Backstepping Design for Time-Delay Nonlinear Systems

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***Abstract*— The backstepping approach is adapted to the problem of globally uniformly asymptotically stabilizing nonlinear systems in feedback form with a delay arbitrarily large in the input. The strategy of design relies on the construction of a Lyapunov-Krasovskii functional. Continuously differentiable control laws are constructed.**

I. INTRODUCTION

One of the most popular nonlinear techniques of control design is the backstepping approach. It is presented for instance in [16], [2] and in [7, Chapter 13]. The key ideas of the approach are the following. If for a system in feedback form, i.e. of the form

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g(x)z \ , \\ \dot{z} &=& u + h(x,z) \ , \end{array} \right. \tag{1}$$

with $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}$, where $u \in \mathbb{R}$ is the input and $f(x), g(x), h(x,z)$ are continuous functions, there exists a continuously differentiable function $z_s(x)$ such that the system

$$\dot{x} \ = \ f(x) + g(x)z_s(x) \tag{2}$$

is globally asymptotically stable and if besides is known a positive definite and radially unbounded function $V(x)$ of class C^1 such that

$$W(x) := -\frac{\partial V}{\partial x}(x)[f(x) + g(x)z_s(x)] \tag{3}$$

is positive definite, then the system (1) is globally asymptotically stabilized by

$$\begin{array}{rcl} u(x,z) &=& -k(z - z_s(x)) - h(x,z) \\ && + \frac{\partial z_s}{\partial x}(x)[f(x) + g(x)z] - \frac{\partial V}{\partial x}(x)g(x) \end{array} \tag{4}$$

where k is a positive real number. Moreover, the derivative of the Lyapunov function

$$U(x,z) \ = \ V(x) + \frac{1}{2}[z - z_s(x)]^2 \tag{5}$$

along the trajectories of (2) in closed-loop with (4) satisfies

$$\dot{U}(x,z) \ = \ -W(x) - k[z - z_s(x)]^2 \ . \tag{6}$$

Many extensions of this basic result have been proved. The multiple advantages offered by this approach are well-known. Observe in particular that this technique yields a wide family of globally asymptotically stabilizing control laws, allows to address robustness issues and to solve adaptive problems. The objective of the present work is to show how the backstepping approach can be adapted to the problem of stabilizing systems in feedback form with a delay in the input. More precisely, we give sufficient conditions ensuring that a nonlinear system of the form

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g(x)z \ , \\ \dot{z} &=& u(t-\tau) + h(x(t-\tau), z(t-\tau)) \ , \end{array} \right. \tag{7}$$

with $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}$, where $u \in \mathbb{R}$ is the input and where τ is a positive real number is globally uniformly asymptotically stabilizable by continuously differentiable state feedback. This work completes the families of recent papers devoted to the control of nonlinear systems with delay. In [9] and [11], the technique of [13], [14] is adapted to the problem of stabilizing chains of integrators with bounded controls when there is a delay arbitrarily large in the input. In [10], the problem of stabilizing an oscillator by bounded feedback when there is a delay in the input is solved. In [12], the interconnection of nonlinear systems with delay is studied. In [5], [6], the concept of control Lyapunov function is extended to the case of nonlinear systems with delay through the Razumikhin theorem. In [15], connections between Razumikhin-type theorems and the ISS nonlinear small gain theorems are exposed. The present work is distinguished from the papers mentioned above because on the one hand it is devoted to systems in feedback form and on the other hand the key tool we use to prove the main result is the Lyapunov Krasovskii's functional (see [8], [3]). Surprisingly enough, this family of Lyapunov functionals has been used so far mainly in the context of the stabilization of linear systems through linear control laws, see for instance [1], [4]. We show in this work that it can be also fruitfully exploited to carry out control design for nonlinear systems. The potential advantages of the knowledge of such a functional are multiple and appealing. Observe in particular that strict Lyapunov functions are known to be very efficient tools for robustness analysis, but this issue is beyond the scope of our work. The stabilizability result we obtain is a global asymptotic stabilizability result for an arbitrarily large delay. The expressions of control laws we exhibit depend on the value the delay. We want to emphasize that we do not assume that the systems (7) are locally exponentially stabilizable. The example we give in Section III to illustrate our control design shows that indeed this assumption is not needed.

Organization of the work. In Section II the backstepping approach is adapted to the case of systems with delay in the input. Some illustrating examples are presented in Section III. Some concluding remarks in Section IV end the work.

