Backstepping Design for Time-Delay Nonlinear Systems

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***Abstract*— The backstepping approach is adapted to the problem of globally uniformly asymptotically stabilizing nonlinear systems in feedback form with a delay arbitrarily large in the input. The strategy of design relies on the construction of a Lyapunov-Krasovskii functional. Continuously differentiable control laws are constructed.**

***Index Terms*— Nonlinear, backstepping, delay, control, Lyapunov.**

I. Introduction

One of the most popular nonlinear techniques of control design is the backstepping approach. It is presented for instance in [2], [15], [3]. The key ideas of the approach are the following. If for a system in feedback form, i.e. of the form

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x) + g(x)z, \\ \dot{z} & = & u + h(x,z), \end{array} \right. \tag{1}$$

with $x \in R^{n_x}$, $z \in R$, where $u \in R$ is the input and $f(x), g(x), h(x,z)$ are continuous functions, there exists a continuously differentiable function $z_s(x)$ such that $z_s(0) = 0$ and the x-subsystem of (1) with z replaced by $z_s(x)$ is globally asymptotically stable and if besides is known a positive definite and radially unbounded function $V(x)$ of class C^1 such that

$$W(x) := -\frac{\partial V}{\partial x}(x)[f(x) + g(x)z_s(x)] \tag{2}$$

is positive definite, then the system (1) is globally asymptotically stabilized by

$$\begin{array}{rcl} u(x,z) & = & -k(z - z_s(x)) - h(x,z) \\ & & +\frac{\partial z_s}{\partial x}(x)[f(x) + g(x)z] - \frac{\partial V}{\partial x}(x)g(x) \end{array} \tag{3}$$

where k is a positive real number. Moreover, the derivative of the Lyapunov function

$$U(x,z) \ = \ V(x) + \frac{1}{2}[z - z_s(x)]^2 \tag{4}$$

along the trajectories of (1) in closed-loop with (3) satisfies

$$\dot{U}(x,z) \ = \ -W(x) - k[z - z_s(x)]^2. \tag{5}$$

Many extensions of this basic result have been proved. The multiple advantages offered by this approach are well-known. Observe in particular that this technique yields a wide family of globally asymptotically stabilizing control laws, allows to address robustness issues and to solve adaptive problems.

The objective of the present work is to show how the backstepping approach can be adapted to the problem of stabilizing systems in feedback form with a delay in the input. More precisely, we give sufficient conditions ensuring that a nonlinear system of the form

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t)) + g(x(t))z(t), \\ \dot{z}(t) & = & u(t-\tau) + h(x(t-\tau), z(t-\tau)), \end{array} \right. \tag{6}$$

with $x \in R^{n_x}$, $z \in R$, where $u \in R$ is the input and where τ is a positive real number is globally uniformly asymptotically stabilizable by continuously differentiable state feedback. Roughly speaking, these conditions ensure that the x-subsystem with z as virtual input is globally uniformly asymptotically stabilizable by a control law with a delay τ and that, for a specific family of control laws, the finite escape time phenomenon does not occur. This work completes the families of recent papers devoted to the control of nonlinear systems with delay. In [9] and [11], the technique of [13] is adapted to the problem of stabilizing chains of integrators with bounded controls when there is a delay arbitrarily large in the input. The main result of [9] is extended in [8] to a family of feedforward nonlinear systems. In [10], the problem of stabilizing an oscillator by bounded feedback when there is a delay in the input is solved. In [12], the interconnection of nonlinear systems with delay is studied. In [6], the concept of control Lyapunov function is extended to the case of nonlinear systems with delay through the Razumikhin theorem. In [14], connections between Razumikhin-type theorems and the ISS nonlinear small gain theorems are exposed. The present work is distinguished from the papers mentioned above because on the one hand it is devoted to systems in feedback form and on the other hand the key tool we use to prove the main result is the Lyapunov-Krasovskii functional (see [7], [4]). Surprisingly enough, this family of Lyapunov functionals has been used so far mainly in the context of the stabilization of linear systems through linear control laws, see for instance [1], [5]. We show in this work that it can be also fruitfully exploited to carry out control designs for nonlinear systems. The potential advantages of the knowledge of such a functional are multiple and appealing. Observe in particular that strict Lyapunov functions are known to be very efficient tools for robustness analysis, but this issue is beyond the scope of our work. The stabilizability result we obtain is a global uniform asymptotic stabilizability result for an arbitrarily large delay. The expressions of control laws we exhibit depend on the value of the delay. We want to emphasize that we do not assume that the systems (6) are locally exponentially stabilizable. The example we give in Section IV to illustrate our control design shows that indeed this assumption is not needed.

