Stability of leaderless discrete-time multi-agent systems

David Angeli* Pierre-Alexandre Bliman†

May 18, 2006 – 2nd revised version

Abstract

The paper presents a result which relates connectedness of the interaction graphs in multi-agent discrete-time systems with the capability for global convergence to a common equilibrium of the system. In particular we extend previously known results by Bertsekas and Tsitsiklis and by Moreau, by including the possibility of arbitrary bounded time-delays in the communication channels and relaxing the convexity of the allowed regions for the state transition map of each agent.

Keywords. Multi-agent systems, rendezvous problem, asymptotic stability, discrete-time systems, time delays, directed graphs.

1 Introduction

Recent years have witnessed a growing interest in the study of the dynamical behaviour of the so called multi-agent systems. Roughly speaking these can be thought of as complex dynamical systems composed by a high number of simpler units, the agents. Each of them updates its state according to some rule, whose Input-Output dynamics are typically much simpler and much better understood, and on the basis of the available information coming from the other agents. All of them, though not necessarily identical, share in fact some common feature of interest (say for instance a given output variable) and are coupled together by communication channels. The focus of the current research is precisely on how the global behaviour of the system, (for instance questions concerning the global stability or the overall synchronization) is influenced by the topology of the coupling on one hand (this is an analysis problem in many respects) or the dual question of how to induce a certain desired property of the ensemble based on some form of local coupling for the agents. Problems of this nature arise in many different fields such as in the theory of coupled oscillators [G, MM, JMB, SPL], in neural networks [H], in economics or in the manoeuvring of groups of vehicles [AOSY, LF, YPP]. For instance in [LMA1, LMA2] the so called *rendezvous* problem is considered, namely how to design a local updating rule, based on nearest neighbor interactions, which would ensure convergence of all of the agents to an unspecified common meeting point. Emergence of a global behaviour is therefore a consequence of the local updating rule, without the need for a leader nor of centralized directions being broadcasted.

In problems linked to formation flight or group manoeuvring, agents are usually coupled when located at small enough distance from one another [AOSY, JLM, YPP, LMA1, LMA2, S, CMB]. In such situations, the arrangement of the information network depends upon the state configuration of the system, and the displacements are designed in such a way to constrain the agents already in contact to always stay close enough in the future. This usually gives rise to symmetric (undirected) communication graphs, monotonically evolving to complete graph. Hereby we take a slightly different approach and consider the configuration of the information network as an independent input. We believe that "opening the loop" may lead to fruitful analysis results, allowing e.g. to relax the distance constraints in the examples of coordinated motion cited

*Dipartimento di Sistemi e Informatica, University of Florence, Via di S. Marta 3, 50139 Firenze, Italy. Email: angeli@dsi.unifi.it

†INRIA, Rocquencourt BP105, 78153 Le Chesnay cedex, France. Email: pierre-alexandre.bliman@inria.fr

above. The emphasis is on how the topology of interconnections between agents (possibly time-varying) affects the convergence of all agents to a common equilibrium. This analysis will be carried out for a class of discrete-time multi-agent systems in the presence of limited transmission speed of the information between the agents. In particular, we propose an extension of the contributions by L. Moreau [M1, M2], mainly in two directions:

- The new setting allows the presence of arbitrary bounded communication delays.
- A central assumption in the results [M1, M2], namely that the future evolution of the studied system is constrained to occur in the convex hull of the agents states, is removed.

The first aspect comes as a very natural question both from a practical and a theoretical point of view. Communication delays are in fact ubiquitous in the "real" world and it is well-known their potential destabilizing effect in conjunction with feedback loops, here induced by the graph topology of the communication channels which need not be of a hierarchical type. It is therefore remarkable to see how, at least in the specific set-up we are considering, this destabilizing effect does not take place and the same global behaviour of the multi-agent system in terms of convergence to a common equilibrium follows also in the extended set-up.

The second extension deals with convexity issues; one of the technical tools used in order to enforce a common behaviour in systems whose state takes value in Euclidean space, is to have local evolutions point always inside the convex hull of all variables. This makes life easier in a certain respect, and that is indeed an assumption quite universally made, but it is unnatural in more general contexts, for instance when oscillators networks are considered (these are typically modeled as systems evolving on a torus) or systems evolving in partially obstructed Euclidean spaces (for instance on a plane minus a circle). Relaxing convexity is meant as a first step in the quest for stability conditions which can work in more general spaces. It provides an original solution to rendezvous problems for populations of agents evolving in a nonconvex set — a task untractable within the frameworks previously developed in the literature. Such a situation is presented in Example 6 below.

The work we present here also extends results on *partially asynchronous iterative methods* published by D.P. Bertsekas and J.N. Tsitsiklis [BT, Chapter 7, especially Section 7.3], see also [BHOT]. In their framework, bounded delays are allowed, whereas some convexity assumption is imposed: the convex hull is indeed considered *componentwise*.

To date, we mention the existence of other interesting aspects in the recent research, such as randomly varying interconnection network [GHM, HM, MM] or quantization of the information [JSZ]: the latter are beyond the scope of the present paper.

Before going on further, we present the main elements of the construction developed below. The multi-agent system under study will be described by a *time-dependent graph* $\mathcal{A}(t)$, describing the transfer of information between the agents at time t, and a *set of rules* according to which each agent updates its state at time $t+1$. The definition of the latter is done by the introduction of two types of objects, which we present now (complete definitions are to be found in Section 2 below).

- A *set-valued map* σ is defined, which associates to any set of present and past states of the agents a certain compact set in the state space common to all the agents. This may be seen as describing the way the agents receive and aggregate the informations at hand to construct a set to estimate the location of the population of agents. In this respect, this may be called a *sensing* or *perception* function. The latter may integrate aspects of the agents functioning, as well as geometric characteristics of the surrounding world.
- It is then necessary to define the rules according to which the agents update their state, given the (possibly delayed) information on the position of the other agents they received. For this, each agent k is attributed a set-valued map e_k which, *given the communication graph* $\mathcal{A}(t)$, defines the set of allowed positions $e_k(\mathcal{A}(t))$.

The main result of the paper may be summarized as follows: if each agent chooses its new position *in the interior* of the set he constructed in the sensing phase, then asymptotic convergence towards a common point occurs, even if each agent has indeed a very poor perception of the global system. Apart from technical conditions, an appropriate connectedness hypothesis on the information graph is of course central. The hypothesis used here (weak connectivity [M1]) will appear a necessary and sufficient requirement for stability. All this material is detailed in the core of the paper.

The paper is organized as follows. The new class of multi-agent systems studied here is constructed and its definition is commented in Section 2, together with examples. Stability is studied afterwards: the main results are given in Section 3, before the Conclusion Section. To facilitate the reading, some technical proofs are gathered at the end of the paper, in the Appendix.

Notations As often as possible, we use notations introduced by Moreau [M1, M2]. Following him, we distinguish between the inclusion, denoted $\subseteq$, and the strict inclusion, denoted $\subset$. The topological interior of a set is denoted `int`, its affine hull `ah` and its closure `cl`.

We study systems with n agents whose position at time t are written as $x_1(t), \dots, x_n(t)$ in the finite-dimensional space X. In the setting introduced in Moreau's contributions, the corresponding overall state variable is $x(t) = (x_1(t), \dots, x_n(t)) \in X^n$. Here, we consider systems with delay smaller than a given integer $h > 0$. In consequence, the complete state variable of the system is $(x_1(t), x_1(t-1), \dots, x_1(t-h+1), \dots, x_n(t), \dots, x_n(t-h+1)) \in X^{hn}$.

We denote $\tilde{x} = (x_1, \dots, x_{hn})$ an arbitrary element of X^{hn} and, when considering the dynamical system, we write $\tilde{x}_k(t) = (x_k(t), x_k(t-1), \dots, x_k(t-h+1))$ for all $k \in \mathcal{N} \doteq \{1, \dots, n\}$ and $\tilde{x}(t) = (\tilde{x}_1(t), \dots, \tilde{x}_n(t))$. We also use the corresponding decomposition of any element $\tilde{x}$ of X^{hn} as $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ (which amounts to identifying X^{hn} to $(X^h)^n$). When needed, any $\tilde{x}_k \in X^h$ is decomposed according to $\tilde{x}_k = (x_{k,0}, \dots, x_{k,h-1})$, in such a way that for the variables of the dynamical systems under study $x_{k,j}(t) = x_k(t-j)$, $k \in \mathcal{N}$, $j \in \mathcal{H} \doteq \{0, \dots, h-1\}$. Similarly we denote by $\mathcal{HN} \doteq \{1, 2, \dots, hn\}$. The previous notation is necessary, in order to be able to distinguish between the delayed and the actual values of the position of the agents. Coherently with the notations introduced above, we sometimes abbreviate $x_{k,0}$ and write simply x_k.

Last, given any $\tilde{x} \in X^{hn}$ we often need to embed it on 2^X, according to the following rule: $\pi(\tilde{x}) \doteq \{x_1, x_2, \dots x_{nh}\}$. In this way the state of the system is mapped to a finite collection of points in the X space.

Finally, for the comfort of the reader we indicate that the Theorems 1, 2, 3 and 5 in reference [M1] are numbered respectively 4, 1, 2 and 5 in [M3].

2 A class of multi-agent dynamical systems

This section is devoted to the presentation of the dynamical system under study. We study here a special class of *nonlinear difference inclusions with delay*, that we write:

$$x_k(t+1) \in e_k(\mathcal{A}(t))(\tilde{x}(t)) \ . \tag{1}$$

Recall that $x_k(t)$ represents the "position" at time t of the agent k. The evolution of the latter depends upon the complete system state $\tilde{x}(t)$ (including delayed components), through the time-varying map $e_k(\mathcal{A}(t))$. For a trajectory of (1), we call *decision set of agent* k at time t the value taken by $e_k(\mathcal{A}(t))(\tilde{x}(t))$. The specificity of the problem lies in these maps: they depend upon the topology of the inter-agent communications, modeled by the *graph* $\mathcal{A}(t)$.

The modeling of the communication network is presented below in Section 2.1. The construction of the decision sets inside which, given the communication network, each agent may update the value of its state, is completed in Section 2.2. Last, we provide some examples in Section 2.3.

2.1 Inter-agent communications modeling

The first ingredient of the construction is the family of *continuous set-valued maps* $e_k(\mathcal{A}) : X^{hn} \rightrightarrows X$ taking on *compact values*, and defined for $k \in \mathcal{N}$ and any directed graph $\mathcal{A}$. The latter will define, according to the position of the other agents, in which subset of X agent k is allowed to choose its future state.

Here, we are concerned by information transfer from the past to the present. In other words, we need to consider graphs in X^{hn} linking some past and/or present values $x_k(t-j)$ of the states of an agent k to another agent l. Consequently, at each time, the communication graph $\mathcal{A}$ is a *weighted, directed multigraph* defined on the set $\mathcal{N}$ of the *nodes,* that is a set of ordered couples of nodes (with possible repetitions), called *arcs.* To each of these arcs is associated a *weight,* chosen in $\mathcal{H}$, to be interpreted as the corresponding information delay[1]. All the considered graphs will contain all the loops of zero weight, corresponding to the ability for each agent to use without delay the knowledge on its own state.

Definition 1. *An* admissible graph *is any weighted, directed multigraph defined on $\mathcal{N}$, with weights in $\mathcal{H}$ and containing all the zero-weight loops.*

We write $i \overset{j}{\leadsto}_{\mathcal{A}} k$ when an arc of weight j links in the admissible graph $\mathcal{A}$ the node i to the node k (with $i, k \in \mathcal{N}$, $j \in \mathcal{H}$).

Definition 2. *A node $k \in \mathcal{N}$ is said to be* connected *to a node $l \in \mathcal{N}$ if there exists a* path *from k to l in the admissible graph $\mathcal{A}$ which respects the orientation of the arcs. Given a sequence of admissible graphs $\mathcal{A}(t)$, $t \in \mathbb{N}$, a node $k \in \mathcal{N}$ is said* connected *to a node $l \in \mathcal{N}$* on an interval $I \subseteq \mathbb{N}$ *if k is connected to l for the graph $\bigcup_{t \in I} \mathcal{A}(t)$.*

An admissible graph $\mathcal{A}$ is called weakly connected *[M1] if there is a node $k \in \mathcal{N}$ connected to all other nodes $l \in \mathcal{N}$. A sequence of admissible graphs $\mathcal{A}(t)$, $t \in \mathbb{N}$, is called* weakly connected across an interval $I \subseteq \mathbb{N}$ *if the graph $\bigcup_{t \in I} \mathcal{A}(t)$ is weakly connected (that is, if there is a node connected across I to all other nodes).*

Figure 1 provides an example of admissible graph. For the graph represented therein, agents 1 and 2 are mutually connected and agent 3 is connected to 1 and 2, but neither 1 nor 2 is connected to 3. Notice that generally speaking there may exist more than one arc between two distinct nodes, and that a node may be connected to itself (via delayed values).

Definition 3. *Consider an admissible graph $\mathcal{A}$ and a nonempty subset $\mathcal{L} \subseteq \mathcal{N}$. The set* $\mathrm{Neighbors}(\mathcal{L}, \mathcal{A})$ *is the set of those nodes $k \in \mathcal{N} \setminus \mathcal{L}$ for which there is $l \in \mathcal{L}$ such that (at least) one arc from k to l exists. When $\mathcal{L}$ is a singleton $\{l\}$, the notation* $\mathrm{Neighbors}(l, \mathcal{A})$ *is used instead of* $\mathrm{Neighbors}(\{l\}, \mathcal{A})$.

