From Lyapunov-Krasovskii Functionals for Delay-Independent Stability to LMI Conditions for μ-Analysis

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Our scope in this note is to give a unified view on different approaches for studying stability of delay systems and parameter-dependent systems, and on estimation methods for some structured singular values. The classical approaches are exposed in Sections 1 to 3. A new result which links them together is given in Section 4, Elements of proof are gathered in Section 5. Comments are provided in Section 6. Finally, Section 7 proposes some open problems. For sake of space, exposure is kept to minimum, the reader is refered to the cited literature for more details.

0 Notations, representation of polynomials

By $\overline{\mathbb{C}^+}$ is meant the closed set of complex numbers with nonnegative real part. The closed unit ball (resp. circle) in $\mathbb{C}$ is denoted $\overline{\mathbb{D}}$ (resp. $\partial\mathbb{D}$). The symbol $\otimes$ denotes Kronecker product, the power of Kronecker products being used with the natural meaning: $M^{0\otimes} = 1$, $M^{p\otimes} \stackrel{\text{def}}{=} M^{(p-1)\otimes} \otimes M$. The transpose and transconjugate of a matrix are respectively denoted with a superscript ^{T} and ^{H}. We study here the stability of linear systems with m independent delays $h_1, \dots, h_m$. In the whole note, we write $h \stackrel{\text{def}}{=} (h_1, \dots, h_m)$, $\nabla \stackrel{\text{def}}{=} (\nabla_1, \dots, \nabla_m)$, where ∇_i is the delay operator associated to delay h_i, acting on any convenient space of time functions. Also, $z \stackrel{\text{def}}{=} (z_1, \dots, z_m)$ denotes a free variable in $\mathbb{C}^m$, and for simplicity, the notation $e^{-sh} \stackrel{\text{def}}{=} (e^{-sh_1}, \dots, e^{-sh_m})$ is used in the transfers, where s is the Laplace variable.

For any integer n, let $\mathbb{R}^{n\times n}[z]$ (resp. $\mathbb{R}^{n\times n}[z, \bar{z}]$) be the ring of polynomials in $z \in \mathbb{C}^m$ (resp. in $z, \bar{z}$) with coefficients in $\mathbb{R}^{n\times n}$. The sets $\mathbb{C}^{n\times n}[z]$, $\mathbb{C}^{n\times n}[z, \bar{z}]$ are defined similarly. With $\mathcal{S}^n$ the subset of symmetric matrices in $\mathbb{R}^{n\times n}$, one defines analogously the set $\mathcal{S}^n[z, \bar{z}]$. An important subset of $\mathcal{S}^n[z, \bar{z}]$ is the set of those $M(z)$ such that $\forall z \in \mathbb{C}^m$, $M(z)^H = M(z)$; it is denoted $\mathcal{S}^n_H[z, \bar{z}]$.

To be able to represent and manipulate matrix-valued polynomials, define, for $l \in \mathbb{N}$, $i = 1, \dots, m$ and for any $v \in \mathbb{C}$,

$$v^{[l]} \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ v \\ \vdots \\ v^{l-1} \end{pmatrix}, \quad \nabla_i^{[l]} \stackrel{\text{def}}{=} \begin{pmatrix} \mathrm{Id} \\ \nabla_i \\ \vdots \\ \nabla_i^{l-1} \end{pmatrix} .$$

Notice that we denote in the same way the powers of complex numbers and the powers of delay operators (for the composition product), along the rule: $\nabla_i^2 \stackrel{\text{def}}{=} \nabla_i \circ \nabla_i \dots$ This will permit in the sequel to apply polynomials in $\mathbb{C}^m$ to the operator ∇.

