On robust semidefinite programming

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Abstract

This paper is devoted to the study of robust semidefinite programming. We show that to the issue of computing the worst-case optimal value of semidefinite programs depending polynomially upon a finite number of bounded scalar parameters, one may associate a countable family of standard semidefinite programs, whose optimal values converge monotonically towards the requested quantity. The results is linked to representation formula and positiveness criterion for matrix-valued polynomials.

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1 Introduction

Semidefinite programming has become a powerful unifying framework for expressing and solving many problems, especially in optimization and control theory [12, 5]. This class of convex optimization problems, solved by efficient interior-point methods, has spread widely, see an up-to-date panorama of the theoretical, applicative and algorithmic aspects in [13]. Among other applications in control (where semidefinite programs are often referred to as linear matrix inequalities, abbreciated LMIs), stability, stabilizability, detectability, H^2 and H^∞ performance analysis, and various related design issues may be stated as LMIs, see e.g. recent progress in [6].

A natural extension was to introduce *robust semidefinite programming*, adapted to semidefinite programming problems with data subject to uncertainties [3]. In the context of control [1], this type of problems appears for example when studying control techniques robust against parametric uncertainty, or gain-scheduling methods, as these issues amount to check solvability of LMIs obtained for different values of some parameters.

Robust semidefinite programming is linked to a difficult problem of algebraic geometry, namely the determination of extrema of multivariate polynomials. Recently, Lasserre [7] and Parrilo [9] have shown independently that to every problem of the latter type may be associated a sequence of standard semidefinite programming relaxations, whose optimal values converge monotonically to the requested worst-case optimal value. Both approaches make large use of techniques and results of algebraic geometry. Solution of robust definite programming problems with polynomial dependence of the parameters may be obtained as a by-product of these results, basically adding the initial decision variables to the set of variables introduced in the relaxed problems.

In the present paper, we present alternative method to solve the robust semidefinite programming method. The result is similar in spirit, as it also provides a sequence of semidefinite relaxations of the initial problem, with increasing precision. However, it is directly obtained from the matrix inequality, without introducing additional variables to obtain scalar inequalities. Moreover, the proof is obtained by a completely different approach, based essentially on Kalman-Yakubovich-Popov lemma. together with a result on existence of polynomial solutions to parameter-dependent SDP problems taken from [4].

The paper is orgamized as follows. In §2 are introduced useful notations. Positiveness of matrix-valued matrices is then studied in §3, where the key result is stated and proved (Theorem 2). It is shown in §4 how this result is linked to representation result for polynomial matrices (Theorem 5), in the same way than the results by Lasserre and Parrilo are linked to representation of polynomials by sums of squares. Last, Theorem 2 is applied to the issue of robust semidefinite programming in §5 (Corollaries 6 and 7).

2 Notations

The matrices I_n, 0_n, $0_{n\times p}$ are the $n\times n$ identity matrix and the $n\times n$ and $n\times p$ zero matrices respectively. The symbol $\otimes$ denotes Kronecker product, the power of Kronecker product being used with the natural meaning: $M^{p\otimes} \stackrel{\mathrm{def}}{=} M^{(p-1)\otimes}\otimes M$. A key property is that $(A\otimes B)(C\otimes D) = (AC\otimes BD)$ for matrices of compatible size. The conjugate and transconjugate of M, are denoted M^T and M^H. The unit circle in $\mathbb{C}$ is denoted as the boundary $\partial\mathbb{D}$ of the unit disk, and the set of positive integers $\mathbb{N}$. Last, the set of symmetric real (resp. hermitian complex) matrices of size $n\times n$ is denoted $\mathcal{S}^n$ (resp. $\mathcal{H}^n$).

We now introduce more specific notations. For any $l\in\mathbb{N}$, for any $v\in\mathbb{C}$, let

$$v^{[l]} \stackrel{\mathrm{def}}{=} \begin{pmatrix} 1 \\ v \\ \vdots \\ v^{l-1} \end{pmatrix} . \tag{1}$$

This notation will permit manipulation of polynomials. Notice in particular that, for a free variable $z\in\mathbb{C}^m$, the vector $(z_m^{[l]}\otimes\cdots\otimes z_1^{[l]})$ contains exactly the l^m monomials in $z_1,\dots,z_m$ of degree at most $l-1$ in each variable.

Last, for any $l\in\mathbb{N}$, let

$$\hat{J}_l \stackrel{\mathrm{def}}{=} (I_l \quad 0_{l\times 1}),\ \check{J}_l \stackrel{\mathrm{def}}{=} (0_{l\times 1} \quad I_l) \ . \tag{2}$$

The previous matrices are fixed elements of $\mathbb{R}^{l\times(l+1)}$.

3 Positiveness of matrix-valued polynomials

Our first result, the key result of the paper, studies the following problem: for given map $G(\delta)$ taking on values in $\mathcal{S}^n$, and polynomial in the components of a vector $\delta\in\mathbb{R}^m$, check whether:

$$\forall\delta\in[-1;+1]^m,\ G(\delta)>0 \ . \tag{3}$$

The polynomial G will be represented as follows. Let us achieve the change of variables

$$\delta = \frac{z+\bar{z}}{2} \ . \tag{4a}$$

When z covers $(\partial\mathbb{D})^m$, then δ varies in the whole set $[-1;+1]^m$. Without loss of generality, one may write

$$G(\delta) = G(\frac{z+\bar{z}}{2}) = (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H G_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) \ , \tag{4b}$$

where $k-1$ is the maximum of the degrees of G in the variables $\delta_1, \dots, \delta_m$ separately, and where G_k is a fixed matrix in $\mathcal{S}^{k^m n}$, called the *coefficient matrix* of G. The expression of G_k in (4) may be deduced easily from the similar expansion of $G(\delta)$ in powers of δ, see the Appendix, where the adequate techniques are developed.

By (4), the initial problem has thus been transformed, without loss of generality, into checking whether

$$\forall z \in (\partial\mathbb{D})^m,\ (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H G_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) > 0 \ , \tag{5}$$

for a fixed coefficient matrix $G_k \in \mathcal{S}^{k^m n}$.