We impose to the maps e_k the following assumption.

Assumption A. *For all $k \in \mathcal{N}$ and all admissible graphs $\mathcal{A}$, the set-valued map e_k is continuous and takes on compact values. Moreover,*

- $e_k(\mathcal{A})(\tilde{x}) = \{x_k\}$ *if* $\{x_{i,j} \ : \ i \overset{j}{\leadsto}_{\mathcal{A}} k\} = \{x_k\}$;
- $e_k(\mathcal{A})(\tilde{x}) \subset \mathtt{ri}\ \sigma \left(\{x_k\} \cup \{x_{i,j} \ : \ i \overset{j}{\leadsto}_{\mathcal{A}} k\} \right)$ *otherwise.*

The exact meaning and the properties of the set-valued map $\mathtt{ri}\ \sigma$ are the subject of Section 2.2. However, we may already make some remarks on the form of the right-hand side of problem (1). Clearly, Assumption A implies that the evolution of each agent depends only upon the possibly delayed information received from its neighbors. The case where $\{x_{i,j} \ : \ i \overset{j}{\leadsto}_{\mathcal{A}} k\} = \{x_k\}$ is realized when either the agent k has no neighbor and the set involved in the formula is empty, or all the (possibly delayed) positions received from the neighboring agents are also equal to the present position x_k of agent k; in this case, no motion is allowed. We shall see below that in the present framework the use by each agent of the present value of its own position is mandatory for stability, see counterexample in Example 8.

[1] Recall that $\mathcal{N} = \{1, \dots, n\}$, $\mathcal{H} = \{0, \dots, h-1\}$, where n is the number of agents and $h-1$ the larger transmission delay.

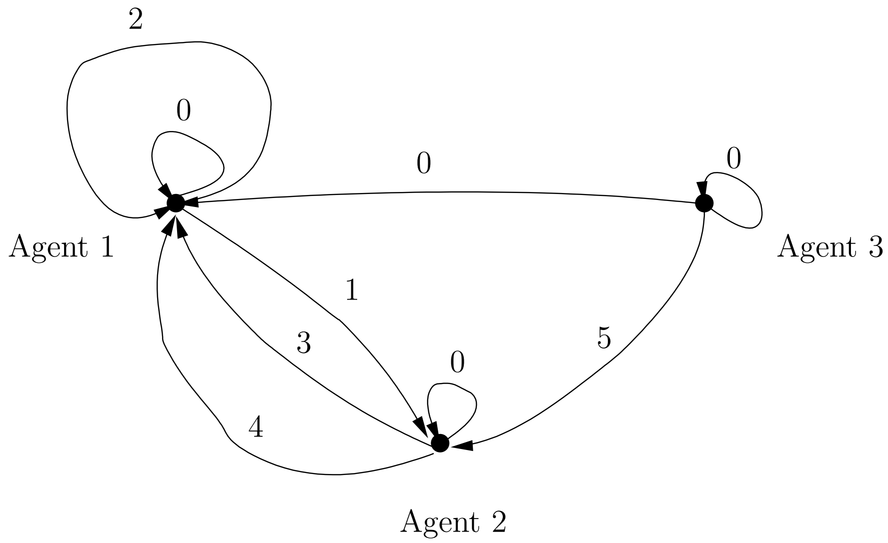

Figure 1: An example of admissible graph for a system with three agents.

2.2 Construction of the decision sets

The second ingredient necessary for the construction of the dynamical system under study is a *set-valued map* $\sigma : 2^X \rightrightarrows X$, taking on *compact values*. According to the interpretation presented in Section 1, we call σ a *sensing function*. It has a central role in the definition of the dynamics: as said before, it describes the way the agents aggregate the different informations they have at hand to estimate the location of the total population. In a complementary way, the functions e_k previously introduced define the *evolution policy* adopted by each agent. The latter has to be *compatible* with the results of the *sensing function*: this is the meaning of Assumption A. It will be shown afterwards (cf. in particular the proof of Theorem 4) that $t \mapsto \sigma(\pi(\tilde{x}(t))) = \sigma(\{x_1(t), \dots, x_1(t-h+1), \dots, x_k(t), \dots, x_k(t-h+1)\})$ plays the supplementary role of a "set-valued Lyapunov function" for the studied system.

In order to state the properties that σ should fulfil, we have to introduce beforehand some notions. First of all, define $\mathcal{S}$, a set of subsets of X in which σ will be compelled to take on its values, as:

$$\mathcal{S} \doteq \{S \subset X \ : \ S \text{ compact and } \exists \varphi : X \to X, \varphi \text{ bijective}, \varphi, \varphi^{-1} \text{ Lipschitz and } \varphi(S) \text{ convex}\} \ . \qquad (2)$$

Important consequences will proceed from the fact that σ takes on values in $\mathcal{S}$, inherited from properties of $\mathcal{S}$ summarized in the following result.

Lemma 1. *Let $\mathcal{S}$ be defined by* (2)*.*

1. *for any $S \in \mathcal{S}$, the function $d_S(x^0, x^1) : S \times S \to [0, +\infty)$ defined as*

$$d_S(x^0, x^1) \doteq \inf \left\{ \mathtt{length}(\psi) \ : \ \psi : [0,1] \xrightarrow{\textit{Lipschitz}} S, \psi(0) = x^0, \psi(1) = x^1 \right\}$$

is well-defined and continuous. Define $\mu : \mathcal{S} \to \mathbb{R}^+$ by:

$$\mu(S) \doteq \max_{x^0, x^1 \in S} d_S(x^0, x^1). \qquad (3)$$

Then, for all $S \in \mathcal{S}$,

- $\mu(S) < +\infty$.

- *$\mu(S) = 0$ if and only if S is a singleton.*
- *$\mu(S)$ is at least equal to the (euclidian) diameter of S, and equal to this value if S is convex.*
- *μ is lower semicontinuous in S, but nowhere continuous.*

2. *for any $S \in \mathcal{S}$, let φ be as in (2) and*

$$\mathtt{ri}(S) \doteq \varphi^{-1}\left(\mathtt{ri}(\varphi(S))\right) \ ,$$

where $\mathtt{ri}(\varphi(S))$ designates the relative interior of the convex set $\varphi(S)$, i.e. its interior when regarded as a topological subspace of its affine hull $\mathtt{ah}\ \varphi(S)$. Then, for all $S \in \mathcal{S}$,

- *$\mathtt{ri}(S)$ is independent of the choice of φ.*
- *$\mathtt{ri}(S) = \emptyset$ if and only if S is a singleton.*
- *$\mathtt{int}\ S \subseteq \mathtt{ri}\ S \subset S$.*
- *$\mathtt{ri}(S)$ is the relative interior of S if S is convex.*

The proof of Lemma 1 is given in Appendix A. Lemma 1 permits to measure the distance between points of a set $S \in \mathcal{S}$ "along the arcs". It permits to define extended notions of diameter, dimension and of relative interior, which coincide with the usual ones for convex subsets of X. By definition, we call *relative boundary* of sets S in $\mathcal{S}$ the set

$$\mathtt{r}\partial(S) \doteq S \setminus \mathtt{ri}(S) \ ,$$

and *dimension of S* the nonnegative integer[2]

$$\dim S \doteq \dim \mathtt{ah}\ \varphi(S) \ .$$

Also, according to the definition of d_S in Lemma 1, we define, for any subsets S', S'' of a set S in $\mathcal{S}$ the *S-distance from S' to S''* as:

$$d_S(S', S'') \doteq \inf_{x^0 \in S', x^1 \in S''} d_S(x^0, x^1) \ . \tag{4}$$

Notice that $d_S(\{x^0\}, \{x^1\}) = d_S(x^0, x^1)$.

We now gather the properties that σ must fulfil, and afterwards comment on their meaning and consequences.

Assumption B. *The set-valued map $\sigma : 2^X \rightrightarrows X$ is continuous with respect to the topology induced by Hausdorff metric and maps the bounded subsets of X to $\mathcal{S}$. Moreover, the following should hold:*

1. *$S \subseteq \sigma(S)$ with equality if S is a singleton.*
2. *$\sigma(S) = \sigma \circ \sigma(S)$ for all $S \in 2^X$.*
3. *$S' \subseteq S \ \Rightarrow\ \sigma(S') \subseteq \sigma(S)$ for all $S, S' \in 2^X$.*
4. *If S is bounded and not a singleton, for all $x \in S$, there exists $\Sigma_x \subseteq \mathtt{r}\partial\sigma(S)$ such that $\Sigma_x \cap S \neq \emptyset$ and $x \notin \Sigma_x$. Moreover, if $S' \subseteq \sigma(S)$:*
   - *(a) if $\mathtt{ri}\ \sigma(S') \cap \Sigma_x \neq \emptyset$, then $S' \subseteq \Sigma_x$ (and in particular, $x \notin S'$).*
   - *(b) if $d_{\sigma(S)}(S', \Sigma_x) > 0$, then $\mu(\sigma(S')) < \mu(\sigma(S))$.*
5. *$\mu \circ \sigma$ is continuous.*

[2] Recall that $\mathtt{ah}$ denotes the affine hull. The dimension of the affine hull of a convex set is uniquely defined [AC].

Remark that at this point, the problem under study is fully understandable: our goal is to find stability conditions for systems defined by (1), where the maps e_k verify Assumption A for a given map σ fulfilling Assumption B, and where the meaning of the relative interior $\mathtt{ri}$ has been defined previously by Lemma 1.

We shall see further — see Theorem 3 — that Assumptions B.1–B.3 are indeed sufficient to forbid increase along time of the natural set-valued Lyapunov function of the system $t \mapsto \sigma(\pi(\tilde{x}(t)))$ that we already mentioned. The additional Assumptions B.4–B.5 induce *strict* decrease of the set-valued Lyapunov function — see Theorem 4.

The properties stated in Assumptions B.1 to B.3 are quite natural for a "sensing function", and require few comments. However, they are far from fixing univocally the form of σ. As an example, can the form of the sets $\sigma(S)$ become more and more "complicated" when S becomes smaller and smaller? Can a set S exists with void intersection with the boundary of $\sigma(S)$? The remaining hypotheses in Assumption B limit these possibilities, and in particular answer the previous questions.

With the aim of explaining and illustrating further the signification and implications of Assumption B, we gather in the next proposition various consequences. Comments are in the sequel.

Proposition 1. *Assume Assumption B is fulfilled. Let S, S', $S' \subseteq S$, be bounded subsets of X, S not a singleton, and $x \in S$. The following properties are verified.*

1. $\mathtt{ri}\ \sigma(S) \neq \emptyset$ *and* $\mu(\sigma(S)) > 0$.
2. $\mathrm{card}\ (S \cap \mathtt{r}\partial\sigma(S)) \geq 2$.
3. $\mu(\sigma(S')) \leq \mu(\sigma(S))$.
4. *The family of sets Σ_x fulfilling the properties stated in Assumption B.4 is closed under union.*

This allows to univocally define Σ_x (by maximality): from now on, Σ_x will denote the uniquely defined maximal set exhibited in Point 4. We also write $\Sigma_x|_S$ for Σ_x (in the situation described in B.4), in order to stress with respect to which set it is considered.

5. *The set Σ_x verifies $\Sigma_x = \mathtt{cl}\ \Sigma_x \setminus \{x\}$. In particular, Σ_x is closed if and only if $d_{\sigma(S)}(x, \Sigma_x) > 0$.*
6. $\Sigma_x|_S = \Sigma_x|_{\sigma(S)}$.
7. *Assume $x \in S'$. Then, $\Sigma_x|_S \cap \sigma(S') \subseteq \Sigma_x|_{S'}$.*

The proof of Proposition 1 is detailed in Appendix B.

Proposition 1.2 indicates that the sensing process operated by σ is *sharp* in a certain sense: $\sigma(S)$ contains S in such a tight way, that at least two points of S are on the relative boundary $\mathtt{r}\partial\sigma(S)$. For *convex* sets S, S' in X, $S' \subseteq S$ implies $\mu(S') \leq \mu(S)$, but this is not always true for general sets in $\mathcal{S}$. Proposition 1.3 shows however that this order relation is preserved by σ.

Propositions 1.4 allows to consider Σ_x as a uniquely defined subset of $\mathtt{r}\partial\sigma(S)$. Generally speaking, the form of this set, which defines a critical part of the relative boundary of $\sigma(S)$ relative to x, looks like an union of "faces" of $\mathtt{r}\partial\sigma(S)$. Although in all the examples developed below in Section 2.3, $\Sigma_x = \mathtt{r}\partial\sigma(S)$ whenever $x \in \mathtt{ri}\ \sigma(S)$, it is unknown whether this property is true or not. It is possible to characterize easily Σ_x in a particular case, when σ fulfills Assumption B and is such that $\mathtt{ri}\ \sigma(S') \cap \mathtt{r}\partial\sigma(S) = \emptyset$ for every $S' \subseteq S$. In this case, one verifies easily that

$$\forall x \in \sigma(S), \Sigma_x = \mathtt{r}\partial\sigma(S) \text{ if } x \in \mathtt{ri}\ \sigma(S),\ \Sigma_x = \mathtt{r}\partial\sigma(S) \setminus \{x\} \text{ if } x \in \mathtt{r}\partial\sigma(S)\ . \tag{5}$$

The proof of this fact is direct and its details are left to the reader (it stems from the fact that Assumption B.4a is then automatically fulfilled, as $\mathtt{ri}\ \sigma(S') \cap \Sigma_x \neq \emptyset$ is incompatible with $\Sigma_x \subseteq \mathtt{r}\partial\sigma(S)$). This occurs especially when $\dim S = \dim X$ ($\dim S$ has been defined after Lemma 1) for any bounded S with at least two elements. As an example, take for σ the convex hull (a paradigm considered in more details in Example 1 below) and the sphere for S.

