Stabilization of LPV Systems

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***Abstract*— We study in this paper the static state-feedback stabilization of linear finite dimensional systems depending polynomially upon a finite set of real, bounded, parameters. These parameters are a priori unknown, but available in real-time for control.**

We state two main results. First, we show that stabilizability of the class of systems obtained for frozen values of the parameters, may be expressed equivalently by some LMI conditions, linked to certain class of parameter-dependent Lyapunov functions. Second, we show that existence of such a Lyapunov function for the LPV systems subject to bounded rate of variation of the parameters with respect to time, may be in the same manner expressed equivalently by some LMI conditions. In both cases, the method provides explicitly parameter-dependent stabilizing gain.

I. INTRODUCTION

Linear parameter-varying (LPV) systems have recently received much attention, in connection with the gain-scheduling control design methodologies, see [5], [7] for recent surveys and bibliography on the subject. LPV systems are linear systems that depend upon time-varying real parameters. The latter are not known in advance, but may be used in real-time for control purposes. However, they are usually constrained to lie inside a known bounded set.

The issue of checking the stabilizability and determining a parameter-dependent stabilizing gain for every frozen admissible value of the parameters, is already a difficult task. As an example, a coarse application of the Lyapunov-based synthesis techniques available for linear systems is impossible, as it leads to solve an infinite number of linear matrix inequalities (LMIs). At this point, two types of methods are usually used (see the recent works [8], [6] on LPV systems): either the controller gain is first computed for a bunch of parameter values, and then interpolated between the nodes of this grid (but the stability, and possibly performance, results are not guaranteed between the nodes); or the solution of the parameter-dependent LMIs involved is sought for with pre-specified dependence with respect to the parameters, usually constant or affine (at the cost of adding conservatism). Of course, the stabilization issue is still more complicated when the parameters are time-varying.

In this paper, we show that, in principle, the previous task may be realized without conservatism, to provide guaranteed stability results. More precisely, we state two main results, Theorems 1 and 2, whose contribution may be summarized as follows.

1. The stabilization of all the systems obtained for *constant values* of the parameters in the admissible hypercube is equivalent to the existence for the closed-loop system of a quadratic Lyapunov function *polynomial* with respect to the parameters. For fixed value of the degree, the coefficients of this polynomial are solutions of a LMI.

2. The existence of a similar quadratic Lyapunov function (depending in the same way upon the parameters) for the corresponding LPV system with *restricted rate of variation* of the parameters, is also equivalent, for fixed degree, to the solvability of a LMI.

3. In both cases, a parameter-dependent stabilizing gain is deduced from the solution of the LMIs.

The originality of the results presented here lies in the nonconservative nature of the LMI conditions proposed. They constitute a systematization of the approaches based on parameter-dependent Lyapunov functions. Further work should consider dynamic controller synthesis and performace verification.

Effective use of the results given here is subordinate to powerful LMI solvers. A general idea for reducing the computation complexity consists in performing first a subdivision of the admissible parameter set in subdomains and applying the results presented below on these smaller regions. The present paper provides a stage towards such an hybrid control, which in principle could lead to sensible diminution of the (off-line) computational burden, but whose study is out of our scope here.

The paper is organized as follows. The problem is presented in Section II. Notations are provided in Section III. The result on systems with frozen parameters (Theorem 1) is stated in Section IV. The results on systems with parameters with bounded derivative (Theorem 2) is stated in Section V. Elements of proof are displayed in Section VI. Some technical results related to the computations involved are gathered in Appendix.

II. PROBLEM STATEMENT

We consider here the issue of state-feedback stabilization for the class of linear systems

$$\dot{x} = A(\sigma(t))x + B(\sigma(t))u \ . \tag{1}$$

In (1), the matrix $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}$ are supposed polynomial of partial degree (at most) k with respect to the components of the vector $\sigma \overset{\text{def}}{=} (\sigma_1,\ldots,\sigma_m)$ of m real parameters.

We are interested in the design of stabilizing static state-feedback for (1), under the assumption that $\forall t \geq 0$, $\sigma(t) \in [-1;+1]^m$. In the special case where the components of σ are constant ($\dot\sigma \equiv 0$), this is equivalent to find, for any $\sigma \in [-1;+1]^m$, a gain $K(\sigma)$ such that $A(\sigma)+B(\sigma)K(\sigma)$ is Hurwitz. This leads to study the following property.

Property I. *There exist mappings $P : [-1;+1]^m \to \mathcal{S}^n$, $N : [-1;+1]^m \to \mathbb{R}^{p\times n}$ such that,* $\forall\sigma \in [-1;+1]^m$, $P(\sigma) > 0_n$, $A(\sigma)P(\sigma)+P(\sigma)A(\sigma)^T+B(\sigma)N(\sigma)+N(\sigma)^TB(\sigma)^T < 0_n$.

In this formula, $\mathcal{S}^n$ represents the set of symmetric matrices of size $n\times n$. Property I is *equivalent* to the stabilizability of (1) for every admissible choice of the parameters.

As is well-known, the previous condition, guaranteeing stability for the frozen parameter systems, is not enough to guarantee stability of the systems with time-varying parameters. An attempt to extend the previous ideas to stabilization of LPV systems with parameters having variation rate constrained by $|\dot\sigma_i| \leq \bar\rho_i$ a.e., $i = 1,\dots,m$, leads to the following interesting issue.

Property II. *There exist mappings $P : [-1;+1]^m \to \mathcal{S}^n$, $N : [-1;+1]^m \to \mathbb{R}^{p\times n}$, P differentiable, such that,* $\forall\sigma \in [-1;+1]^m$, $\forall\rho_i \in [-\bar\rho_i;\bar\rho_i]$, $P(\sigma) > 0_n$, $A(\sigma)P(\sigma) + P(\sigma)A(\sigma)^T+B(\sigma)N(\sigma)+N(\sigma)^TB(\sigma)^T-\sum_{i=1}^m \rho_i\frac{\partial P(\sigma)}{\partial\sigma_i} < 0_n$.

