STABILITY ANALYSIS OF DISCRETE-TIME SWITCHED SYSTEMS THROUGH LYAPUNOV FUNCTIONS WITH NONMINIMAL STATE

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Abstract: In this paper we investigate stability analysis for discrete-time switched systems. We first consider quadratic Lyapunov functions defined over a nonminimal state encompassing the past history of the state trajectory over a finite horizon. This allows us to state necessary and sufficient conditions for testing uniform exponential stability. Quite remarkably, such conditions can be recast into suitable Linear Matrix Inequalities (LMIs). Next, we consider more general Lyapunov functions dependent also on the past of the switch trajectory. We show that, despite the increased flexibility, this class is no more powerful in capturing stability than the previous class of quadratic Lyapunov functions. However, the associated LMI-based tests may be computationally more advantageous than the ones derived in the quadratic case.

Keywords: Switched systems, stability, Lyapunov functions, linear matrix inequalities.

1. INTRODUCTION

This paper focuses on stability analysis for discrete-time autonomous switched systems defined as

$$x(k+1) = A_{\delta(k)}x(k),\ \delta(k) : \mathbb{N} \mapsto \mathcal{I}. \qquad (1)$$

where $\mathcal{I} = \{1, 2, \ldots, s\}$, is the set of *modes*, $\delta(k)$ is the switching trajectory and $A_\sigma \in \mathbb{R}^{n\times n}$, $\sigma \in \mathcal{I}$. In the recent years, switched systems attracted the interest of the control community because they provide an effective modeling framework for describing hybrid phenomena arising in many real-world application fields such as mechanics, automotive, switching power converters (see references in (Liberzon and Morse, 1999)), and chaos synchronization (Daafouz *et al.*, 2002). At a more abstract level, switched systems arise when considering adaptive control systems based on a finite family of regulators and control systems involving the use of multiple models (Narendra and Xiang, 2000).

Despite the fact that the dynamics of each mode is linear, the problem of assessing the stability of a switched system is challenging for two main reasons. First, the stability/instability of the overall system cannot be inferred from the stability/instability of each mode (Branicky, 1998). Second, the stability problem is either $\mathcal{NP}$-complete or undecidable (Blondel and Tsitsiklis, 1999). These results led many researchers to look for sufficient stability conditions based on special choices of Lyapunov functions that, ultimately, can be computed by solving suitable LMIs. Among them,

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one may cite simultaneous quadratic Lyapunov functions (Liberzon and Morse, 1999) and the more general class of switch-dependent quadratic Lyapunov functions (Daafouz *et al.*, 2002)

$$V_\delta(k,x) = x^T P_{\delta(k)} x, \quad P_\sigma > 0,\ \sigma \in \mathcal{I} \tag{2}$$

The corresponding sufficient criterion reads as

$$\begin{aligned} &\forall \sigma \in \mathcal{I}, \exists P_\sigma = P_\sigma^T > 0, \text{ such that} \\ &A_{\delta(k)}^T P_{\delta(k+1)} A_{\delta(k)} < P_{\delta(k)}\ , \end{aligned} \tag{3}$$

for any possible pair $(\delta(k), \delta(k+1))$ chosen along all the possible trajectories. These conditions ensure asymptotic stability of system (1), but are, generally speaking, far from being necessary.

Independently, a new method for studying stability of delay systems has been given in (Bliman, 2002*a*). It is based on the idea of considering quadratic Lyapunov functions defined on the nonminimal state

$$x^{[p]}(k) \stackrel{\text{def}}{=} \begin{pmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-p+1) \end{pmatrix} \tag{4}$$

where $p \in \mathbb{N}$ is the number of the past time instants considered (in (Bliman, 2002*a*), k is a continuous-time variable, and the delay unit is supposed equal to one). In (Bliman, 2002*a*) it is shown that some stability and performance properties can be characterized by the existence of quadratic Lyapunov functions, provided that a p large enough is considered. Indeed, as p increases, more and more precise sufficient conditions are obtained, which asymptotically become necessary. From the computational point of view, these conditions can be recast into suitable LMIs. We also point out that the same technique can be adapted for studying stability of systems with scalar parameters (Bliman, 2002*b*).