An important point is that the set Σ_x cannot be deduced from the geometry of $\sigma(S)$ alone. Rather, it characterizes the way the sets $\sigma(S)$ vary when S changes. This is the main interest of Example 4 below to illustrate this point.

Concerning Proposition 1.6, notice that the right-hand side of the identity therein is meaningful: if $x \in S$ and S is not a singleton, then $\sigma(S)$ shares the same properties, because of Assumption B.1 ($S \subseteq \sigma(S)$). Alternatively, one sees that indeed $\Sigma_x|_S$ depends only upon $\sigma(S)$ and x: it may be defined for all x in $\sigma(S)$, and not only in S.

Last, Proposition 1.7 says that, if a point x' is on the critical part of the boundary of $\sigma(S)$ and is also in $\sigma(S')$ for a given subset S' of S, then it is still on the (new) critical boundary, a rather natural property. In particular, $\Sigma_x|_S \cap S' \subseteq \Sigma_x|_{S'}$. Notice how it is *not* true in general that $\mathtt{r}\partial\sigma(S) \cap \sigma(S') \subseteq \mathtt{r}\partial\sigma(S')$, except for example when $\dim S = \dim X$ for every bounded non-singleton subsets S of X. This is shown e.g. by the following counterexample. In the plane $X \doteq \mathbb{R}^2$, let S be the square with vertices $(\pm 2, \pm 2)$, and S' be the segment with extremities $(2, \pm 1)$. Clearly, $S' \subset S$. Taking for σ the convex hull (see Example 1 below), one has $\sigma(S) = S$, $\sigma(S') = S'$. Thus, $\mathtt{r}\partial\sigma(S) \cap \sigma(S') = S'$, but $\mathtt{r}\partial\sigma(S')$ only contains the two extremities $\{(2, \pm 1)\}$ (notice however that, for any $x \in S'$, $\Sigma_x|_S \cap \sigma(S') = \emptyset$, so this set is always included in $\Sigma_x|_{S'}$, and this does not contradict Proposition 1.7).

As a last remark on Assumption B notice that Lemma 1 and the continuity assumption on σ imply that the map $\mu \circ \sigma$ is already lower semicontinuous on X^{hn}. Assumption B.5 thus represents a slightly stronger regularity assumption.

The role of Assumption B is central to deduce the stability results below. It applies to arbitrary (but non trivial) groups of agents S, which may comprise indifferently true agents or "virtual" agents, viz. informations relative to the position of a true agent at previous sampling times. More closely, it will imply that, for each agent x, the agents located on the portion of the boundary of $\sigma(S)$ denoted Σ_x, are *irreversibly* attracted outside of it when using information received from any agent not in Σ_x (such as x itself) according to the rule edicted in Assumption A. This point is sufficiently important to be formalized now, in Proposition 2. Complementarily, the second part of Assumption B.4 imposes that the irreversible escape from Σ_x comes with a *strict decrease* of the diameter of the set-valued Lyapunov function of the system.

Proposition 2. *Assume Assumptions A and B hold. If $i \overset{j}{\sim}_{\mathcal{A}(t_0)} k$ for $i, k \in \mathcal{N}$, $j \in \mathcal{H}$, then any trajectory of* (1) *fulfills:*

$$\forall t \geq t_0 + 1,\ x_k(t) \notin \Sigma_{x_i(t_0 - j)}|_{\sigma(\pi(\tilde{x}(t_0)))} \ .$$

Proof. If $x_k(t) = x_i(t_0 - j)$, by definition, $x_k(t) \notin \Sigma_{x_i(t_0-j)}$. Otherwise, Assumption A implies that

$$x_k(t_0 + 1) \in e_k(\mathcal{A}(t_0))(\tilde{x}(t_0)) \subset \mathtt{ri}\ \sigma\left(\{x_k(t_0)\} \cup \{x_{i',j}(t_0)\ :\ i' \overset{j}{\sim}_{\mathcal{A}(t_0)} k\}\right) \ .$$

Now, $x_{i,j}(t_0) = x_i(t_0 - j) \in \{x_{i',j}(t_0)\ :\ i' \overset{j}{\sim}_{\mathcal{A}(t_0)} k\}$. Fix $S = \pi(\tilde{x}(t_0))$, $S' = \{x_k(t_0)\} \cup \{x_{i',j}(t_0)\ :\ i' \overset{j}{\sim}_{\mathcal{A}(t_0)} k\}$ and $x = x_i(t_0 - j)$. As $S' \subseteq \sigma(S)$, one may apply Assumption B.4a. Notice that $x \in S'$, so that $S' \nsubseteq \Sigma_x$. In this situation, Assumption B.4a yields $\mathtt{ri}\ \sigma(S') \cap \Sigma_x = \emptyset$. In particular, $x_k(t_0 + 1) \in \mathtt{ri}\ \sigma(S')$, so $x_k(t_0 + 1) \notin \Sigma_{x_i(t_0-j)}|_{\sigma(\pi(\tilde{x}(t_0)))}$. This establishes the desired property for $t = t_0 + 1$.

The argument is similar to prove recursively the property for $t > t_0 + 1$. Assume that $x_k(t) \notin \Sigma_{x_i(t_0-j)}|_{\sigma(\pi(\tilde{x}(t_0)))}$. Either $x_k(t + 1) = x_k(t)$, or $x_k(t + 1) \in \mathtt{ri}\ \sigma\left(\{x_k(t)\} \cup \{x_{i',j}(t)\ :\ i' \overset{j}{\sim}_{\mathcal{A}(t)} k\}\right)$. In the latter case, one applies Assumption B.4a with $S' = \{x_k(t)\} \cup \{x_{i',j}(t)\ :\ i' \overset{j}{\sim}_{\mathcal{A}(t)} k\}$ and the same S and x, using the fact that $x_k(t) \in \mathtt{ri}\ \sigma(S') \setminus \Sigma_x \neq \emptyset$. In both cases, one gets $x_k(t + 1) \notin \Sigma_x|_S$. □

2.3 Examples

We present here different examples and counter-examples of maps σ fulfilling the properties previously defined.

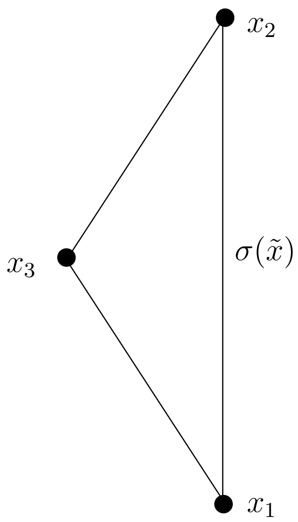

Figure 2: The convex-hull, Moreau's set-valued Lyapunov function, see Example 1.

Example 1 (convex hull). In Moreau's work, $\sigma(S)$ is taken to be the convex hull of S, see Figure 2. One may check easily that Assumptions B.1 to B.5 are all fulfilled. Here, the sets Σ_x involved in Assumption B.4 can be defined as follows:

$$\Sigma_x \doteq \bigcup_{c \in TC_{\sigma(S)}(x), |c|=1} x + \max\{t : x + ct \in \sigma(S)\}c \ ,$$

where $TC_{\sigma(S)}(x)$ denotes the Bouligand contingent cone to the set $\sigma(S)$ at x (otherwise called tangent cone, as $\sigma(S)$ is convex here; see [AC, pp. 176–177 and 219] for details).

Example 2 (a different convex example). A close, but different, map is employed in [BT]. For a given basis e_j, $j = 1, \dots, p$ of X, take

$$\sigma(S) \doteq \left[\min_{x \in S} e_1^T x, \max_{x \in S} e_1^T x\right] \times \cdots \times \left[\min_{x \in S} e_p^T x, \max_{x \in S} e_p^T x\right] \ .$$

In this example, the convex hull is applied "componentwise", see Figure 3. Remark that $\mathrm{conv}(S) \subseteq \sigma(S)$ for this case, but this relation is not mandatory, see Example 6 below.

In the example depicted on Figure 3, one may check that the choice consisting in taking for $\Sigma_x \doteq \bigcup_{c \in TC_{\sigma(S)}(x), |c|=1} x + \max\{t : x + ct \in \sigma(S)\}c$, fulfills the Assumptions.

Example 3 (other convex examples). One may also define $\sigma(S)$ as the smallest set containing S and with boundary parallel to given $p+1$ non-parallel hyperplanes (where $X = \mathbb{R}^p$), see Figure 4. More precisely, let $\Sigma = \mathrm{conv}(S)$ and $e_1, \dots, e_{p+1}$ be $(p+1)$ vectors in X such that for some positive $\lambda \in \mathbb{R}^{p+1}$ we have $\sum_j \lambda_j e_j = 0$. The set $\sigma(S)$ is a polytope defined as: $\left\{x \in X \ : \ e_j^T x \le \max_{x' \in \Sigma} e_j^T x', \ j = 1, \dots, p+1\right\}$, containing the points $x_1, \dots, x_{hn}$. Symmetrically we may define $\sigma(S) = \left\{x \in X \ : \ e_j^T x \ge \min_{x' \in \Sigma} e_j^T x', \ j = 1, \dots, p+1\right\}$. Similarly to what occurs in Example 2, one may take for Σ_x the portion of the boundary obtained by following the vectors coming out from the tangent cone at x all the way to their extreme intersection point with the boundary of $\sigma(S)$, and the Assumptions B.1-B.5 are fulfilled.

Example 4 (a last convex example). A somehow artificial example is now provided, with the aim of illustrating the non-purely geometric nature of the sets Σ_x. For S bounded in $X \doteq \mathbb{R}^2$, define $\sigma(S)$ as the region containing S and included in the intersection of the two cones $\{(x, y) \in \mathbb{R}^2 \ : \ -(x - x^*) \le |y - y^*| \le x - x^*\}$ and $\{(x, y) \in \mathbb{R}^2 \ : \ x - x^{**} \le |y - y^*| \le -(x - x^{**})\}$, where first, x^* is maximal (and y^* is then

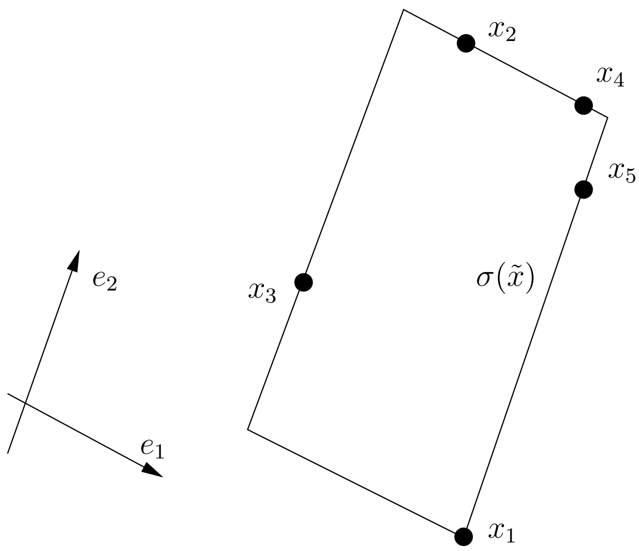

Figure 3: Illustration of Example 2.

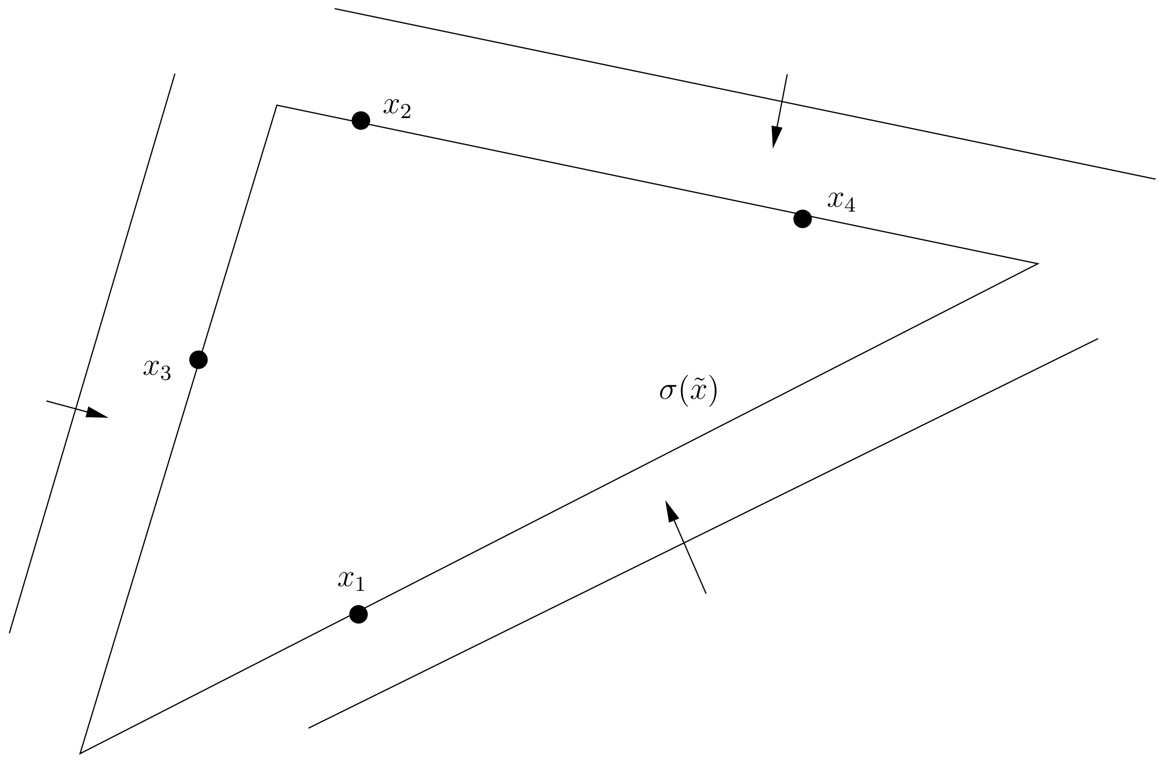

Figure 4: Other convex examples of set-valued Lyapunov function, see Example 3.