Property II is *equivalent* to the existence of a quadratic Lyapunov function depending upon the present values of the parameters. This property is thus a priori stronger than the stability of the systems (1) attached to every admissible trajectories of the parameters.

III. NOTATIONS AND PRELIMINARIES

• The matrices I_n, 0_n, $0_{n\times p}$ are the $n\times n$ identity matrix and the $n\times n$ and $n\times p$ zero matrices respectively. The symbol $\otimes$ denotes Kronecker product, the power of Kronecker product being used with the natural meaning: $M^{0\otimes} = 1$, $M^{p\otimes} \stackrel{\text{def}}{=} M^{(p-1)\otimes}\otimes M$. Key properties are: $(A\otimes B)^T = A^T\otimes B^T$, $(A\otimes B)(C\otimes D) = (AC\otimes BD)$ for matrices of compatible size. The conjugate and transconjugate of M, are denoted M^T and M^H. The unit circle in $\mathbb{C}$ is denoted as the boundary $\partial\mathbb{D}$, and the set of positive integers $\mathbb{N}$. Also, the set of symmetric real (resp. hermitian complex) matrices of size $n\times n$ is denoted $\mathcal{S}^n$ (resp. $\mathcal{H}^n$).

Last, we introduce some spaces of matrix-valued polynomials. $\mathbb{R}^{n\times n}[\sigma]$ (resp. $\mathcal{S}^n[\sigma]$) will denote the set of polynomials in the variable $\sigma\in\mathbb{R}^m$, with coefficients in $\mathbb{R}^{n\times n}$ (resp. $\mathcal{S}^n$). We shall also consider in the sequel the set, denoted $\mathbb{R}^{n\times n}[z,\bar z]$, of polynomials in z and $\bar z$, $z\in\mathbb{C}$, with coefficients in $\mathbb{R}^{n\times n}$. The sets $\mathcal{S}^n[z,\bar z]$, $\mathcal{H}^n[z,\bar z]$ are defined similarly.

• We now introduce specific notations. For any $l\in\mathbb{N}$, for any $v\in\mathbb{C}$, let

$$v^{[l]} \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ v \\ \vdots \\ v^{l-1}\end{pmatrix} . \tag{2}$$

This notation permits to manipulate polynomials. Notice in particular that, for a free variable $z\in\mathbb{C}^m$, the vector $(z_m^{[l]}\otimes\cdots\otimes z_1^{[l]})$ contains exactly the l^m monomials in $z_1,\dots,z_m$ of degree at most $l-1$ in each variable.

Using this notation, any element $M(z)$ in $\mathbb{R}^{p\times n}[z,\bar z]$ may be represented as

$$M(z) = (z_m^{[l]}\otimes\cdots\otimes z_1^{[l]}\otimes I_p)^H M_l (z_m^{[l]}\otimes\cdots\otimes z_1^{[l]}\otimes I_n) . \tag{3}$$

In this formula, the integer $l\in\mathbb{N}$ and the matrix $M_l\in\mathbb{R}^{l^mp\times l^mn}$ are unique, provided that l is minimal. Independently of minimality, the matrix M_l is called the *coefficient matrix* of this representation of $M(z)$, $l-1$ its *degree*.

In the sequel, we shall use the following change of variables ($j^2=-1$):

$$\begin{gathered}\varphi : [-1;+1]^m \to (\partial\mathbb{D})^m, \sigma\mapsto z = \varphi(\sigma) \\ \text{where } z_i \stackrel{\text{def}}{=} \sigma_i + j\sqrt{1-\sigma_i^2}, i = 1,\dots,m .\end{gathered} \tag{4}$$

Basically (see the developments below), this will permit to use Kalman-Yakubovich-Popov lemma, "replacing" the free variables z_i by matrix multipliers in the parameter-dependent LMIs appearing in Properties I and II. In particular, for z in the range of φ, $\varphi^{-1}(z) = \frac{z+\bar z}{2}$. When $z = (z_1,\dots,z_m)$ covers $(\partial\mathbb{D})^m$, $\frac{z+\bar z}{2}$ varies in the whole set $[-1;+1]^m$.

Generally speaking, for M defined as in (3) and the change of variable φ as in (4), $M(\varphi(\sigma))$ is a polynomial in σ_i, $\sqrt{1-\sigma_i^2}$, $i=1,\dots,m$. Among these polynomials, some will be of particular interest, those leading to polynomials in the σ_i only. It may checked easily that these are the polynomials whose coefficients in $z_i^j\bar z_i^{j'}$ and in $z_i^{j'}\bar z_i^j$ are equal, for any $i=1,\dots,m$: indeed, up to factorization by powers of $|z_i|^2$ (which is equal to 1 on $\partial\mathbb{D}$), those polynomials are functions of $z_i+\bar z_i = 2\sigma_i$ only. This property corresponds to matrices $M_l\in\mathbb{R}^{l^mp\times l^mn}$ in (3) having a particular *mirror* block structure, those pertaining to the set $\mathbb{R}_M^{p\times n,l^m} \stackrel{\text{def}}{=} \{M_l\in\mathbb{R}^{l^mp\times l^mn} : \forall i_1,\dots,i_m,i'_1,\dots,i'_m\in\{0,\dots,l-1\}, (e_{i_m}\otimes\cdots\otimes e_{i_1}\otimes I_p)^TM_l(e_{i'_m}\otimes\cdots\otimes e_{i'_1}\otimes I_n) = (e_{i'_m}\otimes\cdots\otimes e_{i'_1}\otimes I_p)^TM_l(e_{i_m}\otimes\cdots\otimes e_{i_1}\otimes I_n)\}$, where we put $e_i^T \stackrel{\text{def}}{=} \begin{pmatrix}0_{1\times i} & 1 & 0_{1\times(l-i-1)}\end{pmatrix}$. The definition of $\mathbb{R}_M^{p\times n,l^m}$ is such that $M_l\in\mathbb{R}_M^{p\times n,l^m}$ *iff* for $M(z)$ defined by (3), $M(\varphi(\sigma))$ is polynomial in $\sigma\in[-1;+1]^m$. The subset of those maps $M(z)$ of $\mathbb{R}^{p\times n}[z,\bar z]$ such that $M(\varphi(\sigma))$ is polynomial in $\sigma\in[-1;+1]^m$, will be denoted $\mathbb{R}_M^{p\times n}[z,\bar z]$. Also, we define $\mathcal{S}_M^n[z,\bar z] \stackrel{\text{def}}{=} \mathbb{R}_M^{n\times n}[z,\bar z]\cap\mathcal{S}^n[z,\bar z]$.

