A CONVEX APPROACH TO ROBUST STABILITY FOR LINEAR SYSTEMS WITH UNCERTAIN SCALAR PARAMETERS

PIERRE-ALEXANDRE BLIMAN*

Abstract. In this paper, robust stability for linear systems with several uncertain (complex and/or real) scalar parameters, is studied. A countable family of conditions sufficient for robust stability is given, in terms of solvability of some simple linear matrix inequalities (LMIs). These conditions are of increasing precision, and it is shown conversely that robust stability implies solvability of these LMIs, from a certain rank and beyond. This result constitutes an extension of the characterization by solvability of Lyapunov inequality, of the asymptotic stability for usual linear systems. It is based on the search of parameter-dependent quadratic Lyapunov functions, polynomial of increasing degree in the parameters.

Key words. Robust stability, real and complex parametric uncertainty, polytopic uncertainty, parameter-dependent Lyapunov functions, linear matrix inequalities, μ-analysis, structured singular values, Kalman-Yakubovich-Popov lemma.

AMS subject classifications. 93D05, 93D09, 90C22, 15A39

1. Introduction. We study in this paper the robust asymptotic stability of finite-dimensional linear systems subject to several scalar parametric uncertainties, namely:

$$\dot{x} = A(z)x, \quad z \stackrel{\text{def}}{=} (z_1, \dots, z_m), \quad A(z) \stackrel{\text{def}}{=} A_0 + z_1 A_1 + \dots + z_m A_m \ , \tag{1.1}$$

where the fixed matrices $A_0, A_1, \dots, A_m$ are elements of $\mathbb{C}^{n\times n}$. Here, the uncertain scalar parameters z_i may be complex or real numbers. In the latter case, for sake of clarity, we shall rather write r_i.

It is a well-known fact that asymptotic stability of system (1.1) without uncertainty ($z_1 = \dots = z_m = 0$) is equivalent to existence of a hermitian matrix $P \in \mathbb{C}^{n\times n}$ such that

$$P > 0_n, \ A_0^H P + P A_0 < 0_n \ .$$

This is the well-known Lyapunov inequality. This approach is related to the search for a Lyapunov function of the form $x(t)^H P x(t)$, positive definite and decreasing along the trajectories of $\dot{x} = A_0 x$.

This approach has been extended in different ways, in order to consider uncertain systems (1.1). In the various existing variants, one usually considers a set of constant systems, typically compact and convex: the task is to establish whether all the systems in this set are asymptotically stable or not. Various types of parameter sets are in consequence associated to (1.1), usually elliptic or polytopic. In the present paper, we mainly face the case of constant, noncorrelated, parameters, with values in closed unit balls of $\mathbb{R}$ or $\mathbb{C}$. In other words, we wish to test the existence of a hermitian matrix $P(z)$ such that

$$P(z) > 0_n, \ A(z)^H P(z) + P(z)A(z) < 0_n \ , \tag{1.2}$$

for any $z \in \mathbb{C}^m$ with $|z_i| \leq 1$, $z_i \in \mathbb{R}$ or $\mathbb{C}$, $i = 1, \dots, m$. This problem appears as a *parameter-dependent LMI*.

*INRIA, Domaine de Voluceau, Rocquencourt BP 105, 78153 Le Chesnay cedex, FRANCE – Email: pierre-alexandre.bliman@inria.fr – Tel: + 33 1.39.63.55.68 – Fax: + 33 1.39.63.57.86

This problem is decidable, but NP-hard. Indeed, it amounts to evaluate some particular structured singular values [15, 47]. Generally speaking, computing and approximating μ is a hard task [9, 40, 19], and the gap with its usual upper bound is infinite [41, 38]. The more specific problem studied here may be seen equivalently [10] as checking delay-independent stability [28, 29, 24] of a delay system, which has been proved to be NP-hard too [39].

A first method to cope with uncertainty consists in looking for a simultaneous Lyapunov function, i.e. for a constant hermitian positive definite P such that $x(t)^H Px(t)$ decreases along the trajectories of (1.1), for any value of z in the convenient set; see bibliography on quadratic stability in [8, pp. 72–73]. Subsequent developments have led to consider *parameter-dependent Lyapunov functions*: sufficient conditions for existence of affine parameter-dependent functions $P(z)$ in (1.2) are provided in [22, 18, 12, 34], and in [42, 43] for functions quadratic in the parameters. Methods involving piecewise quadratic Lyapunov functions [44, 36] and LMIs with augmented number of variables [23, 33] may also be found.

Another approach is based on the use of *scaling* or *multiplier* in an input/output stability framework. The use of diagonal scaling (D-scaling) [15] permits to obtain upper bound for μ, whereas DG-scaling [16] plays analogue role for real parametric uncertainty. Contributions based on the larger class of LFT-scaling [1] and on multiplier technique [20] have provided less conservative results. Some results are based on mixed methods [13, 21].