Definitions and technical preliminaries.

1. We assume throughout the paper that the functions encountered are sufficiently smooth.
2. The argument of the functions will be omitted or simplified whenever no confusion can arise from the context. For example, one may denote a function $f(x(t))$ by simply $f(x)$ or $f(t)$ or $f(\cdot)$.
3. For a real-valued C^1 function $k(\cdot)$, we denote by $k'(\cdot)$ its first derivative.
4. A real-valued function $k(\cdot)$ is of class $\mathcal{K}_\infty$ if it is zero at zero, strictly increasing and goes to the infinity when its argument goes to the infinity.
5. $C_{n,\tau} = C^1([-\tau,0],\mathbb{R}^n)$ denotes the Banach space of continuously differentiable vector functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$.
6. For a given $t \geq 0$, $x_t(\cdot)$ denotes the restriction of $x(\cdot)$ to the interval $[t-\tau,t]$ translated to $[-\tau,0]$, i.e.

$$x_t(\theta) = x(t+\theta), \quad \forall\, \theta \in [-\tau,0].$$

7. The following norms will be used: $||\cdot||$ refers to the Euclidean vector norm; $||\phi||_c = \sup\limits_{t\in[-\tau,0]} ||\phi(t)||$ stands for the norm of a function $\phi \in C_{n,\tau}$.
8. Recall the Krasovskii Stability Theorem (see [8], [3]).

Consider the functional differential equation of retarded type

$$\begin{cases} \dot{x}(t) = f(t,x_t), & t \geq 0 \\ x_0(\theta) = \phi(\theta), & \forall\, \theta \in [-\tau,0] \end{cases} \tag{8}$$

It is assumed that $\phi \in \mathcal{C}_{n,\tau}$ and the map $f(t,\phi) : \mathbb{R}^+ \times \mathcal{C}_{n,\tau} \mapsto \mathbb{R}^n$ is continuous and Lipschitzian in ϕ and $f(t,0) = 0$. Suppose that the function $f : \mathbb{R} \times \mathcal{C}_{n,\tau} \mapsto \mathbb{R}^n$ takes bounded sets of $\mathcal{C}_{n,\tau}$ in bounded sets of $\mathbb{R}^n$ and suppose that $u(s)$, $v(s)$ and $w(s)$ are continuous, nonnegative and nondecreasing functions with $u(s),\ v(s) > 0$ for $s \neq 0$ and $u(0) = v(0) = 0$.

If there is a continuous function $V : \mathbb{R} \times \mathcal{C}_{n,\tau} \mapsto \mathbb{R}$ such that

$$u(||\phi(0)||) \leq V(t,\phi) \leq v(||\phi||_c) \ , \tag{9}$$

$$\dot{V}(t,\phi) \leq -w(||\phi(0)||) \tag{10}$$

then the solution $x = 0$ of the equation (8) is uniformly stable.

If $u(s) \to \infty$ as $s \to \infty$ the solutions are uniformly bounded.

If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

II. MAIN RESULTS

Consider the nonlinear system (7) and introduce a set of assumptions.

Assumption A1. There exist a state feedback $z_s(x)$, a positive definite and radially unbounded function $V(x)$ and a positive definite function $W(x)$ such that

$$\frac{\partial V}{\partial x}(x)[f(x) + g(x)z_s(x)] \ = \ -W(x) \ . \tag{11}$$

Assumption A2. Let C be a positive real number. For all $x \in \mathbb{R}^{n_x}$, the inequalities

$$\left|\frac{\partial V}{\partial x}(x)g(x)\right|^2 \ \leq \ W(x) \ , \ \left|\frac{\partial z_s}{\partial x}(x)g(x)\right| \ \leq \ C \tag{12}$$

are satisfied.

Assumption A3. Let Ω be a positive real number such that $\Omega \geq 8\tau$. For all $x \in \mathbb{R}^{n_x}$ and $\xi \in C^1([0,2\tau],\mathbb{R})$, the inequality

$$-\frac{1}{4}W(x) - T(x,\xi) - \frac{1}{\Omega}\int_0^{2\tau} W(\xi(l))dl \leq 0 \tag{13}$$

with

$$\begin{array}{rcl} T(x,\xi) & = & \frac{\partial V}{\partial x}(x)g(x)\displaystyle\int_\tau^{2\tau} H(\xi(l),\xi(l-\tau))dl \ , \\ H(a,b) & = & \frac{\partial z_s}{\partial x}(a)[f(a) + g(a)z_s(b)] \ , \end{array} \tag{14}$$

is satisfied.