Let us point out to the reader, that some technical results linked to the matrix transformations induced by operations on polynomials, are gathered in Appendix.

• We finally define some matrices. For $l, l' \in \mathbb{N}$, let $\hat{J}_{l',l}, \check{J}_{l',l} \in \mathbb{R}^{l\times(l+l')}$ be defined by

$$\hat{J}_{l',l} \stackrel{\text{def}}{=} \begin{pmatrix} I_l & 0_{l\times l'} \end{pmatrix},\ \check{J}_{l',l} \stackrel{\text{def}}{=} \begin{pmatrix} 0_{l\times l'} & I_l \end{pmatrix} . \tag{5}$$

A key property of these matrices is that, $\forall v \in \mathbb{C}$, for $v^{[l]}$ defined previously,

$$v^{[l]} = \hat{J}_{l',l} v^{[l+l']},\ v^{l'} v^{[l]} = \check{J}_{l',l} v^{[l+l']} . \tag{6}$$

Last, define $L_l \in \mathbb{R}^{l\times l}$ by:

$$L_l \stackrel{\text{def}}{=} \left(\begin{array}{cccc|c} 0 & & \dots & 0 & 0 \\ \hline 1 & & & & 0 \\ & 2 & & & \\ & & \ddots & & \vdots \\ & & & l-1 & 0 \end{array}\right) . \tag{7}$$

IV. CONSTANT PARAMETERS

In the case where the parameters σ are *constant*, it turns out that Property I is fulfilled *if and only if* it is fulfilled for certain $P(\sigma), N(\sigma)$ depending polynomially upon σ. This naturally introduces as new variables the degree $l-1$ of the polynomials, and the coefficient matrices of P and N. It turns out moreover, that, for given l, the coefficients may be found out by solving an LMI. This permits to find in an explicit way stabilizing controllers, as functions of the parameter σ.

Theorem 1. *The following are* equivalent.

(i) *Property* I *is fulfilled.*

(ii) *There exists* $(P(\sigma), N(\sigma)) \in \mathcal{S}^n[\sigma]\times\mathbb{R}^{p\times n}[\sigma]$ *fulfilling Property* I.

(iii) *There exist an integer* $l \in \mathbb{N}$ *and* $2(m+1)$ *matrices* $P_l \in \mathcal{S}^{l^m n} \cap \mathbb{R}_M^{n\times n, l^m}$, $N_l \in \mathbb{R}_M^{p\times n, l^m}$ *and* $Q_{l,i}^P \in \mathcal{S}^{(l-1)^{m-i+1} l^{i-1} n}$, $Q_{l,i}^R \in \mathcal{S}^{(k+l-1)^{m-i+1}(k+l)^{i-1} n}$, $i = 1, \dots, m$, *such that the system* (8) *of* 2 *LMIs is fulfilled, where* $R_{k+l} = R_{k+l}(P_l, N_l) \in \mathcal{S}^{(k+l)^m n}$ *is the coefficient matrix of* $R(z)$ *defined in* (9), *corresponding to* $P(z), N(z)$ *with coefficient matrices* P_l, N_l.

Moreover,

- *given a solution of LMI* (8), *for* $P(z), N(z)$ *having coefficient matrices* P_l, N_l, $P(\varphi(\sigma)), N(\varphi(\sigma))$ *fulfil Property* I, *and* $K(\sigma) \stackrel{\text{def}}{=} N(\varphi(\sigma))P(\varphi(\sigma))^{-1}$ *is a stabilizing gain, rational in* σ*;*
- *if LMI* (8) *is solvable for the value* l *of the index, then it is also solvable for any larger value.*

The matrices $\hat{J}, \check{J}$ have been defined in (5). Details for a systematic computation of the matrix R_{k+l} and of the gain $K(\sigma)$ may be found in Appendix.

Theorem 1 offers a family of less and less conservative relaxations of Property I (in which $Q_{l,i}^P, Q_{l,i}^R$ play the role of Lagrange multipliers). Asymptotically, the conservatism vanishes, as solvability of (8) for certain l is necessary to have Property I.

Incidentally, stabilizability of a pair (A, B) is equivalent [4, §7.2.1] to the existence of a definite positive matrix P such that $AP+PA^T < BB^T$. This corresponds to the choice $N = -\frac{1}{2}B^T$ in the LMI: $AP + PA^T + BN + N^TB^T < 0$. Similarly, it may be checked that, replacing in (9) the matrix $N(z)$ by $-\frac{1}{2}B^T(\frac{z+\bar{z}}{2})$, provides a simpler stabilizability criterion.

Another particular case is $B(\sigma) = 0$, which provides a *robust stability* criterion, see also [2], [1].

