Stabilization of LPV Systems

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We study here the static state-feedback stabilization of linear finite dimensional systems depending polynomially upon a finite set of real, bounded, parameters. These parameters are a priori unknown, but available in real-time for control. In consequence, it is natural to allow possible dependence of the gain with respect to the parameters (*gain-scheduling*).

We state two main results. First, we show that stabilizability of the class of systems obtained for frozen values of the parameters, may be expressed *equivalently* by linear matrix inequalities (LMIs), linked to certain class of parameter-dependent Lyapunov functions. Second, we show that existence of such a Lyapunov function for the linear parameter-varying (LPV) systems subject to bounded rate of variation of the parameters with respect to time, may be in the same manner expressed equivalently by LMI conditions. In both cases, the method provides explicitly parameter-dependent stabilizing gain. The central arguments are linked to the existence of a decomposition of some symmetric parameter-dependent matrices as sum of positive definite terms.

1 Introduction

Linear parameter-varying (LPV) systems have recently received much attention, in connection with the gain-scheduling control design methodologies, see [5, 12] for recent surveys and bibliography on the subject. LPV systems are linear systems that depend upon time-varying real parameters. The latter are not known in advance, but may be used in real-time for control purposes. However, they are usually constrained to lie inside a known bounded set.

The issue of checking the stabilizability and determining a parameter-dependent stabilizing gain for every frozen admissible value of the parameters, is already a difficult task. As an example, a coarse application of the Lyapunov-based synthesis techniques available for linear systems is impossible, as it leads to solve an infinite number of linear matrix inequalities (LMIs). At this point, two types of methods are usually used (see the recent works [14, 7] on LPV systems): either the controller gain is first computed for a bunch of parameter values, and then interpolated between the nodes of this grid (but the stability, and possibly performance, results are not guaranteed between the nodes); or the solution of the parameter-dependent LMIs involved is sought for with prespecified dependence with respect to the parameters, usually constant or affine (at the cost of adding conservatism). Of course, the stabilization issue is still more complicated when the parameters are time-varying.

In this paper, we show that, in principle, for linear systems depending polynomially upon finite number of bounded parameters, the determination of parameter-dependent stabilizing gain may be achieved without conservatism. More precisely, we state two main results (Theorems 1 and 2 below), whose contribution may be summarized as follows.

1. The stabilization of all the systems obtained for *constant values* of the parameters in the admissible hypercube is equivalent to the existence for the closed-loop system of a quadratic Lyapunov function *polynomial* with respect to the parameters. For fixed value of the degree, the coefficients of this polynomial may be found by solving a LMI.
2. The existence of a similar quadratic Lyapunov function (depending in the same way upon the parameters) for the corresponding LPV system with *restricted rate of variation* of the parameters, is also equivalent, for fixed degree, to the solvability of a LMI.
3. In both cases, a parameter-dependent stabilizing gain is deduced from the solution of the LMIs.

The originality of the results presented here lies in the nonconservative nature of the LMI conditions proposed. They constitute a systematization of the approaches based on parameter-dependent Lyapunov functions. Further work should consider dynamic controller synthesis and performace verification.

Effective use of the results given here is subordinate to powerful LMI solvers. A general idea for reducing the computation complexity consists in performing first a subdivision of the admissible parameter set in subdomains and applying the results presented below on these smaller regions. The present paper provides a stage towards such a *hybrid* control (with switches according to the parameter values), which in principle could lead to sensible diminution of the (off-line) computational burden, but whose study is out of our scope here.

The paper is organized as follows. The problem is presented in Sect. 2. Notations are provided in Sect. 3. The result on systems with frozen parameters (Theorem 1) is stated in Sect. 4. The results on systems with parameters with bounded derivative (Theorem 2) is stated in Sect. 5. Elements of proof are displayed in Sect. 6. Some technical results related to the computations involved are gathered in Appendix.

2 Problem Statement

We consider here the issue of state-feedback stabilization for the class of linear systems

$$\dot{x} = A(\sigma(t))x + B(\sigma(t))u \ . \tag{1}$$

In (1), the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}$ are supposed to be polynomials of partial degree (at most) k with respect to the components of a vector $\sigma \stackrel{\text{def}}{=} (\sigma_1, \ldots, \sigma_m)$ of m real parameters.

We are interested in the design of stabilizing static state-feedback for (1), under the assumption that $\forall t \geq 0$, $\sigma(t) \in [-1; +1]^m$. In the special case where the components of σ are constant ($\dot{\sigma} \equiv 0$), this is equivalent to find, for any $\sigma \in [-1; +1]^m$, a gain $K(\sigma)$ such that $A(\sigma)+B(\sigma)K(\sigma)$ is Hurwitz. This leads to study the following property.

Property I. *There exist mappings* $P : [-1; +1]^m \to \mathcal{S}^n$, $N : [-1; +1]^m \to \mathbb{R}^{p\times n}$ *such that,* $\forall \sigma \in [-1; +1]^m$, $P(\sigma) > 0_n$, $A(\sigma)P(\sigma) + P(\sigma)A(\sigma)^T + B(\sigma)N(\sigma) + N(\sigma)^T B(\sigma)^T < 0_n$.

In this formula, $\mathcal{S}^n$ represents the set of symmetric matrices of size $n \times n$. Property I is *equivalent* to the stabilizability of (1) for every admissible choice of the parameters.

As is well-known, the previous condition, guaranteeing stability for the frozen parameter systems, is not enough to guarantee stability of the systems with time-varying parameters. An attempt to extend the previous ideas to stabilization of LPV systems with parameters having variation rate constrained by $|\dot{\sigma}_i| \leq \bar{\varrho}_i$ a.e., $i = 1, \ldots, m$, leads to the following interesting issue.

Property II. *There exist mappings* $P : [-1; +1]^m \to \mathcal{S}^n$, $N : [-1; +1]^m \to \mathbb{R}^{p\times n}$*, P differentiable, such that,* $\forall \sigma \in [-1; +1]^m$, $\forall \varrho_i \in [-\bar{\varrho}_i; \bar{\varrho}_i]$, $P(\sigma) > 0_n$, $A(\sigma)P(\sigma)+P(\sigma)A(\sigma)^T+B(\sigma)N(\sigma)+N(\sigma)^T B(\sigma)^T - \sum_{i=1}^m \varrho_i \frac{\partial P(\sigma)}{\partial \sigma_i} < 0_n$.

Property II is *equivalent* to the existence of a quadratic Lyapunov function depending regularly upon the present values of the parameters. This property is thus a priori stronger than the stability of the systems (1) attached to every admissible trajectories of the parameters.