The expression $z_m^{[l]} \otimes \dots \otimes z_1^{[l]}$ gathers all the monomials with degree at most $l-1$ in each of the components of z, so for any $M(z)$ in $\mathbb{C}^{n\times n}[z,\bar{z}]$, there exist $l \in \mathbb{N}$ and $M_l \in \mathbb{C}^{l^m n \times l^m n}$ such that, for all $z \in \mathbb{C}^m$,

$$M(z) = (z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n)^H M_l (z_m^{[l]} \otimes \dots \otimes z_1^{[l]} \otimes I_n) \ .$$

The matrix M_l is just a concatenation, in prescribed order, of the matrices coefficients of $M(z)$. In this representation, which will be used as a central tool in the whole paper, l and M_l are unique when taking minimal l. The matrix M_l is called the *coefficient matrix* of $M(z)$, $l-1$ the *degree* of the representation. Remark that, for M under the previous form, $M \in \mathcal{S}_H^n[z,\bar{z}]$ *iff* $M_l \in \mathcal{S}^{l^m n}$ for some $l > 0$.

The following identities are useful for calculations: for any l', $1 \leq l' \leq l$,

$$\begin{aligned} v^{l'-1} &= \begin{pmatrix} 0_{1\times(l'-1)} & 1 & 0_{l-l'} \end{pmatrix} v^{[l]}, \\ \nabla_i^{l'-1} x &= (\begin{pmatrix} 0_{1\times(l'-1)} & 1 & 0_{l-l'} \end{pmatrix} \otimes I_n) \nabla_i^{[l]} x \ , \end{aligned} \tag{1}$$

for any complex v and any time-function x taking values in $\mathbb{R}^n$. Last, let $\hat{J}_l, \check{J}_l \in \mathbb{R}^{l\times(l+1)}$ be defined by:

$$\hat{J}_l \stackrel{\text{def}}{=} \begin{pmatrix} I_l & 0_{l\times 1} \end{pmatrix}, \ \check{J}_l \stackrel{\text{def}}{=} \begin{pmatrix} 0_{l\times 1} & I_l \end{pmatrix} \ . \tag{2}$$

This corresponds to the matrix present in (1), for the values $l' = 1$ and $l' = l$.

1 Delay systems and associated stability properties

The delay system under study is denoted under the quite general form

$$\dot{x} = A(\nabla) x \ , \tag{3}$$

where $A(z) \in \mathbb{R}^{n\times n}[z]$ is a polynomial. By definition, we denote its degree $k-1$ (that is, the maximum of the m partial degrees with respect to $z_1, \dots, z_m$). For example, for the affine map $(k-1=1)$

$$A(z) = A_0 + z_1 A_1 + \dots + z_m A_m \ , \tag{4}$$

this yields: $\dot{x} = A_0 x + A_1 x(t-h_1) + \dots + A_m x(t-h_m)$.

1.1 Basic properties

Let us first recall the following

Theorem 1 (Stability characterization). *System* (3) *is asymptotically stable* iff

$$\forall s \in \overline{\mathbb{C}^+},\ \det(sI_n - A(e^{-sh})) \neq 0\ .$$

As in [20, 21], we put:

Definition 1 (Delay-independent stability (DIS)). *System* (3) *is said delay-independently stable if it is stable for any* $h \in [0, +\infty)^m$.

The previous notion has been introduced in order to study the stability of systems with delays of imperfectly known values. The assumption that no information on the value of the delay is available may be coarse in practice, when bounds are already known. This has necessitated development of *delay-dependent* criteria too. This topic is not treated here.

Extension of results in [17, 16] permits the following claim.

Theorem 2 (Characterization of the delay-independent stability). *System* (3) *is DIS* iff

$$\forall (s,z) \in \overline{\mathbb{C}^+} \setminus \{0\} \times \overline{\mathbb{D}}^m \cup \{(0,1,\dots,1)\},\ \det(sI_n - A(z)) \neq 0\ .$$

Extending [23] leads to introduce the slightly stronger property:

Definition 2 (Strong delay-independent stability (SDIS)). *System* (3) *is said strongly delay-independently stable if*

$$\forall (s,z) \in \overline{\mathbb{C}^+} \times \overline{\mathbb{D}}^m,\ \det(sI_n - A(z)) \neq 0\ . \tag{5}$$

Infinitely close (in terms of a metric on the coefficients of A) from any DIS system which is *not* SDIS, one may find systems which are *not* DIS. In other words, the set of SDIS systems is the *interior* of the set of DIS systems endowed with the topology whose neighborhoods are defined by the choice of a metric on the coefficient matrices [2].