V. TIME-VARYING PARAMETERS WITH BOUNDED DERIVATIVE

Contrary to the constant parameter case, when Property II is fulfilled, there is probably no necessity for existence of a parameter-dependent Lyapunov function of the kind exhibited in Theorem 1. But it is worth noting that, for given degree, the existence of such a Lyapunov function may be expressed without loss of generality as a LMI problem, in a way similar to what was done for Property I in Theorem 1. Analogously, stabilizing controllers are then found explicitly as functions of $\sigma(t)$.

Theorem 2. *The following are* equivalent.

(i) *There exists* $(P(\sigma), N(\sigma)) \in \mathcal{S}^n[\sigma]\times\mathbb{R}^{p\times n}[\sigma]$ *fulfilling Property* II.

(ii) *There exist an integer* $l \in \mathbb{N}$ *and* $(2^m + m + 2)$ *matrices* $P_l \in \mathcal{S}^{l^m n} \cap \mathbb{R}_M^{n\times n, l^m}$, $N_l \in \mathbb{R}_M^{p\times n, l^m}$ *and* $Q_{l,i}^P \in \mathcal{S}^{(l-1)^{m-i+1} l^{i-1} n}$, $Q_{l,i}^{R,\eta} \in \mathcal{S}^{(k+l-1)^{m-i+1}(k+l)^{i-1} n}$, $i = 1, \dots, m$, *such that the system* (10) *of the* (2^m+1) *LMIs obtained for all* η *in* $\{-1, 1\}^m$ *is fulfilled, where* $R_{k+l} = R_{k+l}(P_l, N_l)$ *has the same meaning than in Theorem* 1 *and* $\hat{P}_{k+l,i} \in \mathcal{S}^{(k+l)^m n}$ *is a coefficient matrix of the map* $z \mapsto \frac{\partial P(\varphi(\sigma))}{\partial \sigma_i}\big|_{\sigma = \frac{z+\bar{z}}{2}}$.

Moreover,

- *given a solution of LMI* (10), *for* $P(z), N(z)$ *having coefficient matrices* P_l, N_l, $P(\varphi(\sigma)), N(\varphi(\sigma))$ *fulfil Property* II, *and, for any absolutely continuous* σ *such that* $\sigma(t) \in [-1; +1]^m$, $\dot{\sigma}(t) \in \prod_{i=1}^m [-\bar{\rho}_i; +\bar{\rho}_i]$ *almost everywhere,* $K(\sigma(t)) \stackrel{\text{def}}{=} N(\varphi(\sigma(t)))P(\varphi(\sigma(t)))^{-1}$ *is a stabilizing gain, rational in* $\sigma(t)$*;*
- *if LMI* (10) *is solvable for the value* l *of the index, then it is also solvable for any larger value.*

See Appendix for details on the computation of the terms $\hat{P}_{k+l,i}$.

VI. ELEMENTS OF DEMONSTRATION

A. Sketch of proof of Theorem 1

1. The equivalence between (i) and (ii), i.e. the fact that P, N in Property I may be supposed polynomial without loss of generality, is consequence of a result on existence of

$$P_l+\sum_{i=1}^m\left(\hat{J}_{1,l-1}^{(m-i+1)\otimes}\otimes I_{l^{i-1}n}\right)^T Q_{l,i}^P\left(\hat{J}_{1,l-1}^{(m-i+1)\otimes}\otimes I_{l^{i-1}n}\right) - \sum_{i=1}^m\left(\hat{J}_{1,l-1}^{(m-i)\otimes}\otimes \check{J}_{1,l-1}\otimes I_{l^{i-1}n}\right)^T Q_{l,i}^P\left(\hat{J}_{1,l-1}^{(m-i)\otimes}\otimes \check{J}_{1,l-1}\otimes I_{l^{i-1}n}\right)>0_{l^m n}\ , \tag{8a}$$

$$R_{k+l}+\sum_{i=1}^m\left(\hat{J}_{1,k+l-1}^{(m-i+1)\otimes}\otimes I_{(k+l)^{i-1}n}\right)^T Q_{l,i}^R\left(\hat{J}_{1,k+l-1}^{(m-i+1)\otimes}\otimes I_{(k+l)^{i-1}n}\right) - \sum_{i=1}^m\left(\hat{J}_{1,k+l-1}^{(m-i)\otimes}\otimes \check{J}_{1,k+l-1}\otimes I_{(k+l)^{i-1}n}\right)^T Q_{l,i}^R\left(\hat{J}_{1,k+l-1}^{(m-i)\otimes}\otimes \check{J}_{1,k+l-1}\otimes I_{(k+l)^{i-1}n}\right)<0_{(k+l)^m n}\ , \tag{8b}$$

$$R(z)\stackrel{\text{def}}{=}A(\frac{z+\bar{z}}{2})P(z)+P(z)A(\frac{z+\bar{z}}{2})^T+B(\frac{z+\bar{z}}{2})N(z)+N(z)^TB(\frac{z+\bar{z}}{2})^T<0_n\ , \tag{9}$$

$$P_l+\sum_{i=1}^m\left(\hat{J}_{1,l-1}^{(m-i+1)\otimes}\otimes I_{l^{i-1}n}\right)^T Q_{l,i}^P\left(\hat{J}_{1,l-1}^{(m-i+1)\otimes}\otimes I_{l^{i-1}n}\right) - \sum_{i=1}^m\left(\hat{J}_{1,l-1}^{(m-i)\otimes}\otimes \check{J}_{1,l-1}\otimes I_{l^{i-1}n}\right)^T Q_{l,i}^P\left(\hat{J}_{1,l-1}^{(m-i)\otimes}\otimes \check{J}_{1,l-1}\otimes I_{l^{i-1}n}\right)>0_{l^m n}, \tag{10a}$$

$$R_{k+l}+\sum_{i=1}^m\eta_i\bar{\rho}_i\hat{P}_{k+l,i}+\sum_{i=1}^m\left(\hat{J}_{1,k+l-1}^{(m-i+1)\otimes}\otimes I_{(k+l)^{i-1}n}\right)^T Q_{l,i}^{R,\eta}\left(\hat{J}_{1,k+l-1}^{(m-i+1)\otimes}\otimes I_{(k+l)^{i-1}n}\right) - \sum_{i=1}^m\left(\hat{J}_{1,k+l-1}^{(m-i)\otimes}\otimes \check{J}_{1,k+l-1}\otimes I_{(k+l)^{i-1}n}\right)^T Q_{l,i}^{R,\eta}\left(\hat{J}_{1,k+l-1}^{(m-i)\otimes}\otimes \check{J}_{1,k+l-1}\otimes I_{(k+l)^{i-1}n}\right)<0_{(k+l)^m n}\ , \tag{10b}$$

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polynomial solutions for LMIs depending continuously upon parameters lying in a compact set [3].

