An existence result for polynomial solutions of parameter-dependent LMIs

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Abstract

We show in this paper that any system of Linear Matrix Inequalities depending continuously upon scalar parameters and solvable for any value of the latter in a fixed compact set, admits a branch of solutions *polynomial* with respect to the parameters. This result is useful for studying e.g. parametric robustness or gain-scheduling issues.

1 Introduction

Linear Matrix Inequalities (LMIs) have become a powerful unifying framework for expressing and solving many problems in control theory. This class of convex optimization problems, solved by efficient interior-point methods, has spread widely, in particular since the publication in 1994 of the by now classical monograph by S. Boyd *et al.* [4]. Among other, stability, stabilizability, detectability, H^2 and H^∞ performance analysis, and various related design issues may be stated as LMIs, see e.g. recent progress in [8].

The next important step was to introduce parameter-dependent LMIs, see e.g. the exposition in [1] and the references therein. The latter appear naturally when studying control techniques robust against parametric uncertainty, or gain-scheduling methods, as these issues amount to check solvability of LMIs obtained for different values of some parameters. Generally, these parameters are a priori unknown but assumed to be inside a certain prespecified bounded set, and an attempt to extend the use of LMI solvers to these problems immediately reaches an obstacle: the impossibility to check an infinite (usually uncountable) number of independent LMI conditions.

An early way to circumvent this difficulty has been to assume prescribed, simple, dependence of the LMI solutions with respect to the parameters, see references in [3] in the context of robust stability analysis. According to the admissible parameter set, in the case of constant or affine dependence, but also for general polynomial dependence [3], the coefficients of the involved polynomials may be found (for given value of the degree) as solutions of standard LMIs.

However, very few results exist, guaranteeing existence of solutions with prespecified dependence, to parameter-dependent LMIs. In other words, nothing is known in general on the conservatism of this simplifying assumption, and thus of the derived approaches. For the Lyapunov inequality $P = P^T > 0$, $A^TP + PA < 0$, for given parameter-dependent matrix A, use of the integral form of the Lyapunov equation $A^TP + PA = -I$ permits to show existence of a polynomial solution, in domains where A is analytical and Hurwitz.

In [6, Lemma 1.1], Delchamps established an analyticity result, which permits to conclude that the LMIs which may be transformed into Riccati inequality by Schur transformation possess, if they are solvable, solutions that are polynomial with respect to their coefficients. Stated initially in the real case, this result is extended to the complex case in [9, Chapter 4, Lemma p. 134].

Last in [5, Proposition 2.1], Y.-S. Chou *et al.* provided a result ensuring existence of polynomial solution, for LMIs depending upon one complex parameter lying on the unit circle.

In the present short note, we provide a general result, Theorem 1, ensuring existence of polynomial solution to any solvable LMI depending continuously upon scalar parameters lying in a fixed compact set. Direct consequences are stated in Corollaries 2, 3, 4. Other applications of these results will be developed in further contributions.

2 Main results

We first consider in the sequel the following property:

$$\forall \delta \in K, \exists x \in \mathbb{R}^p,\ G(x,\delta) \stackrel{\text{def}}{=} G_0(\delta) + x_1 G_1(\delta) + \cdots + x_p G_p(\delta) > 0_n \ . \tag{1}$$

Here, $G_0, G_1, \ldots, G_p$ are mappings defined in a *compact* subset K of $\mathbb{R}^m$, and taking values in the set of symmetric matrices of $\mathbb{R}^{n\times n}$. Formula (1) is to be seen as a *parameter-dependent LMI*, depending upon the parameter $\delta \in K$, and with unknown x. It represents indeed the general form of a feasibility problem for a LMI depending upon scalar parameters.

We first state the key result of the paper.

Theorem 1. *Suppose $G_0, G_1, \ldots, G_p$ are continuous. If for every $\delta \in K$, there exists $x(\delta) \in \mathbb{R}^p$ such that $G(x(\delta),\delta) > 0_n$, then there exists a* polynomial *function $x^* : K \to \mathbb{R}^p$, such that, for every $\delta \in K$, $G(x^*(\delta),\delta) > 0_n$.* ■

Before demonstrating Theorem 1, we state and demonstrate three results which are direct consequences. The following result concerns non-strict inequalities.