For $l > k$, define the coefficient matrices $G_l \in \mathcal{S}^{l^m n}$ by the following recursion formula:

$$G_{l+1} \overset{\text{def}}{=} \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\\ \alpha=1,\dots,m}} (J_m \otimes \cdots \otimes J_1 \otimes I_n)^T G_l (J_m \otimes \cdots \otimes J_1 \otimes I_n) \ . \tag{6}$$

Recall that $\hat{J}_l, \check{J}_l$ are defined in (2). Before going on further, we clarify in the next result the link existing between the matrix-valued polynomials $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H G_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$ obtained for different values of $l \geq k$.

Lemma 1. *For all $l \geq k$, for all $z \in \mathbb{C}^m$,*

$$(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H G_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) = \prod_{i=1}^m \left(\frac{1+|z_i|^2}{2} \right)^{l-k} G(\frac{z+\bar{z}}{2}) \ . \tag{7}$$

In particular, for all $l \geq k$, for all $z \in (\partial\mathbb{D})^m$,

$$(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H G_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) = G(\frac{z+\bar{z}}{2}) \ . \qquad \blacksquare$$

Proof. Due to (6) and the basic properties:

$$\forall v \in \mathbb{C}, \forall l \in \mathbb{N},\ v^{[l]} = \hat{J}_l v^{[l+1]},\ v v^{[l]} = \check{J}_l v^{[l+1]} \ , \tag{8}$$

one has, for any $l \geq k$,

$$\begin{aligned}
&(z_m^{[l+1]} \otimes \cdots \otimes z_1^{[l+1]} \otimes I_n)^H G_{l+1} (z_m^{[l+1]} \otimes \cdots \otimes z_1^{[l+1]} \otimes I_n) \\
&= \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\\ \alpha=1,\dots,m}} (J_m z_m^{[l+1]} \otimes \cdots \otimes J_1 z_1^{[l+1]} \otimes I_n)^H G_l (J_m z_m^{[l+1]} \otimes \cdots \otimes J_1 z_1^{[l+1]} \otimes I_n) \\
&= \frac{1}{2^m} \sum_{\substack{j_\alpha \in \{0,1\},\\ \alpha=1,\dots,m}} |z_m|^{2j_1} \dots |z_1|^{2j_m} (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H G_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \\
&= \frac{1}{2^m} (1+|z_m|^2) \dots (1+|z_1|^2) (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H G_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)
\end{aligned}$$

The claimed property (7) is then obtained inductively, and the second formula in Lemma 1 is deduced immediatly. $\square$

We are now ready to state the key result of the paper.

Theorem 2. *Let G a polynomial mapping: $\mathbb{R}^m \to \mathcal{S}^n$ of degree at most $k-1$ in each variable. Define its coefficient matrices G_l, $l \geq k$, by the transformation* (4) *and the recursion* (6). *Then, the following assertions are* equivalent.

(i) Matrix $G(\delta)$ is positive definite for any $\delta \in [-1;+1]^m$.

(ii) There exist $l \in \mathbb{N}$, $l \geq k$, and m matrices $Q_{l,i} \in \mathcal{S}^{(l-1)^{m-i+1} l^{i-1} n}$, $i = 1 \dots m$, fulfilling the semidefinite program:

$$\begin{aligned} G_l &+ \sum_{i=1}^{m} \left(\hat{J}_{l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n}\right)^T Q_{l,i} \left(\hat{J}_{l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n}\right) \\ &- \sum_{i=1}^{m} \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{i-1}n}\right)^T Q_{l,i} \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{i-1}n}\right) > 0_{l^m n} \ . \end{aligned} \tag{9}$$

Moreover, if LMI (9) *is solvable for the index l, then it is also solvable for any larger index.* ■

Theorem 2 provides a family of standard LMIs, indexed by the positive integer l, whose solvability is sufficient to deduce *(i)*. These conditions are more and more precise when l increases. A capital property is that they are "asymptotically necessary", as property *(i)* implies solvability of the LMIs for large enough values of l.

The LMIs above constitute a family of convex relaxations, computationally tractable, of the initial problem, which is nonconvex.

A central technique in Theorem 2 consists in achieving the change of variables (4), in order to take as a departure the auxiliary problem (5), expressed with variables lying on the unit circle. In consequence, Theorem 2, as well as other results in the present paper, may be extended along the same principles, in order to check positiveness of polynomial matrices on sets different from a product of intervals or unit circles, but which, up to polynomial change of variables, may be parametrized by a finite number of independent variables lying on complex unit disks. To date, it is possible to consider sets such as the boundary of an ellipse, the boundary of a hypersphere (using generalized spherical coordinates), or even spheres or hyperspheres themselves (introducing a new complex variable lying on the disk to parametrize the radius). This artifice permits some extensions of Theorem 2, without however attaining the powerfulness of the results by Lasserre and Parillo.

Proof of Theorem 2. The converse implication *(ii)* $\Rightarrow$ *(i)* is the easy part of the proof. Indeed, right-multiplying and left-multiplying both sides of inequality (9) by $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$ and its transconjugate yields, after repeated use of formulas (8),

$$\begin{aligned} &0_n < (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H G_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \\ &+ \sum_{i=1}^{m} (1-|z_i|^2)(z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H Q_{l,i} (z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \ . \end{aligned}$$

This sense of the claimed equivalence is then deduced by letting $|z_i| = 1$, $i = 1, \dots, m$, and using Lemma 1.

The principle of the proof of the implication *(i)* $\Rightarrow$ *(ii)* consists in "removing" one-by-one the m free variables z_i in *(i)*, and "replacing" them by the matrix variable $Q_{l,i}$ in *(ii)*. The intermediate stage, where the first i free variables $z_1, \dots, z_i$ have been removed and the corresponding i matrices $Q_{l,1}, \dots, Q_{l,i}$ have been introduced, is called property $(\mathcal{P}_i)$. The proof is organized in four steps that we now present.