Theorem 1: Assume that the system (7) satisfies the assumptions A1 to A3. Then the system (7) is globally uniformly asymptotically stabilized by the feedback

$$\begin{array}{rcl} u_s(t) & = & -\varepsilon(z(t) - z_s(x(t-\tau))) - h(x(t),z(t)) \\ & & +\frac{\partial z_s}{\partial x}(x(t))[f(x(t)) + g(x(t))z(t)] \end{array} \tag{15}$$

where ε is a positive real number such that $\varepsilon \in \left(0, \frac{1}{2\tau}\right]$.

Discussion of Theorem 1.

• Assumption A1 is not surprising: in the well-known framework of the backstepping for systems without delay, this assumption or a similar one is imposed. Assumptions A2 and A3 are introduced because of the delay in the input.
• Due to the finite escape time phenomenon, Assumption A2 cannot be removed without adding another assumption. This fact is illustrated by the system

$$\begin{cases} \dot{x} & = & -x + x^4 z \ , \\ \dot{z} & = & u(t-\tau) \ , \end{cases} \tag{16}$$

because on the one hand, it is globally asymptotically stabilized by the feedback $u(x,z) = -z - x^5$ when $\tau = 0$, but not globally asymptotically stabilizable when $\tau > 0$ (see Appendix A) and on the other hand, it is of the form (7), satisfies Assumption A1 and A3 with $V(x) = x^2, W(x) = 2x^2, z_s(x) = 0$ but, for any choice of functions $V(x)$ and $z_s(x)$, it does not satisfy Assumption A2.

• We prove in Appendix B that Assumption A3 may be replaced by the slightly more restrictive assumption:

Assumption A3'. For all $t \geq 0$ and $\xi \in C^1([0, 2\tau], \Re)$, the inequality

$$\Omega\tau \int_{\tau}^{2\tau} |H(\xi(l), \xi(l-\tau))|^2 \, dl \leq \int_{0}^{2\tau} W(\xi(l)) dl \quad (17)$$

where $H(\cdot)$ is the function defined in (14) and where Ω is a positive real number such that $\Omega \geq 8\tau$ is satisfied.

Thanks to Assumption A3' and A2 one understands that, roughly speaking, the system (7) is globally asymptotically stabilizable when $z_s(x)$ and its first partial derivatives are sufficiently small in norm. In some cases, one can take advantage of Assumption A3' to determine suitable functions $z_s(\xi)$.

• Many extensions of Theorem 1 can be easily proved. For instance, the approach applies when the x-subsystem of (7) is not affine with respect to z, extensions to multi-input systems in feedback form can be made and Theorem 1 can be extended to systems of the form

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)(z + h_1(x(t-\tau), z(t-\tau))) \ , \\ \dot{z} & = & u(t-\tau) + h_2(x(t-\tau), z(t-\tau)) \ , \end{array} \right. \quad (18)$$

where $h_1(\cdot)$ is a nonlinear function of class C^1. However, in this section we have chosen to focus our attention on systems of the form (7) because the key ideas of our control design in the case where bounded feedbacks are not searched out can be clearly exposed and understood when applied to (7).

• Theorem 1 applies to systems which are not locally linearizable. The assumptions of this theorem do not even ensure that the x-subsystem of (7) with z as virtual input is locally exponentially stabilizable. The example we give in Section III illustrates this remark.

Proof of Theorem 1. The proof consists in constructing a functional allowing to prove by means of the Krasovskii's Theorem that the system (7) in closed-loop with the control law (15) is globally uniformly asymptotically stable. First, observe that the change of variable

$$Z(t) \ = \ z(t) - z_s(x(t-\tau)) \quad (19)$$

transforms (7) into

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)(Z + z_s(x(t-\tau))) \ , \\ \dot{Z} & = & u(t-\tau) \\ & & +M(x(t-\tau), x(t-2\tau), Z(t-\tau)) \ , \end{array} \right. \quad (20)$$

with

$$\begin{array}{rcl} M(a,b,c) & = & h(a, c + z_s(b)) \\ & & -\frac{\partial z_s}{\partial x}(a)[f(a) + g(a)(c + z_s(b))] \ . \end{array}$$