3 Notations and Preliminaries

• The matrices I_n, 0_n, $0_{n\times p}$ are the $n \times n$ identity matrix and the $n \times n$ and $n\times p$ zero matrices respectively. The symbol $\otimes$ denotes Kronecker product, the power of Kronecker product being used with the natural meaning: $M^{0\otimes} = 1$, $M^{p\otimes} \stackrel{\text{def}}{=} M^{(p-1)\otimes} \otimes M$. Key properties are: $(A \otimes B)^T = A^T \otimes B^T$, $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ for matrices of compatible size. The transpose and transconjugate of M are denoted M^T and M^*. The unit circle in $\mathbb{C}$ is denoted as the boundary $\partial\mathbb{D}$, and the set of positive integers $\mathbb{N}$. Diagonal matrices are defined by diag. Also, the set of symmetric real (resp. hermitian complex) matrices of size $n \times n$ is denoted $\mathcal{S}^n$ (resp. $\mathcal{H}^n$).

Last, we introduce some spaces of matrix-valued polynomials. $\mathbb{R}^{n\times n}[\sigma]$ (resp. $\mathcal{S}^n[\sigma]$) will denote the set of polynomials in the variable $\sigma \in \mathbb{R}^m$, with coefficients in $\mathbb{R}^{n\times n}$ (resp. $\mathcal{S}^n$). We shall also consider in the sequel the set, denoted $\mathbb{R}^{n\times n}[z,\bar{z}]$, of polynomials in z and $\bar{z}$, $z \in \mathbb{C}$, with coefficients in $\mathbb{R}^{n\times n}$. The sets $\mathcal{S}^n[z,\bar{z}]$, $\mathcal{H}^n[z,\bar{z}]$ are defined similarly.

• We now introduce specific notations. For any $l \in \mathbb{N}$, for any $v \in \mathbb{C}$, let

$$v^{[l]} \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ v \\ \vdots \\ v^{l-1} \end{pmatrix} . \tag{2}$$

This notation permits to manipulate polynomials. Notice in particular that, for a free variable $z \in \mathbb{C}^m$, the vector $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]})$ contains exactly the l^m monomials in $z_1, \ldots, z_m$ of degree at most $l-1$ in each variable.

Using this notation, any element $M(z)$ in $\mathbb{R}^{p\times n}[z,\bar{z}]$ may be represented as

$$M(z) = (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_p)^* M_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) . \tag{3}$$

In this formula, for given $l \in \mathbb{N}$, the matrix $M_l \in \mathbb{R}^{l^m p \times l^m n}$ is unique, in the sense that: $M(z) = 0$ *for all* $z \in \mathbb{C}^m$ *iff* $M_l = 0$. Independently of minimality, the matrix M_l is called the *coefficient matrix* of this representation of $M(z)$, $l-1$ its *degree*.

In the sequel, we shall use the following *change of variables* ($\mathrm{i}^2 = -1$):

$$\begin{gathered} \varphi : [-1;+1]^m \to (\partial\mathbb{D})^m, \sigma \mapsto z = \varphi(\sigma) \\ \text{where } z_i \stackrel{\text{def}}{=} \sigma_i + \mathrm{i}\sqrt{1-\sigma_i^2}, i = 1, \ldots, m . \end{gathered} \tag{4}$$

Basically (see the developments below), changing σ in z will permit to use Kalman-Yakubovich-Popov lemma, "replacing" the free variables z_i by matrix multipliers in the parameter-dependent LMIs appearing in Properties I and II. In particular, for z in the range of φ, $\varphi^{-1}(z) = \frac{z+\bar{z}}{2}$. When $z = (z_1, \ldots, z_m)$ covers $(\partial\mathbb{D})^m$, $\frac{z+\bar{z}}{2}$ varies in the whole set $[-1;+1]^m$.

Generally speaking, for M defined as in (3) and the change of variable φ as in (4), $M(\varphi(\sigma))$ is a polynomial in σ_i and $\sqrt{1-\sigma_i^2}$, $i = 1, \ldots, m$. Among these polynomials, some will be of particular interest here, those leading to polynomials in the σ_i *only*. It may checked easily that these are the polynomials whose coefficients in the monomials $\prod_{i=1,\ldots,m} z_i^{\alpha_i} \bar{z}_i^{\alpha'_i}$ and $\prod_{i=1,\ldots,m} z_i^{\beta_i} \bar{z}_i^{\beta'_i}$ are equal when $\{\alpha_i, \alpha'_i\} = \{\beta_i, \beta'_i\}$ for any $i = 1, \ldots, m$. Indeed, up to factorization by powers of $|z_i|^2$ (which is equal to 1 on $\partial\mathbb{D}$), those polynomials are functions of $z_i + \bar{z}_i = 2\sigma_i$ only. This property corresponds to matrices $M_l \in \mathbb{R}^{l^m p \times l^m n}$ in (3) having a particular *mirror* block structure, those pertaining to the set

$$\mathbb{R}_M^{p\times n,l^m} \stackrel{\text{def}}{=} \{M_l \in \mathbb{R}^{l^m p\times l^m n} \ : \ \forall \alpha_1,\dots,\alpha_m,\alpha'_1,\dots,\alpha'_m \in \{0,\dots,l-1\},$$
$$(e_{\alpha_m}\otimes\cdots\otimes e_{\alpha_1}\otimes I_p)^T M_l(e_{\alpha'_m}\otimes\cdots\otimes e_{\alpha'_1}\otimes I_n) \text{ depends only upon the sets } \{\alpha_1,\alpha'_1\},\dots,\{\alpha_m,\alpha'_m\}\}\ ,$$

where we put $e_\alpha^T \stackrel{\text{def}}{=} \begin{pmatrix}0_{1\times\alpha} \ 1 \ 0_{1\times(l-\alpha-1)}\end{pmatrix}$.

The definition of $\mathbb{R}_M^{p\times n,l^m}$ is such that $M_l \in \mathbb{R}_M^{p\times n,l^m}$ *iff* $M(\varphi(\sigma))$ is polynomial in $\sigma \in [-1;+1]^m$, for $M(z)$ defined by (3). The subset of those maps $M(z)$ of $\mathbb{R}^{p\times n}[z,\bar{z}]$ such that $M(\varphi(\sigma))$ is polynomial in $\sigma\in[-1;+1]^m$, will be denoted $\mathbb{R}_M^{p\times n}[z,\bar{z}]$. Also, we define $\mathcal{S}_M^n[z,\bar{z}] \stackrel{\text{def}}{=} \mathbb{R}_M^{n\times n}[z,\bar{z}]\cap\mathcal{S}^n[z,\bar{z}]$.

Let us point out to the reader, that some technical results linked to the matrix transformations induced by operations on polynomials, are gathered in Appendix.

