On existence of smooth solutions of parameter-dependent convex programming problems

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Abstract

We show in this paper that, under general conditions, any convex programming problem depending continuously upon scalar parameters, and solvable for any value of the latter in a fixed compact set (resp. open set), admits a branch of solutions which is *polynomial* (resp. *smooth*) with respect to these parameters. This result may be useful to generate tractable approximations of uncertain convex programming problems with vanishing conservativeness.

1 Introduction

In many practical applications of convex optimization, the data of the problem are subject to uncertainties, measurement errors, modelling approximations. The study of parameter-dependent convex programming problems leads basically to *two* types of problems.

The first one consists in finding decision variables fulfilling the convex problem, for all values of the parameters (in a prescribed set). This is the subject of *robust convex programming*, see [8] for a recent survey. It has been established that the robust counterpart of linear programming is equivalent to a standard convex programming problems, under usual constraints on the perturbations [3, 4, 8]. But in general, this nice property does not hold any more for quadratic programming and conic quadratic programming problems [3, 7, 8], and for semidefinite programming problems [3, 8]. Indeed, except for special uncertainty structures, these robust convex programming problems are NP-hard. In these conditions, efforts have been made to exhibit tractable approximations of the latter. For quadratic and conic quadratic programming problems [13, 7, 8] and for semidefinite programming problems [14, 5, 6, 8], such an operation is possible, and in certain cases, astute computations even permit to estimate (from above) some appropriately defined *levels of conservativeness.*

The second sort of problems consists in checking that, for all values of the parameters, there exist decision variables fulfilling the convex problem under study: the difference with the first class of problems lies in the order of the quantifiers. The corresponding problems are also NP-hard in general. The optimal solutions of the considered problem may be seen, generally speaking, as functions of the parameters, with

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an unprescribed regularity. The usual way to obtain relaxation of the latter consists in looking for solutions with prescribed dependency, for example affine with respect to the parameters. However, the works based on this approach already published in the literature do not offer, up to our knowledge, the possibility to decrease, and asymptotically remove, the approximation error.

In an attempt to progress in this direction, we provide here a result on existence of smooth solutions to a general class of convex programming problems depending upon parameters. The results exposed below show that, without loss of generality, provided that the convex program is solvable for any value of the parameters, one may assume that the unknowns (the decision variables) are *indefinitely differentiable* with respect to the parameters. In the case of a *compact* parameter set, this function may even be supposed *polynomial*. This leads to consider new unknowns, instead of the original untractable function: the degree and coefficients of a polynomial solution. Thus, a natural next step to complete this procedure is to consider the theoretically simpler problem, obtained when assuming polynomial dependence with respect to the parameters, of the solution of the studied problem. The results stated herein ensure that the conservativeness of this procedure vanishes when the degree increases.

This idea has been applied successfully to robust semidefinite programming. Based on a result on existence of polynomial solutions for this type of problems [9], this approach has allowed explicit construction of a family of standard semidefinite programming problems approximating with increasing, asymptotically perfect, precision, a given robust semidefinite programming problem [10]. The previous family is indexed by the degree of the underlying polynomial solution, and the coefficients of the latter may be deduced from the solution of the corresponding linear matrix inequality. The results given in this note are indeed based on an extension of the work in [9] to general convex problems.

The reader should be aware that results on existence of smooth solutions for *differential* matrix inequalities, related in spirit to the present contribution, may be found in [15, 16, 17].

The results are provided in Section 2. The central result presented here, Theorem 1, considers robust feasibility problem, for parameters lying in a compact set. It is afterwards extended in Theorem 3 to open, possibly unbounded, parameter sets. An application to estimation of the worst-case optimal value of convex programming problems depending upon parameters is proposed in Corollary 2.

Last, proofs are given in Section 3.

2 Existence of smooth solutions

In all the paper, $\mathcal{C}$ denotes a proper cone in $\mathbb{R}^n$, in other words a closed convex solid and pointed cone. To $\mathcal{C}$ is associated as usual a partial ordering in $\mathbb{R}^n$, denoted $\leq_{\mathcal{C}}$: by definition

$$\forall \alpha, \alpha' \in \mathbb{R}^n,\ \alpha \leq_{\mathcal{C}} \alpha' \Leftrightarrow \alpha' - \alpha \in \mathcal{C}\ .$$