In the present paper, we show that the idea of using Lyapunov functions defined over a nonminimal state can be fruitfully exploited for testing stability of switched systems. In particular, in Section 2, we prove that a *necessary* and sufficient condition for uniform exponential stability is that there exists a finite p and a quadratic Lyapunov function $\mathcal{V}(x^{[p]})$ decreasing along the nonminimal state trajectories. We also provide an equivalent condition that amounts to an LMI feasibility problem. In terms of computational complexity, the necessity of the criterion implies that the problem of checking uniform exponential stability is *semi-decidable*. This means, roughly speaking, that if the property holds, then it can be proved in a finite computational time.

We generalize the previous approach in Section 3, where uniform exponential stability is characterized in terms of quadratic Lyapunov functions dependent on the history of the p past switches. From the theoretical side, we show that, despite the increased generality, these new Lyapunov functions are not more expressive than the quadratic ones, i.e. they capture, at most, the UE stability of the system. However, the associated stability test (that generalize the criterion (3)) can be recast into LMIs that are likely to capture stability for smaller values of the parameter p. This point is illustrated in Section 4 through a simple example.

2. SIMULTANEOUS LYAPUNOV FUNCTIONS

As in Liberzon *et al.* (Liberzon and Morse, 1999), we investigate the stability of (1) for certain classes of switching sequences, more formally, the stability of the switched difference inclusion

$$\begin{aligned} x(k+1) \in \{&A_{\delta(k)} x(k), \\ &\delta(k) \in \mathcal{S}(k, \delta(k-1)) \subset \mathcal{I}\} \end{aligned} \tag{5}$$

where the map $\mathcal{S} : \mathbb{N} \times \mathcal{I} \mapsto 2^{\mathcal{I}}$ describes constraints on the switching trajectories. Note that the switching evolution is independent of the state x of the system. Nevertheless, many different switching rules can be modeled through a proper choice of the constraints.

The case of a single switching trajectory amounts to considering functions $\mathcal{S}$ fulfilling

$$\operatorname{card}\left[\mathcal{S}(k, \delta(k-1))\right] = 1$$

where card$[\mathcal{A}]$ is the cardinality of the set $\mathcal{A}$. Possible uncertainty in the switch is taken into account if card $[\mathcal{S}(k, \delta(k-1))] > 1$ and the maximal uncertainty coincides with $\mathcal{S}(k, \delta(k-1)) \equiv \mathcal{I}$. The modes that can be active at time $k > 0$ belong to the *reach set*, $\mathcal{R}(k) = \bigcup_{\delta(k-1)\in\mathcal{I}} \mathcal{S}(k, \delta(k-1))$. In order to guarantee that the evolution of (5) is not blocked at any time instant, we assume that the sets $\mathcal{S}(0) = \mathcal{R}(0) \neq \emptyset$ are given and that the following rule is satisfied recursively, for $k > 0$

$$\mathcal{S}(k, \delta(k-1)) = \emptyset \Leftrightarrow \delta(k-1) \notin \mathcal{R}(k-1)\ .$$

For any time $k > p-1$, the vector $\xi^{[p]}(k) = \begin{pmatrix} \xi_1 & \dots & \xi_p \end{pmatrix}^T$ is called a *switch register of length* p if $\xi_i \in \mathcal{S}(k-i+1, \delta(k-i))$ and $\delta(k-i) \in \mathcal{R}(k-i)$, for any $i = 1, \dots, p$. Note that the elements of $\xi^{[p]}(k)$ are ordered from the current switch to the previos ones. Moreover, we shall use the sets

$$\Xi^{[p]}(k,\sigma) \stackrel{\text{def}}{=} \left\{ \xi^{[p]}(k) : \delta(k-p+1) = \sigma \right\}\ ,$$

$$\Xi^{[p]}(k) \stackrel{\text{def}}{=} \bigcup_{\sigma\in\mathcal{I}} \Xi^{[p]}(k,\sigma)\ ,$$

$$\Xi^{[p]} \stackrel{\text{def}}{=} \bigcap_{k'=p-1}^{\infty} \bigcup_{k=k'}^{\infty} \Xi^{[p]}(k)\ .$$