The contributions presented previously provide sufficient conditions for *robust stability* of (1.1), that is for asymptotic stability for any value of the parameters in the adequate set. However, they are far from being necessary and, due to their conservatism, may fail to detect robust stability. On the other hand, they may be checked easily. Indeed, most of them reduce to testing the solvability of LMI problems, a standard convex optimization problem [8], achievable in polynomial-time. Efficient interior-point methods have been developed and are available as toolboxes in widely-spread control-oriented scientific softwares, such as Matlab or Scilab.

From a theoretical point of view, the connection between the two methods has been enlightened by Iwasaki [25] and Iwasaki and Hara [27]. Both may be interpreted as special cases of the *quadratic separator*, separating in an appropriate space a graph associated to the "system" from a graph associated to the "perturbation", here the parameters. Roughly speaking, the previous results are obtained when looking for such a separator with prespecified, "simple", dependency, either with respect to the frequency (frequency-dependent scaling matrix in μ-analysis), or to the parameters (parameter-dependent Lyapunov functions). Clearly, taking small separator classes yields gain in computational simplicity. On the other hand, increasing the separator class size reduces the conservatism of the obtained criterion.

The existing exact methods of resolution of the problem are based on the use of upper and lower bounds on (smaller and smaller) subdomains of the parameter space, see [11, 3, 46]. Due to the computational complexity of the task, they lead to prohibitive growth in computation cost with the problem size, at least in the worst case. The main problem is to find an acceptable trade-off between precision and computational burden.

The results in the present paper provide a systematic way for the use of parameter-dependent Lyapunov functions and their related LMI criteria. The general principle for their derivation may be explained as follows.

First, the solution $P(z)$ of the Lyapunov equation $A(z)^H P(z) + P(z)A(z) = -I_n$ is *analytic* with respect to the vector of parameters z and its conjugate $\bar{z}$ (this fact may be checked from the explicit form $P(z) = \int_0^{+\infty} e^{A(z)^H t} e^{A(z)t}\, dt$). This suggests that for systems which are robustly stable, there always exists a parameter-dependent Lyapunov function $x(t)^H P(z)x(t)$ with P fulfilling (1.2) which is *polynomial with respect to* $z, \bar{z}$. Basically, this comes from the the fact that one may truncate the latter expansion, due to convergence uniform in z. One hence takes as new unknowns of the problem a positive integer k, such that $k-1$ represents the maximal power in the variables $z, \bar{z}$ of the polynomial $P(z)$, plus the k^m coefficients themselves, which are hermitian matrices of size $n \times n$. Second, it turns out that the conditions that must be verified by the previous coefficients (including the global condition of positivity of $P(z)$ for all z) may be transformed into a set of linear matrix inequalities in a total of $m+1$ unknown hermitian matrices. The main tool for this operation is the application, repeated m times, of the discrete-time Kalman-Yakubovich-Popov lemma.

This two-step procedure motivates the form of the results presented in the core of the paper, which we now summarize. A *family of LMIs* is exhibited, indexed by the positive integer k (roughly speaking, the degree in the $z, \bar{z}$ of a solution of (1.2)), and whose solvability implies robust stability of system (1.2). Also, it is shown that solvability for rank k implies solvability for $k' \geq k$, so these sufficient conditions are more and more precise (less and less conservative), as the degree of the polynomial solution increases. A key issue is that a *necessity* property also holds, in the precise sense that: if robust stability holds, then the corresponding LMIs are fulfilled *from a certain rank k and beyond*. Thus, the conservatism vanishes asymptotically. Robust stability of system (1.1) is hence *characterized* by solvability of LMI problems. The originality of the proposed method is to associate to a sequence of increasing classes of candidate parameter-dependent Lyapunov functions, whose existence for a precise problem may be checked by solving a LMI, a *completeness* result, ensuring that robust stability implies existence of a Lyapunov function in at least one of the classes. Related idea for generation of parameter-dependent Lyapunov functions based on nonminimal state is used in [26], without however insight into the necessity part.

The paper is organized as follows. In §2 are given some notations necessary to the statement of the results. In §3 are stated the two results corresponding to m complex parameters (Theorem 4.1) and m real parameters (Theorem 4.3). The mixed case may be written down easily, and is not extensively developed here. In the sequel, we provide as a straightforward consequence a result on robust stability of systems with polytopic uncertainties (Corollary 4.4). A numerical example is presented further on, in §5. Comments on the status of the results are given in §6. Complete proof of Theorems 4.1 and 4.3 is given in §7. Last, concluding remarks are made in §8.