1. The property $(\mathcal{P}_i)$ is first defined, and it is shown that *(i)* and *(ii)* are just $(\mathcal{P}_0)$ and $(\mathcal{P}_m)$ respectively.

2. Departing from property $(\mathcal{P}_i)$, Kalman-Yakubovich-Popov lemma is applied. It results in the suppression of the free-variable z_{i+1} and the introduction of a new matrix.

3. It is shown – using basically Theorem 4 below, a result on existence of polynomial solutions for parameter-dependent LMIs established in [4] – that the previous matrix, which depends upon the remaining free-variables $z_{i+2}, \dots, z_m$, may be supposed polynomial with respect to the latter and their conjugates. Therefore, it may be represented by its coefficient matrix. This new constant matrix, denoted $Q_{l,i+1}$, is precisely the $(i+1)$-th matrix variable in the LMI (9).

4. Some matrix manipulations permit finally to establish that $(\mathcal{P}_i)$ is equivalent to $(\mathcal{P}_{i+1})$. At this point, an induction demonstrates that $(\mathcal{P}_0)$ and $(\mathcal{P}_m)$ are equivalent. This ends the proof of the equivalence between *(i)* and *(ii)*. The fact that solvability of (9) for l implies the same for larger value, is obtained as a by-product of **3.**, see Remark 1 below.

• **1.** For $i \in \{0, \dots, \dots, m\}$, define the property $(\mathcal{P}_i)$ as follows: $\exists l \in \mathbb{N}, l \geq k, \exists Q_{l,1} \in \mathcal{H}^{(l-1)^m n}$, $\dots$, $\exists Q_{l,i} \in \mathcal{H}^{(l-1)^{m-i+1} l^{i-1} n}$, $\forall (z_{i+1}, \dots, z_m) \in (\partial\mathbb{D})^{m-i}$,

$$\left(z_m^{[l]} \otimes \dots \otimes z_{i+1}^{[l]} \otimes I_{l^i n}\right)^H \left[G_l + \sum_{j=1}^{i} \left(\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n}\right)^T Q_{l,j} \left(\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n}\right) \right.$$
$$\left. - \sum_{j=1}^{i} \left(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1} n}\right)^T Q_{l,j} \left(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1} n}\right) \right] \left(z_m^{[l]} \otimes \dots \otimes z_{i+1}^{[l]} \otimes I_{l^i n}\right) > 0_{l^i n} \ .$$

Property $(\mathcal{P}_0)$ writes

$$\exists l \in \mathbb{N}, l \geq k, \forall z \in (\partial\mathbb{D})^m,\ (z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n)^H G_l (z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n) > 0_n \ .$$

and it is thus clear, in view of Lemma 1, that $(\mathcal{P}_0)$ is equivalent to (5), and to the initial problem *(i)*.

On the other hand, $(\mathcal{P}_m)$ is just the LMI (9) in the statement of Theorem 2, except that the matrices $Q_{l,i}$ are not restricted to be real, but are allowed to be complex. In the former case, the matrices G_l being real themselves, one may indeed assume without loss of generality, that the $Q_{l,i}$ are real symmetric: otherwise, one may consider their real part ... In brief, $(\mathcal{P}_m)$ is hence equivalent to solvability of (9).

At this point, it thus remains, in order to achieve the proof of *(i)* $\Rightarrow$ *(ii)*, to demonstrate the equivalence between $(\mathcal{P}_0)$ and $(\mathcal{P}_m)$. In the sequel, we shall establish that $(\mathcal{P}_i) \Leftrightarrow (\mathcal{P}_{i+1})$ for any $i = 0, \dots, m$, this leads by induction to the desired equivalence.

• **2.** Now,

$$\left(z_m^{[l]} \otimes \dots \otimes z_{i+1}^{[l]} \otimes I_{l^i n}\right) = \left(z_m^{[l]} \otimes \dots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1} n}\right) (z_{i+1}^{[l]} \otimes I_{l^i n}) \ ,$$

and

$$(z_{i+1}^{[l]} \otimes I_{l^i n}) = \begin{pmatrix} I_{l^i n} \\ z_{i+1}(z_{i+1}^{[l-1]} \otimes I_{l^i n}) \end{pmatrix} = \begin{pmatrix} I_{l^i n} \\ z_{i+1} \left(I_{(l-1)l^i n} - z_{i+1}(F_{l-1} \otimes I_{l^i n}) \right)^{-1} (f_{l-1} \otimes I_{l^i n}) \end{pmatrix} ,$$

where the matrices $F_l \in \mathbb{R}^{l \times l}$, $f_l \in \mathbb{R}^{l \times 1}$ are defined by

$$F_l \stackrel{\text{def}}{=} \begin{pmatrix} 0_{1\times(l-1)} & 0 \\ I_{l-1} & 0_{(l-1)\times 1} \end{pmatrix}, \ f_l \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 0_{(l-1)\times 1} \end{pmatrix} .$$

Indeed, to establish the previous identity, it suffices to verify that

$$\left(I_{(l-1)l^i n} - z_{i+1}(F_{l-1} \otimes I_{l^i n}) \right) (z_{i+1}^{[l-1]} \otimes I_{l^i n}) = (f_{l-1} \otimes I_{l^i n}) ,$$

which is straightforward, as $(I_{l-1} - vF_{l-1})v^{[l-1]} = f_{l-1}$, for any complex number v.

At this point, recall the discrete-time version of Kalman-Yakubovich-Popov lemma. This fundamental result, initially due to Yakubovich [14] for the continuous-time case, has been adapted to discrete time by Szegö and Kalman [11]. We use the statement as expressed e.g. in [10]. A proof of the result in the complex case (and for the continuous-time case) may be found in [8, Theorem 1.11.1 and Remark 1.11.1].