1.2 The Lyapunov-Krasovskii functionals approach

For $P, Q_1, \dots, Q_m \in \mathcal{S}^n$, define for any $\phi \in \mathcal{C}([-(k-1)\max h_i, 0]; \mathbb{C}^n)$ the functional V by (see [22, 15, 6]):

$$V(\phi)(t) \overset{\text{def}}{=} \phi(0)^T P \phi(0) + \int_{-(k-1)h_1}^{0} \phi(\tau)^T Q_1 \phi(\tau)\ d\tau + \dots + \int_{-(k-1)h_m}^{0} \phi(\tau)^T Q_m \phi(\tau)\ d\tau\ . \tag{6}$$

Denoting abusively $V(x|_{[t-(k-1)\max h_i,t]})$ by $V(x)(t)$, one has, along the trajectories of (3):

$$\frac{d[V(x)(t)]}{dt} = ((\nabla_m^{[k]} \otimes \cdots \otimes \nabla_1^{[k]} \otimes I_n)x)(t)^T R((\nabla_m^{[k]} \otimes \cdots \otimes \nabla_1^{[k]} \otimes I_n)x)(t) \ ,$$

where the exact value of the matrix $R = R(P, Q_1, \dots, Q_m) \in \mathcal{S}^{(k+1)^m n}$ may be written using the formulas in (1). It is important to remark that R is *affine* in $P, Q_1, \dots, Q_m$ and *independent* of the values of $h_1, \dots, h_m$. Thus, searching for a Lyapunov-Krasovskii functional in the class (6) leads to the following.

Theorem 3 (Sufficient condition for SDIS). *If there exist* $P, Q_1, \dots, Q_m \in \mathcal{S}^n$ *such that*

$$P > 0, Q_1 > 0, \dots, Q_m > 0, R < 0 \ ,$$

then system (3) *is SDIS.*

The sufficient condition in Theorem 3 is a Linear Matrix Inequality (LMI), see [6].

2 Robust stability of parameter dependent systems

Associated to delay system (3) is the system with parameter $z \in \mathbb{C}^m$ given by

$$\dot{x} = A(z)x \ . \tag{7}$$

2.1 Basic properties

Definition 3 (Robust stability). *System* (7) *is said robustly stable if it is asymptotically stable for any* $z \in \overline{\mathbb{D}}^m$.

Trivially, this notion is linked with SDIS:

Theorem 4 (Link with SDIS). *System* (7) *is robustly stable* iff *system* (3) *is SDIS.*

2.2 Sufficient conditions for robust stability and the parameter-dependent Lyapunov functions approach

A number of published contributions have obtained robust stability criteria for systems similar to (7), by use of some prescribed class of parameter-dependent Lyapunov functions. The latter have been chosen *independent of the parameters*, *affine* [14, 11, 8, 24], *quadratic* [29, 30].

Existence of a Lyapunov function in these classes may be recast as solvability problem for certain LMI. Nevertheless, due to the fact that they assume a prespecified dependence of the Lyapunov function with respect to the parameters, they all lead to *sufficient conditions* for robust stability.

3 Structured singular values with repeated scalar blocks

The notion of structured singular values is one of the basic tools of robust control [10].

3.1 Basic properties

Definition 4 (Structured singular values (ssv)). *For fixed* $r_1, \dots, r_m \in \mathbb{N}$*, let* $\Delta \stackrel{\text{def}}{=} \{\text{diag}[\delta_1 I_{r_1}; \dots; \delta_m I_{r_m}] \ : \ (\delta_1, \dots, \delta_m) \in \mathbb{C}^m\}$*. Then, for any* $M \in \mathbb{C}^{r \times r}$*, where* $r \stackrel{\text{def}}{=} r_1 + \cdots + r_m$*, the structured singular value* $\mu_\Delta(M)$ *is null if no matrix* $\varDelta \in \Delta$ *makes* $I_r - M\varDelta$ *singular, and otherwise equal to*

$$(\min\{\bar{\sigma}(\varDelta) \ : \ \varDelta \in \Delta, \det(I_r - M\varDelta) = 0\})^{-1} \ .$$