Note that $\Xi^{[p]}$ collects all possible switch vectors in the asymptotic regime and its cardinality is at most s^p. Moreover, since the number of modes is finite, this means that there exists a time instant k^* such that $\Xi^{[p]} = \bigcup_{k'=k}^{\infty} \Xi^{[p]}(k')$ for any $k \geq k^*$.

We introduce now the notion of stability we consider.

Definition 1. System (1) is *Uniformly Exponentially (UE) stable* if:

$$\begin{aligned}&\exists c > 0, \alpha \in [0,1),\ \forall k, k' \in \mathbb{N}, k \leq k',\\ &\forall x(k) \in \mathbb{R}^n,\ \|x(k')\| \leq c\alpha^{k'-k}\|x(k)\|\ .\end{aligned}$$

■

Note that, in Definition 1, the convergence is uniform both in time and in space. We are now in a position to state the main result of the Section.

Theorem 2. The four following properties are equivalent.

(i) System (1) is UE stable.

(ii) There exist $p \in \mathbb{N}$ and a quadratic function V defined on $\mathbb{R}^n$ such that, along the trajectories of system (5), it holds

$$\begin{aligned}\forall k \in \mathbb{N},\ x(k) \neq 0 \Rightarrow V(x(k)) > 0 \text{ and}\\ V(x(k+p)) < V(x(k))\ .\end{aligned}$$

(ii') There exist $p \in \mathbb{N}$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that, $\forall \xi \in \Xi^{[p]}$, it holds

$$A_{\xi_p}^T A_{\xi_{p-1}}^T \dots A_{\xi_1}^T P A_{\xi_1} \dots A_{\xi_{p-1}} A_{\xi_p} < P \tag{6}$$

(iii) There exist $p \in \mathbb{N}$ and a quadratic function $\mathcal{V}$ defined on $\mathbb{R}^{pn}$ such that, along the trajectories of system (5), it holds

$$\begin{aligned}&\forall k \geq p-1,\ x^{[p]}(k) \neq 0 \Rightarrow \mathcal{V}(x^{[p]}(k)) > 0\\ &\text{and } \mathcal{V}(x^{[p]}(k+1)) < \mathcal{V}(x^{[p]}(k))\end{aligned} \tag{7}$$

where $x^{[p]}(k)$ is defined as in (4). ■

Proof. The proof consists in the steps summarized by the following graph:

$$\begin{array}{c}\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)}\\ \Updownarrow\\ \text{(ii')}\end{array}$$

• (i) $\Rightarrow$ (ii). Suppose (i) holds with the constants c, α defined as in Definition 1, and let p be such that $c\alpha^p < 1$. Then, taking $V(x) = x^T x$, one has: $V(x(k+p)) < V(x(k))$, for any $k \geq p$.

• (ii) $\Rightarrow$ (iii). When (ii) holds, define, for the same p and any $x^{[p]} \in \mathbb{R}^{pn}$, the functional $\mathcal{V}(x^{[p]})$ by

$$\mathcal{V}(x^{[p]}) \stackrel{\text{def}}{=} V(x_1) + \dots + V(x_p)\ ,$$

where the vectors $x_i \in \mathbb{R}^n$, $i = 1, \dots, [p]$, are defined by the decomposition $x^{[p]} = (x_1^T, \dots, x_p^T)^T$. Then, along the trajectories of (1), $\mathcal{V}(x^{[p]}(k+1)) - \mathcal{V}(x^{[p]}(k)) = V(x(k+p)) - V(x(k)) < 0$.