Lemma 3. *Let $F \in \mathbb{C}^{p\times p}, f \in \mathbb{C}^{p\times q}, M = M^H \in \mathbb{C}^{(q+p)\times(q+p)}$. If $\det(I_p - zF) \neq 0$ for any $z \in \partial\mathbb{D}$, then the following are equivalent.*

(i) For any $z \in \partial\mathbb{D}$,

$$0_p < \begin{pmatrix} I_p \\ z(I_p - zF)^{-1} f \end{pmatrix}^H M \begin{pmatrix} I_p \\ z(I_p - zF)^{-1} f \end{pmatrix} .$$

(ii) There exists $Q \in \mathcal{H}^p$ such that

$$0_{p+q} < \begin{pmatrix} f & F \end{pmatrix}^H Q \begin{pmatrix} f & F \end{pmatrix} - \begin{pmatrix} 0_{p\times q} & I_p \end{pmatrix}^H Q \begin{pmatrix} 0_{p\times q} & I_p \end{pmatrix} + M . \qquad \blacksquare$$

Putting $p = (l-1)l^i n$, $q = l^i n$, $F = F_{l-1} \otimes I_{l^i n}$, $f = f_{l-1} \otimes I_{l^i n}$ in the previous statement, we recognize:

$$\begin{pmatrix} f & F \end{pmatrix} = \hat{J}_{l-1} \otimes I_{l^i n}, \ \begin{pmatrix} 0_{p\times q} & I_p \end{pmatrix} = \check{J}_{l-1} \otimes I_{l^i n} ,$$

and this yields equivalence of $(\mathcal{P}_i)$ with: $\exists l \in \mathbb{N}, l \geq k, \exists Q_{l,1} \in \mathcal{H}^{(l-1)^m n}, \dots, \exists Q_{l,i} \in \mathcal{H}^{(l-1)^{m-i+1} l^{i-1} n}$, $\forall (z_{i+2}, \dots, z_m) \in (\partial\mathbb{D})^{m-i-1}$, $\exists \tilde{Q}_{l,i+1}(z_{i+2}, \dots, z_m) \in \mathcal{H}^{(l-1)l^i n}$,

$$\begin{aligned} &\left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1} n} \right)^H \Bigg[G_l + \sum_{j=1}^{i} \left(\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n} \right)^T Q_{l,j} \left(\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n} \right) \\ &- \sum_{j=1}^{i} \left(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1} n} \right)^T Q_{l,j} \left(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1} n} \right) \Bigg] \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1} n} \right) \\ &\qquad + \left(\hat{J}_{l-1} \otimes I_{l^i n} \right)^T \tilde{Q}_{l,i+1} \left(\hat{J}_{l-1} \otimes I_{l^i n} \right) - \left(\check{J}_{l-1} \otimes I_{l^i n} \right)^T \tilde{Q}_{l,i+1} \left(\check{J}_{l-1} \otimes I_{l^i n} \right) > 0_{l^{i+1} n} . \end{aligned} \qquad (10)$$

• **3.** The next step consists in assigning polynomial form to $\tilde{Q}_{l,i+1}$. This is done with the help of the following general result, borrowed from [4].

Theorem 4. *Suppose $G_0, G_1, \ldots, G_p$ are continuous mappings defined in a* compact *subset K of $\mathbb{R}^m$, and taking values in $\mathcal{S}^n$. If, for any $\delta \in K$, there exists a solution $x(\delta) \in \mathbb{R}^p$ to the parameter-dependent LMI*

$$\exists x \in \mathbb{R}^p,\ G(x,\delta) \stackrel{\text{def}}{=} G_0(\delta) + x_1 G_1(\delta) + \cdots + x_p G_p(\delta) > 0 \ , \tag{11}$$

then there exists a polynomial *function $x^* : K \to \mathbb{R}^p$, such that, for any $\delta \in K$, $G(x^*(\delta),\delta) > 0$.* ■

Notice that any LMI depending upon a finite number of scalar parameters may be put under the form (11).

By use of the previous result, $\tilde{Q}_{l,i+1}(z_{i+2},\ldots,z_m)$, being solution of a LMI with parameter in the compact set $(\partial\mathbb{D})^{m-i-1}$, may be chosen *polynomial* in its variables and their conjugates. Let $\tilde{l}-2$ be its degree. If $\tilde{l} \leq l$, then one may write

$$\tilde{Q}_{l,i+1}(z_{i+2},\ldots,z_m) = \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right)^H Q_{l,i+1} \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right) \ , \tag{12}$$

for a certain coefficient matrix $Q_{l,i+1} \in \mathcal{H}^{(l-1)^{m-i} l^i n}$.

Otherwise, we show now that, up to an increase of l, the degree may be supposed the same, so the previous formula still holds. For this, let us form, for $j = 1,\ldots,i$, the matrices

$$Q_{l+1,j} \stackrel{\text{def}}{=} \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\ldots,j-1 \\ J_\alpha \in \{\hat{J}_{l-1}, \check{J}_{l-1}\},\ \alpha=j,\ldots,m}} (J_m \otimes \cdots \otimes J_1 \otimes I_n)^T Q_{l,j} (J_m \otimes \cdots \otimes J_1 \otimes I_n) \ ,$$

and

$$\tilde{Q}_{l+1,i+1} \stackrel{\text{def}}{=} \frac{1}{2^{i+1}} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\ldots,i \\ J_{i+1} \in \{\hat{J}_{l-1}, \check{J}_{l-1}\}}} (J_{i+1} \otimes \cdots \otimes J_1 \otimes I_n)^T \tilde{Q}_{l,i+1} (J_{i+1} \otimes \cdots \otimes J_1 \otimes I_n) \ . \tag{13}$$

By construction, the matrix $\tilde{Q}_{l+1,i+1}$ has the same degree $\tilde{l}-2$ in $z_{i+2},\ldots,z_m,\bar{z}_{i+2},\ldots,\bar{z}_m$ than $\tilde{Q}_{l,i+1}$. Denoting for short R_l the left-hand side of (10) and R_{l+1} the analogue expression, obtained with the definitions of $Q_{l+1,j}$, $j=1,\ldots,i$ and $\tilde{Q}_{l+1,i+1}$ given above, and with the definition of G_{l+1} given in (6), we will now show that

$$R_{l+1} = \frac{1}{2^{i+1}} \sum_{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\ldots,i+1} (J_{i+1} \otimes \cdots \otimes J_1 \otimes I_n)^T R_l (J_{i+1} \otimes \cdots \otimes J_1 \otimes I_n) \ . \tag{14}$$