When $u = u_s(\cdot)$ where $u_s(\cdot)$ is the function defined in (15),

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)(Z + z_s(x(t-\tau))) \ , \\ \dot{Z} & = & -\varepsilon Z(t-\tau) \ . \end{array} \right. \quad (21)$$

The objective is now to determine a Lyapunov-Krasovskii functional for the x-subsystem of (21) which will be used for the construction of a Lyapunov Krasovskii functional for the system (21). Consider the functional

$$U(x_t) \ = \ V(x(t)) + \frac{1}{\Omega} \int_{t-2\tau}^{t} \left(\int_{s}^{t} W(x(l)) dl \right) ds \quad (22)$$

where Ω is the positive real number involved in Assumption A3. The derivative of the functional $U(\cdot)$ along the solutions of the system (21) satisfies

$$\begin{array}{rcl} \dot{U} & = & \frac{\partial V}{\partial x}(x)[f(x) + g(x)(Z + z_s(x(t-\tau)))] \\ & & +\frac{2\tau}{\Omega} W(x) - \frac{1}{\Omega} \int_{t-2\tau}^{t} W(x(s)) ds \\ & = & -W(x) + \frac{\partial V}{\partial x}(x) g(x) Z \\ & & +\frac{\partial V}{\partial x}(x) g(x)[z_s(x(t-\tau)) - z_s(x)] \\ & & +\frac{2\tau}{\Omega} W(x) - \frac{1}{\Omega} \int_{t-2\tau}^{t} W(x(s)) ds \ . \end{array} \quad (23)$$

According to Assumption A3, $\Omega \geq 8\tau$. Consequently, the inequality

$$\begin{array}{rcl} \dot{U} & \leq & -\frac{3}{4} W(x) + \frac{\partial V}{\partial x}(x) g(x) Z \\ & & +\frac{\partial V}{\partial x}(x) g(x)[z_s(x(t-\tau)) - z_s(x)] \\ & & -\frac{1}{\Omega} \int_{t-2\tau}^{t} W(x(s)) ds \end{array} \quad (24)$$

is satisfied. One can easily deduce from Assumption A3 that

$$\begin{array}{rcl} \dot{U} & \leq & -\frac{1}{2} W(x) \\ & & +\frac{\partial V}{\partial x}(x) g(x) \int_{t-\tau}^{t} H(x(l), x(l-\tau)) dl \\ & & +\frac{\partial V}{\partial x}(x) g(x)[z_s(x(t-\tau)) - z_s(x)] \\ & & +\frac{\partial V}{\partial x}(x) g(x) Z \\ & \leq & -\frac{1}{2} W(x) \\ & & +\frac{\partial V}{\partial x}(x) g(x) \\ & & \times \left[Z - \int_{t-\tau}^{t} \frac{\partial z_s}{\partial x}(x(l)) g(x(l)) Z(l) dl \right] \ . \end{array} \quad (25)$$

According to Assumption A2, the inequality

$$\dot{U} \ \leq \ -\frac{1}{4} W(x) + 2 \left[Z^2 + C^2 \left(\int_{t-\tau}^{t} |Z(l)| dl \right)^2 \right] \quad (26)$$

is satisfied. From Cauchy-Schwartz inequality, we deduce that

$$\dot{U} \ \leq \ -\frac{1}{4} W(x) + 2 \left[Z^2 + \tau C^2 \int_{t-\tau}^{t} Z(l)^2 dl \right] \ . \quad (27)$$

We now construct a Lyapunov Krasovskii functional for the Z-subsystem of (21). First observe that the derivative of the quadratic function

$$L(Z) \ = \ \frac{1}{2} Z^2 \quad (28)$$

along the trajectories of (21) satisfies

$$\begin{array}{rcl} \dot{L} & = & -\varepsilon Z Z(t-\tau) \\ & = & -\varepsilon Z^2 + \varepsilon Z[Z - Z(t-\tau)] \\ & = & -\varepsilon Z^2 - \varepsilon^2 Z \displaystyle\int_{t-\tau}^{t} Z(s-\tau) ds \ . \end{array} \quad (29)$$

Using the triangular inequality, one obtains

$$\dot{L} \ \leq \ -\tfrac{1}{2}\varepsilon Z^2 + \tfrac{1}{2}\varepsilon^3 \left(\int_{t-\tau}^{t} Z(s-\tau) ds \right)^2 \ . \quad (30)$$