Organization of the work. Two technical lemmas are stated and proved in Section II. In Section III the backstepping approach

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is adapted to the case of systems with delay in the input. An illustrating example is presented in Section IV.

Definitions and technical preliminaries.

1. We assume throughout the paper that the functions encountered are sufficiently smooth.
2. The argument of the functions and of the functionals will be omitted or simplified whenever no confusion can arise from the context. For example, one may denote a function $f(x(t))$ by simply $f(x)$ or $f(t)$ or $f(\cdot)$ or f.
3. A real-valued function $k(\cdot)$ is of class $\mathcal{K}_\infty$ if it is zero at zero, strictly increasing and unbounded.
4. The following norms will be used: $||\cdot||$ refers to the Euclidean vector norm; $||\phi||_c = \sup_{t\in[-r,0]} ||\phi(t)||$ stands for the norm of a function $\phi \in C^1([-r,0],R^m)$.
5. Let r be a positive real number. For a given $t \geq 0$, $x_t(\cdot)$ denotes the restriction of $x(\cdot)$ to the interval $[t-r,t]$ translated to $[-r,0]$, i.e. $x_t(\theta) = x(t+\theta), \quad \forall\, \theta \in [-r,0]$.

II. Technical lemmas

This section is devoted to the construction of Lyapunov-Krasovskii functionals for two families of systems with delay. These constructions are crucial in establishing the main result of the work.

A. First technical lemma

The following lemma gives conditions ensuring that the x-subsystem of (6) with z as virtual input is globally uniformly asymptotically stabilizable by a control law $z_s(x(t-\tau))$ and provides with a Lyapunov-Krasovskii functional.

Lemma 2.1: Consider the system

$$\dot{x}(t) = f(x(t)) + g(x(t))z_s(x(t-\tau)) \tag{7}$$

where $x \in R^{n_x}$, where $f(x), g(x)$ are continuous functions, where $z_s(x)$ is a continuously differentiable real-valued function such that $z_s(0) = 0$ and τ is a positive real number. Assume that there exist a positive definite and radially unbounded function $V(x)$, a positive definite function $W(x)$ and a positive real number $\Omega \geq 8\tau$ such that the following assumptions are satisfied

H1. For all $x \in R^{n_x}$, the equality

$$\frac{\partial V}{\partial x}(x)[f(x) + g(x)z_s(x)] \;=\; -W(x) \tag{8}$$

holds.

H2. For all $x \in R^{n_x}$ and $\xi \in C^1([0,2\tau],R^{n_x})$, the inequality

$$-\frac{1}{4}W(x) - T(x,\xi) - \frac{1}{\Omega}\int_0^{2\tau} W(\xi(l))dl \leq 0 \tag{9}$$

with

$$\begin{array}{rcl} T(x,\xi) & = & \frac{\partial V}{\partial x}(x)g(x)\displaystyle\int_\tau^{2\tau} H(\xi(l),\xi(l-\tau))dl, \\ H(a,b) & = & \frac{\partial z_s}{\partial x}(a)[f(a)+g(a)z_s(b)], \end{array} \tag{10}$$

holds.

H3. For all function $\xi \in C^1([0,2\tau],R^{n_x})$, there exists a contant $K_\xi \geq 0$ such that, for all $x \in R^{n_x}$ and for all $t \in [0,2\tau]$, the inequality

$$\begin{array}{l} -\frac{1}{2}W(x) + \frac{\partial V}{\partial x}(x)g(x)[z_s(\xi(t)) - z_s(x)] \\ \qquad\qquad \leq K_\xi[V(x)+1] \end{array} \tag{11}$$

holds.