2. Take now (ii) as departure: there exists $(P,N)\in\mathcal{S}_M^n[z,\bar{z}]\times\mathbb{R}_M^{p\times n}[z,\bar{z}]$, with coefficient matrices $P_l\in\mathcal{S}^{l^m n}\cap\mathbb{R}_M^{n\times n,l^m}$, $N_l\in\mathbb{R}_M^{p\times n,l^m}$, such that, $\forall z\in(\partial\mathbb{D})^m$, $(z_m^{[l]}\otimes\cdots\otimes z_1^{[l]}\otimes I_n)^H P_l(z_m^{[l]}\otimes\cdots\otimes z_1^{[l]}\otimes I_n)<0_n$ and $(z_m^{[k+l]}\otimes\cdots\otimes z_1^{[k+l]}\otimes I_n)^H R_{k+l}(z_m^{[k+l]}\otimes\cdots\otimes z_1^{[k+l]}\otimes I_n)<0_n$, for $R_{k+l}(P_l,N_l)$ defined as in the statement. The proof includes joint reduction of these two inequalities to the inequalities in (8). For simplicity, we expose this procedure for one inequality only, say the first one. For $i=0,\dots,m$, denote $(\mathcal{P}_i)$ the property: $\exists l\in\mathbb{N},\exists Q_{l,1}^P\in\mathcal{H}^{(l-1)^m n},\dots,\exists Q_{l,i}^P\in\mathcal{H}^{(l-1)^{m-i+1}l^{i-1}n},\forall(z_{i+1},\dots,z_m)\in(\partial\mathbb{D})^{m-i}$ such that (11) holds. Property $(\mathcal{P}_0)$ is the part of (ii) devoted to P, whereas $(\mathcal{P}_m)$ is just (8a). We indicate in the remaining, how to establish that $(\mathcal{P}_i)\Leftrightarrow(\mathcal{P}_{i+1})$ for any $i=0,\dots,m-1$.

Remark that $(z_m^{[l]}\otimes\cdots\otimes z_{i+1}^{[l]}\otimes I_{l^in})=(z_m^{[l]}\otimes\cdots\otimes z_{i+2}^{[l]}\otimes I_{l^{i+1}n})(z_{i+1}^{[l]}\otimes I_{l^in})$ and $(z_{i+1}^{[l]}\otimes I_{l^in})=\begin{pmatrix}I_{l^in}\\ z_{i+1}\left(I_{(l-1)l^in}-z_{i+1}(F_{l-1}\otimes I_{l^in})\right)^{-1}(f_{l-1}\otimes I_{l^in})\end{pmatrix}$, with $F_l\stackrel{\text{def}}{=}\begin{pmatrix}0_{1\times(l-1)} & 0\\ I_{l-1} & 0_{(l-1)\times1}\end{pmatrix}$, $f_l\stackrel{\text{def}}{=}\begin{pmatrix}1\\ 0_{(l-1)\times1}\end{pmatrix}$.

Applying discrete-time Kalman-Yakubovich-Popov lemma yields equivalence of $(\mathcal{P}_i)$ with: $\exists l\in\mathbb{N},\exists Q_{l,1}^P\in\mathcal{H}^{(l-1)^m n},\dots,\exists Q_{l,i}^P\in\mathcal{H}^{(l-1)^{m-i+1}l^{i-1}n}$, $\forall(z_{i+2},\dots,z_m)\in(\partial\mathbb{D})^{m-i-1}$, $\exists\tilde{Q}_{l,i+1}^P(z_{i+2},\dots,z_m)\in\mathcal{H}^{(l-1)l^in}$ such that (12) holds.

3. Using again the result in [3], $\tilde{Q}_{l,i+1}^P(z_{i+2},\dots,z_m)$, solution of a LMI with parameter in $(\partial\mathbb{D})^{m-i-1}$, may be chosen polynomial in its variables and their conjugates. Let $\tilde{l}-2$ be its degree. If $\tilde{l}\leq l$, then one writes $\tilde{Q}_{l,i+1}^P(z_{i+2},\dots,z_m)=(z_m^{[l-1]}\otimes\cdots\otimes z_{i+2}^{[l-1]}\otimes I_{(l-1)l^in})^H Q_{l,i+1}^P(z_m^{[l-1]}\otimes\cdots\otimes z_{i+2}^{[l-1]}\otimes I_{(l-1)l^in})$, for a coefficient matrix $Q_{l,i+1}^P\in\mathcal{H}^{(l-1)^{m-i}l^in}$. If $\tilde{l}>l$, it may be shown that, up to an increase of l, the degree may be supposed the same, so same formula holds.

At this point, $(\mathcal{P}_i)$ has been proved equivalent to: $\exists l\in\mathbb{N},\exists Q_{l,1}^P\in\mathcal{H}^{(l-1)^m n},\dots,\exists Q_{l,i+1}^P\in\mathcal{H}^{(l-1)^{m-i}l^in}$, such that, $\forall(z_{i+2},\dots,z_m)\in(\partial\mathbb{D})^{m-i-1}$, (13) holds.