2. Notation. The matrices I_n, 0_n, $0_{n\times p}$ are the $n \times n$ identity matrix and the $n \times n$ and $n \times p$ zero matrices respectively. The symbol $\otimes$ denotes Kronecker product, the power of Kronecker products being used with the natural meaning: $M^{0\otimes} = 1$, $M^{p\otimes} \stackrel{\text{def}}{=} M^{(p-1)\otimes} \otimes M$. Recall the important property that $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ for any matrices with compatible size. The spectrum of a square matrix M is written $\sigma(M)$, and applying the operation Re to this set, one denotes by $\operatorname{Re}\sigma(M)$ the set $\{\operatorname{Re} s \ : \ s \in \sigma(M)\}$: $\operatorname{Re}\sigma(M) < 0$ thus means that M is Hurwitz. The spectral radius of a square matrix M is written $\rho(M)$. The conjugate and transconjugate of M, are denoted M^T and M^H. $\mathbb{N}$ is the set of positive integers. By $\overline{\mathbb{D}}$ is denoted the closed unit ball in $\mathbb{C}$. The unit circle is denoted as the boundary $\partial\mathbb{D}$. By $\overline{\mathbb{C}^+}$ is meant the closed set of complex numbers with nonnegative real part. Last, the set of complex hermitian matrices of size $n \times n$ is denoted by $\mathcal{H}^n$.

Let $\hat{J}_k, \check{J}_k \in \mathbb{R}^{k\times(k+1)}$ be defined by

$$\hat{J}_k \stackrel{\text{def}}{=} \begin{pmatrix} I_k & 0_{k\times 1} \end{pmatrix},\ \check{J}_k \stackrel{\text{def}}{=} \begin{pmatrix} 0_{k\times 1} & I_k \end{pmatrix}\ .$$

These matrices will prove essential for polynomial manipulation. In particular, a key property is that, for $u^{[k]} \in \mathbb{C}^k$ defined, for $u \in \mathbb{C}$, by

$$u^{[k]} \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ u \\ \vdots \\ u^{k-1} \end{pmatrix}\ , \tag{2.1}$$

one has

$$\hat{J}_k u^{[k+1]} = u^{[k]},\ \check{J}_k u^{[k+1]} = u u^{[k]}\ . \tag{2.2}$$

Also, we will use the fact that, for any $k \in \mathbb{N}$,

$$\hat{J}_k \check{J}_{k+1} = \check{J}_k \hat{J}_{k+1} = \begin{pmatrix} 0_{k\times 1} & I_k & 0_{k\times 1} \end{pmatrix}\ . \tag{2.3}$$

Finally, one shows directly that, for any matrix $M \in \mathbb{C}^{p\times q}$, for any $u \in \mathbb{C}$,

$$(u^{[k]} \otimes I_p)M = (I_k \otimes M)(u^{[k]} \otimes I_q)\ . \tag{2.4}$$

3. Polynomially parameter-dependent quadratic functions and their evolution. In the study of system (1.1), a crucial role will be played here by the search for parameter-dependent Lyapunov functions chosen within the following class.

DEFINITION 3.1. *We call* polynomially parameter-dependent quadratic function *(PPDQ function for short) any quadratic function $x^H P(z)x$ on $\mathbb{C}^n$ such that*

$$P(z) \stackrel{\text{def}}{=} (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H P_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)\ , \tag{3.1}$$

for a certain $P_k \in \mathcal{H}^{k^m n}$. The integer $k-1$ is called the degree of the PPDQ function P.

Notice that the expression $(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]})$ gathers in a column all the monomials with degree at most $k-1$ in each of the components of z.

The following auxiliary result provides the derivative of a PPDQ function along the trajectories of (1.1).

PROPOSITION 3.2. *The derivative of the PPDQ function* (3.1) *of degree $k-1$ along the trajectories of the system $\dot{x} = A(z)x$ is a PPDQ function R(z) of degree k given by*

$$R(z) \stackrel{\text{def}}{=} (z_m^{[k+1]} \otimes \cdots \otimes z_1^{[k+1]} \otimes I_n)^H R_k (z_m^{[k+1]} \otimes \cdots \otimes z_1^{[k+1]} \otimes I_n)\ , \tag{3.2}$$

where $R_k \in \mathcal{H}^{(k+1)^m n}$ is defined as

$$\begin{aligned} R_k \stackrel{\text{def}}{=} & \left(\left(\hat{J}_k^{m\otimes} \otimes A_0\right) + \sum_{i=1}^m \left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i\right)\right)^H P_k \left(\hat{J}_k^{m\otimes} \otimes I_n\right) \\ & + \left(\hat{J}_k^{m\otimes} \otimes I_n\right)^T P_k \left(\left(\hat{J}_k^{m\otimes} \otimes A_0\right) + \sum_{i=1}^m \left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i\right)\right) \end{aligned} \tag{3.3}$$

and depends linearly upon $P_k \in \mathcal{H}^{k^m n}$.