Denoting $\mathtt{int}\ \mathcal{C}$ the interior of the set $\mathcal{C}$, we also consider the strict partial ordering associated to $\mathcal{C}$:

$$\forall \alpha, \alpha' \in \mathbb{R}^n,\ \alpha <_{\mathcal{C}} \alpha' \Leftrightarrow \alpha' - \alpha \in \mathtt{int}\ \mathcal{C}\ .$$

Such generalized inequalities satisfy nice properties, among which the following will be especially important in the sequel:

$$\text{For any sequence } \alpha_k \leq_{\mathcal{C}} 0_n,\ \alpha_k \to \alpha_\infty \Rightarrow \alpha_\infty \leq_{\mathcal{C}} 0_n\ , \tag{1}$$

and:

$$\text{For any } \alpha <_{\mathcal{C}} 0_n, \quad \text{there exists } \varepsilon > 0,\ \|\alpha'\|_n < \varepsilon \Rightarrow \alpha + \alpha' <_{\mathcal{C}} 0_n\ . \tag{2}$$

Here and in the sequel, 0_n denotes the zero vector in $\mathbb{R}^n$, and $\|\cdot\|_n$ any norm in this space.

The first result of the present contribution is the following.

Theorem 1. *Let K be a compact set of $\mathbb{R}^m$. Let $G : \mathbb{R}^p \times K \to \mathbb{R}^n$ be a continuous function, $\mathcal{C}$-convex with respect to the first variable, that is:*

$$\forall x, x' \in \mathbb{R}^p,\ \forall \delta \in K,\ \forall \lambda \in [0,1],\ G(\lambda x + (1-\lambda)x', \delta) \leq_{\mathcal{C}} \lambda G(x,\delta) + (1-\lambda) G(x', \delta)\ . \tag{3}$$

Assume that:

$$\forall \delta \in K,\ \exists x \in \mathbb{R}^p,\ G(x,\delta) <_{\mathcal{C}} 0_n \ . \tag{4}$$

Then, there exists a polynomial *function* $x^* : K \to \mathbb{R}^p$ *such that*

$$\forall \delta \in K, G(x^*(\delta),\delta) <_{\mathcal{C}} 0_n \ .$$

For fixed value of the parameter δ, to find $x \in \mathbb{R}^p$ such that $G(x,\delta) <_{\mathcal{C}} 0_n$, is a convex programming problem. Thus, problem (4) is a *robust convex program.* Theorem 1 states that, under very general assumptions, solvability of the latter for any value of the perturbation vector δ in K, is *equivalent* to existence of a solution polynomial with respect to the components of δ. Remark in particular that no convexity or connectedness assumption is made on the compact set K.

The proof, detailed in Section 3.1, is based essentially on the construction of a *continuous* solution $x(\delta)$ of (4). From this, the density of the set of polynomial mappings in the space of continuous functions is used to conclude. In consequence, other similar existence results (e.g. of trigonometric polynomial or spline functions) may be deduced in a straightforward way.

We now apply Theorem 1 to the issue of finding the worst-case optimal value of a convex objective under generalized inequality constraints.

Corollary 2. *Let K be a compact set of $\mathbb{R}^m$. Let $G : \mathbb{R}^p \times K \to \mathbb{R}^n$, $g : \mathbb{R}^p \times K \to \mathbb{R}$ be continuous functions, $\mathcal{C}$-convex with respect to the first variable. Then*

$$\sup_{\delta \in K} \inf \{g(x,\delta)\ :\ x \in \mathbb{R}^p,\ G(x,\delta) <_{\mathcal{C}} 0_n\}$$
$$= \sup_{\delta \in K} \inf \{g(x^*(\delta),\delta)\ :\ x^* \textit{ polynomial}\ ,\ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \ .$$

Proof is provided in Section 3.2.

We next state an extension of Theorem 1, valid on non-compact sets.

Theorem 3. *Let Ω be an open subset of $\mathbb{R}^m$. Let $G : \mathbb{R}^p \times \Omega \to \mathbb{R}^n$ be a continuous function, $\mathcal{C}$-convex with respect to the first variable. Assume that:*

$$\forall \delta \in \Omega,\ \exists x \in \mathbb{R}^p,\ G(x,\delta) <_{\mathcal{C}} 0_n \ .$$

Then, there exists a C^∞ function $x^* : \Omega \to \mathbb{R}^p$ such that

$$\forall \delta \in \Omega, G(x^*(\delta),\delta) <_{\mathcal{C}} 0_n \ . \tag{5}$$

See Section 3.3 for proof of Theorem 3. The counterpart of Corollary 2 is evident and not stated explicitly.