Computing μ is generally a NP-hard task [28]. Using the change of variable $s = \frac{1+z_0}{1-z_0}$, which maps $\overline{\mathbb{C}^+}$ into $\overline{\mathbb{D}}$, one may exhibit, for any polynomial $A(z)$, a certain structure Δ_A and a real matrix M_A such that (5) holds *iff*

$$\mu_{\Delta_A}(M_A) < 1 \ . \tag{8}$$

Thus, checking SDIS of (3) or robust stability of (7) amounts to estimate a ssv with $m+1$ repeated scalar blocks. This more specific problem is also NP-hard [27].

Alternatively, this is equivalent [7] to check whether

$$\forall s \in j\mathbb{R}, \ \mu_{\tilde{\Delta}_A}(\tilde{M}_A(s)) < 1 \ , \tag{8'}$$

for certain structure $\tilde{\Delta}_A$ (with m repeated scalar blocks) and transfer $\tilde{M}_A$.

Conversely, let M be a real square matrix and Δ a block structure of compatible size, having $m+1$ repeated complex scalar blocks. Is it possible to find a polynomial $A(z)$ such that (3) is SDIS *iff* $\mu_\Delta(M) < 1$? The answer is no in general, as structured singular values may describe not only polynomial dependences, but also rational ones, via Linear Fractional Transform. As a matter of fact, the whole generality is obtained when considering delay-differential equations of *neutral type*, and not only of *retarded type*. In terms of parameter-dependent systems, this corresponds to *parameter-dependent singular (descriptor) systems*.

3.2 Upper bounds for ssv and the multiplier approach

Various upper bounds for the structured singular values have been proposed. Their principle relies on the use of *multipliers* [12] or *scaling technique* [1]. Some results are based on mixed methods [9, 13].

Interestingly enough, it has been shown [32] that checking SDIS by means of Lyapunov-Krasovskii functionals of the class (6), amounts to use in the previous inequality the conservative evalution of μ provided by D-scalings (the classical "μ upper bound").

Connection between the scaling approach and the parameter-dependent methods has been established by Iwasaki *et al.* [19, 18]. Both approaches may be interpreted as special cases of the *quadratic separator*, separating in an appropriate space a graph associated to the "system" from a graph associated to the "perturbation", here the parameters. Roughly speaking, the previous results are obtained when looking for such a separator with prespecified, "simple", dependence, either with respect to the frequency (frequency-dependent scaling matrix in μ-analysis), or to the parameters (parameter-dependent Lyapunov functions).

4 A key result

For any $l \in \mathbb{N}$, for any $P_l \in \mathcal{S}^{l^m n}$, define $R_l = R_l(P_l) \in \mathcal{S}^{(k+l-1)^m n}$ to be the coefficient matrix[1] of $R(z) \stackrel{\text{def}}{=} A(z)^H P(z) + P(z)A(z)$, where $P(z)$ is defined by its coefficient matrix P_l. As an example [3, 5], for $A(z)$ defined in (4), one has

$$R_l = \left(\left(\hat{J}_l^{m\otimes} \otimes A_0 \right) + \sum_{i=1}^m \left(\hat{J}_l^{(m-i)\otimes} \otimes \check{J}_l \otimes \hat{J}_l^{(i-1)\otimes} \otimes A_i \right) \right)^H P_l \left(\hat{J}_l^{m\otimes} \otimes I_n \right)$$
$$+ \left(\hat{J}_l^{m\otimes} \otimes I_n \right)^T P_l \left(\left(\hat{J}_l^{m\otimes} \otimes A_0 \right) + \sum_{i=1}^m \left(\hat{J}_l^{(m-i)\otimes} \otimes \check{J}_l \otimes \hat{J}_l^{(i-1)\otimes} \otimes A_i \right) \right) .$$

The following result is an extension of [3, 5] to the cases where $k > 2$.