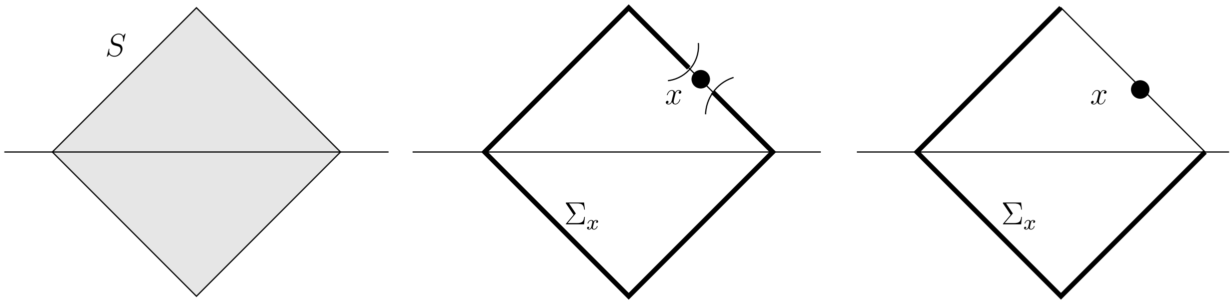

Figure 5: Demonstration of the non-geometric nature of the sets Σ_x. The sets Σ_x corresponding to the maps $\sigma(S)$ of Examples 4 (center) and 1 (right) are different, although in both cases the set S (left) has the same value, equal to $\sigma(S)$.

univocally deduced from this property) and second, x^{**} is chosen in such a way that $\sigma(S)$ is a *square*. In this configuration, $\dim \sigma(S) = \dim X = 2$ when card $S > 1$, so (5) holds. Figure 5 shows a configuration where, although the two values of $\sigma(S)$ coincide, the value of $\Sigma_x|_S$ differs when computed for the present map σ and for the map in Example 1.

Example 5 (a counterexample). The smallest ball or the smallest hypercube containing S does *not* fulfil the requested properties. To see that, consider for instance that the smallest circle containing a triangle never contains the smallest circle containing the shortest of its edges: this violates the monotonicity assumption on the map σ prescribed in Assumption B.3.

Example 6 (nonconvex example). For any bijective transformation $\varphi : X \to X$ which is Lipschitz together with its inverse, one may take

$$\sigma_\varphi(S) \doteq \varphi^{-1}\left(\sigma(\varphi(S))\right) \ ,$$

where σ fulfils all the Assumptions. In general $\sigma_\varphi(S) \not\subseteq \mathrm{conv}(S)$ and is not convex: indeed, this latter property is not essential. Such an example of nonconvex sets is given in Figure 6, obtained for $X = \mathbb{R}^2$, $x_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $x_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$, $\varphi(x) = \begin{pmatrix} \cos\alpha\|x\|^2 & \sin\alpha\|x\|^2 \\ -\sin\alpha\|x\|^2 & \cos\alpha\|x\|^2 \end{pmatrix} x$, $\alpha = 0.04$, and $\sigma(S) = \mathrm{conv}(S)$.

Notice that, generally speaking, the systems generated along this principle are such that the map φ in (2) is identical for *all* the sets $\sigma(S)$. The sets Σ_x may be obtained as for Example 1, up to transformation by φ.

The present example is useful to treat the case of agents located in spaces which are homeomorphic to euclidean space. For instance, this permits to model, and solve, the rendezvous problem for agents constrained within a given domain with complicated uncrossable boundaries, as the "labyrinth" shown in Fig. 7. Of course, φ^{-1} is the bijection that maps the plane to the white area, and therefore, the shape of constraints needs to be apriori known for the agents to make suitable decisions. Notice that this model is topologically very different from the case where obstacles look like "holes" in the space X.

Example 7 (intersection of decision sets). When σ, σ' fulfil the properties stated above, an interesting issue is to see whether $\sigma \cap \sigma'$ does. One verifies easily that Assumptions B.1–B.3 are automatically fulfilled. The validity of B.4 and B.5 depends upon the configuration of the sets Σ_x, Σ'_x corresponding to σ and σ'. In Figure 8 an example is presented where the resulting map fulfills all the properties.

3 Results

Before stating the results of this paper, we recall the notions under discussion below, see [M1, M3]. As in Moreau's papers, we call *equilibrium point* any element of the state space which is the constant value of an

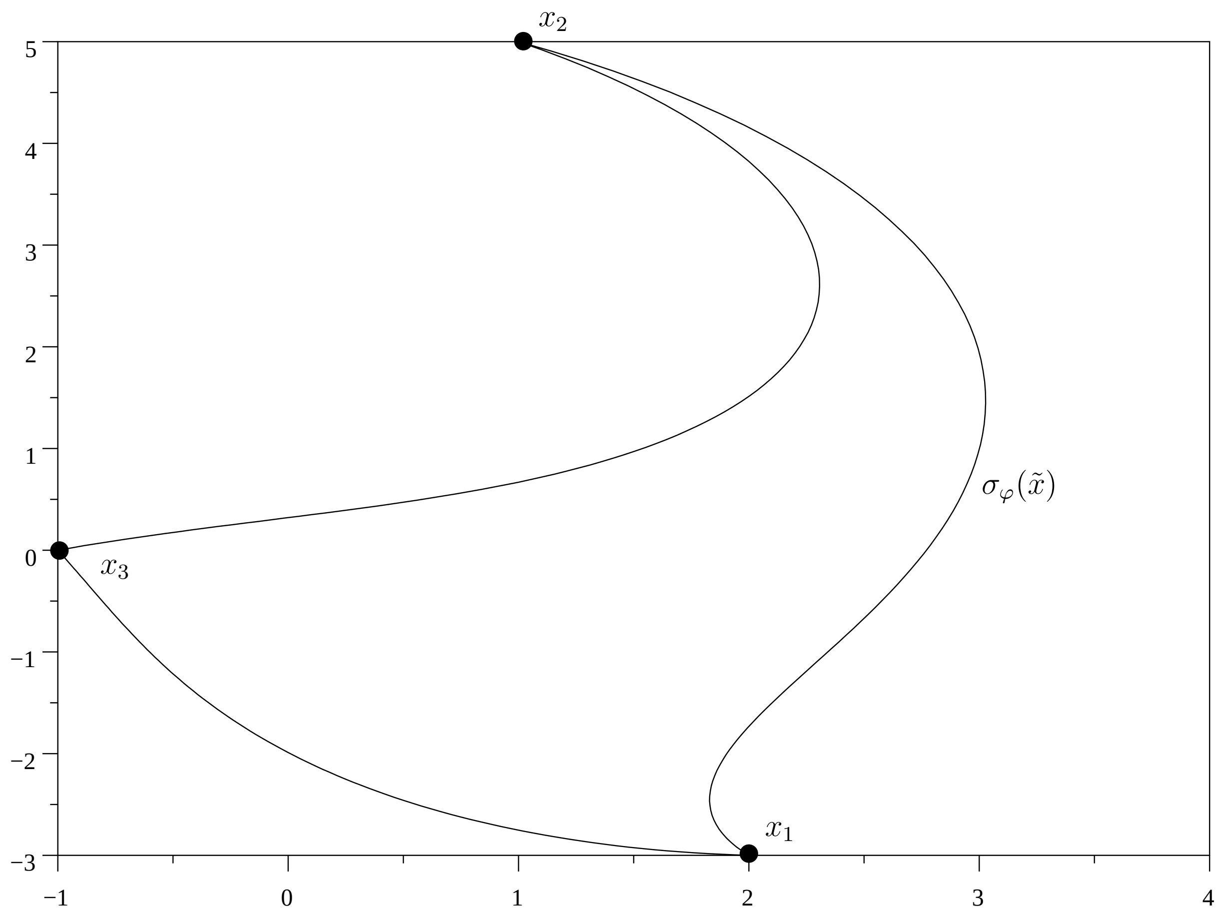

Figure 6: An example of map σ giving rise to nonconvex sets, see Example 6. Notice that $\mathrm{conv}(S) \not\subseteq \sigma(S)$, and that $\mu(\sigma(S))$ is larger than the diameter $\mu(\mathrm{conv}(S))$ of $\mathrm{conv}(S)$.

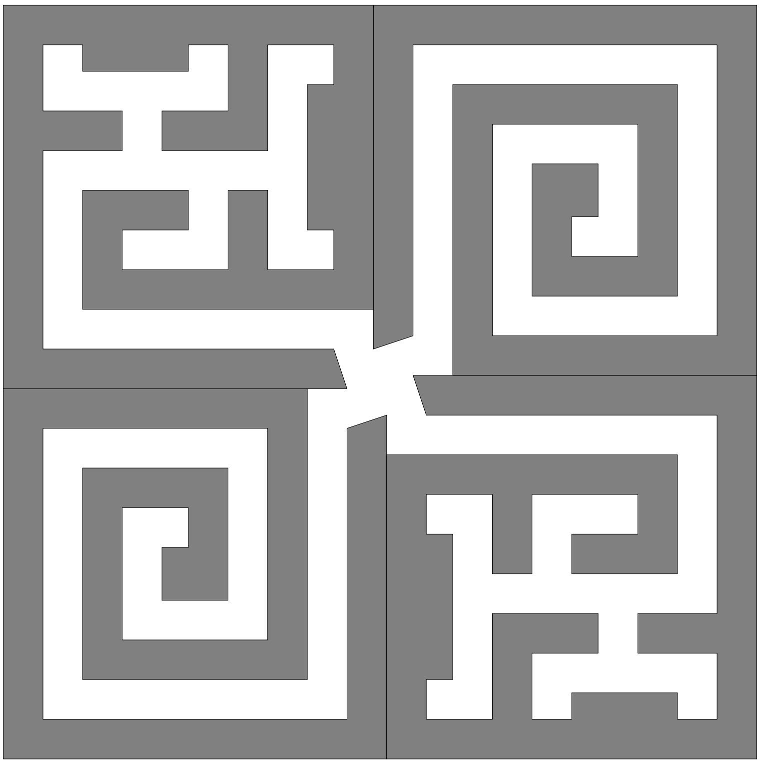

Figure 7: Example of a non-convex state-space homeomorphic to Euclidean space

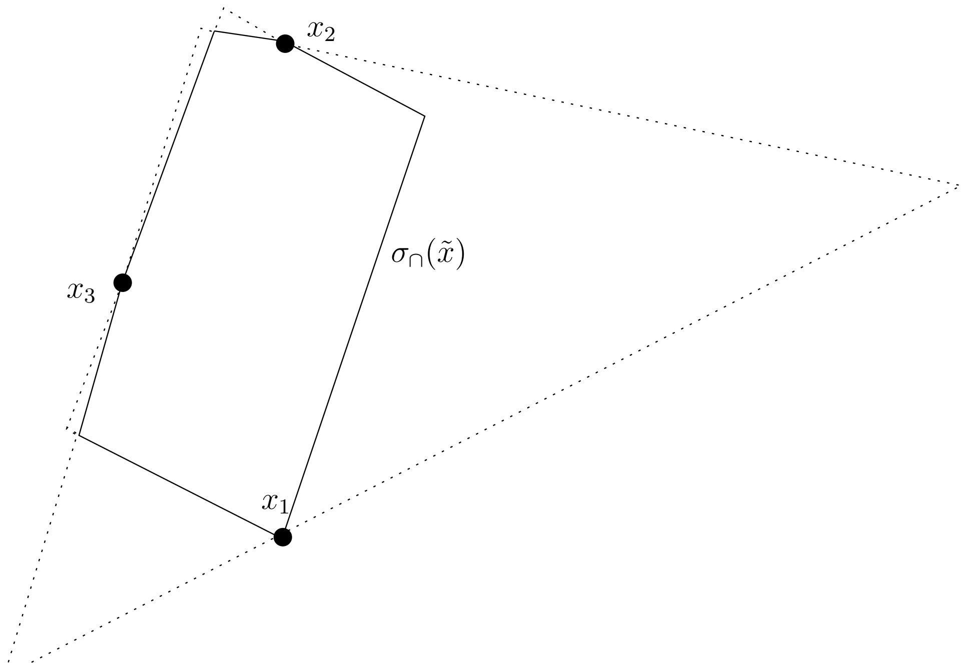

Figure 8: Map obtained by intersection of the maps from Figures 3 and 4.

equilibrium solution.