Cauchy-Schwartz inequality leads to

$$\dot{L} \ \leq \ -\tfrac{1}{2}\varepsilon Z^2 + \tfrac{1}{2}\varepsilon^3 \tau \int_{t-2\tau}^{t-\tau} Z(s)^2 ds \ . \quad (31)$$

This last inequality implies that the derivative of the functional

$$M(Z_t) \ = \ L(Z(t)) + \varepsilon^3 \tau \int_{t-2\tau}^{t} \left(\int_{s}^{t} Z(l)^2 dl \right) ds \quad (32)$$

along the trajectories of (21) satisfies

$$\begin{array}{rcl} \dot{M} & \leq & -\tfrac{1}{2}\varepsilon Z^2 + \tfrac{1}{2}\varepsilon^3 \tau \displaystyle\int_{t-2\tau}^{t-\tau} Z(s)^2 ds \\ & & -\varepsilon^3 \tau \displaystyle\int_{t-2\tau}^{t} Z(s)^2 ds + \varepsilon^3 \tau^2 Z^2 \\ & \leq & \varepsilon \left(-\tfrac{1}{2} + \varepsilon^2 \tau^2 \right) Z^2 - \tfrac{1}{2}\varepsilon^3 \tau \displaystyle\int_{t-2\tau}^{t} Z(s)^2 ds \ . \end{array} \quad (33)$$

From $\varepsilon \in \left(0, \frac{1}{2\tau}\right]$, one deduces that the inequality

$$\dot{M} \ \leq \ -\frac{1}{4}\varepsilon Z^2 - \frac{1}{2}\varepsilon^3 \tau \int_{t-2\tau}^{t} Z(s)^2 ds \quad (34)$$

holds. Combining (26) and (34), one obtains that the derivative of the functional

$$U_f(x_t, Z_t) \ = \ U(x_t) + K M(Z_t) \quad (35)$$

where K is a positive real number such that $K \geq \max\left\{ \frac{12}{\varepsilon}, \frac{4C^2+1}{\varepsilon^3 \tau} \right\}$ satisfies the inequality

$$\dot{U}_f \ \leq \ -\frac{1}{4} W(x(t)) - \int_{t-\tau}^{t} Z(s)^2 ds - Z(t)^2 \ . \quad (36)$$

The right-hand side of (36) is smaller than a negative definite function of $(x(t), Z(t))$. To prove that the other assumptions of the Krasovskii's theorem are satisfied, we have to prove that inequalities of the type (9) are satisfied by U_f. The definitions of U_f, U, M imply that

$$U_f(x_t, Z_t) \ \geq \ V(x(t)) + K L(Z(t)) \ . \quad (37)$$

Since $V(x) + KL(Z)$ is positive definite and radially unbounded, according to [7, Lemma 3.5], there exists $\gamma_1(\cdot)$ of class $\mathcal{K}_\infty$ such that

$$V(x) + KL(Z) \ \geq \ \gamma_1(||(x^\top, Z)^\top||) \ . \quad (38)$$

It follows that

$$U_f(x_t, Z_t) \ \geq \ \gamma_1(||(x(t)^\top, Z(t))^\top||) \ . \quad (39)$$

On the other hand, since $V(x)$ and $W(x)$ are positive definite, then according to [7, Lemma 3.5] there exist $\gamma_2(\cdot)$ and $\gamma_3(\cdot)$ of class $\mathcal{K}_\infty$ such that

$$V(x) \ \leq \ \gamma_2(||x||) \ , \ W(x) \ \leq \ \gamma_3(||x||) \ . \quad (40)$$

It follows that

$$\begin{array}{c} V(x) + \frac{1}{\Omega} \displaystyle\int_{t-2\tau}^{t} \left(\int_{s}^{t} W(x(l)) dl \right) ds \\ \leq \gamma_2(||x||) + \frac{1}{\Omega} \displaystyle\int_{t-2\tau}^{t} \left(\int_{s}^{t} \gamma_3(||x(l)||) dl \right) ds \\ \leq \gamma_2 \left(\displaystyle\sup_{l \in [t-2\tau, t]} ||x(l)|| \right) + \frac{4\tau^2}{\Omega} \gamma_3 \left(\displaystyle\sup_{l \in [t-2\tau, t]} ||x(l)|| \right) \ . \end{array} \quad (41)$$