3 Proof of the results

3.1 Proof of Theorem 1

We first show the existence of a certain $\alpha \in \texttt{int}\ \mathcal{C}$ such that

$$\forall \delta \in K,\ \{x \in \mathbb{R}^p\ :\ G(x,\delta) \leq_{\mathcal{C}} -4\alpha\} \neq \emptyset \ . \tag{6}$$

Otherwise, for any $\alpha >_{\mathcal{C}} 0$, there exists $\delta^\alpha \in K$ such that the previous set is empty. In this case, consider δ^0 an accumulation point of the sequence δ^α, $\alpha \to 0$, and, thanks to (4), there exists $x^0 \in \mathbb{R}^p$ such that $G(x^0,\delta^0) <_{\mathcal{C}} 0_n$. By continuity and (2), there exist points δ^α, $\alpha >_{\mathcal{C}} 0_n$, arbitrarily close from δ^0, and $\alpha^0 >_{\mathcal{C}} 0_n$ such that, say, $G(x^0,\delta^\alpha) \leq_{\mathcal{C}} -4\alpha^0$. Thus, for such an α with $0 <_{\mathcal{C}} \alpha <_{\mathcal{C}} \alpha^0$, we have $x^0 \in \{x \in \mathbb{R}^p\ :\ G(x,\delta^\alpha) \leq_{\mathcal{C}} -4\alpha^0\} \subset \{x \in \mathbb{R}^p\ :\ G(x,\delta^\alpha) \leq_{\mathcal{C}} -4\alpha\} \neq \emptyset$, so we are led to a contradiction. This establishes the validity of (6) for a certain positive $\alpha \in \texttt{int}\ \mathcal{C}$.

Now, for the vector α previously exhibited, define

$$F \ : \ K \to 2^{\mathbb{R}^p}, \ \delta \mapsto F(\delta) = \{x \in \mathbb{R}^p \ : \ G(x,\delta) \leq_{\mathcal{C}} -2\alpha\} \ . \tag{7}$$

Notice that the set-valued map F maps K into the non-void closed convex subsets of $\mathbb{R}^p$. As a matter of fact, for any $\delta \in K$, if $x_k \to x_\infty$ for a sequence $x_k \in F(\delta)$, then $G(x_k,\delta) \to G(x_\infty,\delta)$ by continuity, and $G(x_\infty,\delta) \leq_{\mathcal{C}} -2\alpha$, so $x_\infty \in F(\delta)$: the set $F(\delta)$ is thus closed. On the other hand, $\mathcal{C}$-convexity property (3) implies that, for any $\delta \in K$, any $x, x' \in F(\delta)$ and any $\lambda \in [0,1]$, $G(\lambda x + (1-\lambda)x', \delta) \leq_{\mathcal{C}} \lambda G(x,\delta) + (1-\lambda)G(x',\delta) \leq_{\mathcal{C}} -2\lambda\alpha - 2(1-\lambda)\alpha = -2\alpha$, and this establishes the convexity of the set $F(\delta)$.

At this point, let us establish that F fulfils the following property of *lower semicontinuity*, see e.g. [2].

Definition. *Let X be a topological space, Y a metric space. A set-valued map F from X to Y is said* lower semicontinuous *at $x^0 \in X$ if for any $y^0 \in F(x^0)$ and any neighborhood $N(y^0)$ of y^0, there exists a neighborhood $N(x^0)$ of x^0 such that*

$$\forall x \in N(x^0), \ F(x) \cap N(y^0) \neq \emptyset \ .$$

F is said lower semicontinuous *if it is lower semicontinuous at every point $x^0 \in X$.* ∎

Let $\delta^0 \in K$, $x^0 \in F(\delta^0)$, $\varepsilon > 0$. To prove lower semicontinuity of F at δ^0, we exhibit $\eta > 0$ such that for every $\delta \in K$ with $\|\delta - \delta^0\|_m < \eta$, there exists $x \in F(\delta)$, $\|x - x^0\|_p < \varepsilon$.

Indeed, by assumption, there exists $x^{\delta^0} \in \mathbb{R}^p$ such that $G(x^{\delta^0},\delta^0) \leq_{\mathcal{C}} -4\alpha$. For $\lambda \in (0,1]$ such that

$$\lambda \leq \frac{\varepsilon}{2\|x^{\delta^0} - x^0\|_p} \ , \tag{8}$$

let $x \stackrel{\text{def}}{=} (1-\lambda)x^0 + \lambda x^{\delta^0}$. In particular, this implies $\|x - x^0\|_p = \lambda \|x^{\delta^0} - x^0\|_p \leq \varepsilon/2 < \varepsilon$.