• We finally define some matrices. For $l,l'\in\mathbb{N}$, let $\hat{J}_{l',l},\check{J}_{l',l}\in\mathbb{R}^{l\times(l+l')}$ be defined by

$$\hat{J}_{l',l} \stackrel{\text{def}}{=} \begin{pmatrix}I_l \ 0_{l\times l'}\end{pmatrix},\ \check{J}_{l',l} \stackrel{\text{def}}{=} \begin{pmatrix}0_{l\times l'} \ I_l\end{pmatrix}\ . \tag{5}$$

A key property of these matrices is that, $\forall v\in\mathbb{C}$, for $v^{[l]}$ defined previously,

$$v^{[l]} = \hat{J}_{l',l}v^{[l+l']},\ v^{l'}v^{[l]} = \check{J}_{l',l}v^{[l+l']}\ . \tag{6}$$

Last, define $L_l\in\mathbb{R}^{l\times l}$ by:

$$L_l \stackrel{\text{def}}{=} \left(\begin{array}{cccc|c} 0 & \dots & & 0 & 0 \\ \hline 1 & & & & 0 \\ & 2 & & & \\ & & \ddots & & \vdots \\ & & & l-1 & 0 \end{array}\right)\ . \tag{7}$$

4 Constant Parameters

In the case where the parameters σ are *constant*, it turns out that Property I is fulfilled *if and only if* it is fulfilled for certain $P(\sigma),N(\sigma)$ depending polynomially upon σ (see also [2]). This naturally introduces as new variables the degree $l-1$ of the polynomials, and the coefficient matrices of P and N. It turns out moreover, that, for given l, the coefficients may be found out by solving an LMI. This permits to find in an explicit way stabilizing controllers, as functions of the parameter σ.

Theorem 1. *The following assertions are* equivalent*.*

(i) Property I *is fulfilled.*
(ii) There exists $(P(\sigma),N(\sigma))\in\mathcal{S}^n[\sigma]\times\mathbb{R}^{p\times n}[\sigma]$ *fulfilling Property* I.

(iii) There exist an integer $l \in \mathbb{N}$, 2 matrices $P_l \in \mathcal{S}^{l^m n} \cap \mathbb{R}_M^{n\times n, l^m}$, $N_l \in \mathbb{R}_M^{p\times n, l^m}$ and $2m$ matrices $Q_{l,i}^P \in \mathcal{S}^{(l-1)^{m-i+1} l^{i-1} n}$, $Q_{l,i}^R \in \mathcal{S}^{(k+l-1)^{m-i+1}(k+l)^{i-1} n}$, $i = 1, \dots, m$, such that the system (8) of 2 LMIs is fulfilled, where $R_{k+l} = R_{k+l}(P_l, N_l) \in \mathcal{S}^{(k+l)^m n}$ is the coefficient matrix of $R(z)$ defined in (9), corresponding to $P(z), N(z)$ with coefficient matrices P_l, N_l.

$$0_{l^m n} < P_l + \sum_{i=1}^m \left(\hat{J}_{1,l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n}\right)^T Q_{l,i}^P \left(\hat{J}_{1,l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n}\right) - \sum_{i=1}^m \left(\hat{J}_{1,l-1}^{(m-i)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{i-1}n}\right)^T Q_{l,i}^P \left(\hat{J}_{1,l-1}^{(m-i)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{i-1}n}\right) , \tag{8a}$$

$$0_{(k+l)^m n} > R_{k+l} + \sum_{i=1}^m \left(\hat{J}_{1,k+l-1}^{(m-i+1)\otimes} \otimes I_{(k+l)^{i-1}n}\right)^T Q_{l,i}^R \left(\hat{J}_{1,k+l-1}^{(m-i+1)\otimes} \otimes I_{(k+l)^{i-1}n}\right) - \sum_{i=1}^m \left(\hat{J}_{1,k+l-1}^{(m-i)\otimes} \otimes \check{J}_{1,k+l-1} \otimes I_{(k+l)^{i-1}n}\right)^T Q_{l,i}^R \left(\hat{J}_{1,k+l-1}^{(m-i)\otimes} \otimes \check{J}_{1,k+l-1} \otimes I_{(k+l)^{i-1}n}\right) , \tag{8b}$$

$$R(z) \stackrel{\text{def}}{=} A(\frac{z+\bar{z}}{2})P(z) + P(z)A(\frac{z+\bar{z}}{2})^T + B(\frac{z+\bar{z}}{2})N(z) + N(z)^T B(\frac{z+\bar{z}}{2})^T < 0_n . \tag{9}$$

Moreover,

- *given a solution of LMI (8), for $P(z), N(z)$ having coefficient matrices P_l, N_l, $P(\varphi(\sigma)), N(\varphi(\sigma))$ fulfil Property I, and $K(\sigma) \stackrel{\text{def}}{=} N(\varphi(\sigma))P(\varphi(\sigma))^{-1}$ is a stabilizing gain, rational in σ;*
- *if LMI (8) is solvable for the value l of the index, then it is also solvable for any larger value.*

The matrices $\hat{J}, \check{J}$ have been defined earlier in (5). Details for a systematic computation of the matrix R_{k+l} and of the gain $K(\sigma)$ may be found in Appendix.

Theorem 1 offers a family of relaxations of Property I. These conditions are less and less conservative when the index l increases. Asymptotically, the conservatism vanishes, as solvability of (8) for certain l is also *necessary* to have Property I.

Notice that the two inequalities in (8) correspond respectively to the conditions $P(\frac{z+\bar{z}}{2}) > 0_n$ and $R(\frac{z+\bar{z}}{2}) = A(\frac{z+\bar{z}}{2})P(z) + P(z)A(\frac{z+\bar{z}}{2})^T + B(\frac{z+\bar{z}}{2})N(z) + N(z)^T B(\frac{z+\bar{z}}{2})^T < 0_n$ for all $z \in (\partial\mathbb{D})^m$. Elements of proof of Theorem 1 are provided in Sect. 6, but we briefly indicate here how to prove that feasibility of (8) implies Property I. Right- and left-multiplication of (8a) by $(z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n)$ and its transconjugate yields, using (6) repeatedly:

$$0_n < (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^* P_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) + \sum_{i=1}^{m} (1 - |z_i|^2)$$
$$\times (z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^* Q_{l,i}^P (z_m^{[l-1]} \otimes \cdots \otimes z_i^{[l-1]} \otimes z_{i-1}^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) ,$$

from which one deduces, putting $|z_i| = 1$:

$$\forall z \in (\partial \mathbb{D})^m,\ P(z) \stackrel{\text{def}}{=} (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^* P_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) > 0_n .$$

Applying similar argument on (8b) with $(z_m^{[k+l]} \otimes \cdots \otimes z_1^{[k+l]} \otimes I_n))$ leads to

$$\forall z \in (\partial \mathbb{D})^m,\ R(z) \stackrel{\text{def}}{=} (z_m^{[k+l]} \otimes \cdots \otimes z_1^{[k+l]} \otimes I_n)^* R_{k+l} (z_m^{[k+l]} \otimes \cdots \otimes z_1^{[k+l]} \otimes I_n) < 0_n .$$

It is now evident that solvability of (8) gives rise to a solution (P, N) of Problem I of degree $l-1$ in $z, \bar{z}$, and $K(\sigma)$ as defined in the statement appears as a stabilizing gain, for every admissible value of the parameters.