Theorem 5. *The following properties are* equivalent*.*

1. *System* (3) *is SDIS (resp. system* (7) *is robustly stable, resp. condition* (8) *or* (8') *is fulfilled with adequate structure and matrix choice).*

2. *There exists* $P(z) \in \mathcal{S}_H^n[z, \bar{z}]$ *such that,*

$$\forall z \in \overline{\mathbb{D}}^m,\ P(z) > 0_n,\ A(z)^H P(z) + P(z)A(z) < 0_n\ .$$

3. *There exist* $l \in \mathbb{N}$ *and* $m+1$ *matrices* $P_l \in \mathcal{S}^{l^m n}$ *and* $Q_{l,i} \in \mathcal{S}^{(k+l-2)^{m-i+1}(k+l-1)^{i-1} n}$, $i = 1, \dots, m$, *such that*

$$P_l > 0_{l^m n} \tag{9a}$$

and

[1] There may exist a representation of $R(z)$ with coefficient matrix of size smaller than $(k+l-1)^m n$, this aspect has no incidence on the sequel.

$$S_l(P_l, Q_{l,1}, \ldots, Q_{l,m}) \stackrel{\text{def}}{=} R_l(P_l)$$
$$+ \sum_{i=1}^{m} \left(\hat{J}_{k+l-2}^{(m-i+1)\otimes} \otimes I_{(k+l-1)^{i-1}n} \right)^T Q_{l,i} \left(\hat{J}_{k+l-2}^{(m-i+1)\otimes} \otimes I_{(k+l-1)^{i-1}n} \right)$$
$$-\sum_{i=1}^{m} \left(\hat{J}_{k+l-2}^{(m-i)\otimes} \otimes \check{J}_{k+l-2} \otimes I_{(k+l-1)^{i-1}n} \right)^T Q_{l,i} \left(\hat{J}_{k+l-2}^{(m-i)\otimes} \otimes \check{J}_{k+l-2} \otimes I_{(k+l-1)^{i-1}n} \right)$$
$$< 0_{(k+l-1)^m n} \ , \quad \text{(9b)}$$

where $\hat{J}_k, \check{J}_k$ are defined in (2)*.*

Moreover, if LMI (9) *is solvable for the index l, then it is also solvable for any larger index.*

Thus, the conditions expressed in (9) are more and more precise (less and less conservative) when l increases, and the feasibility of any of them is sufficient to have the properties depicted in **1.** An important point is that *necessity* also holds, in the precise sense that: if the stability properties hold, then the corresponding LMIs are fulfilled *from a certain rank l and beyond.*

5 Elements of proof of Theorem 5

System (3) is SDIS *iff* for any $z \in \overline{\mathbb{D}}^m$, there exists $P(z) > 0$ such that $A(z)^H P(z) + P(z)A(z) < 0$. The previous problem is a *parameter-dependent LMI*, in which z is the parameter vector. The dependence upon the latter being polynomial, and thus continuous, one may apply the result given in [4], and concludes that if (3) is SDIS, then without loss of generality P may be chosen *polynomial in z and $\bar{z}$*. This establishes the implication **1.** $\Rightarrow$ **2.**

To prove that **3.** $\Rightarrow$ **1.**, right- and left- multiply (9a) (resp. (9b)) by $(z_m^{[l]} \otimes \cdots \otimes z_1^{[l]} \otimes I_n)$ (resp. $(z_m^{[k+l-1]} \otimes \cdots \otimes z_1^{[k+l-1]} \otimes I_n)$) and its transconjugate. This yields $P(z) > 0_n$ and

$$R(z) + \sum_{i=1}^{m} (1-|z_i|^2)(z_m^{[k+l-2]} \otimes \cdots \otimes z_i^{[k+l-2]} \otimes z_{i-1}^{[k+l-1]} \otimes \ldots z_1^{[k+l-1]} \otimes I_n)^H Q_{l,i}$$
$$(z_m^{[k+l-2]} \otimes \cdots \otimes z_i^{[k+l-2]} \otimes z_{i-1}^{[k+l-1]} \otimes \ldots z_1^{[k+l-1]} \otimes I_n) < 0_n \ , \quad (10)$$