The $\mathcal{C}$-convexity property (3) implies that, for any $\eta > 0$ and every $\delta \in K$,

$$\begin{aligned}
G(x,\delta) \quad &\leq_{\mathcal{C}} \quad (1-\lambda)G(x^0,\delta) + \lambda G(x^{\delta^0},\delta) \\
&= \quad (1-\lambda)G(x^0,\delta^0) + \lambda G(x^{\delta^0},\delta^0) + (1-\lambda)\left(G(x^0,\delta) - G(x^0,\delta^0)\right) + \lambda\left(G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\right) \\
&\leq_{\mathcal{C}} \quad -2(1-\lambda)\alpha - 4\lambda\alpha + (1-\lambda)\left(G(x^0,\delta) - G(x^0,\delta^0)\right) + \lambda\left(G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\right) \ .
\end{aligned}$$

For $\lambda > 0$, one has $\lambda\alpha \in \texttt{int}\ \mathcal{C}$. Thus, by continuity of G, for any $\varepsilon > 0$ and any λ in $(0,1]$ fulfilling (8), there exists $\eta > 0$ such that

$$\|\delta - \delta^0\|_m < \eta \Rightarrow G(x^0,\delta) - G(x^0,\delta^0) \leq_{\mathcal{C}} 2\lambda\alpha, \ G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0) \leq_{\mathcal{C}} 2\lambda\alpha \ .$$

With this choice for η, one has $G(x,\delta) \leq_{\mathcal{C}} -2(1+\lambda)\alpha + 2\lambda\alpha = -2\alpha$ when $\|\delta - \delta^0\|_m < \eta$. Thus $x \in F(\delta)$, provided that $\delta \in K$ and $\|\delta - \delta^0\|_m < \eta$. We conclude that F is lower continuous at δ^0. This achieves the proof of the lower semicontinuity of F defined in (7).

We now apply to F Michael's Selection Theorem [18], see also [2].

Theorem (Michael's Selection Theorem). *Let X be a metric space, Y a Banach space. Let F, a set-valued map from X into the closed convex subsets of Y, be lower semicontinuous. Then there exists $f : X \to Y$, a continuous selection from F.* ∎

Recall that a selection from F is any single valued map f such that, for any $x \in X$, $f(x) \in F(x)$. Application of the previous result yields existence of a continuous selection $f : K \to \mathbb{R}^p$ from F defined in (7). This function is such that

$$\forall \delta \in K, \ G(f(\delta),\delta) \leq_{\mathcal{C}} -2\alpha \ .$$

It remains to apply to each of the p components of f the following result, see e.g. [12].

Theorem (Weierstrass Approximation Theorem). *Every continuous real-valued function defined on a compact subset K of $\mathbb{R}^m$, is the limit of a sequence of polynomials, which converges uniformly in K.* ■

Thus, the selection f previously exhibited is the uniform limit in K of a sequence of (vector-valued) polynomials in δ. In particular, there exists a polynomial function $x^* \ : \ K \to \mathbb{R}^p$ such that

$$\forall \delta \in K, \ G(x^*(\delta), \delta) \leq_{\mathcal{C}} -\alpha <_{\mathcal{C}} 0_n \ .$$

This achieves the proof of Theorem 1.

3.2 Proof of Corollary 2

Let $\gamma \stackrel{\text{def}}{=} \sup_{\delta \in K} \inf\{g(x,\delta) \ : \ x \in \mathbb{R}^p, \ G(x,\delta) <_{\mathcal{C}} 0_n\}$. Note that $\gamma \leq +\infty$. First, one has, for every $\delta \in K$: $\inf\{g(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \geq \inf\{g(x,\delta) \ : \ x \in \mathbb{R}^p, \ G(x,\delta) <_{\mathcal{C}} 0_n\}$, due to the inclusion of the first set involved in the second one. Thus,

$$\gamma \leq \sup_{\delta \in K} \inf \left\{ g(x^*(\delta),\delta) \ : \ x^* \text{ polynomial} \ , \ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n \right\} \ .$$

If $\gamma = +\infty$, it ends the proof of Corollary 2. Let us assume for the remaining of this proof that $\gamma < +\infty$.