Remark that, writing the positive right-hand side of, say, (8a) as $U^T \Lambda U$ with $U^T = U^{-1}$ and $\Lambda = \mathrm{diag}\{\Lambda_i\}$, the previous computations show that, for any $z \in (\partial \mathbb{D})^m$,

$$P(z) = \sum_{i=1}^{l^m n} \Lambda_i \left(U (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \right)_i^* \left(U (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \right)_i ,$$

which thus appears as a sum of squares of matrix-valued polynomials.

Incidentally, stabilizability of a pair (A, B) is equivalent [4, §7.2.1] to the existence of a definite positive matrix P such that $AP + PA^T < BB^T$. This corresponds to the choice $N = -\frac{1}{2} B^T$ in the LMI: $AP + PA^T + BN + N^T B^T < 0$. Similarly, it may be checked that, replacing in (9) the matrix $N(z)$ by $-\frac{1}{2} B^T (\frac{z+\bar{z}}{2})$, provides a simpler stabilizability criterion. Another particular case is $B(\sigma) = 0$, which provides a *robust stability* criterion, see also [1].

5 Time-Varying Parameters with Bounded Variation

Contrary to the constant-parameter case, when Property II is fulfilled, there is probably no necessity for existence of a parameter-dependent Lyapunov function of the kind exhibited in Theorem 1. See however related results in [8, 9, 10]. But it is worth noting that, for given degree, the existence of such a Lyapunov function may be expressed without loss of generality as a LMI problem, in a way similar to what was done for Property I in Theorem 1. Analogously, stabilizing controllers are then found explicitly as functions of $\sigma(t)$.

Theorem 2. *The following assertions are* equivalent*.*

(i) There exists $(P(\sigma), N(\sigma)) \in \mathcal{S}^n[\sigma] \times \mathbb{R}^{p \times n}[\sigma]$ *fulfilling Property* II*.*

(ii) There exist an integer $l \in \mathbb{N}$, 2 matrices $P_l \in \mathcal{S}^{l^m n} \cap \mathbb{R}_M^{n\times n, l^m}$, $N_l \in \mathbb{R}_M^{p\times n, l^m}$, m matrices $Q_{l,i}^P \in \mathcal{S}^{(l-1)^{m-i+1} l^{i-1} n}$, $i = 1, \dots, m$, and 2^m matrices $Q_{l,i}^{R,\eta} \in \mathcal{S}^{(k+l-1)^{m-i+1}(k+l)^{i-1} n}$, $i = 1, \dots, m$, $\eta \in \{-1, 1\}^m$ such that the system (10) of (2^m+1) LMIs obtained for all η in $\{-1, 1\}^m$ is fulfilled, where $R_{k+l} = R_{k+l}(P_l, N_l)$ has the same meaning than in Theorem 1 and $\hat{P}_{k+l,i} \in \mathcal{S}^{(k+l)^m n}$ is a coefficient matrix of the map $z \mapsto \frac{\partial P(\varphi(\sigma))}{\partial \sigma_i}\big|_{\sigma = \frac{z+\bar{z}}{2}}$.

$$
\begin{aligned}
0_{l^m n} < P_l &+ \sum_{i=1}^{m} \left(\hat{J}_{1,l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n}\right)^T Q_{l,i}^P \left(\hat{J}_{1,l-1}^{(m-i+1)\otimes} \otimes I_{l^{i-1}n}\right) \\
&- \sum_{i=1}^{m} \left(\hat{J}_{1,l-1}^{(m-i)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{i-1}n}\right)^T Q_{l,i}^P \left(\hat{J}_{1,l-1}^{(m-i)\otimes} \otimes \check{J}_{1,l-1} \otimes I_{l^{i-1}n}\right), \qquad \text{(10a)}
\end{aligned}
$$

$$
\begin{aligned}
0_{(k+l)^m n} > R_{k+l} &+ \sum_{i=1}^{m} \eta_i \bar{\varrho}_i \hat{P}_{k+l,i} \\
&+ \sum_{i=1}^{m} \left(\hat{J}_{1,k+l-1}^{(m-i+1)\otimes} \otimes I_{(k+l)^{i-1}n}\right)^T Q_{l,i}^{R,\eta} \left(\hat{J}_{1,k+l-1}^{(m-i+1)\otimes} \otimes I_{(k+l)^{i-1}n}\right) \\
&- \sum_{i=1}^{m} \left(\hat{J}_{1,k+l-1}^{(m-i)\otimes} \otimes \check{J}_{1,k+l-1} \otimes I_{(k+l)^{i-1}n}\right)^T Q_{l,i}^{R,\eta} \left(\hat{J}_{1,k+l-1}^{(m-i)\otimes} \otimes \check{J}_{1,k+l-1} \otimes I_{(k+l)^{i-1}n}\right), \qquad \text{(10b)}
\end{aligned}
$$

Moreover,

- *given a solution of LMI (10), for $P(z), N(z)$ having coefficient matrices P_l, N_l, $P(\varphi(\sigma)), N(\varphi(\sigma))$ fulfil Property II, and, for any absolutely continuous σ such that $\sigma(t) \in [-1; +1]^m$, $\dot{\sigma}(t) \in \prod_{i=1}^m [-\bar{\varrho}_i; +\bar{\varrho}_i]$ almost everywhere, $K(\sigma(t)) \stackrel{\text{def}}{=} N(\varphi(\sigma(t)))P(\varphi(\sigma(t)))^{-1}$ is a stabilizing gain, rational in $\sigma(t)$;*
- *if LMI (10) is solvable for the value l of the index, then it is also solvable for any larger value.*

The LMIs in Theorems 1 and 2 differ only by the presence of the terms in $\hat{P}_{k+l,i}$ in (10b). The latter correspond to the derivative terms $\frac{\partial P(\sigma)}{\partial \sigma_i}$ appearing in the inequality in Property II. See Appendix for details on the computations.

6 Elements of Demonstration

We only give here indications for proving Theorems 1 and 2. Application of the same techniques may be found in [3, 1], under more detailed form.

