Extension of Popov absolute stability criterion to nonautonomous systems with delays.

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Abstract

This paper extends in a simple way the classical absolute stability Popov criterion to multivariable systems with delays and with time-varying memoryless nonlinearities subject to sector conditions. The proposed sufficient conditions are expressed in the frequency domain, a form well-suited for robustness issues, and lead to simple graphical interpretations for scalar systems. Apart from the usual conditions, the results assume basically a generalized sector condition on the derivative of the nonlinearities with respect to time. Results for local and global stability are given, the latter concerning in particular the linear time-varying ones. For rational transfers, the frequency conditions are equivalent to some easy-to-check Linear Matrix Inequalities: this leads to a tractable method of numerical resolution by rational approximation of the transfer. As an illustration, a numerical example is provided.

1 Introduction

This paper deals with an extension of Popov absolute stability criterion to nonstationary delay systems. We consider the multivariable control system given in Figure 1, where H is a strictly proper transfer function matrix of size $p \times p$, $p \in \mathbb{N} \setminus \{0\}$, and $\psi : \mathbb{R}^+ \times \mathbb{R}^p \to \mathbb{R}^p$ a time-dependent nonlinearity.

Here, the transfer function matrix H is supposed to be represented by the following delay differential system

$$\dot{x} = \sum_{l=0}^{L} A_l x(t-h_l) + Bu, \quad u = -\psi(t,y), \quad y = \sum_{l=0}^{L} C_l x(t-h_l), \quad x|_{[-h,0]} = \phi \ , \qquad (1)$$

where

$$n \in \mathbb{N} \setminus \{0\},\ L \in \mathbb{N},\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^p,\ A_l \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times p},\ C_l \in \mathbb{R}^{p\times n},$$
$$0 \le h_l,\ h \stackrel{\text{def}}{=} \max\{h_0, \dots, h_L\} \ ,$$

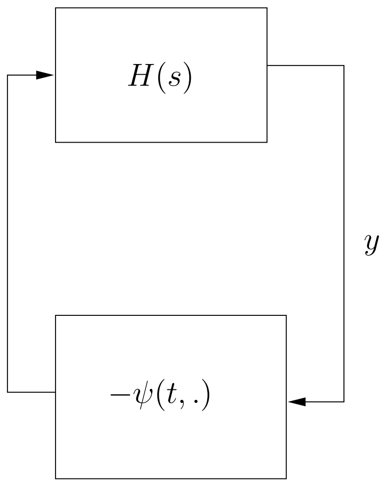

Figure 1: The system under study

in such a way that

$$H(s) = (\sum_{l=0}^{L} C_l e^{-h_l s})(sI - \sum_{l=0}^{L} A_l e^{-h_l s})^{-1} B \ .$$

Assume that ψ is not perfectly known, but rather that it belongs to a certain class of nonlinearities, defined by the following properties: ψ is decentralized [15] (that is: $\forall i \in \{1, \dots p\}$, $\psi_i(t,y) = \psi_i(t, y_i)$) and verifies moreover the following sector condition

$$\forall (t,y) \in \mathbb{R}^+ \times \mathbb{R}^p, \quad \psi(t,y)^T (\psi(t,y) - Ky) \leq 0 \ , \tag{2}$$

for a certain nonnegative diagonal matrix K. The asymptotic stability of all the systems obtained by coupling (1) with a nonlinearity fulfilling (2) is called *absolute stability* of the class of systems (1).

In order for the class of systems (1) to be absolutely stable, asymptotic stability should hold for *all linear time-invariant* choice of ψ compatible with (2), that is for the maps:

$$\psi(t,y) = \text{ diag}\{k_i\} y, \quad 0 \leq k_i \leq K_i \ .$$

In 1947, Aizerman [1] was the first to formulate (for finite sectors and rational systems) the question of the sufficiency of this condition. As is well-known, the answer is negative, even if one restricts oneself to time-invariant nonlinearities, as proved in 1958 by Pliss [24] (see also a counterexample in [2, p. 86-88]). Popov gave in 1959 an elegant sufficient condition for absolute stability of the class of systems (1) with time-invariant nonlinearities [25], known as *Popov criterion* (see e.g. [38, 15, 17]): absolute stability holds, provided that the poles of H have negative real part and that there exists a diagonal matrix η such that

$$I + (I + \eta s) K H(s) \quad \text{is strictly positive real (SPR)} \ . \tag{3}$$

The latter result was initially given for rational systems, and it was extended shortly after by Popov *et al.* [26] to delay systems (and seemingly independently by Li [18]). For rational systems, the result may be proved using a Lyapunov function with a Lur'e term of the form

$$V(t,x) \stackrel{\text{def}}{=} x^T Px + 2\sum_{i=1}^{p} \eta_i K_i \int_0^{y_i} \psi_i(t,z)dz \ , \tag{4}$$

and applying Kalman-Yakubovich-Popov Lemma.

Since the late fifties, a lot of contributions have been published, giving generalizations of Popov criterion to various classes of time-invariant nonlinearities (see [17, 4] for an overview and references, [19] for chinese contributions). As an example, the result has been extended by Yakubovich to some hysteresis nonlinearities [39]. Results have been obtained with stronger conditions on the nonlinearity, especially incremental sector conditions, see references in [22, 17].

Also, attempts have been made to adapt Popov criterion to time-varying rational systems (see [20] for a review of the period 1968-1977, surveying an important number of contributions, especially from Eastern Europe). Recall first that for time-varying nonlinearities fulfilling sector condition (2), circle criterion provides a sufficient condition of absolute stability, namely that the poles of H have negative real part, and that

$$I + KH(s) \quad \text{is SPR} \ . \tag{5}$$

Also, for such a class of time-varying nonlinearities, Pyatnitskii has shown [27] that absolute stability is *equivalent* to the asymptotic stability of all the time-varying linear systems of this class, that is for the maps

$$\psi(t,y) = \ \text{diag}\{k_i(t)\}y, \quad 0 \le k_i(t) \le K_i \ .$$

This result gives rise to a class of sufficient conditions of absolute stability, see e.g. [21]. See also [16, 3] for some frequential conditions of absolute stability without restrictions on the rate of variation of the nonlinearity.

On the other hand, it is possible to obtain conditions of absolute stability for smaller classes of time-varying nonlinearities, especially by making restrictions on $\frac{\partial \psi}{\partial t}$. This limitation may be acceptable e.g. when studying the stability of limit cycles. Narendra *et al.* devoted Chapter VI of their monograph [22] to this question. They obtained conditions for global stability involving two parts: the Popov condition plus a differential (in the case of a so-called separate nonlinearity $\psi(t,y) = k(t)f(y)$) or integrodifferential inequality, linking $\frac{\partial \psi}{\partial t}$ and ψ. These conditions may be not so easy to handle, see examples in [22, Chapter VIII]. In [37], Walker provided conditions for global stability. The assumptions given therein imply that (3) holds and, e.g. when $\eta \ge 0$,

$$\forall (t,y) \in \mathbb{R}^+ \times \mathbb{R}^p, \quad \psi(t,y)^T(\psi(t,y) - Ky) \le -\eta K \frac{d}{dt}\left[\int_0^y \psi(t,z)\ dz\right] \ ,$$

instead of (2). As the a priori knowledge on the right-hand side does not usually permit to consider it as nonnegative, this expresses a restriction of the rate of variation of the nonlinearity, but K does not define anymore the width of the sector, and it is not clear how to check systematically the conditions, especially for multivariable systems. Rekasius *et al.* [33], Hul'chuk *et al.* [14] and Bertoni *et al.* [5] published contributions providing frequency criteria for absolute stability of nonstationary systems. In [33, 14], the authors require for the terms of the form $\int_0^{y_i} \frac{\partial \psi_i}{\partial t}(t,z)\ dz$ appearing in the derivative of the Lyapunov function (4) to be bounded by a quadratic form in y_i and $\psi_i(t,y_i)$. This condition, see (7) below, may be interpreted as a *generalized sector condition*; it is fulfilled e.g.

when a sector condition on $\frac{\partial\psi}{\partial t}(t,y)$ holds. In [5], only this simpler condition is used, and graphical interpretation is given. See also [11] for related approach.

To the best of our knowledge, very few papers have been published on the topic of absolute stability of time-varying delay systems. Some non-frequential criteria are cited in [19, Paragraphs 6.1 and 6.5]. A paper by Răsvan [31] provides an extension of Popov criterion to systems with separate nonlinearities. Achieving an analysis close to the one presented here, it has to assume the monotonicity (wrt time) of the time varying gains, a quite limiting assumption. Also, Walker [36] provides results generalizing the approach of [37] to delay systems.