Then the origin of (7) is globally uniformly asymptotically stable. Moreover, along the trajectories of (7), the derivative of the functional

$$U(x_t) = V(x(t)) + \frac{1}{\Omega}\int_{t-2\tau}^{t}\left(\int_s^t W(x(l))dl\right)ds \tag{12}$$

satisfies, for all $t \geq 2\tau$,

$$\dot{U}(t) \leq -\frac{1}{2}W(x(t)). \tag{13}$$

Proof. Let $\varphi \in C^1([-\tau,0],R^{n_x})$ be any initial condition of the system (7). Let $T \in (0,\tau]$ be such that the solution $x(t)$ is defined on $[0,T)$. Then, for all $t \in [0,T)$, the equality

$$\begin{array}{rcl} \dot{V}(t) & = & -W(x(t)) \\ & & +\frac{\partial V}{\partial x}(x(t))g(x(t))[z_s(\varphi(t)) - z_s(x(t))] \end{array} \tag{14}$$

is satisfied. From the inequality (11) in H3, we deduce that there exists $K_\varphi > 0$ such that, for all $t \in [0,T)$

$$\dot{V}(t) \leq K_\varphi[V(x(t))+1]. \tag{15}$$

It follows that, for all $t \in [0,T)$, the inequality

$$V(x(t)) \leq (V(x(0))+1)e^{K_\varphi t} - 1 \tag{16}$$

is satisfied. From this inequality, one can deduce that the finite escape time phenomenon does not occur on $[0,\tau]$; the solutions are well-defined on $[0,\tau]$. By applying this reasoning repeatedly, one can prove that all the solutions are defined on any interval $[k\tau,(k+1)\tau]$, where k is an integer, and thereby on $[0,+\infty[$.

Now, let us observe that, for all $t \geq 2\tau$, the derivative of the functional U defined in (12) along the trajectories of (7) satisfies

$$\begin{array}{rcl} \dot{U} & = & \dot{V} - \frac{1}{\Omega}\left(\displaystyle\int_{t-2\tau}^{t} W(x(l))dl\right) \\ & & +\frac{2\tau}{\Omega}W(x(t)) \\ & = & \frac{\partial V}{\partial x}(x(t))[f(x(t)) + g(x(t))z_s(x(t-\tau))] \\ & & -\frac{1}{\Omega}\left(\displaystyle\int_{t-2\tau}^{t} W(x(l))dl\right) + \frac{2\tau}{\Omega}W(x(t)). \end{array} \tag{17}$$

From (8), we deduce that

$$\begin{array}{rcl} \dot{U} & = & \left(-1+\frac{2\tau}{\Omega}\right)W(x(t)) \\ & + & \frac{\partial V}{\partial x}(x(t))g(x(t))[z_s(x(t-\tau)) - z_s(x(t))] \\ & - & \frac{1}{\Omega}\left(\displaystyle\int_{t-2\tau}^{t} W(x(l))dl\right) \\ & = & \left(-1+\frac{2\tau}{\Omega}\right)W(x(t)) \\ & - & \frac{1}{\Omega}\left(\displaystyle\int_{t-2\tau}^{t} W(x(l))dl\right) \\ & - & \frac{\partial V}{\partial x}(x(t))g(x(t))\displaystyle\int_{t-\tau}^{t}\lambda(x(s),x(s-\tau))ds \end{array} \tag{18}$$

with

$$\lambda(a,b) = \frac{\partial z_s}{\partial x}(a)[f(a)+g(a)z_s(b)] \tag{19}$$

From the inequality $\Omega \geq 8\tau$ and (9), we deduce that, for all $t \geq 2\tau$, the inequality (13) is satisfied. On the other hand, one can check readily that there exist two functions μ_1, μ_2 of class $\mathcal{K}_\infty$ such that, for all function $\phi \in C^1([-2\tau, 0], R^{n_x})$, the functional U satisfies the inequalities

$$\mu_1(||\phi(0)||) \leq U(\phi) \leq \mu_2(||\phi||_c). \tag{20}$$

Finally, from the Krasovskii stability theorem (see [4], [7]), one can deduce that the origin of (7) is globally uniformly asymptotically stable.

B. Second technical lemma

The following lemma establishes the asymptotic stability of a family of linear systems with delay through a Lyapunov-Krasovskii approach.