where $R(z) \stackrel{\text{def}}{=} A(z)^H P(z) + P(z)A(z)$. Indeed, this is a direct consequence of (1). Thus, $R(z) < 0_n$ if $|z_1| = \cdots = |z_m| = 1$, so the matrix $A(z)$ is Hurwitz for all $z \in (\partial\mathbb{D})^m$. This observation may be extended to the whole $\overline{\mathbb{D}}^m$, basically by subanalyticity, as in [5]. This proves that solvability of (9) implies robust stability of (7). In other terms, **3.** implies **1.**.

The difficult part of the proof is the implication **2.** $\Rightarrow$ **3.**, whose proof is adapted from [5].

First, it may be shown (but this is a non-trivial result) that the coefficient matrix P_l of $P(z)$, which is symmetric as $P \in \mathcal{S}_H^n[z,\bar{z}]$, may be supposed *positive definite, without loss of generality.*

The next stage consists in removing one by one the free variables $z_1, \dots, z_m$ and introducing concomitantly the multipliers $Q_{l,1}, \dots, Q_{l,m}$. Basically, this operation is achieved by applying recursively D-scaling with respect to z_1, $\dots$, z_m. This procedure is lossless for one complex parameter (this is just the discrete-time counterpart of Kalman-Yakubovich-Popov lemma, see [31, 26], and [25] for recent statement and proof). The argument is the same than for the results in [5], up to some technical details. At each step, a new matrix is introduced, which however depends upon the remaining free-variables. Applying again [4], one may assume that this dependence is indeed polynomial, and the coefficient matrix of the latter turns out to be one of the $Q_{l,i}$. Some special care has to be taken, as the degree of the polynomial previously introduced is unknown: indeed, increases of the "degree" l may occur when passing from **2.** to **3.**, this is explained in detail in [5].

6 Interpretation of Theorem 5 and comments

6.1 Link with parameter-dependent systems

Based on a solution $(P_l, Q_{1,l}, \dots, Q_{m,l})$ of LMI (9), construct $P(z) \in \mathcal{S}_H^n[z,\bar{z}]$ with coefficient matrix P_l. Then, along the trajectories of (7),

$$\frac{d[x(t)^T P(z) x(t)]}{dt} = x(t)^T R(z) x(t) \ ,$$

where $R(z) = A(z)^H P(z) + P(z)A(z)$ is defined by its coefficient matrix $R_l(P_l)$. LMI (9) is thus related to the search for a parameter-dependent Lyapunov function for (7) in $\mathcal{S}_H^n[z,\bar{z}]$, the matrices $Q_{1,l}, \dots, Q_{l,m}$ playing the role of Lagrange multipliers.

Remark however that $A(z)$, being polynomial, is analytic, and Hurwitzness of $A(z)$ for $z \in (\partial\mathbb{D})^m$ implies the same property in $\overline{\mathbb{D}}^m$, see [5]. In order to obtain a simpler LMI in the stability criterion, the smallest set has been considered, and the corresponding parameter-dependent Lyapunov function based on a solution of (9) is guaranteed to decrease only for $|z_1| = \dots = |z_m| = 1$. Positivity of the matrices $Q_{l,i}$ would ensure the property for the whole set $\overline{\mathbb{D}}^m$, see inequality (10) above. We conjecture that the previous positivity condition may be added without supplementary conservatism. This assertion is true at least for $m = 1$, $k = 2$ [2].