In the present paper, one proposes an extension of Popov criterion to nonautonomous systems with delays, generalizing the work done in [33, 14, 5]. More precisely, one provides simple conditions ensuring uniform asymptotic stability of the origin. These conditions are expressed in terms of a frequency condition in Theorems 1 and 3. The results provide uniform local asymptotic stability, a generalization of the property of *absolute stability with finite domain* [15], or global stability. The latter results may be applied in particular to linear time-varying operators ψ. The proposed criteria just add some supplementary terms to Popov criterion, depending on $\frac{\partial\psi}{\partial t}(t,y)$ as in [33, 14]. They permit to link circle and Popov criteria: when *no* variation wrt time of ψ is permitted, the nonlinearity is time-invariant, and Popov criterion applies; when *any* variation is permitted, circle criterion applies. The results herein fulfill the gap: they give sufficient conditions of stability adapted to the magnitude of $\frac{\partial\psi}{\partial t}$.

The results, being expressed in the frequency domain, are well fitted to robustness issues, especially in presence of unstructured perturbations. Some graphical interpretations are provided for scalar systems, partly as in [5]. Concerning checkability of the conditions, an attractive feature is the possibility to approximate the transfer function matrix H by rational transfers: classical application of Kalman-Yakubovich-Popov (KYP) Lemma shows that for these systems, the proposed frequency conditions are equivalent to some Linear Matrix Inequalities, a now standard class of problems for which sound numerical methods have been developed [8].

An example of application of the results given here comes from the control of chaos [10]: in order to stabilize an unstable periodic orbit of a strange attractor, Pyragas [28, 29] proposed to use a feedback control law built on the difference between the actual value and the delayed value of the output, with delay equal to the period of the cycle. The analysis of the corresponding closed loop system requires stability results for nonstationnary nonlinear delay systems.

Finally, we want to emphasize the fact that the results could be applied to more general systems (e.g. systems with distributed delays, integral systems), as it is indeed the case for Popov criterion, see [12, §4.6.] and [9]. Using the same assumptions on the variations of the nonlinearity, some delay-independent criteria are given in [6, 7].

The paper is organized as follows. The criteria are stated in Section 2. Computation issues are studied in Section 3. An example is presented in Section 4. Finally, proof of the main result (Theorem 1) is exposed in Section 5. In all the sequel, we assume that there exist global solutions of (1), that is, by definition: for all $\phi \in \mathcal{C}([-h,0];\mathbb{R}^n)$, there exists a continuous function x defined on $[-h,+\infty)$, absolutely continuous [35] on $[0,+\infty)$, such that $x|_{[-h,0]} = \phi$ and (1) is fulfilled almost everywhere on $[0,+\infty)$. The stability results given below concern the asymptotic behavior of these global solutions.

Notations In all the paper, $\|.\|$ denotes the euclidian norm or the induced matrix norm, I_r denotes the $r \times r$ identity matrix (simply I when the context is clear), the asterisk * denotes complex conjugation. For any real diagonal matrix $\eta = \operatorname{diag}\{\eta_i\}$, one denotes

$$|\eta| \stackrel{\text{def}}{=} \operatorname{diag}\{|\eta_i|\}, \quad \operatorname{sgn} \eta \stackrel{\text{def}}{=} \operatorname{diag}\{\operatorname{sgn} \eta_i\} ,$$

where one may take indifferently $\operatorname{sgn} 0 = -1$ or $+1$. When η is a function of $t \in \mathbb{R}^+$, by definition:

$$\sup_{t\geq 0} \eta(t) \stackrel{\text{def}}{=} \operatorname{diag}\{\sup_{t\geq 0} \eta_i(t)\} ,$$

and similarly for the infimum, the essential supremum, ... and so on. For $z \in \mathbb{R}$, one denotes

$$|z|_+ \stackrel{\text{def}}{=} \sup\{z, 0\} = \frac{|z| + z}{2}, \quad |z|_- \stackrel{\text{def}}{=} \sup\{-z, 0\} = \frac{|z| - z}{2} .$$

The same notation is used for diagonal matrices:

$$|\eta|_\pm = \sup\{\pm\eta, 0\} = \operatorname{diag}\{\sup\{\pm\eta_i, 0\}\} = \operatorname{diag}\{|\eta_i|_\pm\} .$$

2 Main results

Theorem 1 (A frequency criterion). *Assume that there exists a convex open neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^p$ for which the following assumptions hold.*

(H0) *The function ψ is measurable and, for any $y \in \mathcal{O}$, $t \mapsto \psi(t, y)$ is locally Lipschitz (and hence t-a.e. differentiable), with a Lipschitz constant locally integrable wrt $y \in \mathcal{O}$.*

(H1) *The nonlinearity ψ is decentralized and there exists a diagonal matrix $K = \operatorname{diag}\{K_i\} \geq 0$ such that,*

$$\forall(t, y) \in \mathbb{R}^+ \times \mathcal{O}, \quad \psi(t, y)^T(\psi(t, y) - Ky) \leq 0 . \tag{6}$$

(H2) *The roots of the equation $\det(sI - \sum_{l=0}^{L} A_l e^{-h_l s}) = 0$ have negative real part.*

Assume that there exist diagonal matrices $\eta = \operatorname{diag}\{\eta_i\}$, $D_j = \operatorname{diag}\{D_{j,i}\}$, $j \in \{1, 2, 3\}$, such that the following Hypothesis is fulfilled

(H3) *There exists $\gamma : \mathbb{R}^+ \to \mathbb{R}$ with $\lim_{z\to 0} \gamma(z) = 0$, such that,*

for almost any $t \in \mathbb{R}^+$, $\forall y \in \mathcal{O}$, $\forall i \in \{1, \dots, p\}$,

$$\eta_i \left(\int_0^{y_i} \frac{\partial \psi_i}{\partial t}(t, z)\ dz - D_{1,i}y_i^2 - D_{2,i}y_i\psi_i(t, y_i) - D_{3,i}\psi_i(t, y_i)^2 \right) \leq \|y\|^2\ \gamma(\|y\|) . \tag{7}$$

If the transfer function matrix

$$I - \eta K D_3 + (I + \eta(sI + D_2))KH(s) - H^*(s) \sup \left\{ \eta D_1; \frac{|\eta| - \eta}{2} K(D_2 + KD_3) \right\} KH(s) \quad \textit{is SPR} , \tag{8}$$

then, the origin of system (1) is uniformly locally asymptotically stable.

Moreover, if $\gamma \equiv 0$ and $\mathcal{O} = \mathbb{R}^p$, then the origin of system (1) is uniformly globally asymptotically stable.

A proof of Theorem 1 is presented in Section 5. It essentially follows the approach of [26, 12], with adequate improvements.

Circle criterion is found as a particular case when $\frac{\partial\psi}{\partial t}$ is unconstrained (in this case, one has to take $\eta = 0$, and (8) reduces to (5)), and Popov criterion when $\frac{\partial\psi}{\partial t} = 0$ (taking $D_j = 0$, (8) reduces to (3)).

Remark that if Hypothesis (H3) holds for a certain η, then it holds (with the same D_j and γ) for any η' such that $\eta\eta' > 0$: only the sign of the η_i's intervenes. In practice, one first determines the sign of the η_i's which lead to an estimate like (7). Under these sign constraints on the η_i's, one then verifies (8). The matrices D_j, $j = 1, 2, 3$, may depend upon $\operatorname{sgn} \eta$.

An important case where Hypothesis (H3) is fulfilled leads to the following result.