6.1 Sketch of Proof of Theorem 1

1. The equivalence between (i) and (ii), i.e. the fact that P, N in Property I may be supposed polynomial without loss of generality, is consequence of a result on existence of polynomial solutions for LMIs depending continuously upon parameters lying in a compact set, see [2].
2. Take now (ii) as departure: there exists $(P,N) \in \mathcal{S}^n_M[z,\bar z] \times \mathbb{R}^{p\times n}_M[z,\bar z]$, with coefficient matrices $P_l \in \mathcal{S}^{l^m n} \cap \mathbb{R}^{n\times n, l^m}_M$, $N_l \in \mathbb{R}^{p\times n, l^m}_M$ for a certain integer l, such that, $\forall z \in (\partial\mathbb{D})^m$, $(z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n)^* P_l (z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n) > 0_n$ and $(z_m^{[k+l]} \otimes \dots \otimes z_1^{[k+l]} \otimes I_n)^* R_{k+l} (z_m^{[k+l]} \otimes \dots \otimes z_1^{[k+l]} \otimes I_n) < 0_n$, for $R_{k+l}(P_l,N_l)$ defined as in the statement. The proof consists in achieving joint reduction of these two inequalities to the LMIs in (8). For simplicity, we expose this procedure for one inequality only, the first one. For $i = 0,\dots,m$, denote $(\mathcal{P}_i)$ the property: $\exists l \in \mathbb{N}, \exists Q^P_{l,1} \in \mathcal{H}^{(l-1)^m n}$, …, $\exists Q^P_{l,i} \in \mathcal{H}^{(l-1)^{m-i+1} l^{i-1} n}$, $\forall (z_{i+1},\dots,z_m) \in (\partial\mathbb{D})^{m-i}$ such that (11) holds:

$$
\begin{aligned}
&\left(z_m^{[l]} \otimes \dots \otimes z_{i+1}^{[l]} \otimes I_{l^i n}\right)^* \Bigg[P_l \\
&\qquad + \sum_{j=1}^{i} \left(\hat J_{1,l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1}n}\right)^T Q^P_{l,j} \left(\hat J_{1,l-1}^{(m-j+1)\otimes} \otimes I_{l^{j-1}n}\right) \\
&\quad - \sum_{j=1}^{i} \left(\hat J_{1,l-1}^{(m-j)\otimes} \otimes \check J_{1,l-1} \otimes I_{l^{j-1}n}\right)^T Q^P_{l,j} \left(\hat J_{1,l-1}^{(m-j)\otimes} \otimes \check J_{1,l-1} \otimes I_{l^{j-1}n}\right) \Bigg] \\
&\qquad\qquad \times \left(z_m^{[l]} \otimes \dots \otimes z_{i+1}^{[l]} \otimes I_{l^i n}\right) > 0_{l^i n} \ . \qquad (11)
\end{aligned}
$$

Property $(\mathcal{P}_0)$ is the part of (ii) devoted to P, whereas $(\mathcal{P}_m)$ is just (8a). We indicate in the remaining, how to establish that $(\mathcal{P}_i) \Leftrightarrow (\mathcal{P}_{i+1})$ for any $i = 0,\dots,m-1$.

Remark that $(z_m^{[l]} \otimes \dots \otimes z_{i+1}^{[l]} \otimes I_{l^i n}) = (z_m^{[l]} \otimes \dots \otimes z_{i+2}^{[l]} \otimes I_{l^{i+1} n})(z_{i+1}^{[l]} \otimes I_{l^i n})$ and $(z_{i+1}^{[l]} \otimes I_{l^i n}) = \begin{pmatrix} I_{l^i n} \\ z_{i+1}\left(I_{(l-1)l^i n} - z_{i+1}(F_{l-1} \otimes I_{l^i n})\right)^{-1} (f_{l-1} \otimes I_{l^i n}) \end{pmatrix}$, with $F_l \overset{\text{def}}{=} \begin{pmatrix} 0_{1\times(l-1)} & 0 \\ I_{l-1} & 0_{(l-1)\times 1} \end{pmatrix}$, $f_l \overset{\text{def}}{=} \begin{pmatrix} 1 \\ 0_{(l-1)\times 1} \end{pmatrix}$.

Applying discrete-time Kalman-Yakubovich-Popov lemma (see [13, 11] and the statement in the complex case for the continuous-time case in [6, Theorem 1.11.1 and Remark 1.11.1]) yields equivalence of $(\mathcal{P}_i)$ with: $\exists l \in \mathbb{N}, \exists Q^P_{l,1} \in \mathcal{H}^{(l-1)^m n}, \dots, \exists Q^P_{l,i} \in \mathcal{H}^{(l-1)^{m-i+1} l^{i-1} n}$, $\forall (z_{i+2},\dots,z_m) \in (\partial\mathbb{D})^{m-i-1}$, $\exists \tilde Q^P_{l,i+1}(z_{i+2},\dots,z_m) \in \mathcal{H}^{(l-1)l^i n}$ such that:

$$
\begin{aligned}
0_{l^{i+1}n} < \left(z_m^{[l]}\otimes\cdots\otimes z_{i+2}^{[l]}\otimes I_{l^{i+1}n}\right)^* \Bigg[P_l \\
+\sum_{j=1}^{i}\left(\hat{J}_{1,l-1}^{(m-j+1)\otimes}\otimes I_{l^{j-1}n}\right)^T Q_{l,j}^P\left(\hat{J}_{1,l-1}^{(m-j+1)\otimes}\otimes I_{l^{j-1}n}\right) \\
-\sum_{j=1}^{i}\left(\hat{J}_{1,l-1}^{(m-j)\otimes}\otimes \check{J}_{1,l-1}\otimes I_{l^{j-1}n}\right)^T Q_{l,j}^P\left(\hat{J}_{1,l-1}^{(m-j)\otimes}\otimes \check{J}_{1,l-1}\otimes I_{l^{j-1}n}\right)\Bigg] \\
\times\left(z_m^{[l]}\otimes\cdots\otimes z_{i+2}^{[l]}\otimes I_{l^{i+1}n}\right) \\
+\left(\hat{J}_{1,l-1}\otimes I_{l^i n}\right)^T\tilde{Q}_{l,i+1}^P\left(\hat{J}_{1,l-1}\otimes I_{l^i n}\right)-\left(\check{J}_{1,l-1}\otimes I_{l^i n}\right)^T\tilde{Q}_{l,i+1}^P\left(\check{J}_{1,l-1}\otimes I_{l^i n}\right) .
\end{aligned}
\tag{12}
$$

3. Using again the result in [2], $\tilde{Q}_{l,i+1}^P(z_{i+2},\dots,z_m)$, solution of a LMI with parameter in $(\partial\mathbb{D})^{m-i-1}$, may be chosen polynomial in its variables and their conjugates. Let $\tilde{l}-2$ be its degree. If $\tilde{l}\leq l$, then $\tilde{Q}_{l,i+1}^P(z_{i+2},\dots,z_m) = (z_m^{[l-1]}\otimes\cdots\otimes z_{i+2}^{[l-1]}\otimes I_{(l-1)l^in})^* Q_{l,i+1}^P(z_m^{[l-1]}\otimes\cdots\otimes z_{i+2}^{[l-1]}\otimes I_{(l-1)l^in})$, for a coefficient matrix $Q_{l,i+1}^P\in\mathcal{H}^{(l-1)^{m-i}l^in}$. If $\tilde{l}>l$, it may be shown that, *up to an increase of* l, the degree may be supposed the same, so same formula holds (see [3, 1] for similar arguments).