First, taking into account the fact that

$$\forall l \in \mathbb{N}, \check{J}_l \hat{J}_{l+1} = \hat{J}_l \check{J}_{l+1},\ \hat{J}_l I_{l+1} = I_l \hat{J}_l,\ \check{J}_l I_{l+1} = I_l \check{J}_l \ , \tag{15}$$

one gets, for $Q_{l+1,j}$ defined previously:

$$\begin{aligned}
&(\hat{J}_l^{(m-j+1)\otimes} \otimes I_{(l+1)^{j-1}n})^T Q_{l+1,j} (\hat{J}_l^{(m-j+1)\otimes} \otimes I_{(l+1)^{j-1}n}) \\
&= \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\ldots,j-1 \\ J_\alpha \in \{\hat{J}_{l-1}, \check{J}_{l-1}\},\ \alpha=j,\ldots,m}} (.)^T (.)^T Q_{l,j} (J_m \otimes \cdots \otimes J_1 \otimes I_n)(\hat{J}_l^{(m-j+1)\otimes} \otimes I_{(l+1)^{j-1}n}) \\
&= \frac{1}{2^m} \sum_{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\ldots,m} (.)^T \left[(.)^T Q_{l,j} (\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1}n})\right] (J_m \otimes \cdots \otimes J_1 \otimes I_n) \ .
\end{aligned} \tag{16}$$

Here and in the sequel, the dots in the formulas stand for terms ensuring the symmetry of the expressions, and which are not repeated for sake of space. The same argument applied to the terms $(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1}n})^T Q_{l,j} (\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1}n})$ in (10), shows that formally, both may be treated as the term in G_l (as formula (16) is formally identical to the relation (6) linking G_l and G_{l+1}).

We may now consider all together the terms under brackets in (10), writing for simplicity only the term in G_l. Arguing as in **1.**, one shows based on formulas (6) and (8), that for any $z \in (\partial\mathbb{D})^m$,

$$\begin{aligned}
&\left(z_m^{[l+1]} \otimes \cdots \otimes z_{i+2}^{[l+1]} \otimes I_{l^{i+1}n}\right)^H G_{l+1} \left(z_m^{[l+1]} \otimes \cdots \otimes z_{i+2}^{[l+1]} \otimes I_{l^{i+1}n}\right) \\
&\quad = \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\}, \\ \alpha=1,\dots,m}} \left(.\right)^H (.)^T G_l (J_m \otimes \cdots \otimes J_1 \otimes I_n) \left(z_m^{[l+1]} \otimes \cdots \otimes z_{i+2}^{[l+1]} \otimes I_{l^{i+1}n}\right) \\
&\quad = \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\dots,i+1 \\ j_\alpha \in \{0,1\},\ \alpha=i+2,\dots,m}} |z_m|^{2j_m} \dots |z_{i+2}|^{2j_{i+2}} \left(.\right)^H G_l \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes J_{i+1} \otimes \cdots \otimes J_1 \otimes I_n\right) \\
&\quad = \frac{1}{2^{i+1}} \sum_{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\dots,i+1} \left(.\right)^H G_l \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes J_{i+1} \otimes \dots J_1 \otimes I_n\right) \\
&\quad = \frac{1}{2^{i+1}} \sum_{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\dots,i+1} (.)^T \left(.\right)^H G_l \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1}n}\right) (J_{i+1} \otimes \cdots \otimes J_1 \otimes I_n)
\end{aligned}$$

In order to establish (14), it now remains to consider the last two terms of (10), involving $\tilde{Q}_{l,i+1}$. Using (15) yields

$$\begin{aligned}
&\left(\hat{J}_l \otimes I_{(l+1)^i n}\right)^T \tilde{Q}_{l+1,i+1} \left(\hat{J}_l \otimes I_{(l+1)^i n}\right) \\
&\quad = \frac{1}{2^{i+1}} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\dots,i \\ J_{i+1} \in \{\hat{J}_{l-1}, \check{J}_{l-1}\}}} \left(.\right)^T (.)^T \tilde{Q}_{l,i+1} (J_{i+1} \otimes \cdots \otimes J_1 \otimes I_n) \left(\hat{J}_l \otimes I_{(l+1)^i n}\right) \\
&\quad = \frac{1}{2^{i+1}} \sum_{J_\alpha \in \{\hat{J}_l, \check{J}_l\},\ \alpha=1,\dots,i+1} (.)^T \left(.\right)^T \tilde{Q}_{l,i+1} \left(\hat{J}_{l-1} \otimes I_{l^i n}\right) (J_{i+1} \otimes \cdots \otimes J_1 \otimes I_n)\ ,
\end{aligned}$$

and a corresponding identity holds for $(\check{J}_l \otimes I_{(l+1)^i n})^T \tilde{Q}_{l+1,i+1} (\check{J}_l \otimes I_{(l+1)^i n})$.

Finally, putting together the previous technical developments establishes identity (14). From this identity, one deduces that $R_{l+1} > 0$ whenever $R_l > 0$: a new solution of (10) may thus be constructed, with l replaced by $l+1$, and for which $\tilde{Q}_{l+1,i+1}$ has clearly the same degree $\tilde{l}-2$ in the variables $z_{i+2}, \dots, z_m$ and their conjugates than $\tilde{Q}_{l,i+1}$ (see formula (13)). Hence, one concludes that, up to an increase of l, there is no loss of generality in assuming that $l = \tilde{l}$ in the decomposition (12) of $\tilde{Q}_{l,i+1}$.