Corollary 2. *Assume that there exists a convex open neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^p$ for which* (H0), (H1), (H2) *hold. Assume that there exist diagonal matrices η and $\Delta = \operatorname{diag}\{\Delta_i\}$, $\Delta_i : \mathbb{R}^+ \to \mathbb{R}$, such that the following Hypothesis is fulfilled*

(H3') *There exists $\gamma : \mathbb{R}^+ \to \mathbb{R}$ with $\lim_{z\to 0} \gamma(z) = 0$, such that,*

$$\text{for almost any } t \in \mathbb{R}^+,\ \forall y \in \mathcal{O}, \quad y^T \eta \left(\frac{\partial \psi}{\partial t}(t, y) - \Delta(t) y \right) \leq \|y\|^2 \ \gamma(\|y\|) \ . \tag{9}$$

If the transfer function matrix

$$I + (I + \eta s) K H(s) - \frac{1}{2} H^*(s) K^{\frac{1}{2}} \ \operatorname*{ess\,sup}_{t\geq 0} \{\eta \Delta(t)\} K^{\frac{1}{2}} H(s) \quad \text{is SPR} , \tag{10}$$

then, the origin of system (1) *is uniformly locally asymptotically stable.*

Moreover, if $\gamma \equiv 0$ and $\mathcal{O} = \mathbb{R}^p$, then the origin of system (1) *is uniformly globally asymptotically stable.*

One checks easily that (H3') implies (H3) with $D_2 = D_3 = 0$ and

$$D_{1,i} = \frac{1}{2} \operatorname*{ess\,sup}_{t\geq 0} \{\Delta_i(t)\} \text{ if } \eta_i \geq 0,\ D_{1,i} = \frac{1}{2} \operatorname*{ess\,inf}_{t\geq 0} \{\Delta_i(t)\} \text{ if } \eta_i \leq 0 \ .$$

Indeed, when (9) holds,

$$\operatorname*{ess\,inf}_{t\geq 0} \Delta(t) \leq 0 \leq \operatorname*{ess\,sup}_{t\geq 0} \Delta(t) \ ,$$

otherwise sector condition (6) would be violated, so

$$\eta D_1 = \operatorname*{ess\,sup}_{t\geq 0} \{\eta \Delta(t)\} = |\eta|_+ \ \operatorname*{ess\,sup}_{t\geq 0} \{\Delta(t)\} + |\eta|_- \ \operatorname*{ess\,inf}_{t\geq 0} \{\Delta(t)\} \geq 0 \ .$$

The additional, quadratic, term in (10) is then nonpositive, indicating clearly that the criterion is more restrictive than Popov conditions (compare with (3)) for the systems with nonautonomous nonlinearities.

Condition (7) is a generalization of (9), whose idea is borrowed from [33, 14]. Condition (H3') is in general a "local" sector condition. It is fulfilled in two important cases. When $\gamma \equiv 0$, (9) writes:

$$\text{for almost any } t \in \mathbb{R}^+,\ \forall y \in \mathcal{O}, \quad y^T \eta \left(\frac{\partial \psi}{\partial t}(t,y) - \Delta(t) y \right) \leq 0 \ ,$$

that is:

$$\forall i \in \{1,\dots,p\},\ \text{for almost any } t \in \mathbb{R}^+,\ \forall y \in \mathcal{O}, \begin{cases} \frac{1}{y_i}\frac{\partial \psi_i}{\partial t}(t,y_i) \leq \Delta_i(t) & t-\text{a.e.} \quad \text{if } \eta_i \geq 0 \\ \frac{1}{y_i}\frac{\partial \psi_i}{\partial t}(t,y_i) \geq \Delta_i(t) & t-\text{a.e.} \quad \text{if } \eta_i \leq 0 \end{cases} .$$

This hence defines a sector condition on the map $y \mapsto \frac{\partial \psi}{\partial t}(t,y)$.

Also, when the inequality in (9) is replaced by an equality, then,

$$\forall i \in \{1,\dots,p\},\ \text{for almost any } t \in \mathbb{R}^+,\ \forall y \in \mathcal{O}, \left| \frac{\partial \psi_i}{\partial t}(t,y_i) - \Delta_i(t) y_i \right| = |y_i| \gamma(|y_i|) \ ,$$

which means that $\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0)$ exists almost everywhere and is equal to $\Delta_i(t)$.

Remark that ψ does not have to be continuous wrt y, except in 0, and the same is true for $\frac{\partial \psi}{\partial t}$. In the conditions of application of Theorem 1, there exist functions $k_i(t,y_i)$ such that $\psi_i(t,y_i) = k_i(t,y_i) y_i$, $i = 1,\dots,p$, and it may be fruitful to express the results in terms of the k_i. As an example, (H1) requires that $0 \leq k_i(t,y_i) \leq K_i$. Also, (9) expresses that

$$\forall i \in \{1,\dots,p\},\ \text{for almost any } t \in \mathbb{R}^+,\ \forall y \in \mathcal{O}, \eta_i \left(\frac{\partial k_i}{\partial t}(t,y_i) - \Delta_i(t) \right) \leq \gamma(\|y_i\|) \ ,$$

and $\frac{\partial^2 \psi_i}{\partial y_i \partial t}(t,0) = \frac{\partial k_i}{\partial t}(t,0)$ when the 2nd derivative exists.

When the solutions of (1) are continuous wrt the initial conditions, one may consider the essential suprema [35] of $\eta\Delta(t)$ on $[t_0,+\infty)$ for any $t_0 \geq 0$, instead of $[0,+\infty)$. Indeed, due to the strict inequality involved, one may even use the upper limit [35] of these expressions when $t_0 \to +\infty$.

For a scalar system, $p = 1$, and condition (8) is equivalent to

$$\exists \eta \in \mathbb{R}\ ,\ \forall \omega \in \mathbb{R}, \quad \frac{1}{K} + \operatorname{Re} H(j\omega) - \eta(D_3 - D_2 \operatorname{Re} H(j\omega) + \omega \operatorname{Im} H(j\omega)) \\ - \sup \left\{ \eta D_1 ; \frac{|\eta| - \eta}{2} K (D_2 + K D_3) \right\} |H(j\omega)|^2 \geq 0 \ . \quad (11)$$

This may be interpreted graphically, as in [5]:

> *If, apart from the regularity, sector and stability conditions* (H0), (H1), (H2), (H3), *there exists a line of slope* $1/\eta$ *passing through the point* $(-\frac{1}{K}, 0)$ *and lying to the left of the locus* $(\operatorname{Re} H(j\omega), D_3 - D_2 \operatorname{Re} H(j\omega) + \omega \operatorname{Im} H(j\omega) + \ \operatorname{sgn}\ \eta \cdot \sup\{\ \operatorname{sgn}\ \eta \cdot D_1 ; \frac{1 - \operatorname{sgn}\ \eta}{2} K(D_2 + K D_3)\}\ |H(j\omega)|^2)$ *without intersecting it, then the uniform local stability property holds. If* $\gamma \equiv 0$ *and* $\mathcal{O} = \mathbb{R}^p$*, then the uniform global stability property holds.*

An interesting problem is, given K, to determinate the largest incertitude on $\frac{\partial\psi}{\partial t}$ allowed by Theorem 1, for example under condition (H3'). In this case, the previous graphical criterion can hardly be used, as the ordinate changes with the D_j. To overcome this drawback, condition (8) should rather be seen as a geometrical condition in the 3-dimensional space $(\operatorname{Re} H(j\omega), \operatorname{Im} H(j\omega), \omega \operatorname{Im} H(j\omega))$ obtained as the product of Nyquist and Popov planes, a condition not easy to interpret. We present in the sequel a weaker but simpler condition, located in the Popov plane.

Denoting the $\mathcal{H}_\infty$-norm of H by $\|H(s)\|_\infty \stackrel{\text{def}}{=} \sup\{\|H(s)\| \ : \ \operatorname{Re} s > 0\}$ (when H is stable and proper, this is equal to $\sup\{\|H(j\omega)\| \ : \ \omega \in \mathbb{R}\}$), one deduces easily the following result.

Theorem 3 (A weaker frequency criterion). *Assume that there exists a convex open neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^p$ for which Hypotheses (H0), (H1), (H2) hold. Assume that there exist diagonal matrices η and $D_j, j \in \{1,2,3\}$, such that (H3) is fulfilled. If the transfer function matrix*

$$I - \eta K D_3 + (I + \eta(sI + D_2))KH(s) - K \sup\left\{\eta D_1; \frac{|\eta| - \eta}{2} K(D_2 + KD_3); 0\right\} \ \|H(s)\|_\infty^2 \quad \textit{is SPR} \ , \tag{12}$$

then the conclusions of Theorem 1 hold.