• (iii) $\Rightarrow$ (i). Due to the finite number of modes, there exists $\alpha \in (0,1)$ such that (7) remains true with $\mathcal{V}$ replaced by $\alpha\mathcal{V}$ in the right-hand side of the inequality. Then, there exists $c > 0$ such that $\forall k, k' \in \mathbb{N}$, $k' > k$, we have $\mathcal{V}(x^{[p]}(k')) < c\alpha^{k'-k}\mathcal{V}(x^{[p]}(k))$. Since there exist $L, l > 0$ such that $l\|x^{[p]}\|^2 \leq \mathcal{V}(x^{[p]}) \leq L\|x^{[p]}\|^2$, then $\|x^{[p]}(k')\|^2 < (l)^{-1}cL\alpha^{k'-k}\|x^{[p]}(k)\|^2$.

Let $\eta = \max_{\sigma\in\mathcal{I}} \sigma_{max}(A_\sigma)$ where $\sigma_{max}(A)$ is the maximum singular value of the matrix A. Then, $\forall k, k' \in \mathbb{N}$, $k' > k \geq p-1$, we have

$$\begin{aligned}\|x(k'-p+1)\|^2 &\leq \|x^{[p]}(k')\|^2\\ &\leq \frac{L}{l}c^2\alpha^{k'-k}\|x^{[p]}(k)\|^2\\ &\leq \frac{L}{l}c^2\sum_{j=0}^{p-1}\eta^j\alpha^{k'-k}\|x(k-p+1)\|^2\ .\end{aligned} \tag{8}$$

By using the time-indices $\tilde{k} = k-p+1$ and $\tilde{k}' = k'-p+1$ the inequalities (8) give, $\forall \tilde{k}' > \tilde{k} \geq 0$,

$$\|x(\tilde{k}')\|^2 \leq \tilde{c}\alpha^{\tilde{k}'-\tilde{k}}\|x(\tilde{k})\|^2\ . \tag{9}$$

where $\tilde{c} = \frac{L}{l}c^2\sum_{j=0}^{p-1}\eta^j$. Then, it follows that $x(k)$ tends UE towards the origin.

• The equivalence (ii) $\Leftrightarrow$ (ii') is straightforward.

□

Theorem 2 deserves some remarks. First, note that the values of p in *(ii)*, *(ii')* and *(iii)* may be different. More precisely, a careful examination of the proof reveals that $p_{(ii)} = p_{(ii')} \geq p_{(iii)}$. Second, only condition *(iii)* involves a Lyapunov function in the strict sense, because the quadratic function in *(ii)* decreases only with a delay of p time units. On the other hand, only *(ii')* offers computational facilities since it defines the family of LMI-based tests (parametrized in p), whose success guarantees UE stability. Note also that the specific structure of the constraints $\mathcal{S}(k,\sigma)$ influences the test since it contributes in defining the set $\Xi^{[p]}$.

From the theoretical side, Theorem 2 means that switched systems that are asymptotically stable but that do not possess a quadratic Lyapunov function (in the sense of *(ii)*) are somehow "pathological": the convergence either is not uniform in time or in space, or is not exponential. In other words, the class of Lyapunov functions used is *universal* (Blanchini and Miani, 1999) for the UE stability of system (5).

A computational drawback of condition *(ii')*, is that the number of LMIs involved scales as card($\Xi^{[p]}$) and it may become prohibitive for large values of p. In the next section we provide a UE stability test that, by exploiting a larger number of unknowns, is expected to capture UE stability for a smaller p.

3. SWITCH-DEPENDENT QUADRATIC LYAPUNOV FUNCTIONS

In this Section we consider Lyapunov functions dependent on the past p switches that provide a generalization of (2). Since this class of Lyapunov functions encompasses the quadratic ones considered in Section 2, one may guess that it can be used in order to characterize a notion of stability more general than UE stability. However, Theorem 4 shows that this conjecture is false.