Since $\gamma_2(\cdot)$ and $\gamma_3(\cdot)$ are increasing, we deduce that

$$\begin{array}{rcl} U_f(x_t, Z_t) & \leq & \gamma_2\left(||x_t||_c\right) + \frac{4\tau^2}{\Omega} \gamma_3\left(||x_t||_c\right) + \frac{K}{2} Z^2 \\ & \leq & \Gamma(||(x_t^\top, Z_t)^\top||_c) \end{array} \quad (42)$$

with

$$\Gamma(r) \ = \ \gamma_2(r) + \frac{4\tau^2}{\Omega} \gamma_3(r) + \frac{K}{2} r^2 \ , \ \forall r \geq 0 \ . \quad (43)$$

The function $\gamma_1(\cdot)$ and $\Gamma(\cdot)$ are of class $\mathcal{K}_\infty$: the assumptions of Krasovskii's theorem are satisfied. It follows that the system (21) is globally uniformly asymptotically stable which implies that the system (7) in closed-loop with the control law (15) is globally uniformly asymptotically stable as well. This concludes the proof.

III. EXAMPLE

To illustrate Theorem 1, we determine a globally uniformly asymptotically stabilizing feedback for the two-dimensional system

$$\left\{ \begin{array}{rcl} \dot{x} & = & xz \ , \\ \dot{z} & = & u(t-\tau) \ , \end{array} \right. \quad (44)$$

where τ is an arbitrary positive real number, by applying this theorem. Observe that the linear approximation at the origin of (44) is not asymptotically stabilizable which implies that the system (44) is not locally exponentially stabilizable. It follows that linear techniques cannot be of any help for proving the local asymptotic stabilizability of this system.

First, let us check that Assumptions A1 to A3 are satisfied by (44) for functions $V(x)$ and $z_s(x)$ suitably chosen. Consider the function

$$V(x) \ = \ \eta \ln(1 + x^2) \quad (45)$$

where η is a positive real number. This function is a candidate Lyapunov function: it is a positive definite, radially unbounded function of x and is zero at the origin. Consider the function

$$z_s(x) \ = \ -\frac{\omega x^2}{1+x^2} \tag{46}$$

where ω is a positive real number. Then, with the notations of Theorem 1, $f(x)=0, g(x)=x$ and

$$\begin{array}{rcl} \frac{\partial V}{\partial x}(x)g(x) &=& \frac{2\eta x^2}{1+x^2} \ , \ \frac{\partial z_s}{\partial x}(x) \ = \ -\frac{2\omega x}{(1+x^2)^2} \ , \\ W(x) &=& \frac{2\eta\omega x^4}{(1+x^2)^2} \ . \end{array} \tag{47}$$

It follows that Assumption A1 is satisfied for all $\eta > 0, \omega > 0$ and Assumption A2 is satisfied when $2\eta \leq \omega$. To prove that Assumption A3 is satisfied, one has to establish that the functional

$$\zeta(x,\xi) = -\tfrac{1}{4}W(x) - T(x,\xi) - \tfrac{1}{\Omega}\int_0^{2\tau} W(\xi(l))dl \tag{48}$$

with $x \in \Re^{n_x}$, $\xi \in C^1([0,2\tau],\Re^{n_x})$ and where $T(x,\xi)$ is the functional defined in (14) is nonpositive. One can check readily that it satisfies

$$\begin{array}{rcl} \zeta(x,\xi) &=& -\frac{\eta\omega x^4}{2(1+x^2)^2} \\ && -\frac{4\eta\omega^2 x^2}{1+x^2}\int_\tau^{2\tau} \frac{\xi(l)^2}{(1+\xi(l)^2)^2}\frac{\xi(l-\tau)^2}{1+\xi(l-\tau)^2}dl \\ && -\frac{1}{2}\int_0^{2\tau}\frac{\eta\omega\xi(l)^4}{(1+\xi(l)^2)^2}dl \\ &\leq& -\frac{\eta\omega x^4}{4(1+x^2)^2} \\ && +16\eta\omega^3\left[\int_\tau^{2\tau}\frac{\xi(l)^2}{(1+\xi(l)^2)^2}\frac{\xi(l-\tau)^2}{1+\xi(l-\tau)^2}dl\right]^2 \\ && -\frac{\eta\omega}{2}\int_0^{2\tau}\frac{\xi(l)^4}{(1+\xi(l)^2)^2}dl \ . \end{array} \tag{49}$$