As an example let us examine the case of a scalar system fulfilling (H3'), the general case (H3) is similar. Formula (12) is equivalent to (13a) (resp. (13b)) for $\eta \geq 0$ (resp. $\eta \leq 0$), where

$$\exists \eta \geq 0, \ \forall \omega \in \mathbb{R}, \ \frac{1}{K} + \operatorname{Re} H(j\omega) - \eta\left(\omega \operatorname{Im} H(j\omega) + \frac{1}{2}\|H(s)\|_\infty^2 \ \operatorname*{ess\,sup}_{t\geq 0}\{\Delta(t)\}\right) \geq 0 \ , \tag{13a}$$

$$\exists \eta \leq 0, \ \forall \omega \in \mathbb{R}, \ \frac{1}{K} + \operatorname{Re} H(j\omega) - \eta\left(\omega \operatorname{Im} H(j\omega) + \frac{1}{2}\|H(s)\|_\infty^2 \ \operatorname*{ess\,inf}_{t\geq 0}\{\Delta(t)\}\right) \geq 0 \ , \tag{13b}$$

and this has a clear interpretation:

> *If, apart from the regularity, sector and stability conditions* (H0), (H1), (H2), (H3') *with $\eta \geq 0$ (resp. $\eta \leq 0$), a line of slope $1/\eta$ passing through the point $(-\frac{1}{K}, 0)$ lies above (resp. below) the Popov locus and may be translated vertically towards the locus by a distance*
>
> $$\frac{1}{2}\|H(s)\|_\infty^2 \ \operatorname*{ess\,sup}_{t\geq 0}\{\Delta(t)\} \quad \textit{(resp.} \ -\frac{1}{2}\|H(s)\|_\infty^2 \ \operatorname*{ess\,inf}_{t\geq 0}\{\Delta(t)\}\textit{)}$$
>
> *without intersecting it, then the uniform local stability property holds. If $\gamma \equiv 0$ and $\mathcal{O} = \mathbb{R}^p$, then the uniform global stability property holds.*

This is illustrated in Figure 2: in the left (resp. right) Popov diagram, (13a) (resp. (13b)) holds if

$$\operatorname*{ess\,sup}_{t\geq 0} \Delta(t) < \frac{2d}{\|H(s)\|_\infty^2} \quad (\text{resp.} \ \operatorname*{ess\,inf}_{t\geq 0} \Delta(t) > -\frac{2d}{\|H(s)\|_\infty^2}) \ . \tag{14}$$

As an example, (14) holds if

$$\forall y \in \mathcal{O}, \quad \frac{1}{y}\frac{\partial\psi}{\partial t}(t,y) < \frac{2d}{\|H(s)\|_\infty^2} \ t-\text{a.e.} \quad (\text{resp.} \ \frac{1}{y}\frac{\partial\psi}{\partial t}(t,y) > -\frac{2d}{\|H(s)\|_\infty^2} \ t-\text{a.e.}) \ ,$$

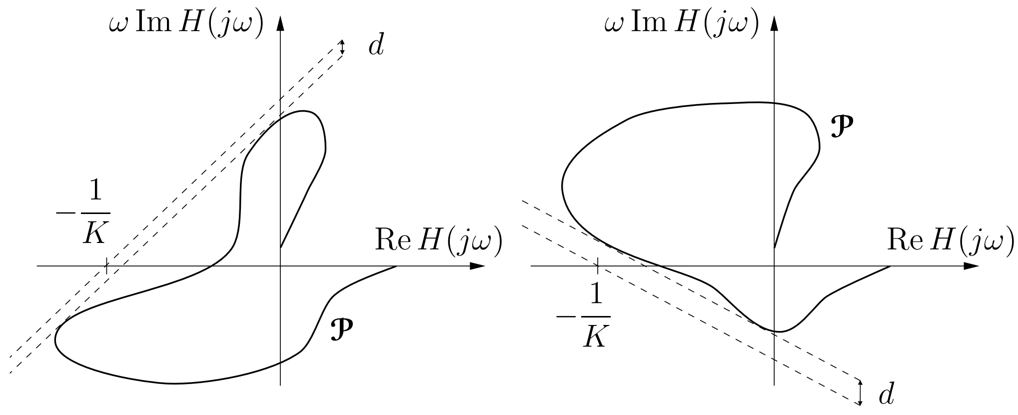

Figure 2: Graphical stability criterion in the Popov plane

or if

$$\frac{\partial^2\psi}{\partial y\partial t}(t,0) < \frac{2d}{\|H(s)\|_\infty^2}\ t-\text{a.e.} \quad (\text{resp. } \frac{\partial^2\psi}{\partial y\partial t}(t,0) > -\frac{2d}{\|H(s)\|_\infty^2}\ t-\text{a.e.})\ .$$

In the configurations shown in Figure 2, the quantity d involved is indeed *the least* $z>0$ *such that one of the points* $(-1/K, \pm z)$ *belongs to the convex hull of the Popov locus* $\boldsymbol{\mathcal{P}}$. To show this, remark that, e.g. for the right diagram,

$$d = -\inf_{\eta\in\mathbb{R}} \sup_{\omega\in\mathbb{R}} \left(\eta\left(\operatorname{Re} H(j\omega) + \frac{1}{K}\right) - \omega \operatorname{Im} H(j\omega)\right)\ .$$

Now, $\sup_{\omega\in\mathbb{R}} \eta(\operatorname{Re} H(j\omega) + 1/K) - \omega \operatorname{Im} H(j\omega)$ may be seen as the value of the support function of the set $\boldsymbol{\mathcal{P}}+1/K$ applied to the vector $(\eta,-1)$ [34]. One may hence replace the set $\boldsymbol{\mathcal{P}}$ by its convex hull conv $\boldsymbol{\mathcal{P}}$, and then reverse the order of inf and sup. One gets:

$$d = -\sup\left\{\inf_{\eta\in\mathbb{R}} \eta(z_1+1/K) - z_2\ :\ (z_1,z_2)\in \text{ conv } \boldsymbol{\mathcal{P}}\right\} = \inf\left\{z_2\ :\ (-1/K, z_2)\in \text{ conv } \boldsymbol{\mathcal{P}}\right\}\ .$$

3 Computation issues

It turns out that for *rational* systems, frequency condition (8) may be checked easily:

Proposition 4 (LMI condition for rational systems). *Let* $H(s) = C(sI-A)^{-1}B$ *be a rational strictly proper Hurwitz transfer function matrix, let* $\eta \stackrel{\text{def}}{=} \operatorname{diag}\{\eta_i\} \geq 0$. *Condition* (8) *holds if and only if the following LMI is feasible:*

$$P>0,\ R \stackrel{\text{def}}{=} \begin{pmatrix} A^TP+PA+2C^T\eta|D_1|_+KC & -PB+C^TK+A^TC^TK\eta+C^TKD_2\eta \\ -B^TP+KC+\eta KCA+\eta D_2KC & -2I-\eta KCB-B^TC^TK\eta+2\eta D_3K \end{pmatrix} < 0\ . \tag{15}$$

Proposition 4 is a direct consequence of Kalman-Yakubovich-Popov (KYP) Lemma [17, 30].

In order to apply Theorem 1, it then suffices to achieve approximation by rational transfers, see e.g. [23, 13] for the techniques of approximation. The following result states this properly.

Proposition 5 (Transfer approximation). *Let H, H_m be strictly proper Hurwitz transfer function matrices. Let η be a diagonal matrix.*

Suppose that there exists $\varepsilon \in (0,1)$ such that

$$\begin{aligned}(1-\varepsilon)I - \eta K D_3 + (I + \eta(sI + D_2))KH_m(s)\\ - H_m^*(s)K \sup\left\{\eta D_1; \frac{|\eta|-\eta}{2}K(D_2 + KD_3)\right\} H_m(s) \quad \textit{is SPR,}\end{aligned} \tag{16a}$$

$$\begin{aligned}\varepsilon I + \left(I + \eta(sI + D_2) - 2H^*(s)\sup\left\{\eta D_1; \frac{|\eta|-\eta}{2}K(D_2 + KD_3)\right\}\right) K(H - H_m)(s)\\ + (H - H_m)^*(s)\sup\left\{\eta D_1; \frac{|\eta|-\eta}{2}K(D_2 + KD_3)\right\} K(H - H_m)(s) \quad \textit{is SPR.}\end{aligned} \tag{16b}$$

Then condition (8) *holds.*

Conversely, suppose that there exists $\varepsilon > 0$ such that

$$\begin{aligned}(1+\varepsilon)I - \eta K D_3 + (I + \eta(sI + D_2))KH_m(s)\\ - H_m^*(s)K \sup\left\{\eta D_1; \frac{|\eta|-\eta}{2}K(D_2 + KD_3)\right\} H_m(s) \quad \textit{is } \text{NOT } \textit{SPR,}\end{aligned} \tag{17a}$$

$$\begin{aligned}\varepsilon I - \left(I + \eta(sI + D_2) - 2H^*(s)\sup\left\{\eta D_1; \frac{|\eta|-\eta}{2}K(D_2 + KD_3)\right\}\right) K(H - H_m)(s)\\ - (H - H_m)^*(s)\sup\left\{\eta D_1; \frac{|\eta|-\eta}{2}K(D_2 + KD_3)\right\} K(H - H_m)(s) \quad \textit{is SPR.}\end{aligned} \tag{17b}$$

Then condition (8) *does not hold.*

The proof is left to the reader. Assumption (16a) (resp. assumption (17a)) is slightly stronger than the assumption needed to apply Theorem 1 to H_m (resp. slightly weaker than the negation of this assumption). Assumptions (16b) or (17b) are fulfilled e.g. when $\|H - H_m\|_\infty$ is small enough.