Lemma 2.2: Consider the system

$$\dot{Z}(t) = -\varepsilon Z(t-\tau) \tag{21}$$

where $Z \in R$ and τ and ε are positive real numbers such that $\varepsilon \in \left(0, \frac{1}{2\tau}\right]$. The origin of this system is globally uniformly asymptotically stable. Moreover, for all $t \geq 2\tau$, the derivative of the functional

$$M(Z_t) \;=\; \frac{1}{2}Z(t)^2 + \varepsilon^3\tau \int_{t-2\tau}^{t} \left(\int_s^t Z(l)^2 dl \right) ds \tag{22}$$

along the trajectories of (21) satisfies, for all $t \geq 2\tau$,

$$\dot{M}(t) \leq -\frac{1}{4}\varepsilon Z(t)^2 - \frac{1}{2}\varepsilon^3\tau \int_{t-2\tau}^{t} Z(s)^2 ds. \tag{23}$$

Proof. The proof of this lemma is omitted. It is similar to the one of Lemma 2.1.

III. Main results

Consider the nonlinear system (6) and introduce a set of assumptions.

Assumption A1. A positive definite, radially unbounded and continuously differentiable function $V(x)$, a continuous and positive definite function $W(x)$, a positive real number $\Omega \geq 8\tau$, a continuously differentiable function $z_s(x)$, satisfying $z_s(0) = 0$, such that the system

$$\dot{x}(t) = f(x(t)) + g(x(t))z_s(x(t-\tau)) \tag{24}$$

satisfies the assumptions H1 to H3 of Lemma 2.1 are known.

Assumption A2. Let C be a positive real number. For all $x \in R^{n_x}$, the inequalities

$$\left| \frac{\partial V}{\partial x}(x)g(x) \right|^2 \;\leq\; W(x)\;,\; \left| \frac{\partial z_s}{\partial x}(x)g(x) \right| \;\leq\; C \tag{25}$$

are satisfied.

Theorem 3.1: Assume that the system (6) satisfies the assumptions A1 and A2. Then the system (6) is globally uniformly asymptotically stabilized by the feedback

$$\begin{array}{rcl} u_s(t) & = & -\varepsilon(z(t) - z_s(x(t-\tau))) - h(x(t), z(t)) \\ & & +\frac{\partial z_s}{\partial x}(x(t))[f(x(t)) + g(x(t))z(t)] \end{array} \tag{26}$$

where ε is a positive real number such that $\varepsilon \in \left(0, \frac{1}{2\tau}\right]$.

Discussion of Theorem 3.1.

• Assumption A1 is only concerned with the x-subsystem of (6). Assumption H1, which ensures the stabilizability of this system with z as virtual input when there is no delay, is not surprising. In the well-known framework of the backstepping for systems without delay, this assumption, or a similar one, is imposed. Assumptions H2 and H3 are introduced to ensure the stabilizability of the x-subsystem of (6) with z as virtual input by a control law with the delay τ.

• The family of control laws (26) and the family of control laws (3) are different, even when $\tau = 0$. The term $-\frac{\partial V}{\partial x}(x)g(x)$ is present in (3) whereas no corresponding term is in (26). The reason is the following. In absence of delay, the system (1) in closed-loop with the control law (26) can be interpreted as a cascade of two passive systems so that the classical damping term $-\frac{\partial V}{\partial x}(x)g(x)$ can be introduced in the control law to turn the passivity into global asymptotic stability. But control designs based on passivity property in general cannot be used when a delay is present: in order to overcome the obstacle, the way we have chosen consists in selecting the control law (26) which results into the cascade of a globally asymptotically stable system and a globally asymptotically stable system with a coupling term. In order to ensure that this coupling term does not destabilize the system, we need to impose Assumption A2.

• In the proof of Theorem 3.1, we shall see that, thanks to Assumption A2, one can construct a Lyapunov-Krasovskii functional for the system (6) in closed-loop with the control law (26) by taking advantage of the two functionals provided by Lemma 2.1 and Lemma 2.2. However, this assumption is not just a convenient tool motivated only by the wish to carry out a Lyapunov construction: It ensures that the finite escape time phenomenon does not occur. It cannot be removed without being replaced by another assumption of a similar type. This fact is enlightened by the system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & -x(t) + x(t)^4 z(t), \\ \dot{z}(t) & = & u(t-\tau), \end{array} \right. \tag{27}$$

because on the one hand, it is globally asymptotically stabilized by the feedback $u(x,z) = -z - x^5$ when $\tau = 0$, but is not globally uniformly asymptotically stabilizable when $\tau > 0$ (see Appendix I) and on the other hand, it is of the form (6), satisfies Assumption A1 with $V(x) = x^2, W(x) = 2x^2, z_s(x) = 0$ but, for any choice of functions $V(x)$ and $z_s(x)$, it does not satisfy Assumption A2.