Definition 4. *Let $\mathcal{X}$ be a finite-dimensional Euclidean space and consider a continuous set-valued map $e : \mathbb{N} \times \mathcal{X} \rightrightarrows \mathcal{X}$ taking on closed values, giving rise to the difference inclusion*

$$x(t+1) \in e(t, x(t)) \ . \tag{6}$$

Consider a collection of equilibrium solutions of this equation and denote the corresponding set of equilibrium points by Φ. By definition, $\varphi \in \Phi$ if and only if $\varphi \in e(t, \varphi)$ for all $t \in \mathbb{N}$.

With respect to the considered collection of equilibrium solutions, the dynamical system is called

1. stable *if for each $\varphi \in \Phi$, for all $c_2 > 0$ and for all $t_0 \in \mathbb{N}$, there is $c_1 > 0$ such that every solution ζ of* (6) *satisfies: if $|\zeta(t_0) - \varphi| < c_1$ then $|\zeta(t) - \varphi| < c_2$ for all $t \geq t_0$.*

2. bounded *if for each $\varphi \in \Phi$, for all $c_1 > 0$ and for all $t_0 \in \mathbb{N}$, there is $c_2 > 0$ such that every solution ζ of* (6) *satisfies: if $|\zeta(t_0) - \varphi| < c_1$ then $|\zeta(t) - \varphi| < c_2$ for all $t \geq t_0$.*

3. globally attractive *if for each $\varphi_1 \in \Phi$, for all $c_1, c_2 > 0$ and for all $t_0 \in \mathbb{N}$, there is $T \geq 0$ such that every solution ζ of* (6) *satisfies: if $|\zeta(t_0) - \varphi_1| < c_1$ then there is $\varphi_2 \in \Phi$ such that $|\zeta(t) - \varphi_2| < c_2$ for all $t \geq t_0 + T$.*

4. globally asymptotically stable *if it is stable, bounded and globally attractive.*

If c_1 (respectively c_2 and T) may be chosen independently of t_0 in Item 1 (respectively Items 2 and 3) then the dynamical system is called uniformly stable (respectively uniformly bounded and uniformly globally attractive) with respect to the considered collection of equilibrium solutions.

Notice that the above notions are uniform with respect to all trajectories of (6).

We now state a first result on boundedness and (simple) stability, analogous to [M1, Theorem 2].

Theorem 3. *Assume that Assumptions A and B.1–B.3 are fulfilled. Then the discrete-time system* (1) *is uniformly globally bounded and uniformly globally stable with respect to the collection of equilibrium solutions $x_1(t) \equiv \cdots \equiv x_n(t) \equiv \text{constant}$.*

Proof. The proof of Theorem 3 is based on the evolution of the following set-valued function $\tilde{V} : X^{hn} \rightrightarrows X$,

$$\tilde{V}(\tilde{x}) \doteq \sigma(\pi(\tilde{x})) \tag{7}$$

along the solutions of (1). The fact that $t \mapsto \tilde{V}(\tilde{x}(t))$ is non-increasing is stated in the following result.

Lemma 2. *Let x be a solution of equation* (1)*. Then, for all $t \in \mathbb{N}$,*

$$\tilde{V}\left(\tilde{x}(t+1)\right) \subseteq \tilde{V}\left(\tilde{x}(t)\right) \ .$$

Let us first prove Lemma 2. For any $k \in \mathcal{N}$, for any $t \in \mathbb{N}$,

$$x_k(t+1) \in \sigma\left(\{x_k(t)\} \cup \{x_i(t-j) \ : \ i \overset{j}{\sim}_{\mathcal{A}(t)} k\}\right) \subseteq \tilde{V}(\tilde{x}(t))$$

successively by Assumption A and Assumption B.3, and one concludes the demonstration of Lemma 2 by use of Assumptions B.3 and B.2. The proof of Theorem 3 is then obtained as a direct consequence. □

In view of Lemma 2, one may now have a clearer understanding of the fact that the map σ has a double role: it is necessary to define the flow, but also serves as a set-valued Lyapunov function of the systems. Indeed, Assumption A states that each agent has to remain in the set $\tilde{V}(\tilde{x}(t))$, of which it has only an imperfect knowledge, and does its best to come closer to the other agents it has detected (this is the meaning of the use of the relative interior). In particular, when no new information is received, the only possible choice is to stay at the same place. Notice that Lemma 2 implies, together with Proposition 1.3, that $\mu(\tilde{V}(\tilde{x}(t+1))) \leq \mu(\tilde{V}(\tilde{x}(t)))$, but this is not of central use here.

As detailed in Section 2.2, contrary to σ, the map $\texttt{ri}\ \sigma$ is not monotone: violation of this rule may occur when $S' \subset S$ and the σ-hulls $\sigma(S), \sigma(S')$ have different topological dimensions as spheres. Up to this subtlety, a consequence of Assumption A is that, in general, *the larger the quantity of information received by agent k from its neighborhood, the larger the set of possible updates it may choose* (see the monotony property in Assumption B.3). Although this may sound paradoxical at first glance, this increase of the decision possibilities is quite natural: it means that supplementary information either leads to make a choice which was previously possible (it is ignored or makes more valuable the decision), or it is effectively used and allows to adopt choices which were impossible without this additional information. The "subtlety" comes from the fact that, when the information available to an agent is poor, some decisions are taken which would not have been possible with richer data. For example, the possibility of staying in the same place, which occurs when an agent, say agent 1, is isolated from the other world, disappears when the position of another agent located elsewhere, agent 2, is received. However, the unique choice $\sigma(\{x_1\}) = \{x_1\}$ is then located "on the boundary" of the decision set $\texttt{ri}\ \sigma(\{x_1, x_2\})$, see Proposition 1.2.

The key result of the paper is now stated. It provides a *necessary and sufficient stability condition* for system (1), which extends [M1, Theorem 3].

Theorem 4. *Assume that Assumptions A and B are fulfilled. Then the discrete-time system* (1) *is uniformly globally attractive with respect to the collection of equilibrium solutions* $x_1(t) \equiv \cdots \equiv x_n(t) \equiv \text{constant}$ if and only if *there exists $T \geq 0$ such that for all $t_0 \in \mathbb{N}$ the sequence of communication graphs is weakly connected across* $[t_0, t_0 + T]$.

The uniformity which is meant in the statement of Theorems 3 and 4 is with respect to *time*. One may check from the proofs that it is also valid with respect to the different trajectories of (1).

Theorem 4 is an analysis result. In a control synthesis perspective, certainly the first step is to construct a sensing map σ which fulfils Assumption B. This may be one of the two basic choices in Examples 1 and 2, or a more complicated one (as in Example 6) if necessitated e.g. by the geometry of the environment. Then, the decision policy e_k of each agent should be determined, according to Assumption A. This choice, together with the characteristics of the sensing function σ, probably influences crucially the convergence rate of the solutions. We leave this issue for future work.

Proof of Theorem 4. (Only if.) The proof consists in an adaptation of the contraposition argument developed by Moreau [M1, Proof of Theorem 3]. Assume that for every $T \geq 0$ there is $t_0 \in \mathbb{N}$ such that the sequence of admissible graphs has no node connected to each other across the interval $[t_0, t_0 + T]$. This implies that for every $T \geq 0$ there is $t_0 \in \mathbb{N}$ and nonempty, disjoint subsets $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{N}$ such that $\text{Neighbors}(\mathcal{L}_1, \mathcal{A}(t)) = \text{Neighbors}(\mathcal{L}_2, \mathcal{A}(t)) = \emptyset$ for all $t \in [t_0, t_0 + T]$. The proof of this fact consists in checking that the proof of [M1, Theorem 5] holds also in the case of admissible graph as defined above.

Let $y, \bar{y} \in X$ and consider any solution ζ of (1) departing from initial data defined by:

$$\zeta_k(t_0 - j) \begin{cases} = y & \forall (k,j) \in \mathcal{L}_1 \times \mathcal{H}, \\ = \bar{y} & \forall (k,j) \in \mathcal{L}_2 \times \mathcal{H}, \\ \in \sigma(\{y, \bar{y}\}) & \forall (k,j) \in (\mathcal{N} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2)) \times \mathcal{H}\ . \end{cases}$$

As in the proof of [M1, Theorem 5], we still have the same relation at time $t_0 + T + 1$, since $\text{Neighbors}(\mathcal{L}_1, \mathcal{A}(t)) = \text{Neighbors}(\mathcal{L}_2, \mathcal{A}(t)) = \emptyset$ for all $t \in [t_0, t_0 + T]$. As the time T may be chosen arbitrarily large, this contradicts uniform global attractivity of (1) with respect to the equilibrium solutions $x_1(t) \equiv \cdots \equiv x_n(t) \equiv \text{constant}$.

(If.) As in [M1, Theorem 3], the proof is based on a (strict) decrease property of the set-valued function $\tilde{V}$ introduced in (7).

Let $T \geq 0$ chosen as in the statement of Theorem 4, x an arbitrary solution of (1), and $t_0 \in \mathbb{N}$ for which the values of $x_k(t - j)$, $k \in \mathcal{N}$, $j \in \mathcal{H}$ are not all equal. For all $k \in \mathcal{N}$, define the integer-valued function $\alpha_k(t)$, $t \geq t_0$, by:

$$\alpha_k(t) \doteq \text{card}\,\{j \in \mathcal{N} :\; x_j(t) \in \Sigma_{x_k(t_0)}\}\ .$$

where Σ_x is meant relative to the set $S = \pi(\tilde{x}(t_0))$. In words, this is the number of agents which at time $t \geq t_0$ are still belonging to the critical portion of the boundary $\Sigma_{x_k(t_0)}$. Assume by contradiction that at time $t_1 > t_0$ a new agent x_α enters $\Sigma_{x_k(t_0)}$ which was not there at time $t_1 - 1$ ($x_\alpha(t_1 - 1) \notin \Sigma_{x_k(t_0)}$). Let S' denote the set of points in X used by x_α at time $t_1 - 1$ in order to update its state. Of course, by Assumption A, S' comprises $x_\alpha(t_1 - 1)$ itself. Moreover, by monotonicity of the set-valued Lyapunov function $\tilde{V}$, $S' \subset \sigma(S)$. Now, by the updating rule A, $x_\alpha(t_1) \in \Sigma_{x_k(t_0)}$ is only possible provided that $\mathtt{ri}(\sigma(S')) \cap \Sigma_{x_k(t_0)} \neq \emptyset$, and therefore, application of Assumption B.4a yields $x_\alpha(t_1 - 1) \in S' \subset \Sigma_{x_k(t_0)}$, which contradicts what was previously stated. Hence, we can conclude that agents can only leave $\Sigma_{x_k(t_0)}$, but never get back in. In particular then, the functions α_k satisfy the inequality:

$$\forall t \in [t_0, +\infty), \forall\, k \in \mathcal{N},\;\; \alpha_k(t+1) \leq \alpha_k(t)$$

and

$$\alpha_k(t_0) = \alpha_k(t_1) \Leftrightarrow \{j \in \mathcal{N} :\, x_j(t_0) \in \Sigma_{x_k(t_0)}\} = \{j \in \mathcal{N} :\, x_j(t) \in \Sigma_{x_k(t_0)}\} \quad \forall\, t \in \{t_0, \ldots, t_1\}. \tag{8}$$

On the other hand, if $t \geq t_0$ is such that $\alpha_k(t) = 0$ for a certain k in $\mathcal{N}$, then Assumption B.4b implies that $\mu(\tilde{V}(\tilde{x}(t))) < \mu(\tilde{V}(\tilde{x}(t_0)))$. An important step consists in showing that:

$$t > t_0 + T' \Rightarrow \exists k \in \mathcal{N}, \alpha_k(t) < \alpha_k(t_0), \quad T' \doteq h + T\ . \tag{9}$$

There are at most n different sets $\Sigma_{x_k(t_0)}$ in $\sigma(\pi(\tilde{x}(t_0)))$, and $\alpha_k(t_0) \leq n - 1$. Consequently, the repetition of the argument used to get implication (9) (if allowed) will yield:

$$t > t_0 + (n-1)^2 T' \Rightarrow \exists k \in \mathcal{N}, \alpha_k(t) = 0\ .$$

As a consequence, the estimate:

$$t > t_0 + T'' \Rightarrow \mu(\tilde{V}(\tilde{x}(t))) < \mu(\tilde{V}(\tilde{x}(t_0))), \quad T'' \doteq (n-1)^2 T'\ , \tag{10}$$

will be deduced from Assumption B.4b, because $\pi(\tilde{x}(t))$ being a *finite* set of points located in $\sigma(\pi(\tilde{x}(t_0))) \setminus \Sigma_{x_k(t_0)}$, is thus at a *nonzero* distance (more precisely a $\sigma(\pi(\tilde{x}(t_0)))$-distance, see (4)) from $\Sigma_{x_k(t_0)}$. In order to get (10), let us now prove (9).

Assume one has $\alpha_k(t_0 + h + T) = \alpha_k(t_0)$ for all k in $\mathcal{N}$ and, by virtue of (8), the set $\mathcal{L}_k \doteq \{j \in \mathcal{N} : x_j(t) \in \Sigma_{x_k(t_0)}\}$ has not changed for $t \in \{t_0, \ldots, t_0 + h + T\}$.