4. Some matrix intervertions in the last two terms of (13) finally yields equivalence between $(\mathcal{P}_i)$ and $(\mathcal{P}_{i+1})$.

5. The same argument is applied to (8b), with some detail variations. Application to (8a) and (8b) has to be done together, because of the coupling term P_l. Due to the fact that solvability of (8a), resp. (8b), implies solvability for every

$$0_{l^i n} < \left(z_m^{[l]} \otimes \cdots \otimes z_{i+1}^{[l]} \otimes I_{l^i n}\right)^H \left[P_l + \sum_{j=1}^{i} \left(\hat{J}_{1,l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n}\right)^T Q_{l,j}^P \left(\hat{J}_{1,l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n}\right) - \sum_{j=1}^{i} \left(\hat{J}_{1,l-1}^{(m-j)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{j-1} n}\right)^T Q_{l,j}^P \left(\hat{J}_{1,l-1}^{(m-j)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{j-1} n}\right)\right] \left(z_m^{[l]} \otimes \cdots \otimes z_{i+1}^{[l]} \otimes I_{l^i n}\right) . \tag{11}$$

$$0_{l^{i+1} n} < \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1} n}\right)^H \left[P_l + \sum_{j=1}^{i} \left(\hat{J}_{1,l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n}\right)^T Q_{l,j}^P \left(\hat{J}_{1,l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n}\right) - \sum_{j=1}^{i} \left(\hat{J}_{1,l-1}^{(m-j)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{j-1} n}\right)^T Q_{l,j}^P \left(\hat{J}_{1,l-1}^{(m-j)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{j-1} n}\right)\right] \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1} n}\right) + \left(\hat{J}_{1,l-1} \otimes I_{l^i n}\right)^T \tilde{Q}_{l,i+1}^P \left(\hat{J}_{1,l-1} \otimes I_{l^i n}\right) - \left(\check{J}_{1,l-1} \otimes I_{l^i n}\right)^T \tilde{Q}_{l,i+1}^P \left(\check{J}_{1,l-1} \otimes I_{l^i n}\right) . \tag{12}$$

$$0_{l^{i+1} n} < \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1} n}\right)^H \left[P_l + \sum_{j=1}^{i} \left(\hat{J}_{1,l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n}\right)^T Q_{l,j}^P \left(\hat{J}_{1,l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1} n}\right) - \sum_{j=1}^{i} \left(\hat{J}_{1,l-1}^{(m-j)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{j-1} n}\right)^T Q_{l,j}^P \left(\hat{J}_{1,l-1}^{(m-j)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{j-1} n}\right)\right] \left(z_m^{[l]} \otimes \cdots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1} n}\right) + \left(\hat{J}_{1,l-1} \otimes I_{l^i n}\right)^T \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right)^H Q_{l,i+1}^P \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right) \left(\hat{J}_{1,l-1} \otimes I_{l^i n}\right) - \left(\check{J}_{1,l-1} \otimes I_{l^i n}\right)^T \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right)^H Q_{l,i+1}^P \left(z_m^{[l-1]} \otimes \cdots \otimes z_{i+2}^{[l-1]} \otimes I_{(l-1)l^i n}\right) \left(\check{J}_{1,l-1} \otimes I_{l^i n}\right) . \tag{13}$$

larger value of the index, taking a value of the index for which both are solvable yields equivalence of (ii) and (iii).

6. A simple way to show that $K(\sigma)$ is a stabilizing feedback, is to right- and left-multiply (8a) (resp. (8b)) by $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$ (resp. $(z_m^{[k+l]} \otimes \cdots \otimes z_1^{[k+l]} \otimes I_n)$) and its transconjugate, and use (6) ...

Last, the assertion that solvability of (8) for index l implies the same property for every larger index, is proved using the same techniques than the one evoked (but not displayed) in point **3.**, to increase the size of the solution ...

B. Sketch of proof of Theorem 2

The demonstration is copied from the demonstration of the previous Theorem. Due to the affine dependence upon the ρ_i in Property II, it is enough to consider only the extremal values $\pm\bar{\rho}_i$. It is hence required that: $\exists l \in \mathbb{N}$, $\exists P_l \in \mathcal{S}^{l^m n}$, $\exists N_l \in \mathbb{R}_M^{p\times n, l^m}$, $\forall \eta \in \{-1,1\}^m$, $\forall z \in (\partial\mathbb{D})^m$, $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^H P_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) < 0_n$ and $(z_m^{[k+l]} \otimes \cdots \otimes z_1^{[k+l]} \otimes I_n)^H \left(R_{k+l} + \sum_{i=1}^m \eta_i \bar{\rho}_i \hat{P}_{k+l,i}\right) (z_m^{[k+l]} \otimes \cdots \otimes z_1^{[k+l]} \otimes I_n) < 0_n$.

The argument then essentially follows the proof of Theorem 1. One has to check carefully that the process of increase of the degree (point **3.** in Section VI-A) still works.

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VIII. APPENDIX ON POLYNOMIAL MATRICES

We give here details on the computations necessary for systematic use of Theorems 1 and 2. It is explained in Sections VIII-A, VIII-B how to compute $R_{k+l}(P_l, N_l)$, that is how to determine the coefficient matrices of the terms in (9). Then in Section VIII-C are provided formulas for explicit computation of $K(\sigma)$ as a function of σ, that is of $P(\varphi(\sigma))$ and $N(\varphi(\sigma))$ for $P(z), N(z)$ defined by their coefficient matrix P_l, N_l. Last, the computation of the term $\hat{P}_{k+l,i}$ in (10) is explained in Section VIII-D.