Proof of Proposition 3.2. Clearly, $R(z) = A(z)^H P(z) + P(z)A(z)$. As an example, let us evaluate $P(z)A(z)$. One has

$$\begin{aligned}
P(z)A(z) &= (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H P_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) A(z) \\
&= (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H P_k (I_{k^m} \otimes A(z))(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) \\
&\qquad \text{(due to (2.4))} \\
&= (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H P_k \Big[(I_{k^m} \otimes A_0)(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) \\
&\quad + (I_{k^m} \otimes A_1)(z_m^{[k]} \otimes \cdots \otimes z_1 z_1^{[k]} \otimes I_n) + \ldots + (I_{k^m} \otimes A_m)(z_m z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) \Big] ,
\end{aligned}$$

and the second term in (3.3) is obtained by repeated use of the two formulas in (2.2). □

To study systems with real parameters $\dot{x} = A(r)x$, we use the change of variables $r = \frac{z+\bar{z}}{2}$, which maps $\overline{\mathbb{D}}^m$ onto $[-1; +1]^m$. It turns out that the formulas are of smaller size when one is directly looking for a Lyapunov function parametrized by z and not by r. The analogue of Proposition 3.2 for this case is given below, and its proof, using the same techniques, is left to the reader:

PROPOSITION 3.3. *The derivative of the PPDQ function* (3.1) *of degree* $k-1$ *along the trajectories of the system* $\dot{x} = A(\frac{z+\bar{z}}{2})x$ *is a PPDQ function R(z) of degree* k *given as* (3.2), *where* $R_k \in \mathcal{H}^{(k+1)^m n}$ *is now defined by*

$$\begin{aligned}
R_k \stackrel{\text{def}}{=} & \frac{1}{2} \left(\left(\hat{J}_k^{m\otimes} \otimes A_0 \right) + \sum_{i=1}^m \left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i \right) \right)^H P_k \left(\hat{J}_k^{m\otimes} \otimes I_n \right) \\
+ & \frac{1}{2} \left(\left(\hat{J}_k^{m\otimes} \otimes A_0 \right)^H P_k \left(\hat{J}_k^{m\otimes} \otimes I_n \right) + \sum_{i=1}^m \left(\hat{J}_k^{m\otimes} \otimes A_i \right)^H P_k \left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes I_n \right) \right) \\
+ & \frac{1}{2} \left(\hat{J}_k^{m\otimes} \otimes I_n \right)^T P_k \left(\left(\hat{J}_k^{m\otimes} \otimes A_0 \right) + \sum_{i=1}^m \left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes A_i \right) \right) \\
+ & \frac{1}{2} \left(\left(\hat{J}_k^{m\otimes} \otimes I_n \right)^T P_k \left(\hat{J}_k^{m\otimes} \otimes A_0 \right) + \sum_{i=1}^m \left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes \hat{J}_k^{(i-1)\otimes} \otimes I_n \right)^T P_k \left(\hat{J}_k^{m\otimes} \otimes A_i \right) \right)
\end{aligned} \tag{3.4}$$

and depends linearly upon $P_k \in \mathcal{H}^{k^m n}$.

4. Main results. We are now in a position to state the main results of the paper.

THEOREM 4.1 (Robust stability of systems with complex parameters). *The following three properties are* equivalent.

(i) *The matrix* $A(z)$ *in* (1.1) *is Hurwitz for any* $z \in \overline{\mathbb{D}}^m$.

(ii) *There exists a PPDQ Lyapunov function* $x^H P(z) x$ *for the class of systems* $\dot{x} = A(z)x$ *with* $A(z)$ *defined by* (1.1), *i.e. such that*

$$\forall z \in \overline{\mathbb{D}}^m, \; P(z) > 0, \; R(z) < 0 \; ,$$

where $R(z)$ *is defined by* (3.2), (3.3).

(iii) *There exist a positive integer* k *and* $(m+1)$ *matrices*

$$P_k \in \mathcal{H}^{k^m n} \text{ and } Q_{k,i} \in \mathcal{H}^{k^{m-i+1}(k+1)^{i-1} n}, \; i = 1 \ldots m \; ,$$

which solve the following LMI:

$$
\begin{cases}
P_k > 0_{k^m n} \ , \\
R_k + \displaystyle\sum_{i=1}^{m} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n} \right)^T Q_{k,i} \left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n} \right) \\
\qquad - \displaystyle\sum_{i=1}^{m} \left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{i-1}n} \right)^T Q_{k,i} \left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{i-1}n} \right) < 0_{(k+1)^m n} \ ,
\end{cases}
\tag{LMI$_k$}
$$

with $R_k = R_k(P_k)$ *defined in* (3.3).