4 An example: computation of stability margin for a 4th order system with delay

One will consider the following system:

$$16\frac{d^4y}{dt^4} + 32\frac{d^3y}{dt^3} + 24\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + y = -\psi(t, y(t-1)) \ , \tag{18}$$

corresponding to the previously studied framework with the transfer function

$$H(s) = \frac{e^{-s}}{(1+2s)^4} \ ,$$

which may easily be realized as in (1).

One supposes that $\psi(t,0) \equiv 0$ and that there exists a constant K such that

$$y \neq 0 \Rightarrow 0 \leq \frac{\psi(t,y)}{y} \leq K \ .$$

One verifies either graphically on Figure 3, or by computations, that asymptotic stability of system (18) cannot be guaranteed if

$$K \geq \frac{1}{0.3613} \simeq 2.768 \ .$$

This corresponds to the smallest positive value of K for which the equation $1 + KH(s) = 0$ has some purely imaginary roots.

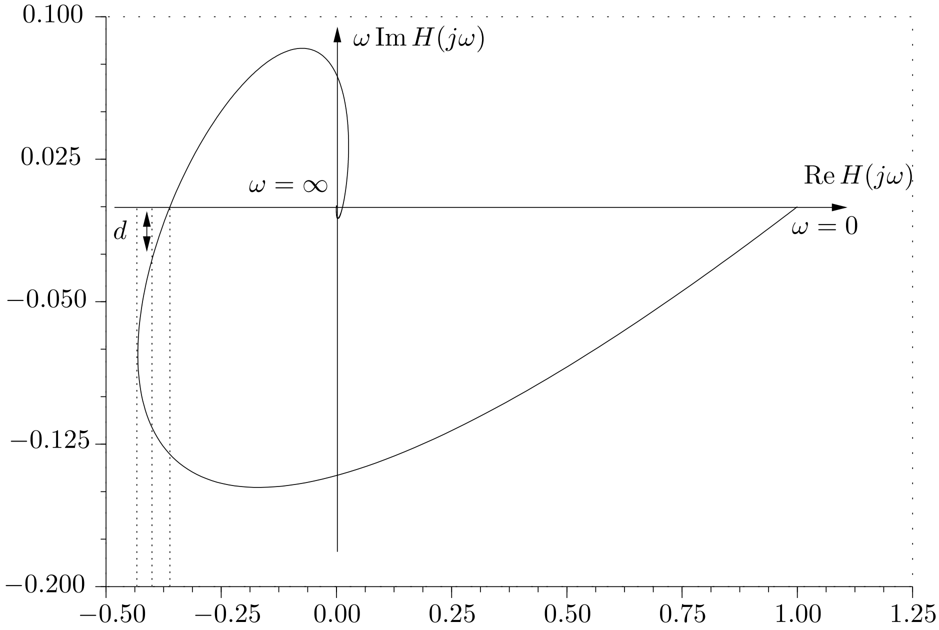

Figure 3: Popov locus of system (18)

Application of circle criterion provides stability if

$$K < \frac{1}{0.4310} \simeq 2.320 \ .$$

On the other hand, Popov criterion guarantees stability if $\psi(t,y) = \psi(y)$ and

$$K < 2.768 \ .$$

In the remaining of the section, one studies the stability of (18) for time-varying nonlinearities fulfilling the sector condition with

$$K = 1/0.4 = 2.5 \in [2.320, 2.768] \ .$$

More precisely, let us assume that condition (H3') holds; one is looking for restrictions on $\Delta(t)$ permitting to ensure stability. The results previously exposed permit to compute measures of the stability robustness wrt time-variations of ψ. In the sequel, we provide lower estimates of the largest number δ such that system (18) is absolutely stable whenever

$$\operatorname*{ess\,sup}_{t\geq 0} \Delta(t) \leq \delta \ .$$

All computations to be presented have been achieved using the Scilab package LMITOOL[1].

Using the ideas of Section 3, one approximates the delay following [23], and considers the sequence of approximants:

$$H_m(s) = \left(\frac{1 - \frac{s}{2m}}{1 + \frac{s}{2m}} \right)^m \frac{1}{(1+2s)^4} \ .$$

For each value of m, one computes the largest value of $\operatorname*{ess\,sup}_{t\geq 0} \Delta(t)$ for which the analog of LMI (15) with the transfer H_m and $D_1 = \frac{1}{2} \operatorname*{ess\,sup}_{t\geq 0} \{\Delta(t)\}$, $D_2 = D_3 = 0$ is feasible. This process provides a lower estimate of δ, denoted δ_1. The results are presented in Table 1. The exact value of δ_1 (when $m \to +\infty$) may be computed numerically for scalar systems, as (10) is fulfilled for a positive value of η if and only if

Order of approximation	Successive estimates of δ_1
$m = 1$	0.3444
$m = 2$	0.3365
$m = 3$	0.3350
$m = 4$	0.3345
$m = 5$	0.3342
$m = +\infty$	0.3338

Table 1: Computation of the robustness measure δ_1

$$\operatorname*{ess\,sup}_{t\geq 0} \Delta(t) < \sup_{\eta > 0} \inf_{\omega \in \mathbb{R}} \frac{2}{|H(j\omega)|^2} \left(\frac{1}{\eta} \left(\frac{1}{K} + \operatorname{Re} H(j\omega) \right) - \omega \operatorname{Im} H(j\omega) \right) \ .$$

The optimal value of η is $\eta_{\text{opt}} \simeq 1.473$. In conclusion, condition (10) is fulfilled if

$$\operatorname*{ess\,sup}_{t\geq 0} \Delta(t) \leq \delta_1 \stackrel{\text{def}}{=} 0.3338 \ .$$

The corresponding graphical interpretation may be read on Figure 4.

One now shows how to compute a bound on δ with the help of Theorem 3. First the quantity d is evaluated, either graphically on Figure 3, or using a solver of algebraic equations. One gets:

$$d \simeq 0.03325 \ .$$

[1]Scilab is a free software developed by INRIA, which is distributed with all its source code. For the distribution and details, see Scilab's homepage on the web at the address `http://www-rocq.inria.fr/scilab/`

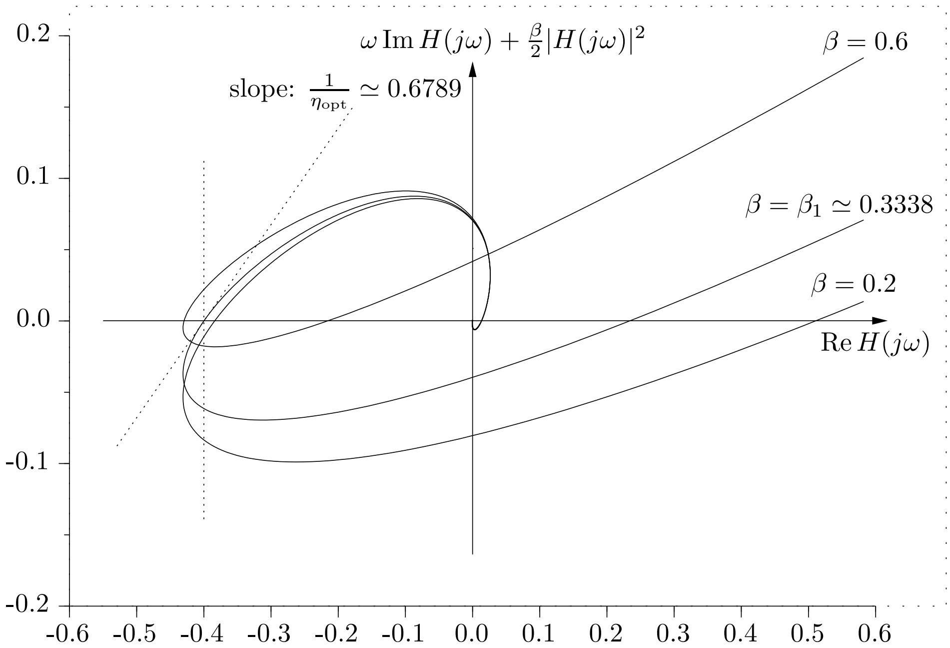

Figure 4: Graphical interpretation of Theorem 1 for system (18)

One then computes $\|H\|_\infty$. This step may also be achieved by use of rational approximations. One obtains here:

$$\|H\|_\infty = 1 \ .$$

Condition (12) is hence fulfilled if

$$\operatorname*{ess\,sup}_{t \geq 0} \Delta(t) \leq \delta_2 \stackrel{\mathrm{def}}{=} \frac{2d}{\|H\|_\infty^2} \simeq 0.06650 \ .$$

One verifies that the ordering

$$\delta_1 > \delta_2$$

is consistent with the increasing conservativeness of the criteria.