Definition 3. Consider two switch registers $\xi, \xi^+ \in \Xi^{[p]}$. We say that ξ^+ is consecutive to ξ if $\exists k \in \mathbb{N}$, such that $[\xi^+\ \xi] \in \Xi^{[2p]}(k)$. ■

Note that only switch register that are consecutive in the asymptotic regime are considered (see the definition of $\Xi^{[p]}$).

Theorem 4. The four following properties are equivalent.

(i) System (1) is UE stable.

(ii) There exist $p \in \mathbb{N}$ and a switch-dependent quadratic function V_p defined on $\Xi^{[p]} \times \mathbb{R}^n$ such that, along the trajectories of system (5), it holds

$$\forall k \in \mathbb{N},\ x(k) \neq 0 \Rightarrow V_p(\xi^{[p]}(k), x(k)) > 0$$
$$\text{and } V_p(\xi^{[p]}(k+p), x(k+p)) < V_p(\xi^{[p]}(k), x(k))\ .$$

(ii') There exist $p \in \mathbb{N}$ and symmetric positive definite matrices P_ξ, $\xi \in \Xi^{[p]}$, such that, for all $\xi^+ \in \Xi^{[p]}$ consecutive to ξ, it holds

$$A^T_{\xi^+_p} A^T_{\xi^+_{p-1}} \ldots A^T_{\xi^+_1} P_{\xi^+} A_{\xi^+_1} \ldots A_{\xi^+_{p-1}} A_{\xi^+_p} < P_\xi \tag{10}$$

(iii)) There exist $p \in \mathbb{N}$ and a switch-dependent quadratic function $\mathcal{V}_p$ defined on $\Xi^{[p]} \times \mathbb{R}^{pn}$ such that, along the trajectories of system (5), it holds

$$\begin{aligned} &\forall k \geq p-1, x^{[p]}(k) \neq 0 \\ &\Rightarrow \mathcal{V}_p(\xi^{[p]}(k), x^{[p]}(k)) > 0 \text{ and} \\ &\mathcal{V}_p(\xi^{[p]}(k+1), x^{[p]}(k+1)) \\ &< \mathcal{V}_p(\xi^{[p]}(k), x^{[p]}(k)) \end{aligned} \tag{11}$$

■

Proof. The structure of the proof is the following

$$\begin{array}{c} \text{(i)} \Rightarrow \text{(ii) in Theorem 2} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)} \\ \Updownarrow \\ \text{(ii')} \end{array}$$

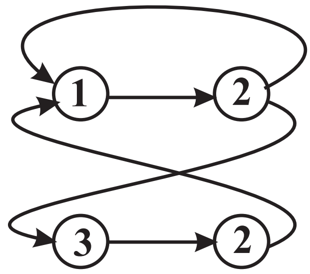

Fig. 1. Automaton generating the admissible switching trajectories for the example in Section 4

The implication "(i) $\Rightarrow$ (ii) in Theorem 2" has already been proved. The step "(ii) in Theorem 2 $\Rightarrow$ (ii)", and the equivalence between (ii) and (ii') are straightforward.

- (ii) $\Rightarrow$ (iii). For the same value of p, let

$$\mathcal{V}_p(\xi^{[p]}(k), x^{[p]}(k)) \stackrel{\text{def}}{=} V_p(\xi^{[p]}(k), x_1) + \cdots + V_p(\xi^{[p]}(k-p+1), x_p)$$

where we put $x^{[p]} = (x_1^T, \ldots, x_p^T)^T$. Then, along the trajectories of (1), we have $\mathcal{V}_p(k+1, x^{[p]}(k+1)) - \mathcal{V}_p(k, x^{[p]}(k)) = V_p(k, x(k)) - V_p(k-p, x(k-p)) < 0$.

- (iii) $\Rightarrow$ (i). This step is conducted in a way similar to the corresponding step in the proof of Theorem 2.