Moreover, if (LMI$_k$) *with* (3.3) *is solvable for the index* k*, then it is also solvable for all indices* $k' \geq k$*. Finally, if the matrices* A_i*,* $0 \leq i \leq m$*, are real, then the statement holds with real, symmetric, matrices* $P_k, Q_{k,i}$*,* $1 \leq i \leq m$*.*

The proof of Theorem 4.1 is given in §7.1.

From Theorem 4.1, one deduces in particular that, for any positive integer k,

$$
\begin{aligned}
(\text{LMI}_k) \text{ is solvable} &\Rightarrow (\text{LMI}_{k'}) \text{ is solvable for } k' \geq k \\
&\Rightarrow \text{system (1.1) is robustly stable against any } z \in \overline{\mathbb{D}}^m.
\end{aligned}
\tag{4.1}
$$

In other words, any of the conditions (LMI$_k$) is sufficient for robust stability, and they are more and more precise. Necessity of the condition is obtained asymptotically, for large enough k.

The sufficiency result (4.1) is central, and turns out to be the "easy" part of the proof. Before commenting further on Theorem 4.1, we provide indications on its demonstration, leaving the details for the complete proof in §7.1.

Sketch of proof for (4.1). Left- and right-multiplication of the second inequality in (LMI$_k$) by $(z_m^{[k+1]} \otimes \dots \otimes z_1^{[k+1]} \otimes I_n)$ and its transconjugate yields $R(z) + \sum_{i=1}^m (1 - |z_i|^2)(z_m^{[k]} \otimes \dots \otimes z_i^{[k]} \otimes z_{i-1}^{[k+1]} \otimes \dots z_1^{[k+1]} \otimes I_n)^H Q_{k,i} (z_m^{[k]} \otimes \dots \otimes z_i^{[k]} \otimes z_{i-1}^{[k+1]} \otimes \dots z_1^{[k+1]} \otimes I_n) < 0_n$. Indeed, this is a direct consequence of (2.2). Thus, $R(z) < 0_n$ if $|z_1| = \dots = |z_m| = 1$, so the matrix $A(z)$ is Hurwitz for all $z \in (\partial\mathbb{D})^m$, and this may be extended to the whole $\overline{\mathbb{D}}^m$; see the details in §7.1.1. This proves that solvability of (LMI$_k$) is sufficient for robust stability.

To prove that solvability of (LMI$_k$) implies solvability of (LMI$_{k+1}$), one constructs directly a new solution $P_{k+1}, Q_{k+1,1}, \dots, Q_{k+1,m}$, by taking

$$
P_{k+1} \stackrel{\text{def}}{=} \sum_{M_i \in \{\hat{J}_k, \check{J}_k\},\ i=1,\dots,m} (M_m \otimes \dots \otimes M_1 \otimes I_n)^T P_k (M_m \otimes \dots \otimes M_1 \otimes I_n),
$$

$$
Q_{k+1,i} \stackrel{\text{def}}{=} \sum_{\substack{M_l \in \{\hat{J}_{k+1}, \check{J}_{k+1}\},\ l=1,\dots,i-1, \\ M_l \in \{\hat{J}_k, \check{J}_k\},\ l=i,\dots,m}} (M_m \otimes \dots \otimes M_1 \otimes I_n)^T Q_{k,i} (M_m \otimes \dots \otimes M_1 \otimes I_n),
$$

for $i = 1, \dots, m$. One then shows that the matrix R_{k+1} obtained from P_{k+1} by formula (3.3) verifies:

$$
R_{k+1} \stackrel{\text{def}}{=} \sum_{M_i \in \{\hat{J}_{k+1}, \check{J}_{k+1}\},\ i=1,\dots,m} (M_m \otimes \dots \otimes M_1 \otimes I_n)^T R_k (M_m \otimes \dots \otimes M_1 \otimes I_n) \ .
$$

As a matter of fact, one may show that this amounts to multiply $P(z)$ and $R(z)$ by $(1 + |z_1|^2) \dots (1 + |z_m|^2)$ in (ii). Based on properties (2.3), (2.4), the two inequalities of (LMI_{k+1}) are then deduced from the inequalities of (LMI_k), see details in §7.1.3 below. $\square$

REMARK 4.2. *Paradoxically, the positive definite PPDQ function $x^H P(z)x$ of degree $k-1$ formed from a solution of* (LMI_k) *is not ensured to decrease along the trajectories of the system. The argument developed above consisted in showing that $R(z) < 0_n$ for $z \in (\partial\mathbb{D})^m$. As said before, this yields Hurwitzness of $A(z)$ for $z \in (\partial\mathbb{D})^m$, which implies the same property in $\overline{\mathbb{D}}^m$, basically by an analyticity result, see* §7.1.1. *However, in general $R(z) \not< 0_n$ for $z \in \overline{\mathbb{D}}^m$, unless $Q_{k,i} > 0_{k^{m-i+1}(k+1)^{i-1}n}$ for all $i = 1,\dots,m$. In the case of a* unique *scalar uncertainty ($m=1$), there is no loss of generality to add this positivity condition on $Q_{k,1}$ in the LMI, see* [5]. *We conjecture that the same remains true for $m > 1$.*