• We prove in Appendix II that Assumption A1 may be replaced by the assumption consisting of H1, H3 and the slightly more restrictive assumption:

Assumption H2'. For all $\xi \in C^1([0, 2\tau], R^{n_x})$, the inequality

$$\Omega\tau \int_{\tau}^{2\tau} |H(\xi(l), \xi(l-\tau))|^2 dl \leq \int_0^{2\tau} W(\xi(l)) dl \tag{28}$$

where $H(\cdot)$ is the function defined in (10) and where Ω is a positive real number such that $\Omega \geq 8\tau$.

Thanks to Assumptions H2' and A2 one understands that, roughly speaking, a necessary condition for the uniform asymptotic stabilizability of the system (6) is that $z_s(x)$ and its first partial derivatives be sufficiently small in norm. In some

cases, one can take advantage of Assumption H2' to determine suitable functions $z_s(x)$.

• Theorem 3.1 applies to systems which are not locally linearizable. The assumptions of this theorem do not even ensure that the x-subsystem of (6) with z as virtual input is locally exponentially stabilizable. The example we give in Section IV illustrates this remark.

Proof of Theorem 3.1.

The proof consists in constructing a Lyapunov-Krasovskii functional whose derivative along the trajectories the system (6) in closed-loop with the control law (26) is smaller than a negative definite function. To simplify the construction, we perform the change of variable

$$Z(t) \;=\; z(t) - z_s(x(t-\tau)) \tag{29}$$

which transforms (6) into

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & f(x(t)) \\ & + & g(x(t))(Z(t)+z_s(x(t-\tau))), \\ \dot{Z}(t) & = & u(t-\tau) \\ & + & \Upsilon(x(t-\tau), x(t-2\tau), Z(t-\tau)), \\ \Upsilon(a,b,c) & = & h(a, c+z_s(b)) \\ & - & \frac{\partial z_s}{\partial x}(a)[f(a)+g(a)(c+z_s(b))]. \end{array}\right. \tag{30}$$

This system in closed-loop with the control law $u = u_s(\cdot)$ where $u_s(\cdot)$ is the function defined in (26), is

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & f(x(t)) \\ & & +g(x(t))(Z(t)+z_s(x(t-\tau))), \\ \dot{Z}(t) & = & -\varepsilon Z(t-\tau). \end{array}\right. \tag{31}$$

We will take advantage of Lemma 2.1 and Lemma 2.2 to prove that the derivative of the Lyapunov-Krasovskii functional (12) along the trajectories of (31) is smaller than a negative definite function of $(x(t), Z(t))$. Before this analysis, we prove that the finite escape time phenomenon does not occur as follows. (i) The Z-subsystem of (31) is linear: Its solutions are defined for all $t \geq 0$. (ii) From Assumption H3 in Lemma 2.1 and $z_s(0) = 0$, we deduce that there exists $K_0 > 0$ such that, for all $T \in [0, \tau]$, such that the solution is defined on the interval $[0, T)$, the following

$$\begin{array}{rcl} \dot{V}(t) & = & -W(x(t)) \\ & & +\frac{\partial V}{\partial x}(x(t))g(x(t))[z_s(0) - z_s(x(t))] \\ & & +\frac{\partial V}{\partial x}(x(t))g(x(t))Z(t) \\ & \leq & -\frac{1}{2}W(x(t)) + K_0[V(x(t))+1] \\ & & +\frac{\partial V}{\partial x}(x(t))g(x(t))Z(t) \end{array} \tag{32}$$

holds. From Assumption A2, we deduce that

$$\dot{V}(t) \leq K_0[V(x(t))+1] + \frac{1}{2}Z(t)^2. \tag{33}$$

Thanks to this inequality, invoking arguments similar to those used in the proof of Lemma 2.1 to establish that the solutions of (7) are defined for all $t \geq 0$, one can prove that the solutions of the x-subsystem of (31) are defined for all $t \geq 0$. It follows that the solutions of (31) are defined for all $t \geq 0$.