At this point, the last two terms in inequality (12) have been transformed in:

$$
\begin{aligned}
&\left(\hat{J}_{1,l-1}\otimes I_{l^in}\right)^T\left(z_m^{[l-1]}\otimes\cdots\otimes z_{i+2}^{[l-1]}\otimes I_{(l-1)l^in}\right)^* Q_{l,i+1}^P\left(z_m^{[l-1]}\otimes\cdots\otimes z_{i+2}^{[l-1]}\otimes I_{(l-1)l^in}\right)\left(\hat{J}_{1,l-1}\otimes I_{l^in}\right) \\
-&\left(\check{J}_{1,l-1}\otimes I_{l^in}\right)^T\left(z_m^{[l-1]}\otimes\cdots\otimes z_{i+2}^{[l-1]}\otimes I_{(l-1)l^in}\right)^* Q_{l,i+1}^P\left(z_m^{[l-1]}\otimes\cdots\otimes z_{i+2}^{[l-1]}\otimes I_{(l-1)l^in}\right)\left(\check{J}_{1,l-1}\otimes I_{l^in}\right) ,
\end{aligned}
$$

for a certain matrix $Q_{l,i+1}^P\in\mathcal{H}^{(l-1)^{m-i}l^in}$.

4. Some matrix intervertions in the last two terms of the previous formula finally yields equivalence between $(\mathcal{P}_i)$ and $(\mathcal{P}_{i+1})$.

5. The assertion that solvability of (8) for index l implies the same property for every larger index, is proved using the same techniques than the one evoked (but not displayed) in point **3.**, to increase the size of the solution.

6. Last, the same argument is applied to (8b), with detail variations. Application to (8a) and (8b) has to be done together, because of the coupling term P_l. Due to the fact that solvability of (8a), resp. (8b), for a value l of the index implies solvability for every larger value, taking a value for which both inequalities are solvable yields equivalence of (ii) and (iii).

6.2 Sketch of Proof of Theorem 2

The demonstration is copied from the demonstration of the previous Theorem. Due to the affine dependence upon the ϱ_i in Property II, it is enough to consider only the extremal values $\pm\bar{\varrho}_i$. It is hence required that: $\exists l \in \mathbb{N}$, $\exists P_l \in \mathcal{S}^{l^m n}$, $\exists N_l \in \mathbb{R}_M^{p\times n, l^m}$, $\forall \eta \in \{-1,1\}^m$, $\forall z \in (\partial\mathbb{D})^m$, $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)^* P_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) < 0_n$ and $(z_m^{[k+l]} \otimes \cdots \otimes z_1^{[k+l]} \otimes I_n)^* \left(R_{k+l} + \sum_{i=1}^m \eta_i \bar{\varrho}_i \hat{P}_{k+l,i} \right) (z_m^{[k+l]} \otimes \cdots \otimes z_1^{[k+l]} \otimes I_n) < 0_n$.

The argument then essentially follows the proof of Theorem 1. One has to check carefully that the process of increase of the degree (point **3.** in Sect. 6.1) still works.

References

1. P.-A. Bliman (2004) A convex approach to robust stability for linear systems with uncertain scalar parameters, *SIAM J. on Control and Optimization* **42** no 6, 2016–2042
2. P.-A. Bliman (2004) An existence result for polynomial solutions of parameter-dependent LMIs, *Systems and Control Letters* **51** no 3-4, 165–169
3. P.-A. Bliman (2004) On robust semidefinite programming, *Proc. of 16th International Symposium on Mathematical Theory of Networks and Systems (MTNS2004)*, Leuven (Belgium), July 2004
4. S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan (1994) *Linear matrix inequalities in system and control theory*, SIAM Studies in Applied Mathematics vol. 15
5. D.J. Leith, W.E. Leithead (2000) Survey of gain-scheduling analysis and design, *Int. J. control* **73** no 11, 1001–1025
6. G.A. Leonov, D.V. Ponomarenko, V.B. Smirnova (1996) *Frequency-domain Methods for Nonlinear Analysis. Theory and Applications*, World Scientific Publishing Co.
7. S. Lim, J.P. How (2002) Analysis of linear parameter-varying systems using a non-smooth dissipative systems framework, *Int. J. Robust Nonlinear Control* **12**, 1067–1092
8. I. Masubuchi (1998) Spline-type solutions to parameter-dependent LMIs, *Proc. of the 37th IEEE CDC* Tampa, Florida (USA)
9. I. Masubuchi (1999) An exact solution to parameter-dependent convex differential inequalities, *Proc. of the European Control Conference* Karlsruhe (Germany)
10. I. Masubuchi, T. Akiyama, M. Saeki (2003) Synthesis of Output Feedback Gain-Scheduling Controllers Based on Descriptor LPV System Representation, *Proc. of the 42nd IEEE CDC* Maui, Hawaii (USA)
11. A. Rantzer (1996) On the Kalman-Yakubovich-Popov lemma, *Syst. Contr. Lett.* **28** no 1, 7–10
12. W.J. Rugh, J.S. Shamma (2000) Research on gain scheduling, *Automatica* **36**, 1401–1425
13. G. Szegö, R.E. Kalman (1963) Sur la stabilité absolue d'un système d'équations aux différences finies, *Comp. Rend. Acad. Sci.* **257** no 2, 338–390
14. F. Wu (2001) A generalized LPV system analysis and control synthesis framework, *Int. J. control* **74** no 7, 745–759

A Appendix on Polynomial Matrices

We give here details on the computations necessary for systematic use of Theorems 1 and 2. It is explained in Sects. A.1 and A.2 how to compute $R_{k+l}(P_l, N_l)$, that is how to determine the coefficient matrices of the terms in (9). Then in Sect. A.3 are provided formulas for explicit computation of $K(\sigma)$ as a function of σ, that is of $P(\varphi(\sigma))$ and $N(\varphi(\sigma))$ for $P(z), N(z)$ defined by their coefficient matrix P_l, N_l. Last, the computation of the term $\hat{P}_{k+l,i}$ in (10) is explained in Sect. A.4.

We first extend the notations defined in (5). For $l, l' \in \mathbb{N}$, $l \leq l'$, $\alpha = 0, 1, \dots, l'$, define $J_{\alpha,l,l'} \in \mathbb{R}^{l\times(l+l')}$ by:

$$J_{\alpha,l,l'} \stackrel{\text{def}}{=} \begin{pmatrix} 0_{l\times\alpha} & I_l & 0_{l\times(l'-\alpha)} \end{pmatrix} .$$

Then $\hat{J}_{l',l} = J_{0,l,l'}$, $\check{J}_{l',l} = J_{l',l,l'}$, and $v^\alpha v^{[l]} = J_{\alpha,l,l'} v^{[l+l']}$.