Remark 1. *Applying the previous argument to* $(\mathcal{P}_m)$ *proves that solvability of* (9) *implies the same property for the larger values of the index.*

To summarize, it has been established until now that $(\mathcal{P}_i)$ is equivalent to: $\exists l \in \mathbb{N}, l \geq k, \exists Q_{l,1} \in$

$\mathcal{H}^{(l-1)^m n}, \ldots, \exists Q_{l,i+1} \in \mathcal{H}^{(l-1)^{m-i} l^i n}$, $\forall (z_{i+2}, \ldots, z_m) \in (\partial\mathbb{D})^{m-i-1}$,

$$0_{l^{i+1}n} < \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1}n}\right)^H \left[G_l + \sum_{j=1}^{i} \left(\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1}n}\right)^T Q_{l,j} \left(\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1}n}\right)\right.$$
$$\left. - \sum_{j=1}^{i} \left(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1}n}\right)^T Q_{l,j} \left(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1}n}\right)\right] \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1}n}\right)$$
$$+\left(\hat{J}_{l-1} \otimes I_{l^i n}\right)^T \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right)^H Q_{l,i+1} \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right) \left(\hat{J}_{l-1} \otimes I_{l^i n}\right)$$
$$-\left(\check{J}_{l-1} \otimes I_{l^i n}\right)^T \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right)^H Q_{l,i+1} \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right) \left(\check{J}_{l-1} \otimes I_{l^i n}\right) .$$

• **4.** It remains now to achieve some matrix intervertions in the last two terms of the previous inequality. Using the following formula (obtained with the help of (8)):

$$\begin{aligned}
&\left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right) \left(\hat{J}_{l-1} \otimes I_{l^i n}\right) \\
&= \left(I_{(l-1)^{m-i-1}} \otimes \hat{J}_{l-1} \otimes I_{l^i n}\right) \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{l^{i+1}n}\right) \\
&= \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes I_{l^i n}\right) \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1}n}\right) ,
\end{aligned}$$

and similarly:

$$\begin{aligned}
&\left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right) \left(\check{J}_{l-1} \otimes I_{l^i n}\right) \\
&\qquad = \left(\hat{J}_{l-1}^{(m-i-1)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^i n}\right) \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1}n}\right) ,
\end{aligned}$$

one finally proves that $(\mathcal{P}_i)$ is equivalent to: $\exists l \in \mathbb{N}, l \geq k, \exists Q_{l,1} \in \mathcal{H}^{(l-1)^m n}, \ldots, \exists Q_{l,i+1} \in \mathcal{H}^{(l-1)^{m-i} l^i n}$, $\forall (z_{i+2}, \ldots, z_m) \in (\partial\mathbb{D})^{m-i-1}$,

$$\left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1}n}\right)^H \left[G_l + \sum_{j=1}^{i+1} \left(\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1}n}\right)^T Q_{l,j} \left(\hat{J}_{l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1}n}\right)\right.$$
$$\left. -\sum_{j=1}^{i+1} \left(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1}n}\right)^T Q_{l,j} \left(\hat{J}_{l-1}^{(m-j)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{j-1}n}\right)\right] \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1}n}\right) > 0_{l^{i+1}n} .$$

One recognizes property $(\mathcal{P}_{i+1})$. In other words, $(\mathcal{P}_i) \Leftrightarrow (\mathcal{P}_{i+1})$ for all $i = 0, \ldots, m-1$, so $(\mathcal{P}_0) \Leftrightarrow (\mathcal{P}_m)$. This achieves the proof of Theorem 2.

□

4 Representation results for matrix-valued polynomials

In the same fashion that the results by Lasserre and Parrilo are closely related to representation result for polynomials (as sums of squares), Theorem 4 is sustended by a representation result, for matrix-valued polynomials of complex variables.

Theorem 5. *Let function $\tilde{G}(z) : \mathbb{C}^m \to \mathcal{H}^n$ be polynomial of degree $k-1$ in $z, \bar{z}$, and such that*

$$\forall z \in (\partial\mathbb{D})^m,\ \tilde{G}(z) > 0_n \ .$$

Then, there exist an integer $l \geq k$, a positive definite *matrix $R_l \in \mathcal{H}^{l^m n}$, and m polynomials matrices $Q_i(z, \bar{z})$ taking values in $\mathcal{H}^n$, such that $\forall z \in \mathbb{C}^m$,*

$$\tilde{G}(z) = \prod_{i=1}^{m} \frac{1}{(1+|z_i|^2)^{l-k}} \left[(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H R_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) + \sum_{i=1}^{m} (1-|z_i|^2) Q_i(z, \bar{z}) \right] \ .$$

$\blacksquare$

Proof. There exist an integer k and $\tilde{G}_k \in \mathcal{H}^{k^m n}$, such that $\tilde{G}(z)$ nay be represented, for any $z \in \mathbb{C}^m$, as $(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H \tilde{G}_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)$. Arguing as in the proof of Theorem 2 and defining $\tilde{G}_l \in \mathcal{H}^{l^m n}$, $l \geq k$, as in (6), one shows that $\forall z \in (\partial\mathbb{D})^m$, $\tilde{G}(z) > 0_n$ *if and only if* there exist $l \in \mathbb{N}$ and m matrices $Q_{l,i} \in \mathcal{H}^{(l-1)^{m-i+1} l^{i-1} n}$, $i = 1 \ldots m$, such that

$$\begin{aligned} R_l \stackrel{\text{def}}{=} \tilde{G}_l &+ \sum_{i=1}^{m} \left(\hat{J}_{l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n} \right)^T Q_{l,i} \left(\hat{J}_{l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n} \right) \\ &- \sum_{i=1}^{m} \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{i-1}n} \right)^T Q_{l,i} \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{i-1}n} \right) > 0_{l^m n} \end{aligned}$$

(compare with (9)). Right-multiplying and left-multiplying, as in the beginning of the proof of Theorem 2, both sides of this inequality by $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$ and its transconjugate yields, for any $z \in \mathbb{C}^m$,

$$\begin{aligned} &(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H R_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) = (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H \tilde{G}_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \\ &+ \sum_{i=1}^{m} (1-|z_i|^2)(z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H Q_{l,i} (z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \ . \end{aligned}$$

Using formula (7) in Lemma 1 then permits to end the proof of Theorem 5. $\square$

5 Robust semidefinite programming

Application of Theorem 2 to robust semidefinite programming problems is direct, due to the fact that the coefficient matrices defined in (6) are then affine with respect to the decision variables.