In the case $m = 0$, the problem (LMI_k) simply states that: $\exists P \in \mathcal{H}^n$, $P > 0_n$, $A_0^H P + PA_0 < 0_n$. For $m = 1$, one gets the following family of LMIs indexed by $k \in \mathbb{N}$: $\exists P_k \in \mathcal{H}^{kn}$, $P_k > 0_{kn}$, $\exists Q_k \in \mathcal{H}^{kn}$,

$$\begin{aligned}(\hat{J}_k \otimes A_0)^H P_k(\hat{J}_k \otimes I_n) &+ (\check{J}_k \otimes A_1)^H P_k(\hat{J}_k \otimes I_n)\\ &+ (\hat{J}_k \otimes I_n)^T P_k(\hat{J}_k \otimes A_0) + (\hat{J}_k \otimes I_n)^T P_k(\check{J}_k \otimes A_1)\\ &+ (\hat{J}_k \otimes I_n)^T Q_k(\hat{J}_k \otimes I_n) - (\check{J}_k \otimes I_n)^T Q_k(\check{J}_k \otimes I_n) < 0_{(k+1)n},\end{aligned}$$

that is:

$$\begin{pmatrix}\hat{J}_k \otimes I_n\\ \check{J}_k \otimes I_n\end{pmatrix}^T \begin{pmatrix}(I_k \otimes A_0)^H P_k + P_k(I_k \otimes A_0) + Q_k & P_k(I_k \otimes A_1)\\ (I_k \otimes A_1)^H P_k & -Q_k\end{pmatrix}\begin{pmatrix}\hat{J}_k \otimes I_n\\ \check{J}_k \otimes I_n\end{pmatrix} < 0_{(k+1)n}\ . \tag{4.2}$$

For two parameters ($m = 2$), one obtains: $\exists P_k \in \mathcal{H}^{k^2n}, P_k > 0_{k^2n}$, $\exists Q_{k,1} \in \mathcal{H}^{k^2n}$, $\exists Q_{k,2} \in \mathcal{H}^{k(k+1)n}$,

$$\begin{aligned}(\hat{J}_k^{2\otimes} \otimes A_0)^H &P_k(\hat{J}_k^{2\otimes} \otimes I_n) + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k(\hat{J}_k^{2\otimes} \otimes A_0)\\ &+ (\hat{J}_k \otimes \check{J}_k \otimes A_1)^H P_k(\hat{J}_k^{2\otimes} \otimes I_n) + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k(\hat{J}_k \otimes \check{J}_k \otimes A_1)\\ &+ (\check{J}_k \otimes \hat{J}_k \otimes A_2)^H P_k(\hat{J}_k^{2\otimes} \otimes I_n) + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k(\check{J}_k \otimes \hat{J}_k \otimes A_2)\\ &+ (\hat{J}_k^{2\otimes} \otimes I_n)^T Q_{k,1}(\hat{J}_k^{2\otimes} \otimes I_n) - (\hat{J}_k \otimes \check{J}_k \otimes I_n)^T Q_{k,1}(\hat{J}_k \otimes \check{J}_k \otimes I_n)\\ &+ (\hat{J}_k \otimes I_{(k+1)n})^T Q_{k,2}(\hat{J}_k \otimes I_{(k+1)n}) - (\check{J}_k \otimes I_{(k+1)n})^T Q_{k,2}(\check{J}_k \otimes I_{(k+1)n})\\ &\qquad < 0_{(k+1)^2n}\ ,\end{aligned}$$

or again

$$\begin{aligned}&\begin{pmatrix}\hat{J}_k \otimes \hat{J}_k \otimes I_n\\ \hat{J}_k \otimes \check{J}_k \otimes I_n\\ \check{J}_k \otimes \hat{J}_k \otimes I_n\end{pmatrix}^T \begin{pmatrix}(I_{k^2} \otimes A_0)^H P_k + P_k(I_{k^2} \otimes A_0) + Q_{k,1} & P_k(I_{k^2} \otimes A_1) & P_k(I_{k^2} \otimes A_2)\\ (I_{k^2} \otimes A_1)^H P_k & -Q_{k,1} & 0_{k^2n}\\ (I_{k^2} \otimes A_2)^H P_k & 0_{k^2n} & 0_{k^2n}\end{pmatrix}\begin{pmatrix}\hat{J}_k \otimes \hat{J}_k \otimes I_n\\ \hat{J}_k \otimes \check{J}_k \otimes I_n\\ \check{J}_k \otimes \hat{J}_k \otimes I_n\end{pmatrix}\\ &\quad + \begin{pmatrix}\hat{J}_k \otimes I_{(k+1)n}\\ \check{J}_k \otimes I_{(k+1)n}\end{pmatrix}^T \begin{pmatrix}Q_{k,2} & 0_{k(k+1)n}\\ 0_{k(k+1)n} & -Q_{k,2}\end{pmatrix}\begin{pmatrix}\hat{J}_k \otimes I_{(k+1)n}\\ \check{J}_k \otimes I_{(k+1)n}\end{pmatrix} < 0_{(k+1)^2n}\ .\end{aligned} \tag{4.3}$$