A.1 Representation of Polynomial Matrices

A rather natural representation for a matrix-valued polynomial $M : \mathbb{R}^m \to \mathbb{R}^{p\times n}$ (such as $A(\sigma)$ and $B(\sigma)$) of degree $l-1$ is

$$M(\sigma) = \tilde{M}_l(\sigma_m^{[l]} \otimes \cdots \otimes \sigma_1^{[l]} \otimes I_n) \ , \tag{13}$$

for a certain, uniquely defined, matrix $\tilde{M}_l \in \mathbb{R}^{p\times l^m n}$. From this, one should be able to deduce the coefficient matrix of the map $M(\frac{z+\bar{z}}{2})$, in order to apply Theorems 1 and 2. The effect of the corresponding change of variable (4) is summarized by Lemma A.1.

Lemma A.1. *Let* $\tilde{M}_l \in \mathbb{R}^{p\times l^m n}$, *then* $\tilde{M}_l\left(\left(\frac{z_m+\bar{z}_m}{2}\right)^{[l]} \otimes \cdots \otimes \left(\frac{z_1+\bar{z}_1}{2}\right)^{[l]} \otimes I_n\right) = (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_p)^* M_l (z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$, *where the matrix* $M_l \in \mathbb{R}_M^{p\times n, l^m}$ *is given by the formula* $M_l \stackrel{\text{def}}{=} \sum_{0\leq\alpha_i\leq l-1} (J_{\alpha_m,1,l-1} \otimes \cdots \otimes J_{\alpha_1,1,l-1} \otimes I_p)^T \tilde{M}_l (K_{l,\alpha_m} \otimes \cdots \otimes K_{l,\alpha_1} \otimes I_n)$, *in which, by definition, the* i*-th line of the matrix* $K_{l,\alpha} \in \mathbb{R}^{l\times l}$ *is equal to* $2^{-i+1}\begin{pmatrix} C_{i-1}^{i-1} & C_{i-1}^{i-2} & \dots & C_{i-1}^0 & 0 & \dots & 0 \end{pmatrix}$, $C_i^\alpha \stackrel{\text{def}}{=} \frac{i!}{\alpha!(i-\alpha)!}$.

Proof. $K_{l,\alpha}$ defined in the statement is such that $\forall v \in \mathbb{C}$, $\left(\frac{v+\bar{v}}{2}\right)^{[l]} = \sum_{\alpha=0}^{l-1} \bar{v}^\alpha K_{l,\alpha} v^{[l]}$. Thus,

$$\begin{aligned} &\tilde{M}_l\left(\left(\frac{z_m+\bar{z}_m}{2}\right)^{[l]} \otimes \cdots \otimes \left(\frac{z_1+\bar{z}_1}{2}\right)^{[l]} \otimes I_n\right) \\ &\quad = \sum_{0\leq\alpha_i\leq l-1} \bar{z}_1^{\alpha_1} \dots \bar{z}_m^{\alpha_m} \tilde{M}_l (K_{l,\alpha_m} \otimes \cdots \otimes K_{l,\alpha_1})(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n) \ . \end{aligned}$$

The conclusion then follows from the fact that $\forall v \in \mathbb{C}$, $v^\alpha = v^\alpha v^{[1]} = J_{\alpha,1,l-1} v^{[l]}$, so $v^{\alpha*} = v^{[l]*} J_{\alpha,1,l-1}^T$.

A.2 Products of Polynomial Matrices

Solving the LMIs in Theorems 1 and 2 necessitates to be able to express the coefficient matrix R_{k+l} of $R(z)$ defined in (9), given the coefficient matrices P_l, N_l of $P(z)$, $N(z)$. This in turn necessitates to express the coefficient matrix of a product of matrix-valued polynomials, as function of the coefficient matrices of the factors. This is the goal of Lemma A.2.

Lemma A.2. *Let $l, l' \in \mathbb{N}$, and $M(z), M'(z)$ with coefficient matrices $M_l \in \mathbb{R}^{l^m p \times l^m n}$, $M'_{l'} \in \mathbb{R}^{l'^m n \times l'^m q}$. Then, $M''(z)$ has coefficient matrix $M''_{l''}$, where $l'' = l + l' - 1$ and $M''_{l''} \stackrel{\text{def}}{=} \sum_{\substack{0\leq \alpha_i \leq l-1, 0\leq \alpha'_i \leq l'-1 \\ 1\leq i\leq m}} (J_{\alpha'_m,l,l'-1} \otimes \cdots \otimes J_{\alpha'_1,l,l'-1} \otimes I_p)^T M_l (J_{\alpha_m,1,l-1} \otimes \cdots \otimes J_{\alpha_1,1,l-1} \otimes I_n)^T (J_{\alpha'_m,1,l'-1} \otimes \cdots \otimes J_{\alpha'_1,1,l'-1} \otimes I_n) M'_{l'} (J_{\alpha_m,l',l-1} \otimes \cdots \otimes J_{\alpha_1,l',l-1} \otimes I_q)$.*

Proof. One has, $\forall v \in \mathbb{C}$,

$$v^{[l]} = \sum_{\alpha=0}^{l-1} v^\alpha J_{\alpha,1,l-1}^T, \quad v^{[l]} v^{[l']*} = \sum_{\substack{0\leq\alpha\leq l-1,\\ 0\leq\alpha'\leq l'-1}} v^\alpha \bar{v}^{\alpha'} J_{\alpha,1,l-1}^T J_{\alpha',1,l'-1} ,$$

and the proof is achieved by using the fact that $v^\alpha v^{[l']} = J_{\alpha,l',l-1} v^{[l+l'-1]}$, $\bar{v}^{\alpha'} v^{[l]*} = v^{[l+l'-1]*} J_{\alpha',l,l'-1}^T$.

A.3 Formulas Attached to the Inversion of the Map φ

Once the LMI (8) or (10) has been solved successfully (for a given l), one has to express explicitly $P(\varphi(\sigma))$ and $N(\varphi(\sigma))$ to obtain the gain $K(\sigma) = N(\varphi(\sigma))^{-1} P(\varphi(\sigma))$, departing from the coefficient matrices P_l, N_l of $P(z), N(z)$. This is done with the help of the following result.