Corollary 2. *Suppose $G_0, G_1, \ldots, G_p$ are continuous. Let E be a continuous function, mapping K into the set of positive definite symmetric matrices in $\mathbb{R}^{n\times n}$. If for every $\delta \in K$, there exists $x(\delta) \in \mathbb{R}^p$ such that $G(x(\delta),\delta) \geq 0_n$, then there exists a* polynomial *function $x^* : K \to \mathbb{R}^p$, such that, for every $\delta \in K$, $G(x^*(\delta),\delta) > -E(\delta)$.* ■

To prove Corollary 2, apply Theorem 1 to the parameter-dependent LMI:

$$\exists x \in \mathbb{R}^p, G(x,\delta) + E(\delta) > 0_n \ .$$

Other direct consequence of Theorem 1, the following result gives more informations on the existence of solutions in the level sets defined by $G(x,\delta)$.

Corollary 3. *Suppose $G_0, G_1, \ldots, G_p$ are continuous. Let $\underline{E}, \overline{E}$ be continuous functions, mapping K into the set of symmetric matrices in $\mathbb{R}^{n\times n}$. If for every $\delta \in K$, there exists $x(\delta) \in \mathbb{R}^p$ such that $\overline{E}(\delta) > G(x(\delta),\delta) > \underline{E}(\delta)$, then there exists a* polynomial *function $x^* : K \to \mathbb{R}^p$, such that, for every $\delta \in K$, $\overline{E}(\delta) > G(x^*(\delta),\delta) > \underline{E}(\delta)$.* ■

The proof of Corollary 3 consists in applying Theorem 1 to the parameter-dependent LMI:

$$\exists x \in \mathbb{R}^p,\ \begin{pmatrix} G(x,\delta) - \underline{E}(\delta) & 0_n \\ 0_n & \overline{E}(\delta) - G(x,\delta) \end{pmatrix} > 0_{2n} \ .$$

For given real scalar-valued mappings $c_0, c_1, \ldots, c_p$ defined in K, computing the worst-case (with respect to the parameter δ) extremal value of the affine objective

$$c(x,\delta) \stackrel{\text{def}}{=} c_0(\delta) + x_1 c_1(\delta) + \cdots + x_p c_p(\delta)$$

under the LMI constraint $G(x,\delta) > 0_n$, is also an issue of interest. This is the object of our last result.

Corollary 4. *Suppose $G_0, G_1, \dots, G_p$ and $c_0, c_1, \dots, c_p$ are continuous. Then,*

$$\sup_{\delta\in K} \inf\{c(x,\delta) \ : \ x \in \mathbb{R}^p,\ G(x,\delta) > 0_n\} = \sup_{\delta\in K} \inf\{c(x^*(\delta),\delta) \ : \ x^* \textit{ polynomial}, \ \forall\delta' \in K, G(x^*(\delta'),\delta') > 0_n\} \ .$$

■

Proof. Let $\gamma \stackrel{\text{def}}{=} \sup_{\delta\in K} \inf\{c(x,\delta) \ : \ x \in \mathbb{R}^p,\ G(x,\delta) > 0_n\}$, and assume that $\gamma < +\infty$ (the case $\gamma = +\infty$, which requires straightforward adaptations, is left to the reader). First, one has, for every $\delta \in K$: $\inf\{c(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall\delta' \in K, G(x^*(\delta'),\delta') > 0_n\} \geq \inf\{c(x,\delta) \ : \ x \in \mathbb{R}^p,\ G(x,\delta) > 0_n\}$, due to the inclusion of the first set involved in the second one. Thus,

$$\gamma \leq \sup_{\delta\in K} \inf\{c(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall\delta' \in K, G(x^*(\delta'),\delta') > 0_n\} \ .$$

On the other hand, by definition, $\inf\{c(x,\delta) \ : \ x \in \mathbb{R}^p,\ G(x,\delta) > 0_n\} \leq \gamma$ for every $\delta \in K$. Thus, for any $\varepsilon > 0$, for any $\delta \in K$, there exists $x \in \mathbb{R}^p$ such that $G(x,\delta) > 0_n$ and $c(x,\delta) < \gamma + \varepsilon$. In other words, for any $\varepsilon > 0$, the following parameter-dependent LMI is feasible:

$$\forall\delta \in K, \exists x \in \mathbb{R}^p,\ \begin{pmatrix} \gamma + \varepsilon - c(x,\delta) & 0_{1\times n} \\ 0_{n\times 1} & G(x,\delta) \end{pmatrix} > 0_{n+1} \ .$$