We study now the sign properties of the derivative of the functional U defined in (12) along the trajectories of (31). Using the key ideas of the proof of Lemma 2.1, one can deduce that, for all $t \geq 2\tau$,

$$\begin{array}{rcl} \dot{U}(t) & \leq & -\frac{1}{2}W(x(t)) + \frac{\partial V}{\partial x}(x(t))g(x(t))Z(t) \\ & & -\frac{\partial V}{\partial x}(x(t))g(x(t)) \\ & & \times \displaystyle\int_{t-\tau}^{t} \frac{\partial z_s}{\partial x}(x(s))g(x(s))Z(s)ds. \end{array} \tag{34}$$

According to Assumption A2, the inequality

$$\begin{array}{rcl} \dot{U}(t) & \leq & -\frac{1}{4}W(x(t)) \\ & & +2\left[Z(t)^2 + C^2\left(\displaystyle\int_{t-\tau}^{t}|Z(s)|ds\right)^2\right] \end{array} \tag{35}$$

is satisfied. From Cauchy-Schwartz inequality, we deduce that

$$\begin{array}{rcl} \dot{U}(t) & \leq & -\frac{1}{4}W(x(t)) \\ & & +2\left[Z(t)^2 + \tau C^2\displaystyle\int_{t-\tau}^{t} Z(s)^2 ds\right]. \end{array} \tag{36}$$

On the other hand, Lemma 2.2 applies to the Z-subsystem of (31) because $\varepsilon \in \left(0, \frac{1}{2\tau}\right]$. Therefore, for all $t \geq 2\tau$,

$$\dot{M}(t) \leq -\frac{1}{4}\varepsilon Z(t)^2 - \frac{1}{2}\varepsilon^3\tau\int_{t-2\tau}^{t} Z(s)^2 ds. \tag{37}$$

Combining (36) and (37), one obtains that the derivative of the functional

$$U_f(x_t, Z_t) \;=\; U(x_t) + KM(Z_t), \tag{38}$$

where K is a positive real number such that $K \geq \max\left\{\frac{12}{\varepsilon}, \frac{4C^2+1}{\varepsilon^3\tau}\right\}$, satisfies the inequality

$$\dot{U}_f(t) \leq -\frac{1}{4}W(x(t)) - Z(t)^2. \tag{39}$$

The right-hand side of (39) is smaller than a negative definite function of $(x(t), Z(t))$.

Finally, we can easily prove that there exist two functions γ_1 and Γ_1 of class $\mathcal{K}_\infty$ such that, for all function $(\phi_x, \phi_Z) \in C^1([-2\tau, 0], R^{n_x} \times R)$,

$$\begin{array}{rcl} \gamma_1(||(\phi_x^\top(0), \phi_Z(0))^\top||) & \leq & U_f(\phi_x, \phi_Z) \\ U_f(\phi_x, \phi_Z) & \leq & \Gamma_1(||(\phi_x^\top, \phi_Z)^\top||_c). \end{array} \tag{40}$$

Therefore, from the Krasovskii stability theorem, one can deduce that the origin of (6) in closed-loop with the feedback (26) is globally uniformly asymptotically stable.

IV. Example

To illustrate Theorem 3.1, we determine a globally uniformly asymptotically stabilizing feedback for the two-dimensional system

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & x(t)z(t), \\ \dot{z}(t) & = & u(t-\tau), \end{array}\right. \tag{41}$$

where τ is an arbitrary positive real number, by applying this theorem. Observe that the linear approximation at the origin of (41) is not asymptotically stabilizable, which implies that the system (41) is not locally exponentially stabilizable by continuous feedback. It follows that linear techniques cannot be of any help for proving the local asymptotic stabilizability of this system.

First, let us check that Assumptions A1 and A2 are satisfied by (41) for functions $V(x)$ and $z_s(x)$ suitably chosen. Consider the functions

$$V(x) = \eta \ln(1+x^2)\,,\; z_s(x) = -\frac{\omega x^2}{1+x^2} \tag{42}$$

where η and ω are positive real numbers. The function V is a candidate Lyapunov function: It is a positive definite, radially unbounded function of x. With the notations of Theorem 3.1, $f(x) = 0, g(x) = x$ and

$$\begin{array}{c} \frac{\partial V}{\partial x}(x)g(x) \;=\; \frac{2\eta x^2}{1+x^2}, \\ \frac{\partial z_s}{\partial x}(x) \;=\; -\frac{2\omega x}{(1+x^2)^2}\;,\; W(x) \;=\; \frac{2\eta\omega x^4}{(1+x^2)^2}. \end{array} \tag{43}$$