The first application concerns robust feasibility.

Corollary 6. *Let $G_0, G_1, \ldots, G_p$ be polynomial mappings: $\mathbb{R}^m \to \mathcal{S}^n$ of degree at most $k-1$ in each variable. Define their coefficient matrices $G_{0,l}$, $G_{1,l}$, $\ldots$, $G_{p,l}$, $l \geq k$, by operations similar to* (4) *and* (6)*. Then, the following assertions are* equivalent.

(i) There exists $x \in \mathbb{R}^p$ such that

$$\forall \delta \in [-1;+1]^m,\ G(x,\delta) \stackrel{\text{def}}{=} G_0(\delta) + x_1 G_1(\delta) + \cdots + x_p G_p(\delta) > 0_n \ .$$

(ii) There exist $x \in \mathbb{R}^p$, $l \in \mathbb{N}$, $l \geq k$, and m matrices $Q_{l,i} \in \mathcal{S}^{(l-1)^{m-i+1}l^{i-1}n}$, $i = 1 \dots m$, fulfilling the LMI:

$$G_{0,l} + x_1 G_{1,l} + \cdots + x_p G_{p,l} + \sum_{i=1}^{m} \left(\hat{J}_{l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n}\right)^T Q_{l,i} \left(\hat{J}_{l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n}\right)$$

$$- \sum_{i=1}^{m} \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{i-1}n}\right)^T Q_{l,i} \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{i-1}n}\right) > 0_{l^m n} \ . \quad (17)$$

Moreover, if this LMI is solvable for the index l, then it is also solvable for any larger index. ■

The proof of Corollary 6 is immediatly deduced from Theorem 2. The latter result provides a sequence of (more and more precise) inner approximations of the set $\{x \ : \ \forall \delta \in [-1;+1]^m, G(x,\delta) > 0_n\}$.

The next application concerns robust evaluation of the worst-case optimum under LMI constraint. For any positive integer l, define $1_l \in \mathcal{S}^{l^m}$ by:

$$1_{l+1} \stackrel{\text{def}}{=} \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\}, \\ \alpha = 1,\dots,m}} (J_m \otimes \cdots \otimes J_1 \otimes I_n)^T 1_l (J_m \otimes \cdots \otimes J_1 \otimes I_n), \quad 1_1 \stackrel{\text{def}}{=} 1 \ .$$

Corollary 7. *Let $G_0, G_1, \dots, G_p : \mathbb{R}^m \to \mathcal{S}^n$, resp. $g_0, \dots, g_p : \mathbb{R}^m \to \mathbb{R}$, be polynomial mappings of degree at most $k-1$ in each variable. Define their coefficient matrices $G_{0,l}$, $G_{1,l}$, $\dots$, $G_{p,l}$, resp. $g_{0,l}$, $g_{1,l}$, $\dots$, $g_{p,l}$, $l \geq k$. Let $g(x,\delta) \stackrel{\text{def}}{=} g_0(\delta) + x_1 g_1(\delta) + \cdots + x_p g_p(\delta)$, and define the, possibly infinite, constants γ_∞ and γ_l, $l \geq k$, by:*

$$\gamma_\infty \stackrel{\text{def}}{=} \inf\{\gamma \in \mathbb{R} \ : \ \exists x \in \mathbb{R}^p, \forall \delta \in [-1;+1]^m, G(x,\delta) > 0_n, g(x,\delta) < \gamma\} \ ,$$

$$\gamma_l \stackrel{\text{def}}{=} \inf\{\gamma \in \mathbb{R} \ : \ \exists x \in \mathbb{R}^p, \exists (Q_{l,i}, q_{l,i}) \in \mathcal{S}^{(l-1)^{m-i+1}l^{i-1}n} \times \mathcal{S}^{(l-1)^{m-i+1}l^{(i-1)}}, i = 1 \dots m, \text{ such that (17) and (18) hold}\} \ ,$$

where

$$\gamma 1_l - g_{0,l} - x_1 g_{1,l} - \cdots - x_p g_{p,l} + \sum_{i=1}^{m} \left(\hat{J}_{l-1}^{(m-i+1)\otimes} \otimes I_{l^{(i-1)}}\right)^T q_{l,i} \left(\hat{J}_{l-1}^{(m-i+1)\otimes} \otimes I_{l^{(i-1)}}\right)$$

$$- \sum_{i=1}^{m} \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{(i-1)}}\right)^T q_{l,i} \left(\hat{J}_{l-1}^{(m-i)\otimes} \otimes \check{J}_{l-1} \otimes I_{l^{(i-1)}}\right) > 0_{l^m} \ . \quad (18)$$

Then, the sequence γ_l, $l \geq k$, is nonincreasing and its limit is equal to γ_∞. ■

Proof. We assume that all the constants γ_∞, γ_l, $l \geq k$, are finite, otherwise feasibility does not hold and Corollary 6 applies.

To show that $\gamma_l \geq \gamma_\infty$, $l \geq k$, we use the same techniques than to prove *(ii)* $\Rightarrow$ *(i)* in the beginning of the proof of Theorem 2: right-multiply and left-multiply inequality (17) (resp. (18)) by $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$ (resp. $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]})$) and its transconjugate, to obtain: $\forall z \in \mathbb{C}^m$,

$$0_n < (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H (G_{0,l} + x_1 G_{1,l} + \cdots + x_p G_{p,l})(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$$

$$+ \sum_{i=1}^{m} (1 - |z_i|^2)(z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H Q_{l,i} (z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$$

and

$$0 < (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H (\gamma 1_l - g_{0,l} - x_1 g_{1,l} - \cdots - x_p g_{p,l})(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$$
$$+ \sum_{i=1}^{m} (1 - |z_i|^2)(z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]})^H q_{l,i} (z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]}) \ .$$

Taking $|z_i| = 1$, $i = 1, \dots, m$, and using Lemma 1 gives finally:

$$\forall \delta \in [-1; +1]^m, \ G_0(\delta) + x_1 G_1(\delta) + \cdots + x_p G_p(\delta) > 0_n, \ g_0(\delta) + x_1 g_1(\delta) + \cdots + x_p g_p(\delta) < \gamma \ .$$

Thus, for any real γ, $\gamma \geq \gamma_l$ implies that $\gamma \geq \gamma_\infty$, and this proves that $\gamma_l \geq \gamma_\infty$.