□

As for Theorem 2, in Theorem 4, the values of p in *(ii)*, *(ii')* and *(iii)* may be different and it holds $p_{(ii)} = p_{(ii')} \geq p_{(iii)}$. From the computational point of view, condition *(ii')* can be translated into card$(\Xi^{[2p]})$ LMIs in card$(\Xi^{[p]})$ unknowns, instead of card$(\Xi^{[p]})$ LMIs in one unknown, as required by condition *(ii')* in Theorem 2.

4. EXAMPLE

In order to highlight the advantages of the LMI test (10) with respect to (6), we consider scalar system (1) with $s = 3$, $A_1 = \frac{1}{10}$, $A_2 = 3$ and $A_3 = \frac{1}{2}$. The admissible switching trajectory are generated by the automaton in Figure 1. Note that the switch sequences $(2,3,2,3)$ and $(3,2,3,2)$ are not allowed. Moreover, the dynamics of mode 2 is unstable and $A_2A_3 > 1$ whereas $A_2A_3A_2A_1 < 1$. On the basis of the previous observations it is easy to realize that $|x(k+4)| \leq \frac{9}{20}|x(k)|$ so motivating the fact that the system is UE stable.

Let $p = 2$. From Figure 1, one obtains $\Xi^{[2]} = \{(2,1), (3,2), (1,2), (2,3)\}$ (note that in a switch register $\xi^{[2]}(k) = (\xi_1, \xi_2)$ the most recent switch is represented by ξ_1).

Consider test (6) that amounts to find a scalar $P > 0$ satisfying

$$\frac{9}{100}P < P \text{ (for } \xi = (2,1) \text{ and } \xi = (1,2))$$
$$\frac{9}{4}P < P \text{ (for } \xi = (3,2) \text{ and } \xi = (2,3)) \quad (12)$$

It is apparent that the inequality (12) is unfeasible, so showing that the test fails in checking UE stability for $p = 2$.

Consider now test (10). Always from Figure 1, the pairs (ξ^+, ξ) of consecutive switch registers are

$$\begin{aligned} &((2,3),(2,1)), \ ((2,1),(2,1)), \\ &((1,2),(3,2)), \ ((1,2),(1,2)), \\ &((3,2),(1,2)), \ ((2,1),(2,3)). \end{aligned} \quad (13)$$

The LMI test (10) amounts to find positive scalars $P_{(2,1)}, \ P_{(1,2)}, \ P_{(3,2)}, \ P_{(2,3)}$ satisfying

$$\frac{9}{4}P_{\xi^+} < P_\xi \text{ for } (\xi^+, \xi) \in \mathcal{W}_1, \quad (14)$$
$$\frac{9}{100}P_{\xi^+} < P_\xi \text{ for } (\xi^+, \xi) \in \mathcal{W}_2 \quad (15)$$

where

$$\begin{aligned} \mathcal{W}_1 &= \{((2,3),(2,1)), \ ((3,2),(1,2))\} \\ \mathcal{W}_2 &= \{((2,1),(2,1)), \ ((1,2),(3,2)), \\ &\quad ((1,2),(1,2)), \ ((2,1),(2,3))\} \end{aligned}$$

By direct calculation, one finds that the inequalities (14), can be verified by any choice of the unknowns fulfilling

$$\frac{9}{100} < \frac{P_{(2,3)}}{P_{(2,1)}} < \frac{4}{9}, \quad \frac{9}{100} < \frac{P_{(3,2)}}{P_{(1,2)}} < \frac{4}{9}, \quad (16)$$

so proving that test (10) is successful in checking UE stability with $p = 2$.

5. CONCLUSION

In this paper, the issue of stability analysis for discrete-time switched systems has been investigated. The main result consists in showing that uniform exponential stability may be characterized by the existence of some quadratic Lyapunov functions, constructed on a sufficiently large number of past states. This property may be expressed as a feasibility problem for some Linear Matrix Inequalities. An example of stability analysis for a simple system is provided. The synthesis of stabilizing control law is naturally the next step to explore.

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