By using Cauchy-Schwartz inequality, one obtains

$$\begin{array}{rcl} \zeta(x,\xi) &\leq& -\frac{\eta\omega x^4}{4(1+x^2)^2} \\ && +16\eta\omega^3\tau\left[\int_\tau^{2\tau}\frac{\xi(l)^4}{(1+\xi(l)^2)^4}\frac{\xi(l-\tau)^4}{(1+\xi(l-\tau)^2)^2}dl\right] \\ && -\frac{\eta\omega}{2}\int_0^{2\tau}\frac{\xi(l)^4}{(1+\xi(l)^2)^2}dl \\ &\leq& -\frac{\eta\omega x^4}{4(1+x^2)^2} + 16\eta\omega^3\tau\int_\tau^{2\tau}\frac{\xi(l)^4}{(1+\xi(l)^2)^4}dl \\ && -\frac{\eta\omega}{2}\int_0^{2\tau}\frac{\xi(l)^4}{(1+\xi(l)^2)^2}dl \ . \end{array} \tag{50}$$

It follows that

$$\zeta(x,\xi) \ \leq \ 0 \tag{51}$$

when $\omega \in \left(0, \frac{1}{\sqrt{32\tau}}\right]$ which implies that, in this case, Assumption A3 is satisfied.

It follows from the above analysis and Theorem 1 that the feedback

$$u_s(t) \ = \ -\varepsilon\left(z(t) + \frac{\omega x(t-\tau)^2}{1+x(t-\tau)^2}\right) - \frac{2\varepsilon x(t)^2}{(1+x(t)^2)^2}z(t) \tag{52}$$

when $\omega \in \left(0, \frac{1}{\sqrt{32\tau}}\right], 0 < \varepsilon \leq \frac{1}{2\tau}$ globally uniformly asymptotically stabilizes the system (44). A possible choice is $\omega = \frac{1}{\sqrt{32\tau}}, \varepsilon = \frac{1}{2\tau}$. It yields the following expression of control law

$$\begin{array}{rcl} u_s(t) &=& -\frac{1}{2\tau}\left(z(t) + \frac{1}{\sqrt{32\tau}}\frac{x(t-\tau)^2}{1+x(t-\tau)^2}\right) \\ && -\frac{1}{\tau}\frac{x(t)^2}{(1+x(t)^2)^2}z(t) \ . \end{array} \tag{53}$$

IV. CONCLUSION

We have carried out the design of globally uniformly asymptotically stabilizing feedback for a family of nonlinear systems in feedback form with a delay in the input arbitrarily large. One of the features of the method is that it does not only apply to systems which are locally exponentially stabilizable. The proposed control laws depend explicitly on the value of the delay. We conjecture that the approach can be adapted to the case where an exact knowledge of this value is not available. This issue, as well as robustness and disturbance attenuation issues, discrete-time versions of Theorem 1, are some issues that we will pursue in future works.

V. REFERENCES

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APPENDIX

A. Delay in the input may cause finite escape time

We prove here that system (16) is not globally asymptotically stabilizable when $\tau > 0$. We proceed by contradiction: suppose that the system (16) is globally asymptotically stabilized by a continuous feedback $u_s(\phi_x, \phi_y)$.

Consider $\delta > 0$ and choose as initial conditions functions $\varphi_x(\cdot), \varphi_z(\cdot)$ such that $(\varphi_x(t), \varphi_z(t)) = (0,0)$ for all $t \leq \frac{\tau}{3}$, $(\varphi_x(t), \varphi_z(t)) = (\delta, \delta)$ for all $t \in [\frac{2\tau}{3}, \tau]$ and $\varphi_x(\cdot)$ and $\varphi_z(\cdot)$ are nondecreasing.