An interesting comparison may be made, concerning the simplest criterion, obtained for $k = 1$. In the case $m = 1$, see (4.2), (LMI_1) writes

$$P_1 = P_1^H > 0,\ Q_{1,1} = Q_{1,1}^H,\ \begin{pmatrix}A_0^H P_1 + P_1 A_0 + Q_{1,1} & P_1 A_1\\ A_1^H P_1 & -Q_{1,1}\end{pmatrix} < 0\ ,$$

which matches the conditions for quadratic stability with D-scalings. For the case $m=2$ of two parameters, see (4.3), the inequalities are

$$P_1 = P_1^H > 0,\ Q_{1,1} = Q_{1,1}^H,\ Q_{1,2} = Q_{1,2}^H,$$
$$\begin{pmatrix} A_0^H P_1 + P_1 A_0 + Q_{1,1} & P_1 A_1 & P_1 A_2 & 0_n \\ A_1^H P_1 & -Q_{1,1} & 0_n & 0_n \\ A_2^H P_1 & 0_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & 0_n \end{pmatrix} + \begin{pmatrix} Q_{1,2} & 0_{2n} \\ 0_{2n} & -Q_{1,2} \end{pmatrix} < 0,$$

where here the size of $Q_{1,2}$ is twice that of $Q_{1,1}$. This is clearly less restrictive that the conditions obtained with D-scalings, namely

$$P_1 = P_1^H > 0,\ Q_{1,1} = Q_{1,1}^H,\ Q_{1,2} = Q_{1,2}^H,$$
$$\begin{pmatrix} A_0^H P_1 + P_1 A_0 + Q_{1,1} + Q_{1,2} & P_1 A_1 & P_1 A_2 \\ A_1^H P_1 & -Q_{1,1} & 0_n \\ A_2^H P_1 & 0_n & -Q_{1,2} \end{pmatrix} < 0.$$

For larger values of m, (LMI$_1$) is obtained by introduction of the remaining multipliers $Q_{1,i}$, along the same principles. The obtained conditions are related to, but less conservative than, the ones obtained with D-scalings.

The result for systems with real parameters is analogous to Theorem 4.1:

Theorem 4.3 (Robust stability of systems with real parameters). *The following three properties are* equivalent.

(i) *The matrix $A(r)$ in* (1.1) *is Hurwitz for any $r \in [-1;+1]^m$.*

(ii) *There exists a PPDQ Lyapunov function $x^H P(r) x$ for the class of systems $\dot{x} = A(r)x$ with $A(r)$ defined by* (1.1), *i.e. such that*

$$\forall r \in [-1;+1]^m,\ P(r) > 0,\ R(r) < 0\ ,$$

where $R(r)$ is defined as in (3.2), *with R_k given by* (3.4).

(iii) *There exist a positive integer k and $(m+1)$ matrices*

$$P_k \in \mathcal{H}^{k^m n} \text{ and } Q_{k,i} \in \mathcal{H}^{k^{m-i+1}(k+1)^{i-1} n},\ i = 1 \dots m\ ,$$

which solve the (LMI$_k$) *with $R_k = R_k(P_k)$ defined in* (3.4).

Moreover, if (LMI$_k$) *with* (3.4) *is solvable for the index k, then it is also solvable for all indices $k' \geq k$. Finally, if the matrices A_i, $0 \leq i \leq m$, are real, then the statement holds with real, symmetric, matrices $P_k, Q_{k,i}$, $1 \leq i \leq m$.*

The proof of Theorem 4.3 is given in §7.2.