Note that, by definition, $\inf\{g(x,\delta) \ : \ x \in \mathbb{R}^p, \ G(x,\delta) <_{\mathcal{C}} 0_n\} \leq \gamma$ for every $\delta \in K$. Thus, for any $\varepsilon > 0$, for any $\delta \in K$, there exists $x \in \mathbb{R}^p$ such that $G(x,\delta) <_{\mathcal{C}} 0_n$ and $g(x,\delta) < \gamma + \varepsilon$. In other words, for any $\varepsilon > 0$, the following parameter-dependent LMI is feasible:

$$\forall \delta \in K, \exists x \in \mathbb{R}^p, \ \begin{pmatrix} g(x,\delta) - \gamma - \varepsilon & 0_{1\times n} \\ 0_{n\times 1} & G(x,\delta) \end{pmatrix} <_{\mathbb{R}^+ \times \mathcal{C}} 0_{n+1} \ .$$

Here, we denote by $<_{\mathbb{R}^+ \times \mathcal{C}}$ the product order relation, defined on $\mathbb{R}^{n+1}$ by: $(a,\alpha) <_{\mathbb{R}^+ \times \mathcal{C}} \Leftrightarrow a < 0$ and $\alpha <_{\mathcal{C}} 0_n$. The cone $\mathbb{R}^+ \times \mathcal{C}$ is proper in $\mathbb{R}^{n+1}$, and, by use of Theorem 1, for any $\varepsilon > 0$, there exists a *polynomial* map $x^*_\varepsilon : K \to \mathbb{R}^p$ such that

$$\forall \delta \in K, \ G(x^*_\varepsilon(\delta),\delta) <_{\mathcal{C}} 0_n \text{ and } g(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ .$$

Thus, for any $\varepsilon > 0$, for every $\delta \in K$,

$$\inf\{g(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \leq g(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ ,$$

so, for any $\varepsilon > 0$,

$$\sup_{\delta \in K} \inf\{g(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \leq \max_{\delta \in K} g(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ .$$

This results finally in:

$$\sup_{\delta \in K} \inf\{g(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall \delta' \in K, G(x^*(\delta'),\delta') <_{\mathcal{C}} 0_n\} \leq \gamma \ ,$$

whence the claimed equality. This ends the proof of Corollary 2.

3.3 Proof of Theorem 3

For all $\delta \in \Omega$, let us denote the open ball centered at δ with radius 1 by $B(\delta, 1)$. Let us choose a locally finite covering $B(\delta_i, 1)_{i\in\mathbb{N}}$ of Ω. Due to [11, Th. V.4.4], there exists a partition of unity $(\Psi_i)_{i\in\mathbb{N}}$ subordonate to this open covering, that is

- For all $i \in \mathbb{N}$, Ψ_i: $\mathbb{R}^m \to [0, +\infty)$ is a smooth (that is C^∞) function, with support in $B(\delta_i, 1)$.

- For all $\delta \in \Omega$, the set $\{i \in \mathbb{N} \ : \ \delta \in B(\delta_i, 1)\}$ is finite, and $\sum_{i\in\mathbb{N}} \Psi_i(\delta) = 1$.

For each $i \in \mathbb{N}$, consider a polynomial function x_i defined on the closure $\mathtt{clos}(B(\delta_i, 1))$ of $B(\delta_i, 1)$ and taking values in $\mathbb{R}^p$, such that, for all $\delta \in \mathtt{clos}(B(\delta_i, 1)) \cap \Omega$,

$$G(x_i(\delta), \delta) < 0_n \ .$$

Such polynomial function exists, due to Theorem 1 and compactness of $\mathtt{clos}(B(\delta_i, 1))$. Let us now define the C^∞ function $x^* : \Omega \to \mathbb{R}^n$ by:

$$x^*(\delta) = \sum_{i\in\mathbb{N}} \Psi_i(\delta) x_i(\delta) \ .$$

One verifies easily that

$$G(x^*(\delta), \delta) = G\left(\sum_{i\in\mathbb{N}} \Psi_i(\delta) x_i(\delta), \delta\right) \leq_{\mathcal{C}} \sum_{i\in\mathbb{N}} \Psi_i(\delta) G(x_i(\delta), \delta) <_{\mathcal{C}} 0_n \ ,$$

due to the $\mathcal{C}$-convexity of G and the fact that $\sum_i \Psi_i \equiv 1$. The smooth function x^* thus fulfills the desired inequality, and this ends the proof of Theorem 3.

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