We first extend (5).For $l, l' \in \mathbb{N}$, $l \leq l'$, $\alpha = 0, 1, \dots, l'$, define $J_{\alpha,l,l'} \in \mathbb{R}^{l\times(l+l')}$ by:

$$J_{\alpha,l,l'} \stackrel{\text{def}}{=} \begin{pmatrix} 0_{l\times\alpha} & I_l & 0_{l\times(l'-\alpha)} \end{pmatrix} \ .$$

Then $\hat{J}_{l',l} = J_{0,l,l'}$, $\check{J}_{l',l} = J_{l',l,l'}$, and $v^\alpha v^{[l]} = J_{\alpha,l,l'} v^{[l+l']}$.

A. Representation of polynomial matrices

A natural representation for a matrix-valued polynomial $M : \mathbb{R}^m \to \mathbb{R}^{p\times n}$ (such as $A(\sigma)$ and $B(\sigma)$) is

$$M(\sigma) = M_l(\sigma_m^{[l]} \otimes \dots \otimes \sigma_1^{[l]} \otimes I_n) \ , \qquad (14)$$

for a certain matrix $M_l \in \mathbb{R}^{p\times l^m n}$. The effect of the change of variable (4) is then summarized by Lemma 1.

Lemma 1. *Let* $M_l \in \mathbb{R}^{l^m p\times l^m n}$, *then* $M_l\left(\left(\frac{z_m+\bar{z}_m}{2}\right)^{[l]} \otimes \dots \otimes \left(\frac{z_1+\bar{z}_1}{2}\right)^{[l]} \otimes I_n\right) = (z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_p)^H \tilde{M}_l (z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n)$, *where the matrix* $\tilde{M}_l \in \mathbb{R}^{p\times n, l^m}$ *is given by the formula* $\tilde{M}_l \stackrel{\text{def}}{=} \sum_{0\leq\alpha_i\leq l-1} (J_{\alpha_m,1,l-1} \otimes \dots \otimes J_{\alpha_1,1,l-1} \otimes I_p)^T M_l (K_{l,\alpha_m} \otimes \dots \otimes K_{l,\alpha_1} \otimes I_n)$, *in which the matrices* $K_{l,\alpha} \in \mathbb{R}^{l\times l}$ *are defined by:* $(K_{l,\alpha})_{i,i-\alpha} = 2^{-i+1} C_{i-1}^\alpha$, *with* $C_i^\alpha \stackrel{\text{def}}{=} \frac{i!}{\alpha!(i-\alpha)!}$ *if* $i \geq \alpha \geq 0$, $C_i^\alpha = 0$ *otherwise.*

Proof: $K_{l,\alpha}$ defined in the statement is such that $\forall v \in \mathbb{C}$, $\left(\frac{v+\bar{v}}{2}\right)^{[l]} = \sum_{\alpha=0}^{l-1} \bar{v}^\alpha K_{l,\alpha} v^{[l]}$. Thus, $M_l\left(\left(\frac{z_m+\bar{z}_m}{2}\right)^{[l]} \otimes \dots \otimes \left(\frac{z_1+\bar{z}_1}{2}\right)^{[l]} \otimes I_n\right) = \sum_{0\leq\alpha_i\leq l-1} \bar{z}_1^{\alpha_1} \dots \bar{z}_m^{\alpha_m} M_l (K_{l,\alpha_m} \otimes \dots \otimes K_{l,\alpha_1})(z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n)$. The conclusion then follows from the fact that $\forall v \in \mathbb{C}$, $v^\alpha = v^\alpha v^{[1]} = J_{\alpha,1,l-1} v^{[l]}$. ■

B. Products of polynomial matrices

Lemma 2 is useful to express the coefficient matrix R_{k+l}, appearing in (8) or (10), of $R(z)$ defined in (9) for given $P(z)$, $N(z)$ with coefficient matrices P_l, N_l.

Lemma 2. *Let* $l, l' \in \mathbb{N}$, *and* $M(z), M'(z)$ *with coefficient matrices* $M_l \in \mathbb{R}^{l^m p\times l^m n}$, $M'_{l'} \in \mathbb{R}^{l'^m n\times l'^m q}$. *Then,* $M''(z)$ *has coefficient matrix* $M''_{l''}$, *where* $l'' = l + l' - 1$ *and* $M''_{l''} \stackrel{\text{def}}{=} \sum_{\substack{0\leq\alpha_i\leq l-1, 0\leq\alpha'_i\leq l'-1\\ 1\leq i\leq m}} (J_{\alpha'_m,l,l'-1} \otimes \dots \otimes J_{\alpha'_1,l,l'-1} \otimes I_p)^T M_l (J_{\alpha_m,1,l-1} \otimes \dots \otimes J_{\alpha_1,1,l-1} \otimes I_n)^T (J_{\alpha'_m,1,l'-1} \otimes \dots \otimes J_{\alpha'_1,1,l'-1} \otimes I_n) M'_{l'} (J_{\alpha_m,l',l-1} \otimes \dots \otimes J_{\alpha_1,l',l-1} \otimes I_q)$.

Proof: One has: $\forall v \in \mathbb{C}$, $v^{[l]} = \sum_{\alpha=0}^{l-1} v^\alpha J_{\alpha,1,l-1}^T$, $v^{[l]} v^{[l']H} = \sum_{\substack{0\leq\alpha\leq l-1,\\ 0\leq\alpha'\leq l'-1}} v^\alpha \bar{v}^{\alpha'} J_{\alpha,1,l-1}^T J_{\alpha',1,l'-1}$, and the proof is achieved by using the fact that $v^\alpha v^{[l']} = J_{\alpha,l',l-1} v^{[l+l'-1]}$. ■

C. Formulas attached to inversion of the map φ

Lemma 3. *Let* $N(z) \in \mathbb{R}_M^{p\times n}[z, \bar{z}]$ *with coefficient matrix* $N_l \in \mathbb{R}_M^{p\times n, l^m}$. *Then,* $N(\varphi(\sigma)) = \sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ i=1,\dots,m}} p_{\alpha_1-\alpha'_1}(\sigma_1) \dots p_{\alpha_m-\alpha'_m}(\sigma_m)(J_{\alpha'_m,1,l-1} \otimes \dots \otimes J_{\alpha'_1,1,l-1} \otimes I_p) N_l (J_{\alpha_m,1,l-1} \otimes \dots \otimes J_{\alpha_1,1,l-1} \otimes I_n)^T$, *where by definition, the polynomials* p_α *are such that, for any* $\phi \in \mathbb{R}$, $\cos(\alpha\phi) = p_\alpha(\cos\phi)$.