For all $t \leq \frac{4}{3}\tau$, the solution of (16) satisfies

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x + x^4 z \ , \\ \dot{z} & = & u_s(0,0) \ . \end{array} \right. \tag{54}$$

So, for all $t \in \left[\tau, \frac{4}{3}\tau\right]$,

$$\left\{ \begin{array}{rcl} \dot{x} & = & -x + x^4 z \ , \\ z & = & u_s(0,0) + z(\tau) = u_s(0,0) + \delta \ . \end{array} \right. \tag{55}$$

Choosing δ such that $u_s(0,0) + \delta \geq \frac{1}{2}\delta$, we have

$$\dot{x} \geq -x + \frac{1}{2}\delta x^4 \tag{56}$$

for all $t \in \left[\tau, \frac{4}{3}\tau\right]$. It follows that $X = e^t x$ satisfies, for all $t \in \left[\tau, \frac{4}{3}\tau\right]$,

$$\dot{X} \geq \frac{1}{2}\delta e^{-3t} X^4 \geq \frac{1}{2}\delta e^{-4\tau} X^4 \ . \tag{57}$$

It follows that for all $t \in \left[\tau, \inf\left\{\frac{4}{3}\tau, \frac{2e^\tau}{3\delta^4} + \tau\right\}\right)$,

$$X(t)^3 \geq \frac{1}{\frac{1}{X(\tau)^3} - \frac{3\delta}{2}e^{-4\tau}(t-\tau)} = \frac{\delta^3 e^{3\tau}}{1 - \frac{3\delta^4}{2}e^{-\tau}(t-\tau)} \ . \tag{58}$$

It follows that when $\delta \geq \left(\frac{2}{\tau}\right)^{\frac{1}{4}} e^{\frac{\tau}{4}}$, then $\frac{4\tau}{3}$ is the smallest of the two quantities in the infimum, and for all $t \in \left[\tau, \frac{2e^\tau}{3\delta^4} + \tau\right)$,

$$X(t)^3 \geq \frac{\delta^3}{1 - \frac{3\delta^4}{2}e^{-\tau}(t-\tau)} \ . \tag{59}$$

On the other hand,

$$\lim_{t \to \frac{2e^\tau}{3\delta^4} + \tau} \frac{\delta^3}{1 - \frac{3\delta^4}{2}e^{-\tau}(t-\tau)} = +\infty \ . \tag{60}$$

It follows that

$$\lim_{t \to \frac{2e^\tau}{3\delta^4} + \tau} x(t) = +\infty \ . \tag{61}$$

Thus, the finite escape time phenomenon occurs and (16) is not globally asymptotically stabilized by $u_s(\phi_x, \phi_y)$. This contradicts the initial assumption. This concludes the proof.

B. Assumption A3' implies Assumption A3

Assume that the system (7) satisfies Assumptions A1, A2 and Assumption A3'. Consider the function

$$\begin{array}{rcl} \zeta(x,\xi) & = & -\frac{1}{4}W(x) \\ & & -\frac{\partial V}{\partial x}(x)g(x) \displaystyle\int_\tau^{2\tau} H(\xi(l), \xi(l-\tau))dl \\ & & -\frac{1}{\Omega} \displaystyle\int_0^{2\tau} W(\xi(l))dl \ . \end{array} \tag{62}$$

Using the triangular inequality, one can prove that it satisfies

$$\begin{array}{rcl} \zeta(x,\xi) & \leq & -\frac{1}{4}W(x) + \frac{1}{4}\left|\frac{\partial V}{\partial x}(x)g(x)\right|^2 \\ & & + \left| \displaystyle\int_\tau^{2\tau} H(\xi(l), \xi(l-\tau))dl \right|^2 \\ & & -\frac{1}{\Omega} \displaystyle\int_0^{2\tau} W(\xi(l))dl \ . \end{array} \tag{63}$$

From Assumption A2, it follows that

$$\begin{array}{rcl} \zeta(x,\xi) & \leq & \left| \displaystyle\int_\tau^{2\tau} H(\xi(l), \xi(l-\tau))dl \right|^2 \\ & & -\frac{1}{\Omega} \displaystyle\int_0^{2\tau} W(\xi(l))dl \ . \end{array} \tag{64}$$

Cauchy-Schwartz inequality implies that

$$\begin{array}{rcl} \zeta(x,\xi) & \leq & \tau \displaystyle\int_\tau^{2\tau} \left| H(\xi(l), \xi(l-\tau)) \right|^2 dl \\ & & -\frac{1}{\Omega} \displaystyle\int_0^{2\tau} W(\xi(l))dl \ . \end{array} \tag{65}$$

Assumption A3' implies that

$$\zeta(x,\xi) \ \leq \ 0 \ . \tag{66}$$

Therefore Assumption A3 is satisfied. This concludes the proof.