Let us give a sample of the results that may be obtained. Let $\mathcal{O}$ be a convex open neighborhood of 0 in $\mathbb{R}$, such that

- There exists $L \in L^1_{loc}(\mathcal{O})$ such that, for all $t, t' \in \mathbb{R}^+$, for all $y \in \mathcal{O}$, $|\psi(t,y) - \psi(t',y)| \leq L(y)|t - t'|$ (condition (H0)).
- For all $t \in \mathbb{R}^+$, for all $y \in \mathcal{O} \setminus \{0\}$, $0 \leq \dfrac{\psi(t,y)}{y} \leq 2.5$, and $\psi(t,0) \equiv 0$ (condition (H1)).

Assume that there exist global solutions to system (18). The origin of system (18) is uniformly locally stable if

$$\limsup_{y\to 0} \frac{1}{y}\frac{\partial \psi}{\partial t}(t,y) \le \delta_1 \;\; t-\text{a.e.}\ ,$$

for example if $\frac{\partial^2\psi}{\partial y \partial t}(t,0)$ exists t-a.e. and verifies:

$$\frac{\partial^2\psi}{\partial y\partial t}(t,0) \le \delta_1 \;\; t-\text{a.e.}$$

The origin of system (18) is uniformly globally stable if $\mathcal{O}=\mathbb{R}$ and

$$\forall y\in\mathbb{R}\setminus\{0\}\ ,\quad \frac{1}{y}\frac{\partial\psi}{\partial t}(t,y)\le \delta_1 \;\; t-\text{a.e.}$$

5 Proof of Theorem 1

One assumes, without loss of generality, that γ is nonnegative, nondecreasing.

• Let us first suppose that $\eta \ge 0$. The demonstration begins as in [26, 12].

Let $T>0$. Define ψ_T, x_T, y_T as follows

$$\psi_T(t) = \begin{cases} \psi(t,y(t)) & \text{if } 0\le t\le T \\ 0 & \text{if } -h\le t<0 \text{ or } t>T \end{cases}\ , \tag{19}$$

$$\dot{x}_T = \sum_{l=0}^{L} A_l x_T(t-h_l) - B\psi_T,\quad y_T = \sum_{l=0}^{L} C_l x_T(t-h_l),\quad x_T|_{[-h,0]} = 0\ . \tag{20}$$

By linearity, we have

$$\dot{x}-\dot{x}_T = \sum_{l=0}^{L} A_l (x(t-h_l) - x_T(t-h_l)) \quad \text{for } t\in[0,T],\quad (x-x_T)|_{[-h,0]} = \phi\ , \tag{21}$$

We shall denote in the sequel by c_j, $j=1,2,\dots$, various positive constants, *independent of ϕ and T*.

One may deduce from Hypothesis (H2), that there exist $c_1>0$, $\alpha>0$, independent of ϕ and T, such that

$$\forall t\in[-h,T],\quad \|x(t)-x_T(t)\| \le c_1 e^{-\alpha t}\|\phi\|_{\mathcal{C}([-h,0])}\ , \tag{22a}$$

$$\forall t\ge T,\quad \|x_T(t)\| \le c_1 e^{-\alpha(t-T)}\|x_T(T+\cdot)\|_{\mathcal{C}([-h,0])}\ , \tag{22b}$$

$$\forall t\in[0,T],\quad \|y(t)\| \le c_1 e^{-\alpha t}\|\phi\|_{\mathcal{C}([-h,0])} + \|y_T(t)\|\ . \tag{22c}$$

The last inequality is obtained by use of the triangle inequality.

From (H1) and the fact that $\eta, K\ge 0$, one gets that, if $y(t)\in\mathcal{O}$, $t\in[0,T]$, then

$$\forall i\in\{1,\dots,p\},\quad \int_0^T (K_i y_i(t) - \psi_i(t,y_i(t)))\psi_i(t,y_i(t))\ dt + \eta_i K_i \int_0^{y_i(T)} \psi_i(T,z)\ dz \ge 0\ . \tag{23}$$

Now, the map $T \mapsto \int_0^{y_i(T)} \psi_i(T,z)\ dz$ is absolutely continuous, due to (H0) and (H1), and the assumed absolute continuity of the solution wrt time. Indeed, $y(t), y(t') \in \mathcal{O}$ implies that, $\forall i \in \{1,\dots,p\}$,

$$\left|\int_0^{y_i(t)} \psi_i(t,z)\ dz - \int_0^{y_i(t')} \psi_i(t',z)\ dz\right|$$
$$\leq |t-t'| \int_0^{y_i(t)} \lambda_i(z)\ dz + K_i \max\{|y_i(t)|, |y_i(t')|\}\ |y_i(t) - y_i(t')|\ ,$$

where λ_i is the Lipschitz constant of ψ, defined by Hypothesis (H1). One may hence write

$$\begin{aligned}
\int_0^{y_i(T)} \psi_i(T,z)\ dz &= \int_0^T \frac{d}{dt}\left[\int_0^{y_i(t)} \psi_i(t,z)\ dz\right]\ dt + \int_0^{y_i(0)} \psi_i(0,z)\ dz \\
&= \int_0^T \left(\dot{y}_i(t)\psi_{T,i}(t) + \int_0^{y_i(t)} \frac{\partial \psi_i}{\partial t}(t,z)\ dz\right)\ dt + \int_0^{y_i(0)} \psi_i(0,z)\ dz\ . \quad (24)
\end{aligned}$$

Now, if $y(t) \in \mathcal{O}$ for any $t \in [0,T]$, one has, using (H3),

$$\begin{aligned}
&\eta_i \int_0^T \int_0^{y_i(t)} \frac{\partial \psi_i}{\partial t}(t,z)\ dz\ dt \\
&\qquad \leq \int_0^T \left(\eta_i\left(D_{1,i}y_i^2(t) + D_{2,i}y_i(t)\psi_{T,i}(t) + D_{3,i}\psi_{T,i}(t)^2\right) + \gamma(|y_i(t)|)y_i^2(t)\right)\ dt\ , \quad (25)
\end{aligned}$$

because γ is nonnegative and nondecreasing, and $\mathcal{O}$ is convex. One hence deduces from (23), (24), (25)

$$\begin{aligned}
&\int_0^T \left(K_i(y_i(t) - y_{T,i}(t))\psi_{T,i}(t) + K_i\gamma(|y_i(t)|)y_i^2(t)\right)\ dt + \eta_i K_i \int_0^{y_i(0)} \psi_i(0,z)\ dz \\
&\geq\ -\int_0^T \Bigg(\left(K_i y_{T,i}(t) - \psi_{T,i}(t)\right)\psi_{T,i}(t) \\
&\quad +\eta_i K_i\left(\dot{y}_i(t)\psi_{T,i}(t) + D_{1,i}y_i^2(t) + D_{2,i}y_i(t)\psi_{T,i}(t) + D_{3,i}\psi_{T,i}(t)^2\right)\Bigg)\ dt \\
&=\ -\int_0^T \Bigg(\left(K_i y_{T,i}(t) - \psi_{T,i}(t)\right)\psi_{T,i}(t) \\
&\quad +\eta_i K_i\left(\dot{y}_{T,i}(t)\psi_{T,i}(t) + D_{1,i}y_{T,i}^2(t) + D_{2,i}y_{T,i}(t)\psi_{T,i}(t) + D_{3,i}\psi_{T,i}(t)^2\right)\Bigg)\ dt \\
&\quad +\eta_i K_i \int_0^T \left((\dot{y}_{T,i}(t) - \dot{y}_i(t))\psi_{T,i}(t) + D_{1,i}(y_{T,i}^2(t) - y_i^2(t)) + D_{2,i}(y_{T,i}(t) - y_i(t))\psi_{T,i}(t)\right)\ dt \\
&\geq\ -\int_0^{+\infty} \Bigg(\left(K_i y_{T,i}(t) - \psi_{T,i}(t)\right)\psi_{T,i}(t) + |\eta_i D_{1,i}|_+\ K_i y_{T,i}^2(t) \\
&\quad +\eta_i K_i\left(\dot{y}_{T,i}(t)\psi_{T,i}(t) + D_{2,i}y_{T,i}(t)\psi_{T,i}(t) + D_{3,i}\psi_{T,i}(t)^2\right)\Bigg)\ dt \\
&\quad +\eta_i K_i \int_0^T \left((\dot{y}_{T,i}(t) - \dot{y}_i(t))\psi_{T,i}(t) + D_{1,i}(y_{T,i}^2(t) - y_i^2(t)) + D_{2,i}(y_{T,i}(t) - y_i(t))\psi_{T,i}(t)\right)\ dt\ ,
\end{aligned}$$

as $K \geq 0$, $\psi_T(t) = 0$ for $t \geq T$, and because $y_{T,i}^2$ is summable over $\mathbb{R}^+$, due to (22b). In the previous integral over $\mathbb{R}^+$, the terms with $\psi_T(t)$ vanishe on $[T, +\infty)$, and one has bounded D_1 by $|D_1|_+$. Denoting $\tilde{y}_T, \tilde{\psi}_T$ the Fourier transform of y_T, ψ_T, the first integral term of the previous expression is proved to be equal to