Monotony of the sequence γ_l, $l \geq k$, is obtained by using techniques similar to the point **3.** in the proof of Theorem 2: one shows that, if $(\gamma, x, Q_{l,i}, q_{l,i})$ constitute a solution of (17), (18), then $(\gamma, x, Q_{l+1,i}, q_{l+1,i})$ solves the same inequality with index $l+1$, where

$$Q_{l+1,j} \stackrel{\text{def}}{=} \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\}, \ \alpha = 1, \dots i-1 \\ J_\alpha \in \{\hat{J}_{l-1}, \check{J}_{l-1}\}, \ \alpha = i, \dots m}} (J_m \otimes \cdots \otimes J_1 \otimes I_n)^T Q_{l,j} (J_m \otimes \cdots \otimes J_1 \otimes I_n),$$
$$q_{l+1,j} \stackrel{\text{def}}{=} \frac{1}{2^m} \sum_{\substack{J_\alpha \in \{\hat{J}_l, \check{J}_l\}, \ \alpha = 1, \dots i-1 \\ J_\alpha \in \{\hat{J}_{l-1}, \check{J}_{l-1}\}, \ \alpha = i, \dots m}} (J_m \otimes \cdots \otimes J_1)^T q_{l,j} (J_m \otimes \cdots \otimes J_1) \ .$$

To show finally that γ_l tends towards γ_∞, let $\varepsilon > 0$. By the very definition of γ_∞, there exists $x \in \mathbb{R}^p$ such that

$$\forall \delta \in [-1; +1]^m, \begin{pmatrix} G(x, \delta) & 0_{n \times 1} \\ 0_{1 \times n} & \gamma_\infty + \varepsilon - g(x, \delta) \end{pmatrix} > 0_{n+1} \ .$$

Corollary 6 then ensures existence of a solution to (17), (18) with $\gamma_\infty + \varepsilon$ instead of γ, for a certain value of $l \geq k$. For this value of l and beyond, one has $\gamma_l \leq \gamma_\infty + \varepsilon$, so $\liminf \gamma_l \leq \gamma_\infty$. From the properties of the sequence γ_l previously demonstrated, one concludes that $\lim \gamma_l = \gamma_\infty$, and this ends the proof of Corollary 7. □

A Appendix

We indicate here how to obtain decomposition (4). A natural representation for a matrix-valued polynomial $G(\delta) : \mathbb{R}^m \to \mathbb{R}^{p \times n}$ is

$$G(\delta) = G_l(\delta_m^{[l]} \otimes \cdots \otimes \delta_1^{[l]} \otimes I_n) \ , \tag{19}$$

for a certain matrix $G_l \in \mathbb{R}^{p \times l^m n}$. The effect of the change of variable (4) is then summarized by Lemma 8.

Lemma 8. *Let $G_l \in \mathbb{R}^{p \times l^m n}$, then*

$$G_l\left(\left(\frac{z_m + \bar{z}_m}{2}\right)^{[l]} \otimes \cdots \otimes \left(\frac{z_1 + \bar{z}_1}{2}\right)^{[l]} \otimes I_n\right) = (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_p)^H \tilde{G}_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \ ,$$

where the matrix $\tilde{G}_l \in \mathbb{R}^{l^m p \times l^m n}$ is given by the formula

$$\tilde{G}_l \stackrel{\text{def}}{=} \sum_{0 \leq \alpha_i \leq l-1} (L_{l,\alpha_m} \otimes \dots \otimes L_{l,\alpha_1} \otimes I_p)^T G_l (K_{l,\alpha_m} \otimes \dots \otimes K_{l,\alpha_1} \otimes I_n) \ ,$$

in which

- *the matrices $K_{l,\alpha} \in \mathbb{R}^{l \times l}$ are defined by: $(K_{l,\alpha})_{i,i-\alpha} = 2^{-i+1} C_{i-1}^{\alpha}$, with $C_i^{\alpha} \stackrel{\text{def}}{=} \frac{i!}{\alpha!(i-\alpha)!}$ if $i \geq \alpha \geq 0$, $C_i^{\alpha} = 0$ otherwise;*
- *the matrices $L_{l,\alpha} \in \mathbb{R}^{1 \times l}$ are defined by: $L_{l,\alpha} = \begin{pmatrix} 0_{1 \times \alpha} & 1 & 0_{1 \times (l-\alpha-1)} \end{pmatrix}$.* ■

Proof. One may check that $K_{l,\alpha}$ defined in the statement is such that $\forall v \in \mathbb{C}$,

$$\left(\frac{v + \bar{v}}{2} \right)^{[l]} = \sum_{\alpha=0}^{l-1} \bar{v}^{\alpha} K_{l,\alpha} v^{[l]} \ .$$

Thus,

$$G_l \left(\left(\frac{z_m + \bar{z}_m}{2} \right)^{[l]} \otimes \dots \otimes \left(\frac{z_1 + \bar{z}_1}{2} \right)^{[l]} \otimes I_n \right) \\ = \sum_{0 \leq \alpha_i \leq l-1} \bar{z}_1^{\alpha_1} \dots \bar{z}_m^{\alpha_m} G_l (K_{l,\alpha_m} \otimes \dots \otimes K_{l,\alpha_1}) (z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n) \ .$$

The conclusion then follows from the fact that $\forall v \in \mathbb{C}$, $v^{\alpha} = v^{\alpha} v^{[1]} = L_{l,\alpha} v^{[l]}$. □

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