6.2 Link with delay systems

For any $l_1, \dots, l_m \in \mathbb{N}$, define

$$x^{[l_1,\dots,l_m]} \stackrel{\text{def}}{=} (\nabla_m^{[l_m]} \otimes \dots \otimes \nabla_1^{[l_1]} \otimes I_n)x \ ,$$

which takes values in $\mathbb{R}^{l_1\dots l_m n}$. Consider the following functional (compare with (6)), parametrized by $(m+1)$ hermitian matrices P_l, $Q_{l,i}$ having the same size than in Theorem 5:

$$\begin{aligned}
V_l(x)(t) &\stackrel{\text{def}}{=} x^{[l,\dots,l]}(t)^T P_l x^{[l,\dots,l]}(t) \\
&\quad + \int_{t-h_1}^t x^{[l+k-2,\dots,l+k-2]}(\tau)^T Q_{l,1} x^{[l+k-2,\dots,l+k-2]}(\tau)\ d\tau \\
&+ \int_{t-h_2}^t x^{[l+k-1,l+k-2,\dots,l+k-2]}(\tau)^T Q_{l,2} x^{[l+k-1,l+k-2,\dots,l+k-2]}(\tau)\ d\tau + \dots \\
&+ \int_{t-h_m}^t x^{[l+k-1,\dots,l+k-1,l+k-2]}(\tau)^T Q_{l,m} x^{[l+k-1,\dots,l+k-1,l+k-2]}(\tau)\ d\tau \ . \qquad (11)
\end{aligned}$$

The value of $V_l(x)$ at time t depends only upon the values of x on $[t-(k+l-2)\sum h_i; t]$. It turns out that

$$\frac{d[V_l(x)(t)]}{dt} = x^{[l+k-1,\dots,l+k-1]}(t)^T S_l(P_l, Q_{l,1}, \dots, Q_{l,m}) x^{[l+k-1,\dots,l+k-1]}(t) \ ,$$

where S_l is defined in (9a). In fact, one has e.g.

$$\begin{aligned}
&\frac{d}{dt}\left[\int_{t-h_1}^t x^{[l+k-2,\dots,l+k-2]}(\tau)^T Q_{l,1} x^{[l+k-2,\dots,l+k-2]}(\tau)\ d\tau\right] \\
&= x^{[l+k-2,\dots,l+k-2]}(t)^T Q_{l,1} x^{[l+k-2,\dots,l+k-2]}(t) \\
&\quad - x^{[l+k-1,l+k-2,\dots,l+k-2]}(t)^T Q_{l,1} x^{[l+k-1,l+k-2,\dots,l+k-2]}(t) \\
&= x^{[l+k-1,\dots,l+k-1]}(t)^T \left[\left(\hat{J}_{k+l-2}^{m\otimes} \otimes I_n\right)^T Q_{l,i} \left(\hat{J}_{k+l-2}^{m\otimes} \otimes I_n\right) \right. \\
&\quad \left. - \left(\hat{J}_{k+l-2}^{(m-1)\otimes} \otimes \check{J}_{k+l-2} \otimes I_n\right)^T Q_{l,i} \left(\hat{J}_{k+l-2}^{(m-1)\otimes} \otimes \check{J}_{k+l-2} \otimes I_n\right)\right] x^{[l+k-1,\dots,l+k-1]}(t) \ ,
\end{aligned}$$

due to (1). Therefore, the appearance of LMI (9) is also related to the search for a Lyapunov-Krasovskii functional of the form (11). However, no positivity assumption has to be made in (9), see also the remark made previously in Section 6.1 for parameter-dependent systems. In the eventuality where the positivity assumption may be added without loss of enerality (e.g. $k=2, m=1$), strong delay-independent stability is equivalent to the existence of a certain Lyapunov-Krasovskii functional in the class (11) ensuring stability of delay system (3) for any nonnegative value of $h_1, \dots, h_m$.

7 Open problems on μ computation

To conclude, we present two open questions, linked to application and extension of the ideas and methods previously presented.

• Is it possible to extend the method, in order to associate to any problem (8) or (8'), a family of LMIs similar to (9), constituting sufficient conditions with increasing precision?
• How to use practically the above results for numerical estimation of structured singular values? In particular, how to choose in (9) the degree $l-1$ of the underlying parameter-dependent Lyapunov function? In the case $m=1$, $k=2$, an answer has been given in [33], which seems extendable to non affine systems ($k>2$), but the general case is still unsolved.

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