Assumption H1 is satisfied for all $\eta > 0, \omega > 0$. Consider now, in the particular case we are interested in, the functional

$$\zeta(x,\xi) = -\frac{1}{4}W(x) - T(x,\xi) - \frac{1}{\Omega}\int_0^{2\tau} W(\xi(l))dl \tag{44}$$

with $x \in R$, $\xi \in C^1([0,2\tau],R)$ and where $T(x,\xi)$ is the functional defined in the general case in (10). One can check readily that it is nonpositive as follows:

$$\begin{array}{rcl} \zeta(x,\xi) & = & -\frac{\eta\omega x^4}{2(1+x^2)^2} \\ & & -\frac{4\eta\omega^2 x^2}{1+x^2}\int_\tau^{2\tau}\frac{\xi(l)^2}{(1+\xi(l)^2)^2}\frac{\xi(l-\tau)^2}{1+\xi(l-\tau)^2}dl \\ & & -\frac{1}{2}\int_0^{2\tau}\frac{\eta\omega\xi(l)^4}{(1+\xi(l)^2)^2}dl \\ & \leq & \eta\omega\int_0^{2\tau}\varrho(x,\xi(l))dl, \end{array} \tag{45}$$

with

$$\begin{array}{rcl} \varrho(a,b) & = & -\frac{a^4}{4\tau(1+a^2)^2} + \frac{4\omega a^2}{1+a^2}\frac{b^2}{(1+b^2)^2} \\ & & -\frac{1}{2}\frac{b^4}{(1+b^2)^2} \end{array} \tag{46}$$

which implies that $\zeta(x,\xi) \leq 0$ when $\omega \in \left(0, \frac{1}{\sqrt{32\tau}}\right]$. Thus, in this case, Assumption H2 is satisfied. Next, let us observe that, for all $x \in R$, $\xi \in C^1([0,2\tau],R)$, the inequality

$$-W(x) + \frac{\partial V}{\partial x}(x)g(x)[z_s(\xi(t)) - z_s(x)] \leq 4\eta\omega \tag{47}$$

is satisfied. Therefore, Assumption H3 is satisfied with $K_\xi = 4\eta\omega$ for all ξ. Thus, we have have shown that Assumption A1 is satisfied. Next, let us prove that Assumption A2 is satisfies, when $2\eta \leq \omega$. One can prove easily that

$$\left|\frac{\partial V}{\partial x}(x)g(x)\right|^2 = \frac{4\eta^2 x^4}{(1+x^2)^2} = \frac{2\eta}{\omega}W(x) \leq W(x) \tag{48}$$

$$\left|\frac{\partial z_s}{\partial x}(x)g(x)\right| \leq \frac{2\omega x^2}{(1+x^2)^2} \leq 2\omega. \tag{49}$$

We conclude that Assumption A2 is satisfied when $2\eta \leq \omega$. It follows from the above analysis that Theorem 3.1 applies to the system (41) which implies that the feedback

$$u_s(t) \;=\; -\varepsilon\left(z(t) + \frac{\omega x(t-\tau)^2}{1+x(t-\tau)^2}\right) - \frac{2\varepsilon x(t)^2}{(1+x(t)^2)^2}z(t) \tag{50}$$

when $\omega \in \left(0, \frac{1}{\sqrt{32\tau}}\right]$ and $0 < \varepsilon \leq \frac{1}{2\tau}$ globally uniformly asymptotically stabilizes the system (41).

Appendix I
A delay in the input may cause finite escape time

We prove here that the system (27) is not globally stabilizable by a continuous feedback $u_s(x(t-\tau), z(t-\tau), x(t-2\tau), z(t-2\tau))$ when $\tau > 0$. To prove this result, we proceed by contradiction. We suppose that, there exist $\tau > 0$ and a continuous feedback $u_s(x(t-\tau), z(t-\tau), x(t-2\tau), z(t-2\tau))$ such that all the solutions of the system (27) in closed-loop with this feedback are defined on $[0,+\infty[$.