The coefficients of the p_α are easily found. Forming the matrices $T_{l,|\alpha|} \in \mathbb{R}^{1\times l}$ such that $\forall\alpha \in \{-(l-1), \dots, 0, \dots, l-1\}$, $\forall\phi \in \mathbb{R}$, $\cos(\alpha\phi) = T_{l,|\alpha|}(\cos\phi)^{[l]}$, the formula in Lemma 3 writes: $N(\varphi(\sigma)) = \sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ i=1,\dots,m}} (J_{\alpha'_m,1,l-1} \otimes \dots \otimes J_{\alpha'_1,1,l-1} \otimes I_p) N_l (J_{\alpha_m,1,l-1}^T T_{l,|\alpha_m-\alpha'_m|} \otimes \dots \otimes J_{\alpha_1,1,l-1}^T T_{l,|\alpha_1-\alpha'_1|} \otimes I_n)^T$, which is of the form (14).

Proof: We have: $N(z) = \sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ i=1,\dots,m}} z_1^{\alpha_1} \bar{z}_1^{\alpha'_1} \dots z_m^{\alpha_m} \bar{z}_m^{\alpha'_m} (J_{\alpha'_m,1,l-1} \otimes \dots \otimes J_{\alpha'_1,1,l-1} \otimes I_p) N_l (J_{\alpha_m,1,l-1} \otimes \dots \otimes J_{\alpha_1,1,l-1} \otimes I_n)^T$. Taking into account the fact that $|z_i| = 1$, $i = 1, \dots, m$ and that $N_l \in \mathbb{R}^{p\times n, l^m}$, the previous expression is equal to $\sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1, \alpha_1=\alpha'_1\\ i=1,\dots,m}} z_2^{\alpha_2} \bar{z}_2^{\alpha'_2} \dots z_m^{\alpha_m} \bar{z}_m^{\alpha'_m} (J_{\alpha'_m,1,l-1} \otimes \dots \otimes J_{\alpha'_1,1,l-1} \otimes I_p) N_l (J_{\alpha_m,1,l-1} \otimes \dots \otimes J_{\alpha_1,1,l-1} \otimes I_n)^T + \sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ \alpha_1<\alpha'_1,\ i=1,\dots,m}} (z_1^{\alpha'_1-\alpha_1} + \bar{z}_1^{\alpha'_1-\alpha_1}) z_2^{\alpha_2} \bar{z}_2^{\alpha'_2} \dots z_m^{\alpha_m} \bar{z}_m^{\alpha'_m} (J_{\alpha'_m,1,l-1} \otimes \dots \otimes J_{\alpha'_1,1,l-1} \otimes I_p) N_l (J_{\alpha_m,1,l-1} \otimes \dots \otimes J_{\alpha_1,1,l-1} \otimes I_n)^T$. Introducing the functions p_i as defined in the statement, this is also equal to $\sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ i=1,\dots,m}} p_{\alpha_1-\alpha'_1}(\sigma_1) z_2^{\alpha_2} \bar{z}_2^{\alpha'_2} \dots z_m^{\alpha_m} \bar{z}_m^{\alpha'_m} (J_{\alpha'_m,1,l-1} \otimes \dots \otimes J_{\alpha'_1,1,l-1} \otimes I_p) N_l (J_{\alpha_m,1,l-1} \otimes \dots \otimes J_{\alpha_1,1,l-1} \otimes I_n)^T$, because $\sigma_1 = \operatorname{Re} z_1$. The result follows by induction. ■

D. Differentiation of polynomial matrices

Lemma 4 is required to express the coefficient matrix of the terms $\frac{\partial P(\sigma)}{\partial \sigma_i}$ in Property II.

Lemma 4. *Let* $M(\sigma) \stackrel{\text{def}}{=} M_l(\sigma_m^{[l]} \otimes \dots \otimes \sigma_1^{[l]} \otimes I_n)$. *Then, for any nonnegative integer* k, $\frac{\partial M(\sigma)}{\partial \sigma_i} = \hat{M}_{k+l,i}(\sigma_m^{[k+l]} \otimes \dots \otimes \sigma_1^{[k+l]} \otimes I_n)$, *with* $\hat{M}_{k+l,i} \stackrel{\text{def}}{=} M_l(\hat{J}_{k,l}^{(m-i)\otimes} \otimes L_l \hat{J}_{k,l} \otimes \hat{J}_{k,l}^{(i-1)\otimes} \otimes I_n)$.

Proof: Indeed, $\frac{\partial M(\sigma)}{\partial \sigma_i} = M_l(\sigma_m^{[l]} \otimes \dots \otimes \frac{\partial \sigma_i^{[l]}}{\partial \sigma_i} \otimes \sigma_{i-1}^{[l]} \otimes \dots \otimes \sigma_1^{[l]} \otimes I_n) = M_l(I_l^{(m-i)\otimes} \otimes L_l \otimes I_l^{(i-1)\otimes} \otimes I_n)(\sigma_m^{[l]} \otimes \dots \otimes \sigma_1^{[l]} \otimes I_n) = \hat{M}_{k+l,i}(\sigma_m^{[k+l]} \otimes \dots \otimes \sigma_1^{[k+l]} \otimes I_n)$. ■