Using the hypothesis in the statement of Theorem 4, there exists an agent, numbered k, connected to all others across the interval $[t_0 + h, t_0 + h + T]$. By definition, the set $\Sigma_{x_k(t_0)}$ does not contain $x_k(t_0)$, and, since $\mathcal{L}_k$ has not varied in time, then also $x_k(t) \notin \Sigma_{x_k(t_0)}$ for $t = t_0, \ldots, t_0 + h + T$. Moreover, because of the weak connectivity property of the graph put in the statement, $\text{Neighbors}(\mathcal{L}_k, \cup_{t \in [t_0+h, t_0+h+T]} \mathcal{A}(t)) \neq \emptyset$. Let $i \in \mathcal{L}_k$ be such that $\text{Neighbors}(i, \cup_{t \in [t_0+h, t_0+h+T]} \mathcal{A}(t)) \setminus \mathcal{L}_k \neq \emptyset$, viz. an agent which over the time-interval $[t_0 + h, t_0 + h + T]$ is receiving information from outside $\mathcal{L}_k$. In other words, there exists $t_1 \in [t_0, t_0 + T]$ such that $x_i(t_1 + 1) \in e_i(\mathcal{A}(t_1))(\tilde{x}(t_1))$ and $\text{Neighbors}(i, \mathcal{A}(t_1)) \setminus \mathcal{L}_k \neq \emptyset$. Arguing as in Proposition 2, one deduces that $e_i(\mathcal{A}(t_1))(\tilde{x}(t_1)) \cap \Sigma_{x_k(t_0)} = \emptyset$, and thus $x_i(t + 1) \notin \Sigma_{x_k(t_0)}$.

This yields finally: $\alpha_k(t_0 + h + T) < \alpha_k(t_0)$ for the value of k previously exhibited. Inequality (9) is thus proved. Of course this is true only as long as $\Sigma_{x_k(t_0)}$ is non-empty to start with.

One verifies easily that the same argument may be used recursively, because the sets $\Sigma_{x_k(t_0)}$ may be kept unchanged as long as $\alpha_k(t) > 0$ for all $k \in \mathcal{N}$. Thus, (10) is proved.

Considering now $\tilde{x}(t_0) \in X^{hn}$ as a variable, let

$$\beta(\tilde{x}(t_0)) \doteq \inf \mu\left(\tilde{V}(\zeta(0))\right) - \mu\left(\tilde{V}(\zeta(T''))\right)\ ,$$

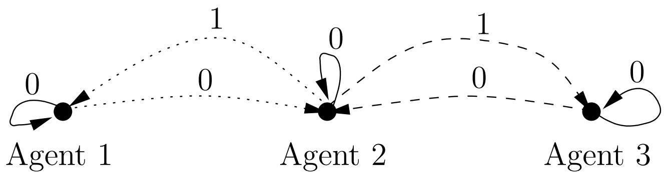

Figure 9: Admissible graph representing the information flow for Example 8: even (dots) and odd (dash) times.

where the infimum is taken over all sequences $\zeta(1),\dots,\zeta(T'')$ in X^{hn} such that $\zeta(0)=\tilde{x}(t_0)$ and, for all $t=1,\dots,T''$, for all $k\in\mathcal{N}$,

$$\zeta_{k,0}(t)\in e_k(\mathcal{A}(t_0+t))(\zeta(t-1)) \text{ and } \zeta_{k,j}(t)=\zeta_{k,j-1}(t-1) \text{ for } j\in\mathcal{H}\setminus\{0\}\ .$$

The meaning of the previous line is precisely that the infimum is computed over *all possible trajectories* of the difference inclusion (1). Now, the collection of $\zeta(t)\in X^{hn}$, $t=1,\dots,T''$ satisfying the previous condition is nonempty and compact for all initial values $\tilde{x}(t_0)\in X^{hn}$. Indeed, by Assumption A, the set-valued functions $e_k(\mathcal{A})$ are continuous and take nonempty, compact values.

The quantity to be minimized is strictly positive when the hn components of $\zeta(0)$ are not all equal, due to the strict decrease property of $\tilde{V}$ established above and Assumption B.4. Also, the expression to be minimized is lower semicontinuous with respect to $\zeta(1),\dots,\zeta(T'')$, as $\tilde{V}$ is continuous (by the continuity hypothesis contained in Assumption B) and μ is lower semicontinuous (by Lemma 1). Thus, $\beta(\tilde{x}(t_0))>0$, except if all the components of $\tilde{x}(t_0)$ are all equal. In other words, β is definite positive with respect to $\{\tilde{x}\in X^{hn}\ :\ x_1=\cdots=x_{hn}\}$.

By Assumption B.5, the map $X^{hn}\to\mathbb{R}^+$, $\tilde{x}\mapsto\mu(\tilde{V}(\tilde{x}))$ is continuous. The proof of Theorem 4 is then achieved as for [M1, Theorem 3], by use of a result on set-valued Lyapunov functions. The latter, Theorem 5, is an extension of [M1, Theorem 1] to differential inclusions, given in Appendix C. □

Example 8. The necessity for each agent to take into account the undelayed values of its own position may be seen by the following counter-example, see Figure 9. Here, $n=3$ and $h=2$. Let the graphs $\mathcal{A}(t)$ be defined by

$$2\overset{1}{\sim}_{\mathcal{A}(2t)}1,\ 1\overset{0}{\sim}_{\mathcal{A}(2t)}2 \text{ and } 2\overset{1}{\sim}_{\mathcal{A}(2t+1)}3,\ 3\overset{0}{\sim}_{\mathcal{A}(2t+1)}2\ .$$

In other words, agent 2 sends alternativaly to agent 1 and 3 the value of its position at the previous instant, and receives in the same time the present position value of the same agent , see Figure 9. Assume the agents use at time t the value of their position at time $t-1$ to elaborate the update applied at time $t+1$. Clearly, for the corresponding admissible graph sequence, the agent 2 is connected to all other agents across any interval $[t,t+1]$. However, one sees easily that provided that the agents 1 and 3 are located initially at different positions, the positions of agent 2 at even and odd times tend in general toward two different values. As indicated by the existence of periodic motions, the strict decrease of the map $t\mapsto\mu(\tilde{V}(\tilde{x}(t)))$ may fail.

4 Conclusion

We studied in this paper convergence of the global behavior for a large class of discrete-time multi-agent systems, containing in particular models previously studied by D.P. Bertsekas and J.N. Tsitsiklis [BT] and by L. Moreau [M1, M2, M3]. The general nonlinear setting introduced permits to model time delays in the inter-agent communications and also to weaken the assumption usually done that the future evolution occurs inside some convex hull of present and past states. This latter feature permits e.g. to solve the rendezvous problem for a population of agents located in a a given domain delimited by some complicated boundary.

The central result of the paper has characterized global convergence by a connectedness property, much in the spirit of Moreau's contributions. Future work will consider quantitative aspects of the convergence, as well as robustness with respect to uncertainty.

Appendix

A Proof of Lemma 1

1. We first recall that the length of a Lipschitz arc $\psi(\lambda)$ defined on $[0,1]$ is equal to

$$\texttt{length}(\psi) \doteq \int_0^1 \left\| \frac{d\psi}{d\lambda} \right\| \cdot d\lambda \ .$$

Notice that, by Rademacher's theorem, ψ Lipschitz implies differentiability almost everywhere, and therefore the previous integral is well defined. Let $x^0, x^1 \in S$. Taking $\varphi : X \to X$ as in (2), define the map $\psi : [0,1] \to X$

$$\psi(\lambda) \doteq \varphi^{-1}\left((1-\lambda)\varphi(x^0) + \lambda\varphi(x^1)\right) \ . \tag{11}$$

As $\varphi(S)$ is convex, ψ maps $[0,1]$ in S. Moreover, due to the regularity assumption on φ, it is a Lipschitz arc, and $\psi(0) = x^0, \psi(1) = x^1$. Thus the set $\left\{\texttt{length}(\psi) \ : \ \psi : [0,1] \overset{\text{Lipschitz}}{\longrightarrow} S, \psi(0) = x^0, \psi(1) = x^1\right\}$ is non-void, and the definition of the map $d_S(x^0, x^1)$ given in the statement is meaningful.

Let us show its continuity with respect to $x^0, x^1 \in S$. Let $(x^0, x^1) \in S \times S$. Consider a sequence $(x^\varepsilon)_{\varepsilon>0}$ of elements of S tending towards x^0. Let $\psi^{\varepsilon|0}$ be a fixed Lipschitz arc linking x^ε to x^0. For any piecewise Lipschitz arc $\psi^{0|1}$ linking x^0 to x^1, one may construct by concatenation of $\psi^{\varepsilon|0}$ and $\psi^{0|1}$ another Lipschitz arc $\psi^{\varepsilon|1}$ linking x^ε to x^1. One has

$$\texttt{length}(\psi^{\varepsilon|1}) = \texttt{length}(\psi^{\varepsilon|0}) + \texttt{length}(\psi^{0|1}) \ ,$$

so

$$d_S(x^\varepsilon, x^1) \leq d_S(x^\varepsilon, x^0) + d_S(x^0, x^1) \ .$$

Arguing similarly, one gets that $|d_S(x^\varepsilon, x^1) - d_S(x^0, x^1)| \leq d_S(x^\varepsilon, x^0)$.

On the other hand, one may take ψ as in (11), in such a way that

$$\inf\left\{\texttt{length}(\psi) \ : \ \psi : [0,1] \overset{\text{Lipschitz}}{\longrightarrow} S, \psi(0) = x^\varepsilon, \psi(1) = x^0\right\} \leq \left\| \frac{d\psi}{d\lambda} \right\|_{L^\infty} \|x^\varepsilon - x^0\| \ ,$$

which shows the desired continuity property.

Defining $\mu(S)$ as in (3), one has $\mu(S) < +\infty$, because the image of a compact set by a continuous function is bounded. Moreover, if $\mu(S) = 0$, then, for any $(x^0, x^1) \in S \times S$, the length defined by the map ψ in (11) is zero, that is $x^0 = x^1$ and S is a singleton. Conversely, if S is a singleton, then $\mu(S) = 0$. Last, for any Lipschitz arc ψ linking x^0 to x^1 and defined as in (11),

$$\texttt{length}(\psi) = \int_0^1 \left\| \frac{d\psi}{d\lambda} \right\| \cdot d\lambda \geq \left\| \int_0^1 \frac{d\psi}{d\lambda} \cdot d\lambda \right\| = \|x^0 - x^1\| \ ,$$

and this shows that $\mu(S)$ is at least equal to the maximal euclidian distance between two points of S, that is its diameter. The equality when S is convex is straightforward, taking the identity for φ in (11).

We now prove the lower semicontinuity of μ. Let $S \in \mathcal{S}$, and a sequence of sets $S_n \in \mathcal{S}$ tending towards S for the topology induced by Hausdorff distance. Our goal is to prove that:

$$\liminf_{n\to+\infty} \mu(S_n) \geq \mu(S) \ . \tag{12}$$

Obviously, in order to establish inequality (12), it is sufficient to consider only sets S_n such that $\mu(S_n)$ is bounded from above by a given constant, say by twice the value of $\mu(S)$. Let $\varepsilon > 0$, and consider two arbitrary sequences $x_n^0, x_n^1 \in S_n$ and a sequence of Lipschitz arcs ψ_n linking x_n^0 to x_n^1 in S_n and of length at most equal to $d_{S_n}(x_n^0, x_n^1) + \varepsilon$. We thus have:

$$\liminf_{n\to+\infty} d_{S_n}(x_n^0, x_n^1) + \varepsilon \geq \liminf_{n\to+\infty} \texttt{length}(\psi_n) \geq \liminf_{n\to+\infty} d_{S_n}(x_n^0, x_n^1) \ . \tag{13}$$

However, due to the previous remark on the boundedness of the sequence $\mu(S_n)$, one may assume without loss of generality that the arcs ψ_n are covered with a rate of variation $\|\frac{d\psi_n}{d\lambda}\|_{L^\infty}$ uniformly bounded. Indeed, if this is not the case, replace ψ_n by the map $\tilde{\psi}_n$ defined by

$$\tilde{\psi}_n \doteq \psi_n \circ \theta^{-1}, \ \theta(t) \doteq \frac{\int_0^t \left\| \frac{d\psi_n}{d\lambda} \right\| \cdot d\lambda}{\int_0^1 \left\| \frac{d\psi_n}{d\lambda} \right\| \cdot d\lambda} \ .$$

The map $\tilde{\psi}_n$ has the same image and length as ψ_n, but the norm of its derivative is equal almost everywhere on $[0, 1]$ to the constant $\int_0^1 \left\| \frac{d\psi_n}{d\lambda} \right\| \cdot d\lambda$. In particular,

$$\liminf_{n\to+\infty} \texttt{length}(\tilde{\psi}_n) = \liminf_{n\to+\infty} \texttt{length}(\psi_n) \ . \tag{14}$$

The arcs considered at this stage are thus equicontinuous. By compactness, one deduces that there exist subsequences (denoted similarly x_n^0, x_n^1 and $\tilde{\psi}_n$) such that

$$x_n^0 \to x^0 \in S, \ x_n^1 \to x^1 \in S, \ \tilde{\psi}_n \to \tilde{\psi} \in \text{Lipschitz}([0, 1]; S) \ .$$

In particular, since by Arzela-Ascoli $\tilde{\psi}_n \to \tilde{\psi}$ uniformly over compact sets, we also have $\texttt{length}(\tilde{\psi}_n) \to \texttt{length}(\tilde{\psi})$. Thus, since $\mu(S_n) \geq d_{S_n}(x_n^0, x_n^1)$, we have:

$$\begin{aligned} \liminf_{n\to+\infty} \mu(S_n) + \varepsilon \ &\geq \ \lim_{n\to+\infty} d_{S_n}(x_n^0, x_n^1) + \varepsilon \\ &\geq \ \liminf_{n\to+\infty} \texttt{length}(\tilde{\psi}_n) = \texttt{length}(\tilde{\psi}) \\ &\geq \ d_S(x^0, x^1) \end{aligned} \tag{15}$$

By arbitrariety of ε:

$$\liminf_{n\to+\infty} \mu(S_n) \geq d_S(x^0, x^1).$$

We finally use arbitrariety of x^0,x^1 in S (arbitrary converging sequences in S_n yield arbitrary limit points in S since $S_n \to S$) to conclude $\liminf_{n\to+\infty} \mu(S_n) \geq \mu(S)$. The lower semicontinuity of μ is demonstrated.