Lemma A.3. *Let $N(z) \in \mathbb{R}_M^{p\times n}[z, \bar{z}]$ with coefficient matrix $N_l \in \mathbb{R}_M^{p\times n, l^m}$. Then,*

$$N(\varphi(\sigma)) = \sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ i=1,\ldots,m}} p_{\alpha_1-\alpha'_1}(\sigma_1) \ldots p_{\alpha_m-\alpha'_m}(\sigma_m)$$
$$\times (J_{\alpha'_m,1,l-1} \otimes \cdots \otimes J_{\alpha'_1,1,l-1} \otimes I_p) N_l (J_{\alpha_m,1,l-1} \otimes \cdots \otimes J_{\alpha_1,1,l-1} \otimes I_n)^T ,$$

where by definition, the polynomials p_α are such that, for any $\phi \in \mathbb{R}$, $\cos(\alpha\phi) = p_\alpha(\cos\phi)$.

The coefficients of the p_α are easily found, allowing effective use of the previous result. For example, $\cos 2\phi = 2\cos^2\phi - 1$, $\cos 3\phi = 4\cos^3\phi - 3\cos\phi$, so $p_0(\sigma) = 1$, $p_1(\sigma) = \sigma$, $p_2(\sigma) = 2\sigma^2 - 1$, $p_3(\sigma) = 4\sigma^3 - 3\sigma$, and so on.

Forming, from the maps p_α, the matrices $T_{l,|\alpha|} \in \mathbb{R}^{1\times l}$ such that $\forall\alpha \in \{-(l-1), \ldots, 0, \ldots, l-1\}$, $\forall\phi \in \mathbb{R}$, $\cos(\alpha\phi) = T_{l,|\alpha|}(\cos\phi)^{[l]}$, the formula in Lemma A.3 writes under matrix form as in (13), with $\tilde{M}_l$ replaced by $\sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ i=1,\dots,m}}(J_{\alpha'_m,1,l-1}\otimes\dots\otimes J_{\alpha'_1,1,l-1}\otimes I_p)N_l(J^T_{\alpha_m,1,l-1}T_{l,|\alpha_m-\alpha'_m|}\otimes\dots\otimes J^T_{\alpha_1,1,l-1}T_{l,|\alpha_1-\alpha'_1|}\otimes I_n)$.

Proof. As a direct consequence of the definition, $N(z)$ is equal to $\sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ i=1,\dots,m}} z_1^{\alpha_1}\bar{z}_1^{\alpha'_1}\dots z_m^{\alpha_m}\bar{z}_m^{\alpha'_m}(J_{\alpha'_m,1,l-1}\otimes\dots\otimes J_{\alpha'_1,1,l-1}\otimes I_p)N_l(J_{\alpha_m,1,l-1}\otimes\dots\otimes J_{\alpha_1,1,l-1}\otimes I_n)^T$. Taking into account the fact that $|z_i|=1$, $i=1,\dots,m$ and that $N_l\in\mathbb{R}_M^{p\times n,l^m}$, the previous expression is equal to $\sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1,\alpha_1=\alpha'_1\\ i=1,\dots,m}} z_2^{\alpha_2}\bar{z}_2^{\alpha'_2}\dots z_m^{\alpha_m}\bar{z}_m^{\alpha'_m}(J_{\alpha'_m,1,l-1}\otimes\dots\otimes J_{\alpha'_1,1,l-1}\otimes I_p)N_l(J_{\alpha_m,1,l-1}\otimes\dots\otimes J_{\alpha_1,1,l-1}\otimes I_n)^T+\sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ \alpha_1<\alpha'_1,\ i=1,\dots,m}}(z_1^{\alpha'_1-\alpha_1}+\bar{z}_1^{\alpha'_1-\alpha_1})z_2^{\alpha_2}\bar{z}_2^{\alpha'_2}\dots z_m^{\alpha_m}\bar{z}_m^{\alpha'_m}(J_{\alpha'_m,1,l-1}\otimes\dots\otimes J_{\alpha'_1,1,l-1}\otimes I_p)N_l(J_{\alpha_m,1,l-1}\otimes\dots\otimes J_{\alpha_1,1,l-1}\otimes I_n)^T$. Introducing the functions p_i as defined in the statement, this is also equal to $\sum_{\substack{0\leq\alpha_i,\alpha'_i\leq l-1\\ i=1,\dots,m}} p_{\alpha_1-\alpha'_1}(\sigma_1)z_2^{\alpha_2}\bar{z}_2^{\alpha'_2}\dots z_m^{\alpha_m}\bar{z}_m^{\alpha'_m}(J_{\alpha'_m,1,l-1}\otimes\dots\otimes J_{\alpha'_1,1,l-1}\otimes I_p)N_l(J_{\alpha_m,1,l-1}\otimes\dots\otimes J_{\alpha_1,1,l-1}\otimes I_n)^T$, because $\sigma_1=\operatorname{Re} z_1$. The result follows by induction on m.

A.4 Differentiation of Polynomial Matrices

Lemma A.4 below permits to express the coefficient matrix of the terms $\frac{\partial P(\sigma)}{\partial\sigma_i}|_{\sigma=\frac{z+\bar{z}}{2}}$ in Property II as function of the coefficient matrix of $P(z)$. Notice that the formula therein provides directly the derivatives as a polynomial of degree $k+l-1$ (instead of $l-2$), ready to be added to the term $A(\sigma)P(\sigma)+P(\sigma)A(\sigma)^T+B(\sigma)N(\sigma)+N(\sigma)^TB(\sigma)^T$ in the matrix inequality in Property II, which has precisely the same degree.

Lemma A.4. *Let* $M(\sigma)\stackrel{\text{def}}{=}M_l(\sigma_m^{[l]}\otimes\dots\otimes\sigma_1^{[l]}\otimes I_n)$. *Then, for any nonnegative integer* k, $\frac{\partial M(\sigma)}{\partial\sigma_i}=\hat{M}_{k+l,i}(\sigma_m^{[k+l]}\otimes\dots\otimes\sigma_1^{[k+l]}\otimes I_n)$, *with* $\hat{M}_{k+l,i}\stackrel{\text{def}}{=}M_l(\hat{J}_{k,l}^{(m-i)\otimes}\otimes L_l\hat{J}_{k,l}\otimes\hat{J}_{k,l}^{(i-1)\otimes}\otimes I_n)$.

Proof. Indeed, $\frac{\partial M(\sigma)}{\partial\sigma_i}=M_l(\sigma_m^{[l]}\otimes\dots\otimes\frac{\partial\sigma_i^{[l]}}{\partial\sigma_i}\otimes\sigma_{i-1}^{[l]}\otimes\dots\otimes\sigma_1^{[l]}\otimes I_n)=M_l(I_l^{(m-i)\otimes}\otimes L_l\otimes I_l^{(i-1)\otimes}\otimes I_n)(\sigma_m^{[l]}\otimes\dots\otimes\sigma_1^{[l]}\otimes I_n)=\hat{M}_{k+l,i}(\sigma_m^{[k+l]}\otimes\dots\otimes\sigma_1^{[k+l]}\otimes I_n)$.