frac{1}{2}\rho_1 + \rho_2$ and $\lambda_2 = -1 - \frac{1}{3}\rho_1 + \frac{1}{2}\rho_2$. Therefore, $A(\rho)$ is Hurwitz for all $(\rho_1, \rho_2) \in [-1, +1]^2$. For this problem $n = 2, m = 2, \bar{k}_1 = \bar{k}_2 = \bar{k} = \frac{1}{2}n(n+1) - 1 = 2$, $\bar{\alpha}_1 = \bar{\alpha}_2 = \bar{\alpha} = \lceil \frac{\bar{k}}{2} \rceil + 1 = 2$. Moreover, $q_1 = (\bar{\alpha} - 1)^m n = 2$ and $q_2 = \bar{\alpha}^m n = 8$. Thus, $K_1 = (\bar{\alpha}, \bar{\alpha})^\intercal = (2, 2)^\intercal$, $K_2 = \left((\bar{\alpha}+1), (\bar{\alpha}+1)\right)^\intercal = (3, 3)^\intercal$. Hence, $P_\Sigma \in \mathbb{R}^{8\times 8}$, $D_1, D_2 \in \mathbb{R}^{4 \times 4}$ and $E_1, E_2 \in \mathbb{R}^{16 \times 16}$. The matrices $\mathcal{C}_{K_1}, \mathcal{C}_{K_2}, \mathcal{J}_{K_1}$ and $\mathcal{J}_{K_2}$ are given by

$$\mathcal{C}_{K_1} = \begin{bmatrix} \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix}, \; \mathcal{J}_{K_1} = \begin{bmatrix} \hat{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \hat{J}_1 \otimes I_2 \\ \check{J}_1 \otimes \check{J}_1 \otimes I_2 \end{bmatrix},$$

$$\mathcal{C}_{K_2} = \begin{bmatrix} \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \hat{J}_2 \otimes \check{J}_2 \otimes I_2 \end{bmatrix}, \; \mathcal{J}_{K_2} = \begin{bmatrix} \hat{J}_2 \otimes \check{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \check{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ \check{J}_2 \otimes \check{J}_2 \otimes I_2 \end{bmatrix},$$

and H_Σ and F_Σ are given by

$$\begin{aligned} H_\Sigma &= \hat{J}_2 \otimes \hat{J}_2 \otimes I_2 \\ F_\Sigma &= \hat{J}_2 \otimes \hat{J}_2 \otimes A_0 + \hat{J}_2 \otimes \check{J}_2 \otimes A_1 + \check{J}_2 \otimes \hat{J}_2 \otimes A_2 \,. \end{aligned}$$

Solving the LMI's of Theorem 4.2 one obtains a possible solution for P_Σ and D_1, D_2 as follows

$$D_1 = \begin{bmatrix} 9.5250 & 6.9870 & -1.0628 & -2.8941 \\ 6.9870 & 65.6791 & 0.0831 & -3.5221 \\ -1.0628 & 0.0831 & 9.9204 & 3.9213 \\ -2.8941 & -3.5221 & 3.9213 & 69.1830 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 9.4393 & 9.0572 & -1.0995 & 1.3359 \\ 9.0572 & 69.1088 & 2.2309 & 2.5806 \\ -1.0995 & 2.2309 & 9.7363 & 6.1760 \\ 1.3359 & 2.5806 & 6.1760 & 71.0502 \end{bmatrix}.$$

and $G_1 = G_2 = 0$. Due to space limitations the numerical values of E_1, E_2, Γ_1 and Γ_2 are omitted. Using the results of Theorem 4.2 we conclude that $A(\rho)$ is Hurwitz for all $\rho \in [-1, +1]^2$.

VI. Conclusions

In this paper we generalize the results of [15] to multi-parameter LTI systems. Specifically, we show that the stability of a multi-parameter, affinely-dependent LTI systems of the form $\dot{x}(t) = A(\rho)x(t)$ is equivalent to the existence of a polynomial, parameter-dependent Lyapunov matrix $P(\rho)$ of a known degree satisfying the matrix inequalities $P(\rho) > 0$ and $A(\rho)P(\rho) + P(\rho)A^\intercal(\rho) < 0$. We also show that testing these two inequalities over a compact, connected set can be cast into a finite-dimensional convex feasibility problem in terms of LMI's. Of course, the dimension of the resulting LMI's is still high (recall that the problem is NP-hard). Comparisons of the computational complexity of the proposed approach with other methods investigating the stability of PDLTI systems will be reported in the future.

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$$P_{\Sigma} = \begin{bmatrix} 27.9276 & 51.0744 & -0.7887 & 16.8161 & -1.4679 & -20.1029 & -1.2207 & 1.5418 \\ 51.0744 & 486.7451 & 13.0354 & 101.0287 & 8.6774 & 8.6872 & 9.1953 & 52.8509 \\ -0.7887 & 13.0354 & 9.1765 & 17.2759 & 0.8705 & -0.8746 & 1.4118 & -4.9874 \\ 16.8161 & 101.0287 & 17.2759 & 244.3918 & 4.3396 & -39.0897 & 6.6145 & 46.1743 \\ -1.4679 & 8.6774 & 0.8705 & 4.3396 & 8.7297 & 8.4651 & -0.8661 & 4.9237 \\ -20.1029 & 8.6872 & -0.8746 & -39.0897 & 8.4651 & 239.4326 & 5.1983 & 13.5890 \\ -1.2207 & 9.1953 & 1.4118 & 6.6145 & -0.8661 & 5.1983 & -0.9111 & -0.3332 \\ 1.5418 & 52.8509 & -4.9874 & 46.1743 & 4.9237 & 13.5890 & -0.3332 & 25.5901 \end{bmatrix}$$

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