Last, to prove that the map μ is nowhere continuous, it suffices to construct, for any $S \in \mathcal{S}$, a sequence of sets $S_n \in \mathcal{S}$ containing S and contained in a $\frac{1}{n}$-ball around S, and whose diameter exceeds n. The construction, although cumbersome, does not present difficulty and is left to the reader.

2. To prove that the definition of $\texttt{ri}(S)$ is independent of φ, consider two set $\texttt{ri}(S)$, obtained by two different bi-Lipschitz maps $\varphi^1, \varphi^2 : X \to X$, such that $\varphi^1(S)$ and $\varphi^2(S)$ are convex. It is sufficient to show that $f \doteq \varphi^2 \circ (\varphi^1)^{-1}$, which is bi-Lipschitz and maps the convex set $\varphi^1(S)$ into the convex set $\varphi^2(S)$, maps the relative interior one-to-one into the relative interior of the image. Otherwise said, we are led to show that: if the image of a compact convex set S by a bijective bi-Lipschitz map $f : X \to X$ is a convex set, then the image of the relative interior $\texttt{ri}(S)$ is the relative interior of the image of S.

Let $x \in \texttt{ri}(S)$. Consider the restriction of f to the affine hull $\texttt{ah}(S)$ of S. The set $\texttt{ri}(S)$ is convex, thus there exists a convex neighborhood $V \subset S$ of x in $\texttt{ah}(S)$. By continuity of f^{-1}, the image of V through f is a neighborhood of $f(x)$ in $f(\texttt{ah}(S))$. By hypothesis, $f(S)$ is convex, thus there exists a convex neighborhood $W \subset f(S)$ of $f(x)$ in $\texttt{ah}(f(S))$. Now, the intersection $f(V) \cap W \subset f(S)$ is a neighborhood of $f(x)$ in

$\mathtt{ah}(f(S))$, so $f(x) \in \mathtt{ri}(f(S))$. We thus get $\mathtt{ri}(S) \subseteq f^{-1}(\mathtt{ri}(f(S)))$, and one shows similarly the converse inclusion.

Let $S \in \mathcal{S}$. If $\mathtt{ri}(S) = \emptyset$, then $\mathtt{ri}(\varphi(S)) = \emptyset$ and S is a singleton. Conversely, if S is a singleton, $\varphi(S)$ is also a singleton and $\mathtt{ri}(\varphi(S)) = \emptyset$.

It is clear that $\mathtt{ri}\ (S) \subset S$. Also, the fact that $\mathtt{int}\ (\varphi(S)) \subseteq \mathtt{ri}\ (\varphi(S)) \subset \varphi(S)$ implies that

$$\begin{aligned}\mathtt{int}\ S &= \mathtt{int}\ \varphi^{-1}(\varphi(S)) \subseteq \varphi^{-1}(\mathtt{int}\ (\varphi(S))) \quad (\text{as } \varphi^{-1} \text{ is continuous}) \\ &\subseteq \varphi^{-1}(\mathtt{ri}\ (\varphi(S))) = \mathtt{ri}\ (S) \quad (\text{by definition of } \mathtt{ri}\ (S))\ .\end{aligned}$$

Last, when S is a convex set, one may take for φ in (2) the identity, and this proves that in this case $\mathtt{ri}(S)$ is equal to the relative interior of S.

This ends the proof of Lemma 1.

B Proof of Proposition 1

1. By Assumption B.1 it follows from card $S > 1$ that card $\sigma(S) > 1$. Since $\sigma(S)$ is not a singleton, it is homeomorphic to a sphere of non-zero dimension. Hence, $\mathtt{ri}\ \sigma(S) \neq \emptyset$. Moreover, $\mu(\sigma(S)) \geq \mathrm{diam}(\sigma(S)) > 0$, since the euclidean diameter of a set vanishes if and only if the set is a singleton.

2. Let $x \in S$. Since S is not a singleton, by Assumption B.4, there exists $\Sigma_x \subset \mathtt{r}\partial\sigma(S)$ so that $\Sigma_x \cap S \neq \emptyset$. Let $y \in \Sigma_x \cap S$. Since $y \in S$ we can apply the same property to y in order to conclude that there exists $\Sigma_y \subset \mathtt{r}\partial\sigma(S)$ and such that $\Sigma_y \cap S \neq \emptyset$. Let $z \in \Sigma_y \cap S$. Since by assumption $y \notin \Sigma_y$ we have $y \neq z$. Therefore y and z both belong to $S \cap \mathtt{r}\partial\sigma(S)$ and are different from each other. This completes the proof of Point 2.

3. Let S, S' as in the statement. Due to Assumption B.3, $\sigma(S') \subseteq \sigma(S)$.

On the other hand, the set $\sigma(S')$ is in $\mathcal{S}$. In addition it is closed and there exists a map φ as in the definition of $\mathcal{S}$ such that $\varphi(\sigma(S'))$ is convex (say a sphere). For sufficiently small $\varepsilon > 0$, the set gathering all the interior points of $\varphi(\sigma(S'))$ located at a distance *at least equal* to ε of its relative boundary is non-void and convex. It is also compact.

Let S'_ε be the image of this set by φ^{-1}. By construction, the set S'_ε is an element of $\mathcal{S}$, and a subset of S', and thus of S too. One has: $d_{\sigma(S)}(S'_\varepsilon, \mathtt{r}\partial\sigma(S)) > d_{\sigma(S)}(S', \mathtt{r}\partial\sigma(S)) \geq 0$. Then, application of Assumption B.4b with any x in $\sigma(S)$ yields: $\mu(\sigma(S'_\varepsilon)) < \mu(\sigma(S))$. This implies that $\limsup_{\varepsilon\to 0^+} \mu(\sigma(S'_\varepsilon)) \leq \mu(\sigma(S))$.

On the other hand, Assumption B.5 on continuity of the map $\mu \circ \sigma$ implies that $\lim_{\varepsilon\to 0^+} \mu(\sigma(S'_\varepsilon))$ exists and is equal to $\mu(\sigma(S'))$, because, due to Lipschitzness of φ^{-1}, S'_ε tends towards S' for the Hausdorff topology. This provides the inequality in Point 3.

4. Let Σ_i, $i \in I$, be a collection of sets fulfilling Assumption B.4, where I is a possibly infinite index set. Let us show that the set

$$\Sigma \doteq \bigcup_{i\in I} \Sigma_i$$

fulfills Assumption B.4 too. Indeed, each set Σ_i is included in $\mathtt{r}\partial\sigma(S)$, so the same is true for Σ. Also, each Σ_i has a non-void intersection with S, so the same holds for the union, and the point x is located outside of each Σ_i, so it is neither an element of Σ.

Now, let $S' \subseteq \sigma(S)$ be such that $\mathtt{ri}\ \sigma(S') \cap \Sigma \neq \emptyset$. There exists at least one index i such that $\mathtt{ri}\ \sigma(S') \cap \Sigma_i \neq \emptyset$. Consequently, $S' \subseteq \Sigma_i$, and $S' \subseteq \Sigma$.

Last, assume that $d_{\sigma(S)}(S', \Sigma) > 0$. ¿From the inclusion $\Sigma_i \subseteq \Sigma$, one deduces that $d_{\sigma(S)}(S', \Sigma_i) \geq d_{\sigma(S)}(S', \Sigma)$ for all $i \in I$. Thus, $d_{\sigma(S)}(S', \Sigma_i) > 0$, and we deduce that $\mu(\sigma(S')) < \mu(\sigma(S))$. This ends the proof of Point 4.

5. Let $x' \neq x$ such that $x' \in \mathtt{cl}\ \Sigma_x$. We shall prove that $\Sigma' \doteq \Sigma_x \cup \{x'\}$ fulfills the hypotheses of Assumption B.4.

First, it is clear that $\Sigma' \subseteq \mathtt{r}\partial\sigma(S)$. Also, $\emptyset \neq \Sigma_x \cap S \subseteq \Sigma' \cap S$. Additionally, Σ' fulfills Assumption B.4b: let $S' \subseteq \sigma(S)$, assume that $d_{\sigma(S)}(S', \Sigma') > 0$. ¿From the inclusion $\Sigma_x \subseteq \Sigma'$, we have $d_{\sigma(S)}(S', \Sigma_x) \geq d_{\sigma(S)}(S', \Sigma')$, and thus $d_{\sigma(S)}(S', \Sigma_x) > 0$, which is sufficient to deduce $\mu(\sigma(S')) < \mu(\sigma(S))$.

We now verify that Assumption B.4a holds for Σ', and that the latter set does not contain x (that is $x' \neq x$). Let $S' \subseteq \sigma(S)$ and assume that $\mathtt{ri}\ \sigma(S') \cap \Sigma' \neq \emptyset$. We intend to show that $S' \subseteq \Sigma'$. Of course, if $\mathtt{ri}\ \sigma(S') \cap \Sigma_x \neq \emptyset$, then $S' \subseteq \Sigma_x \subseteq \Sigma'$. Assume otherwise that $\mathtt{ri}\ \sigma(S') \cap \Sigma_x = \emptyset$ and $x' \in \mathtt{ri}\ \sigma(S')$. Consider the bi-Lipschitz map φ associated to the set $\sigma(S') \in \mathcal{S}$ (see (2)). The point $\varphi(x')$ is contained in the set $\varphi(\mathtt{ri}\ \sigma(S')) = \mathtt{ri}\ \varphi(\sigma(S'))$, which is *open* when considered as a topological subspace of the affine hull $\mathtt{ah}\ \varphi(\sigma(S'))$ of $\varphi(\sigma(S'))$. This point is also located in the intersection $\varphi(\mathtt{cl}\ \Sigma_x) \cap \mathtt{ah}\ \varphi(\sigma(S'))$, which is a closed set in $\mathtt{ah}\ \varphi(\sigma(S'))$, equal to $\mathtt{cl}\ \varphi(\Sigma_x) \cap \mathtt{ah}\ \varphi(\sigma(S'))$. Yet, no other point of $\varphi(\Sigma_x)$ is located in the neighborhood of $\varphi(x')$ (in $\mathtt{ah}\ \varphi(\sigma(S'))$), as $\mathtt{ri}\ \sigma(S') \cap \Sigma_x = \emptyset$. This is impossible. Thus, if $x' \in \mathtt{ri}\ \sigma(S')$, then $\mathtt{ri}\ \sigma(S') \cap \Sigma_x \neq \emptyset$, and $S' \subseteq \Sigma'$, as announced.

In conclusion, $\Sigma' = \Sigma_x \cup \{x'\}$ fulfills all the hypotheses of Assumption B.4, as claimed in the beginning of the proof. Due to the maximality of Σ_x, we deduce $\Sigma_x = \Sigma'$, and finally $\Sigma_x = \mathtt{cl}\ \Sigma_x \setminus \{x\}$, as enounced in the statement of Point 5.

6. It is straightforward to show that $\Sigma_x|_S$ fulfils Assumption B.4 for $\sigma(S)$, so that certainly $\Sigma_x|_S \subseteq \Sigma_x|_{\sigma(S)}$.

To show the reverse inclusion, one must first establish that $\Sigma_x|_{\sigma(S)} \cap S \neq \emptyset$, but that precisely comes from the fact that $\Sigma_x|_S \subseteq \Sigma_x|_{\sigma(S)}$ and that, by definition, $\Sigma_x|_S \cap S \neq \emptyset$. The remaining of the proof of Point 6 is evident.

7. We denote for short Σ, resp. Σ', instead of $\Sigma_x|_S$, resp. $\Sigma_x|_{S'}$. We intend to prove that $\Sigma^* \doteq \Sigma \cap \sigma(S') \subseteq \Sigma'$. The principle of the demonstration consists in establishing that $\Sigma^{**} \doteq \Sigma^* \cup \Sigma'$ is contained in Σ'.

First, remark that $x \notin \Sigma^* = \Sigma \cap \sigma(S')$, because $x \notin \Sigma$.