By use of Theorem 1, for any $\varepsilon > 0$, there exists a *polynomial* map $x^*_\varepsilon : K \to \mathbb{R}^p$ such that

$$\forall\delta \in K,\ G(x^*_\varepsilon(\delta),\delta) > 0_n \text{ and } c(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ .$$

Thus, for any $\varepsilon > 0$, for every $\delta \in K$,

$$\inf\{c(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall\delta' \in K, G(x^*(\delta'),\delta') > 0_n\} \leq c(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ ,$$

so, for any $\varepsilon > 0$,

$$\sup_{\delta\in K} \inf\{c(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall\delta' \in K, G(x^*(\delta'),\delta') > 0_n\} \leq \max_{\delta\in K} c(x^*_\varepsilon(\delta),\delta) < \gamma + \varepsilon \ .$$

This results finally in:

$$\sup_{\delta\in K} \inf\{c(x^*(\delta),\delta) \ : \ x^* \text{ polynomial}, \ \forall\delta' \in K, G(x^*(\delta'),\delta') > 0_n\} \leq \gamma \ ,$$

whence the claimed equality. This ends the proof of Corollary 4. □

It remains now to prove Theorem 1.

Proof of Theorem 1. Under the hypothesis of solvability of (1) for every $\delta \in K$, there exists, by compactness and continuity, a real number $\alpha > 0$ such that

$$\forall\delta \in K,\ \{x \in \mathbb{R}^p \ : \ G(x,\delta) \geq 2\alpha I_n\} \neq \emptyset \ . \tag{2}$$

Otherwise, for any $\alpha > 0$, there exist $\delta^\alpha \in K$ such that the previous set is empty. In this case, consider δ^0 an accumulation point of the sequence δ^α, $\alpha \to 0$, and $x^0 \in \mathbb{R}^p$ such that $G(x^0,\delta^0) > 0_n$. By continuity, there exist points δ^α, $\alpha > 0$, arbitrarily close from δ^0, and a constant $\alpha^0 > 0$ such that, say, $G(x^0,\delta^\alpha) > 2\alpha^0 I_n > 0_n$. Thus, for such an α lying in $(0,\alpha^0)$, we have $x^0 \in \{x \in \mathbb{R}^p \ : \ G(x,\delta^\alpha) \geq 2\alpha^0 I_n\} \subset \{x \in \mathbb{R}^p \ : \ G(x,\delta^\alpha) \geq 2\alpha I_n\} \neq 0$, so we are led to contradiction. This establishes the validity of (2) for a certain positive α.

Now, define

$$F \ : \ K \to 2^{\mathbb{R}^p},\ \delta \mapsto F(\delta) = \{x \in \mathbb{R}^p \ : \ G(x,\delta) \geq \alpha I_n\} \ . \tag{3}$$

The set-valued map F maps K into the non-void closed convex subsets of $\mathbb{R}^p$.

Let us first establish that F fulfils the following property of *lower semicontinuity*, see e.g. [2].

Definition. *Let X be a topological space, Y a metric space. A set-valued map F from X to Y is said* lower semicontinuous *at $x^0 \in X$ if for any $y^0 \in F(x^0)$ and any neighborhood $N(y^0)$ of y^0, there exists a neighborhood $N(x^0)$ such that*

$$\forall x \in N(x^0),\ F(x) \cap N(y^0) \neq \emptyset \ .$$

F is said lower semicontinuous *if it is lower semicontinuous at every point $x^0 \in X$.* ■

Let $\delta^0 \in K$, $x^0 \in F(\delta^0)$, $\varepsilon > 0$. To prove lower semicontinuity of F at δ^0, we exhibit $\eta > 0$ such that for every $\delta \in K$ with $\|\delta - \delta^0\|_m < \eta$, there exists $x \in F(\delta)$, $\|x - x^0\|_p < \varepsilon$.