Let δ be a positive real number. Consider as initial condition the function $\varphi = (\varphi_x, \varphi_z) \in C^1([-2\tau,0],R^2)$ such that $(\varphi_x(t),\varphi_z(t)) = (0,0)$ for all $t \leq -\frac{2\tau}{3}$, $(\varphi_x(t),\varphi_z(t)) = (\delta,\delta)$ for all $t \in [-\frac{\tau}{3},0]$ and φ_x and φ_z are nondecreasing. Then, for all $t \leq \frac{4}{3}\tau$, the solution of the z-subsystem of (27) satisfies $\dot{z}(t) = u_s(0,0,0,0)$. This implies that, for all $t \in \left[0, \frac{4}{3}\tau\right]$,

$$z(t) = u_s(0,0,0,0)t + z(0) = u_s(0,0,0,0)t + \delta. \tag{51}$$

We deduce that when δ is such that $-\frac{4}{3}\tau|u_s(0,0,0,0)| + \delta \geq \frac{1}{2}\delta$, the inequality

$$\dot{x}(t) \geq -x(t) + \frac{1}{2}\delta x(t)^4 \tag{52}$$

is satisfied for all $t \in \left[0, \frac{4}{3}\tau\right]$. It follows that $X(t) = e^t x(t)$ satisfies, for all $t \in \left[0, \frac{4}{3}\tau\right]$,

$$\dot{X}(t) \geq \frac{1}{2}\delta e^{-3t}X(t)^4 \geq \frac{1}{2}\delta e^{-4\tau}X(t)^4. \tag{53}$$

By integrating this inequality, we deduce that, for all $t \in \left[0, \frac{4}{3}\tau\right]$,

$$X(t)^3 \;\geq\; \frac{1}{\frac{1}{X(0)^3} - \frac{\delta}{2}e^{-4\tau}t} \;=\; \frac{\delta^3}{1 - \frac{\delta^4}{2}e^{-4\tau}t}. \tag{54}$$

When $\delta \geq \left(\frac{3}{2\tau}\right)^{\frac{1}{4}}e^{\tau}$, then $\frac{2}{\delta^4}e^{4\tau} \in \left[0, \frac{4}{3}\tau\right]$. It follows that $X(t)$ is not bounded on the interval $\left[0, \frac{4}{3}\tau\right]$. Therefore $x(t)$ cannot be defined over $[0,+\infty[$. This contradicts the initial assumption. The finite escape time phenomenon occurs.

Appendix II
Assumption H2'

We show that if a system (6) satisfies Assumptions H1, H2', H3 and A2, then it satisfies Assumption H2 as well. Consider the functional

$$\begin{array}{rcl} \zeta(x,\xi) & = & -\frac{1}{4}W(x) \\ & & -\frac{\partial V}{\partial x}(x)g(x)\displaystyle\int_\tau^{2\tau} H(\xi(l),\xi(l-\tau))dl \\ & & -\frac{1}{\Omega}\displaystyle\int_0^{2\tau} W(\xi(l))dl. \end{array} \tag{55}$$

Using the triangular inequality, one can prove that the inequality

$$\begin{array}{rcl} \zeta(x,\xi) & \leq & -\frac{1}{4}W(x) + \frac{1}{4}\left|\frac{\partial V}{\partial x}(x)g(x)\right|^2 \\ & & +\left|\displaystyle\int_\tau^{2\tau} H(\xi(l),\xi(l-\tau))dl\right|^2 \\ & & -\frac{1}{\Omega}\displaystyle\int_0^{2\tau} W(\xi(l))dl \end{array} \tag{56}$$

holds. From Assumption A2, it follows that

$$\begin{aligned}\zeta(x,\xi) \quad \leq \quad & \left|\int_{\tau}^{2\tau} H(\xi(l),\xi(l-\tau))dl\right|^2 \\ & -\tfrac{1}{\Omega}\int_{0}^{2\tau} W(\xi(l))dl.\end{aligned} \tag{57}$$

Cauchy-Schwartz inequality implies that

$$\begin{aligned}\zeta(x,\xi) \quad \leq \quad & \tau\int_{\tau}^{2\tau} |H(\xi(l),\xi(l-\tau))|^2\, dl \\ & -\tfrac{1}{\Omega}\int_{0}^{2\tau} W(\xi(l))dl.\end{aligned} \tag{58}$$

Assumption H2' implies that $\zeta(x,\xi) \leq 0$. Therefore Assumption H2 is satisfied.

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