We now prove that $\Sigma^* \subseteq \mathtt{r}\partial\sigma(S')$. If this is not true, then $\Sigma \cap \mathtt{ri}\ \sigma(S') \neq \emptyset$, because $\sigma(S')$ is equal to the union of the two disjoint sets $\mathtt{r}\partial\sigma(S')$ and $\mathtt{ri}\ \sigma(S')$. Now, applying Assumption B.4a yields $\sigma(S') \subseteq \Sigma$. But $x \in S'$, so we deduce $x \in \Sigma$, which contradicts Assumption B.4a.

We now show that Σ^* fulfills the property defined in Assumption B.4a. Let $S'' \subseteq \sigma(S')$ such that $\mathtt{ri}\ \sigma(S'') \cap \Sigma^* \neq \emptyset$. Then $S'' \subseteq \Sigma$, because $\Sigma = \Sigma_x|_S$, and $S'' \subseteq \sigma(S')$, thus $S'' \subseteq \Sigma^*$.

One shows, as in the proof of Proposition 1, that $\Sigma^{**} = \Sigma^* \cup \Sigma'$ fulfills nice properties: first, $\Sigma^{**} \subseteq \mathtt{r}\partial\sigma(S')$ and $x \notin \Sigma^{**}$, because the same is true for Σ' and for Σ^* (see above). Analogously, Σ^{**} fulfills the property defined in Assumption B.4a, as both its components Σ^* and Σ' do. Let us now show that Σ^{**} fulfills Assumption B.4b. Let $S'' \subseteq \sigma(S')$, such that $d_{\sigma(S')}(S'', \Sigma^{**}) > 0$, then $d_{\sigma(S')}(S'', \Sigma') \geq d_{\sigma(S')}(S'', \Sigma^{**}) > 0$, and thus $\mu(\sigma(S'')) < \mu(\sigma(S'))$, which shows the desired property.

Hence, Σ^{**} fulfills *all* the properties of Assumption B.4. Consequently, due to the fact previously established that Σ' is maximal among the sets of this kind, one has $\Sigma^* \cup \Sigma' = \Sigma^{**} \subseteq \Sigma'$. This yields $\Sigma^* \subseteq \Sigma'$ and ends the demonstration of Point 7, and consequently of Proposition 1.

C Stability based on set-valued Lyapunov functions

The following result is an adaptation of [M1, Theorem 1] to difference inclusions. For sake of completeness, a proof is provided, intimately linked to the proof of Moreau's result.

Theorem 5. *Let $\mathcal{X}$ be a finite-dimensional Euclidean space and consider a continuous set-valued map $e : \mathbb{N} \times \mathcal{X} \rightrightarrows \mathcal{X}$ taking on closed values, giving rise to the difference inclusion* (6). *Let Ξ be a collection of equilibrium solutions and denote the corresponding set of equilibrium points by Φ. Consider an upper semicontinuous [AC, p. 41] set-valued function $V : X \rightrightarrows X$ satisfying*

1. $x \in V(x), \forall x \in \mathcal{X}$;
2. $\displaystyle\bigcup_{y \in e(t,x)} V(y) \subseteq V(x), \forall t \in \mathbb{N}, \forall x \in \mathcal{X}.$

If $V(\phi) = \{\phi\}$ for all $\phi \in \Phi$ then the dynamical system is uniformly stable with respect to Φ. If $V(x)$ is bounded for all $x \in \mathcal{X}$ then the dynamical system is uniformly bounded with respect to Φ.

Consider in addition a function $\mu : Image(V) \to \mathbb{R}_{\geq 0}$ and a lower semicontinuous function $\beta : \mathcal{X} \to \mathbb{R}_{\geq 0}$ satisfying

3. *$\mu \circ V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ is bounded uniformly with respect to x in bounded subsets of $\mathcal{X}$;*
4. *β is positive definite with respect to Φ that is, $\beta(\phi) = 0$ for all $\phi \in \Phi$ and $\beta(x) > 0$ for all $x \in \mathcal{X} \setminus \Phi$;*
5. $\sup\limits_{y \in e(t,x)} \mu(V(y)) - \mu(V(x)) \leq -\beta(x), \forall t \in \mathbb{N}, \forall x \in \mathcal{X}$.

If $V(\phi) = \{\phi\}$ for all $\phi \in \Phi$ and $V(x)$ is bounded for all $x \in \mathcal{X}$ then the dynamical system is uniformly globally asymptotically stable with respect to Φ.

The results stated above remain true if, for a fixed $\tau \in \mathbb{N}$, the decrease relations in 2. and 5. occur between $V(y_{i+1})$ and $V(y_i)$, $i = 0, \ldots, \tau - 1$, $y_0 = x$, $y_\tau = y$, instead of $V(y)$ and $V(x)$.

Proof. (Uniform stability.) Consider arbitrary $\varphi \in \Phi$ and $c_2 > 0$. If $V(\varphi) = \{\varphi\}$ then, by upper semicontinuity of V, there is $c_1 > 0$ such that $V(x) \subset B(\varphi, c_2)$ for all $x \in B(\varphi, c_1)$. Consider arbitrary $t_0 \in \mathbb{N}$ and $x_0 \in B(\varphi, c_1)$ and let ζ denote any solution of inclusion (6) with $\zeta(t_0) = x_0$. Conditions 1 and 2 of Theorem 5 imply that, for all $t \geq t_0$,

$$\zeta(t) \in \bigcup_{y \in e(t,x_0)} V(y) \subseteq V(x_0) \subset B(\varphi, c_2) \ .$$

(Uniform boundedness.) Consider arbitrary $\varphi \in \Phi$ and $c_1 > 0$. If $V(x)$ is bounded for all $x \in \mathcal{X}$ then, by upper semicontinuity of V, there is $c_2 > 0$ such that $V(x) \subset B(\varphi, c_2)$ for all $x \in B(\varphi, c_1)$. Consider arbitrary $t_0 \in \mathbb{N}$ and $x_0 \in B(\varphi, c_1)$ and let ζ be any solution of (6) with $\zeta(t_0) = x_0$. Conditions 1 and 2 of Theorem 5 imply that for all $t \geq t_0$,

$$\zeta(t) \in \bigcup_{y \in e(t,x_0)} V(y) \subseteq V(x_0) \subset B(\varphi, c_2) \ .$$

(Uniform global asymptotic stability.) It remains only to prove uniform global attractivity with respect to Ξ.

Consider arbitrary $\varphi_1 \in \Phi$ and $c_1 > 0$. If $V(x)$ is bounded for all $x \in \mathcal{X}$ then, by upper semicontinuity of V, there is a compact set $K \subset \mathcal{X}$ such that $V(x) \subseteq K$ for all $x \in B(\varphi_1, c_1)$. Similarly as above, Conditions 1 and 2 of Theorem 5 imply that every solution of (6) initiated in $B(\varphi_1, c_1)$ remains in K.

Consider in addition arbitrary $c_2 > 0$. If $V(\varphi) = \{\varphi\}$ for all $\varphi \in \Phi$ then, by upper semicontinuity of V, there is $c_3 > 0$ such that for all $x \in B(\Phi \cap K, c_3)$ there is $\varphi_2 \in \Phi$ such that $V(x) \subset B(\varphi_2, c_2)$. Similarly as above, Conditions 1 and 2 of Theorem 5 imply that every solution of (6) entering $B(\Phi \cap K, c_3)$ remains in a c_2-ball around some equilibrium point $\varphi_2 \in \Phi$.

It remains to prove the existence of $T \geq 0$ such that every solution of (6) starting in $B(\varphi_1, c_1)$ cannot remain longer than T subsequent times in K without entering $B(\Phi \cap K, c_3)$. In agreement with Conditions 3 and 4 of Theorem 5 and the lower semicontinuity of β, we introduce two real numbers:

$$M \doteq \sup_{x \in B(\varphi_1, c_1)} \mu(V(x)) < \infty \quad \text{and} \quad \Delta \doteq \min_{x \in K \setminus B(\Phi, c_3)} \beta(x) > 0 \ .$$

Let $T \geq 0$ be such that $T\Delta > M$. Consider arbitrary $t_0 \in \mathbb{N}$ and $x_0 \in B(\varphi_1, c_1)$ and let ζ denote any solution of (6) with $\zeta(t_0) = x_0$. Then Condition 5 of Theorem 5 implies that for some $t_1 \in [t_0, t_0 + T]$, $\zeta(t_1) \in B(\Phi \cap K, c_3)$, since otherwise $\zeta(t) \in K \setminus B(\Phi, c_3)$ for all $t \in [t_0, t_0 + T]$ and

$$\mu(V(\zeta(t_0 + T))) \leq \mu(V(\zeta(t_0))) - T \min_{x \in K \setminus B(\Phi, c_3)} \beta(x) \leq M - T\Delta < 0 \ ,$$

contradicting that μ takes only non-negative values. Putting everything together, we conclude that for some $\varphi_2 \in \Phi$ and for all $t \geq t_0 + T$,

$$\zeta(t) \in V(\zeta(t)) \subseteq \mu(V(\zeta(t_1)) \subset B(\varphi_2, c_2) \ .$$

This achieves the proof of Theorem 5. $\square$

References

[AOSY] H. Ando, Y. Oasa, I. Suzuki and M. Yamashita, Distributed memoryless point convergence algorithm for mobile robots with limited visibility, *IEEE Trans. on Robotics and Automation*, Vol. 15 (5), 1999, 818–828.

[AC] J.P. Aubin and A. Cellina, *Differential Inclusions: set-valued maps and viability theory*, Springer-Verlag, Berlin, 1984.

[BT] D.P. Bertsekas and J.N. Tsitsiklis, *Parallel and distributed computation*, Prentice-Hall International, 1989; republished by Athena Scientific, 1997, and also downloadable at `https://dspace.mit.edu/handle/1721.1/3719`

[BHOT] V.D. Blondel, J.M. Hendrickx, A. Olshevsky and J.N. Tsitsiklis, Convergence in Multiagent Coordination, Consensus, and Flocking, *Proceedings of the Joint European Control Conference/IEEE Conference on Decision and Control*, Sevilla, Spain, 2005.

[CMB] J. Cortés, S. Martínez and F. Bullo, Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions. *IEEE Trans. Automatic Control* 51 (6), 2006 (to appear)

[FM] J.A. Fax and R.M. Murray, Information flow and cooperative control of vehicle formations. *IEEE Trans. Automatic Control* 49 (9), 2004, 1465–1476.

[G] P.M. Gade, Feedback control in coupled map lattices. *Physical Review E*, Vol. 57 (6), 1998, 7309–7312.

[GHM] V. Gupta, B. Hassibi and R.M. Murray, Stability analysis of stochastically varying formations of dynamic agents. *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui HI, USA, 2003, 504–509.

[HM] Y. Hatano and M. Mesbahi, Agreement over random networks. *IEEE Trans. Automatic Control* 50 (11), 2005, 1867–1872.

[H] M.W. Hirsch, Convergent activation dynamics in continuous time networks. *Neural Networks*, Vol. 2, 1989, 331–349.

[JLM] A. Jadbabaie, J. Lin and A.S. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, *IEEE Trans. Automatic Control*, Vol. 48 (6), 2003, 988–1001.

[JMB] A. Jadbabaie, N. Motee and M. Barahona, On the stability of the Kuramoto model of coupled nonlinear oscillators. *Proc. of the American Control Conference*, Boston MS, USA, 2004.

[JSZ] K.H. Johansson A. Speranzon and S. Zampieri, On quantization and communication topologies in multi-vehicle rendezvous. In *Proc. of 16th IFAC World Congress*, Prague, Czech Republic, 2005.

[LF] N.E. Leonard and E. Fiorelli, Virtual leaders, artificial potentials and coordinated control of groups. *Proc. of the 40th IEEE Conf. on Decision and Control*, Orlando, FL, USA, 2001, 2968–2973.

[LMA1] J. Lin, A.S. Morse and B.D.O. Anderson, The multi-agent rendezvous problem. *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui HI, USA, 2003, 1508–1513.

[LMA2] J. Lin, A.S. Morse and B.D.O. Anderson, The multi-agent rendezvous problem. An extended summary. *Cooperative control*, , Lecture Notes in Control and Inform. Sci. 309, Springer, Berlin, 2005, 257–289.

[MM] S.C. Manrubia and A.S. Mikhailov, Mutual synchronization and clustering in randomly coupled chaotic dynamical networks, *Physical Review* E 60, 1579 (1999); also downloadable at `arXiv:cond-mat/9905083`

[M1] L. Moreau, Time-dependent unidirectional communication in multi-agent systems, 2003, `arXiv:math.OC/0306426`

[M2] L. Moreau, A note on leaderless communication via bidirectional and unidirectional time-dependent communication, *Proc. of MTNS'04*, Leuven (Belgium), 2004.

[M3] L. Moreau, Stability of multi-agent systems with time-dependent communication links. *IEEE Trans. Automat. Control* Vol. 50 (2), 2005, 169–182

[S] A.V. Savkin, Coordinated collective motion of groups of autonomous mobile robots: analysis of Vicsk's model. *IEEE Trans. Automat. Control* Vol. 49 (6), 2004, 981–983.

[SPL] R. Sepulchre, D. Paley and N. Leonard, Collective motion and oscillator synchronization. *Proc. of the Block Island Workshop on Cooperative Control*, 2003.

[YPP] L. Yang, K.M. Passino, M.M. Polycarpou, Stability Analysis of M-Dimensional Asynchronous Swarms With a Fixed Communication Topology. In *IEEE Trans. Automat. Control* Vol. 48 (1), 2003, 76–95.