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\tilde{\psi}_{T,i}^*(\omega)\left(K_i\tilde{y}_{T,i}(\omega) - \tilde{\psi}_{T,i}(\omega) + \eta_i K_i\left(j\omega\tilde{y}_{T,i}(\omega) + D_{2,i}\tilde{y}_{T,i}(\omega) + D_{3,i}\tilde{\psi}_{T,i}(\omega)\right)\right)\right.$$
$$\left. +|\eta_i\ D_{1,i}|_+ K_i|\tilde{y}_{T,i}(\omega))|^2\right)\ d\omega\ .$$

Independently, Hypothesis (H3) implies that there exist positive numbers ε and ρ_y, both independent from T and ϕ, such that the open ball in $\mathbb{R}^p$ with centre 0 and radius ρ_y is contained in $\mathcal{O}$ and such that, if

$$\forall t \in [0,T], \quad \|y(t)\| < \rho_y\ , \tag{26}$$

then the following holds:

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\tilde{\psi}_T^*(\omega)\left(K\tilde{y}_T(\omega) - \tilde{\psi}_T(\omega) + \eta K\left(j\omega\tilde{y}_T(\omega) + D_2\tilde{y}_T(\omega) + D_3\tilde{\psi}_T(\omega)\right)\right) + \tilde{y}_T(\omega)|\eta D_1|_+ K\ \tilde{y}_T(\omega))\right)\ d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{\psi}_T^*(\omega)\Big(-I + \eta K D_3 - (I + \eta(j\omega + D_2))KH(j\omega) + H^*(j\omega)|\eta D_1|_+ KH(j\omega)\Big)\tilde{\psi}_T(\omega)\ d\omega\ , \\
&\qquad \text{(using the identity } \tilde{y}_T(\omega) = -H(j\omega)\tilde{\psi}_T(\omega)) \\
&\leq -\frac{\varepsilon}{2\pi}\int_{-\infty}^{+\infty}\|\tilde{\psi}_T(\omega)\|^2\ d\omega = -\varepsilon\int_0^{+\infty}\|\psi_T(t)\|^2\ dt\ .
\end{aligned}$$

Putting together the two inequalities yields (via summation over i)

$$\begin{aligned}
&\frac{\varepsilon}{\|H\|_\infty^2}\int_0^{+\infty}\|y_T(t)\|^2\ dt \leq \varepsilon\int_0^{+\infty}\|\psi_T(t)\|^2\ dt \\
&\leq \int_0^T\Big(\psi_T^*(t)K(y(t) - y_T(t)) + \gamma(\|y(t)\|)\ y^*(t)Ky(t)\Big)\ dt + \sum_{i=0}^p \eta_i K_i\int_0^{y_i(0)}\psi_i(0,z)\ dz \\
&\quad -\int_0^T\Big(\psi_T^*(t)\eta K(\dot{y}_T(t) - \dot{y}(t)) + (y_T^*(t) - y^*(t))\eta K D_1(y_T(t) - y(t)) + \psi_T^*(t)\eta K D_2(y_T(t) - y(t))\Big)\ dt\ .
\end{aligned}$$

Due to (6), (21), (22), one may bound from above the previous expression by

$$c_2\left[\|\phi\|_{\mathcal{C}([-h,0])}\left(\|\phi\|_{\mathcal{C}([-h,0])} + \sup_{t\in[0,T]}\|y_T(t)\|\right) + \sup_{t\in[0,T]}\gamma(\|y(t)\|)\int_0^T\|y_T(t)\|^2\right]\ .$$

To summarize, as long as (26) holds, then $y(t) \in \mathcal{O}$ for any $t \in [0,T]$, and

$$\begin{aligned}
&\frac{\varepsilon}{\|H\|_\infty^2}\int_0^{+\infty}\|y_T(t)\|^2\ dt \\
&\quad \leq c_2\left(\|\phi\|_{\mathcal{C}([-h,0])}\left(\|\phi\|_{\mathcal{C}([-h,0])} + \sup_{t\in[0,T]}\|y_T(t)\|\right) + \sup_{t\in[0,T]}\gamma(\|y(t)\|)\int_0^T\|y_T(t)\|^2\ dt\right)\ .
\end{aligned} \tag{27}$$

Suppose additionally that

$$\forall t \in [0,T],\ c_2\gamma(\|y(t)\|) \leq \frac{\varepsilon}{2\|H\|_\infty^2}\ . \tag{28}$$

Then

$$\int_0^T \|y_T(t)\|^2\ dt,\ \int_0^T \|\dot{y}_T(t)\|^2\ dt \leq c_3\|\phi\|_{\mathcal{C}([-h,0])}\left(\|\phi\|_{\mathcal{C}([-h,0])} + \sup_{t\in[0,T]}\|y_T(t)\|\right)\ . \tag{29}$$

The estimate on y_T in (29) is obtained directly from (27), and the estimate on $\dot{y}_T$ is then deduced, with the help of sector estimate (6) and the fact that H is strictly proper, as

$$\begin{aligned}\int_0^T \|\dot{y}_T(t)\|^2\ dt &\leq \|s(\sum_{l=0}^L C_l e^{-h_l s})(sI - \sum_{l=0}^L A_l e^{-h_l s})^{-1}B\|_\infty^2 \int_0^T \|\psi_T(t)\|^2\ dt \\ &\leq \|s(\sum_{l=0}^L C_l e^{-h_l s})(sI - \sum_{l=0}^L A_l e^{-h_l s})^{-1}B\|_\infty^2 \|K\|^2 \int_0^T \|y(t)\|^2\ dt\ .\end{aligned}$$

One infers, using Cauchy-Schwarz inequality and $y_T(0) = 0$, that (26), (28) imply, for any $t \in [0,T]$:

$$\|y_T(t)\|^2 \leq c_3\|\phi\|_{\mathcal{C}([-h,0])}\left(\|\phi\|_{\mathcal{C}([-h,0])} + \sup_{t\in[0,T]}\|y_T(t)\|\right)\ .$$

Solving this polynomial inequality leads to

$$T \text{ fulfills (26), (28)} \ \Rightarrow\ \sup_{t\in[0,T]}\|y_T(t)\|,\ \sup_{t\in[0,T]}\|y(t)\| \leq c_4\|\phi\|_{\mathcal{C}([-h,0])}\ .$$

Now, let $\rho_x > 0$ be such that (recall that $\gamma(z) \to 0$ when $z \to 0$)

$$\rho_x \leq \frac{\rho_y}{2c_4} \quad \text{and} \quad \forall z \in \mathbb{R}^+,\ z \leq c_4\ \rho_x\ \Rightarrow\ c_2\gamma(z) \leq \frac{\varepsilon}{4\|H\|_\infty^2}\ .$$

For $\phi \in \mathcal{C}([-h,0];\mathbb{R}^n)$ with $\|\phi\|_{\mathcal{C}([-h,0])} < \rho_x$, the previous computations show that, as long as (26), (28) are verified, one has

$$\sup_{t\in[0,T]}\|y(t)\| \leq c_4\|\phi\|_{\mathcal{C}([-h,0])} \leq c_4\ \rho_x\ ,$$

so

$$c_2 \sup_{t\in[0,T]}\gamma(\|y(t)\|) \leq \frac{\varepsilon}{4\|H\|_\infty^2} < \frac{\varepsilon}{2\|H\|_\infty^2}\ ,$$

and

$$\sup_{t\in[0,T]}\|y(t)\| \leq c_4\ \rho_x \leq \frac{\rho_y}{2} < \rho_y\ .$$

Hence, (26), (28) are verified *for any* $T > 0$, so

$$\|\phi\|_{\mathcal{C}([-h,0])} < \rho_x\ \Rightarrow\ \forall T \geq 0\ ,\quad \sup_{t\in[0,T]}\|y_T(t)\| \leq c_4\rho_x\ .$$

This in turn implies by (29), that, for any $T > 0$,

$$\int_0^T \|y_T(t)\|^2\ dt,\ \int_0^T \|\dot{y}_T(t)\|^2\ dt \leq c_5 \|\phi\|^2_{\mathcal{C}([-h,0])}\ .$$

From (22c), one deduces that similar inequalities hold for y and $\dot{y}$. One concludes that $y(t) \to 0$ when $t \to +\infty$, which expresses the uniform local asymptotic stability of the origin.