Indeed, by assumption, there exists $x^{\delta^0} \in \mathbb{R}^p$ such that $G(x^{\delta^0}, \delta^0) \geq 2\alpha I_n$. For $\lambda \in (0, 1]$ to be defined later, let $x \stackrel{\text{def}}{=} (1-\lambda)x^0 + \lambda x^{\delta^0}$. Then, the fact that G is affine with respect to x implies that, for any $\eta > 0$ and every $\delta \in K$ such that $\|\delta - \delta^0\|_m < \eta$,

$$\begin{aligned}
G(x,\delta) &= (1-\lambda)G(x^0,\delta) + \lambda G(x^{\delta^0},\delta) \\
&= (1-\lambda)G(x^0,\delta^0) + \lambda G(x^{\delta^0},\delta^0) + (1-\lambda)\left(G(x^0,\delta) - G(x^0,\delta^0)\right) + \lambda\left(G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\right) \\
&\geq \alpha(1+\lambda)I_n - \left(\sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^0,\delta) - G(x^0,\delta^0)\|_n + \sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\|_n\right) I_n \ .
\end{aligned}$$

On the other hand,

$$\|x - x^0\|_p = \lambda \|x^{\delta^0} - x^0\|_p \ .$$

So, take $\lambda \in (0, 1]$ such that

$$\lambda \leq \frac{\varepsilon}{2\|x^{\delta^0} - x^0\|_p} \ ,$$

and choose $\eta > 0$ such that

$$\sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^0,\delta) - G(x^0,\delta^0)\|_n + \sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^{\delta^0},\delta) - G(x^{\delta^0},\delta^0)\|_n \leq \alpha\lambda \ .$$

With these choices of λ and η, one has $\|x - x^0\|_p \leq \varepsilon/2 < \varepsilon$, and $G(x,\delta) \geq \alpha(1+\lambda)I_n - \alpha\lambda I_n = \alpha I_n$ when $\|\delta - \delta^0\|_m < \eta$. Thus, $x \in F(\delta)$, provided that $\delta \in K$ and $\|\delta - \delta^0\|_m < \eta$. We conclude that F is lower continuous at δ^0. This achieves the proof of lower semicontinuity of F.

We now apply to F defined in (3) Michael's Selection Theorem [10], see also [2].

Theorem (Michael's Selection Theorem). *Let X be a metric space, Y a Banach space. Let F, a set-valued map from X into the closed convex subsets of Y, be lower semicontinuous. Then there exists $f : X \to Y$, a continuous selection from F.* ■

Recall that a selection from F is any single valued map f such that, for any $x \in X$, $f(x) \in F(x)$. Application of the previous result yields existence of a continuous selection $f : K \to \mathbb{R}^p$ from F defined in (3). This function is such that

$$\forall \delta \in K,\ G(f(\delta), \delta) \geq \alpha I_n \ .$$

It remains to apply to each of the p components of f the following result, see e.g. [7].

Theorem (Weierstrass Approximation Theorem). *Every continuous real-valued function defined on a compact subset K of $\mathbb{R}^m$, is the limit of a sequence of polynomials, which converges uniformly in K.* ■

Thus, the selection f previously exhibited is uniform limit in K of a sequence of (matrix-valued) polynomials in x. In particular, there exists a polynomial function $x^* \ :\ K \to \mathbb{R}^p$ such that

$$\forall \delta \in K,\ G(x^*(\delta), \delta) \geq \frac{\alpha}{2} I_n > 0_n \ .$$

We conclude that there exists a polynomial solution to the parameter-dependent LMI (1), and this achieves the proof of Theorem 1. □

Remark. *Assuming $G(x,\delta)$ affine with respect to x is not crucial in Theorem* 1. *One may check directly from the Proof that it suffices that G is continuous and fulfils the following* concavity *hypothesis:*

$$\forall x, x' \in \mathbb{R}^p, \forall \delta \in K, \forall \lambda \in [0,1],\ G(\lambda x + (1-\lambda)x', \delta) - \lambda G(x,\delta) - (1-\lambda)G(x',\delta) \geq 0_n\ .$$

This remark leads to immediate extension of the results previously stated to a class of nonlinear semidefinite programming problems.

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