When $\gamma \equiv 0$ and $\mathcal{O} = \mathbb{R}^p$, one may take $\rho_y = \rho_x = +\infty$.

• We now remove the assumption that $\eta \geq 0$. This part of the proof is similar to the analog enlargement of Popov criterion to nonpositive slopes η, see [2].

By hypothesis, (8) is fulfilled. Consider [32]

$$\varphi(t,y) \stackrel{\text{def}}{=} Ky - \psi(t,y)\ ,$$

Let us choose the input φ_i instead of ψ_i when $\eta_i \leq 0$, and write that (1) is equivalent to

$$\dot{x} = \sum_{l=0}^{L} \hat{A}_l x(t-h_l) + \hat{B}u, \quad u = -\hat{\psi}(t,y(t)) \stackrel{\text{def}}{=} -[\hat{J}\varphi(t,y(t)) + (I-\hat{J})\psi(t,y(t))], \quad y = \sum_{l=0}^{L} C_l x(t-h_l)\ ,$$

where $\hat{I}, \hat{J}, \hat{A}_l, 0 \leq l \leq L, \hat{B}$ are defined by:

$$\hat{I} \stackrel{\text{def}}{=} \text{ sgn } \eta,\ \hat{J} \stackrel{\text{def}}{=} (I - \text{ sgn } \eta)/2,\ \hat{A}_l = A_l - B\hat{J}KC_l,\ \hat{B} = B\hat{I}\ .$$

The following properties will be used repeatedly:

$$\hat{J}^2 = \hat{J} = -\hat{I}\hat{J},\ 2\hat{J} + \hat{I} = \hat{I}^2 = I_p\ .$$

If ψ verifies Hypothesis (H1), then the same holds for $\hat{\psi}$, as

$$\hat{\psi}(t,y)^T(\hat{\psi}(t,y) - Ky) = \psi(t,y)^T(\psi(t,y) - Ky)\ .$$

Replacing ψ by φ, one obtains

$$\begin{aligned}
&\eta_i \left(\int_0^{y_i} \frac{\partial \psi_i}{\partial t}(t,z)\ dz - D_{1,i}y_i^2 - D_{2,i}y_i\psi_i(t,y_i) - D_{3,i}\psi_i(t,y_i)^2 \right) \\
&= -\eta_i \left(\int_0^{y_i} \frac{\partial \varphi_i}{\partial t}(t,z)\ dz + (D_{1,i} + D_{2,i}K_i + D_{3,i}K_i^2)y_i^2 - (D_{2,i} + 2D_{3,i}K_i)y_i\varphi_i(t,y_i)\ + D_{3,i}\varphi_i(t,y_i)^2 \right)\ ,
\end{aligned}$$

In other words, a property similar to (H3) is valid for $\hat{\psi}$, with the values:

$$\hat{\eta} \stackrel{\text{def}}{=} \hat{I}\eta = |\eta|,\ \hat{D}_1 \stackrel{\text{def}}{=} \hat{I}D_1 - \hat{J}K(D_2 + KD_3),\ \hat{D}_2 \stackrel{\text{def}}{=} D_2 + 2\hat{J}KD_3,\ \hat{D}_3 \stackrel{\text{def}}{=} \hat{I}D_3\ . \tag{30}$$

Now, in order to achieve the proof, it is sufficient to show that if formula (8) holds, then the modified system verifies an analogous condition with the new value $\hat{\eta} \geq 0$, that is

$$I - \hat{\eta}K\hat{D}_3 + (I + \hat{\eta}(sI + \hat{D}_2))K\hat{H}(s) - \hat{H}^*(s)|\hat{\eta}\hat{D}_1|_+ K\hat{H}(s) \quad \text{is SPR}\ , \tag{31}$$

where

$$\hat{H}(s) \stackrel{\text{def}}{=} (\sum_{l=0}^{L} C_l e^{-h_l s})(sI - \sum_{l=0}^{L} \hat{A}_l e^{-h_l s})^{-1} \hat{B} \ .$$

Indeed, the first part of the proof may then be applied to the transformed system (as $\hat{\eta} \geq 0$), and this concludes the proof of Theorem 1. It hence remains to prove (31). This is done with the help of the following Lemma.

Lemma 6. *The following identity holds:*

$$\hat{H}(s) = (I + H(s)\hat{J}K)^{-1} H(s)\hat{I} = H(s)(I + \hat{J}KH(s))^{-1}\hat{I} \ .$$

Proof. Lemma 6 is a consequence of the feedback structure involved when changing the input of the linear plant. Indeed, the new input $\hat{\psi}$ being defined as above, one has:

$$\begin{aligned} y = -H\psi = -H\left(\hat{J}\psi + (I - \hat{J})\psi\right) &= -H\left(\hat{J}(Ky - \varphi) + (I - \hat{J})\psi\right) \\ = -H\left(\hat{J}Ky - \hat{J}\hat{\psi} + (I - \hat{J})\hat{\psi}\right) &= -H\left(\hat{I}\hat{\psi} + \hat{J}Ky\right) \ , \end{aligned}$$

and finally:

$$(I + H\hat{J}K)y = -H\hat{I}\hat{\psi} \ ,$$

which gives the 1st equality. Deduction of the 2nd equality is straightforward. ♠

Applying Lemma 6, one gets

$$\begin{aligned} 2(I - \hat{\eta}K\hat{D}_3) + (I + \hat{\eta}(sI + \hat{D}_2))K\hat{H}(s) + \hat{H}^*(s)K(I + \hat{\eta}(s^*I + \hat{D}_2)) - 2\hat{H}^*(s)|\hat{\eta}\hat{D}_1|_+ K\hat{H}(s) \\ = \hat{I}(I + H^*(s)K\hat{J})^{-1} G(s)(I + \hat{J}KH(s))^{-1}\hat{I} \ , \end{aligned}$$

where

$$\begin{aligned} G(s) &= 2(I + H^*(s)K\hat{J})(I - \hat{\eta}K\hat{D}_3)(I + \hat{J}KH(s)) + (I + H^*(s)K\hat{J})\hat{I}(I + \hat{\eta}(sI + \hat{D}_2))KH(s) \\ &\quad + H^*(s)K(I + \hat{\eta}(s^*I + \hat{D}_2))\hat{I}(I + \hat{J}KH(s)) - 2H^*(s)K|\hat{\eta}\hat{D}_1|_+ H(s) \\ &= 2(I - \hat{\eta}K\hat{D}_3) + \left(2\hat{J} + \hat{I} + \hat{\eta}(-2\hat{J}K\hat{D}_3 + \hat{I}(sI + \hat{D}_2))\right) KH(s) \\ &\quad + H^*(s)K\left(I + \hat{\eta}(-2\hat{J}K\hat{D}_3 + \hat{I}(s^*I + \hat{D}_2))\right) \\ &\quad + H^*(s)\left(2K^2\hat{J}(\hat{I} + \hat{J}) + \hat{\eta}K^2\hat{J}\hat{I}(s + s^*) + 2K(-|\hat{\eta}\hat{D}_1|_+ + \hat{\eta}K\hat{I}\hat{J}\hat{D}_2 - \hat{\eta}\hat{J}^2K^2\hat{D}_3)\right) H(s) \ , \end{aligned}$$

using the fact that the (diagonal) matrices $K, \hat{\eta}, \hat{I}, \hat{J}, \hat{D}_j$ commute. From (30), one gets

$$\begin{aligned} G(s) = 2(I - \eta KD_3) + (I + \eta(sI + D_2))KH(s) + H^*(s)K(I + \eta(s^*I + D_2)) \\ - 2H^*(s)\left(|\eta(D_1 + \hat{J}K(D_2 + KD_3))|_+ - \eta\hat{J}K(D_2 + KD_3)\right) KH(s) \ , \end{aligned}$$

which proves that (8) and (31) are equivalent. This achieves the proof of Theorem 1.

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