For $m=1$ and $m=2$ respectively, the two following families of LMIs are obtained: $\exists P_k \in \mathcal{H}^{kn}$, $P_k > 0_{kn}$, $\exists Q_k \in \mathcal{H}^{kn}$,

$$\begin{aligned} &(\hat{J}_k \otimes A_0)^H P_k (\hat{J}_k \otimes I_n) + \tfrac{1}{2}\Big((\hat{J}_k \otimes A_1)^H P_k (\check{J}_k \otimes I_n) + (\check{J}_k \otimes A_1)^H P_k (\hat{J}_k \otimes I_n)\Big) \\ &+ (\hat{J}_k \otimes I_n)^T P_k (\hat{J}_k \otimes A_0) + \tfrac{1}{2}\Big((\check{J}_k \otimes I_n)^T P_k (\hat{J}_k \otimes A_1) + (\hat{J}_k \otimes I_n)^T P_k (\check{J}_k \otimes A_1)\Big) \\ &\qquad + (\hat{J}_k \otimes I_n)^T Q_k (\hat{J}_k \otimes I_n) - (\check{J}_k \otimes I_n)^T Q_k (\check{J}_k \otimes I_n) < 0_{(k+1)^n}\ , \end{aligned}$$

and: $\exists P_k \in \mathcal{H}^{k^2 n}, P_k > 0_{k^2 n}$, $\exists Q_{k,1} \in \mathcal{H}^{k^2 n}$, $\exists Q_{k,2} \in \mathcal{H}^{k(k+1)n}$,

$$\begin{aligned}
&(\hat{J}_k^{2\otimes} \otimes A_0)^H P_k (\hat{J}_k^{2\otimes} \otimes I_n) + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k (\hat{J}_k^{2\otimes} \otimes A_0) \\
&\quad + \tfrac{1}{2}\Big((\hat{J}_k^{2\otimes} \otimes A_1)^H P_k (\hat{J}_k \otimes \check{J}_k \otimes I_n) + (\hat{J}_k \otimes \check{J}_k \otimes A_1)^H P_k (\hat{J}_k^{2\otimes} \otimes I_n)\Big) \\
&\quad + \tfrac{1}{2}\Big((\hat{J}_k^{2\otimes} \otimes A_2)^H P_k (\check{J}_k \otimes \hat{J}_k \otimes I_n) + (\check{J}_k \otimes \hat{J}_k \otimes A_2)^H P_k (\hat{J}_k^{2\otimes} \otimes I_n)\Big) \\
&\quad + \tfrac{1}{2}\Big((\hat{J}_k \otimes \check{J}_k \otimes I_n)^T P_k (\hat{J}_k^{2\otimes} \otimes A_1) + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k (\hat{J}_k \otimes \check{J}_k \otimes A_1)\Big) \\
&\quad + \tfrac{1}{2}\Big((\check{J}_k \otimes \hat{J}_k \otimes I_n)^T P_k (\hat{J}_k^{2\otimes} \otimes A_2) + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k (\check{J}_k \otimes \hat{J}_k \otimes A_2)\Big) \\
&\quad + (\hat{J}_k^{2\otimes} \otimes I_n)^T Q_{k,1} (\hat{J}_k^{2\otimes} \otimes I_n) - (\hat{J}_k \otimes \check{J}_k \otimes I_n)^T Q_{k,1} (\hat{J}_k \otimes \check{J}_k \otimes I_n) \\
&+ (\hat{J}_k \otimes I_{(k+1)n})^T Q_{k,2} (\hat{J}_k \otimes I_{(k+1)n}) - (\check{J}_k \otimes I_{(k+1)n})^T Q_{k,2} (\check{J}_k \otimes I_{(k+1)n}) < 0_{(k+1)^2 n} \,.
\end{aligned}$$

Forms similar to (4.2) and (4.3) may be obtained. For $k = 1$, $m = 1$, one gets for the LMI defined in Theorem 4.3:

$$P_1 = P_1^H > 0,\ Q_{1,1} = Q_{1,1}^H,\ \begin{pmatrix} A_0^H P_1 + P_1 A_0 + Q_{1,1} & \frac{1}{2}(A_1^H P_1 + P_1 A_1) \\ \frac{1}{2}(A_1^H P_1 + P_1 A_1) & -Q_{1,1} \end{pmatrix} < 0 \,,$$

to be compared to the condition obtained by DG-scaling:

$$P_1 = P_1^H > 0,\ D = D^H,\ G + G^H = 0,\ \begin{pmatrix} A_0^H P_1 + P_1 A_0 + D & P_1 A_1 + G \\ A_1^H P_1 + G^H & -D \end{pmatrix} < 0 \,.$$

One may verify that there is no loss of generality to take $G = \frac{1}{2}(A_1^H P_1 - P_1 A_1)$ in the latter inequality, so the two criteria are equivalent. The formulas for larger m are obtained similarly to the complex case, they provide also tighter sufficient conditions than the ones based on DG-scaling.

Theorems 4.1 and 4.3 are easily adapted to treat the mixed complex/real case. The result is not stated completely here, for sake of space. As an example, for stability analysis of $A_0 + zA_1 + rA_2$, for a complex parameter z and a real parameter r, both of norm less or equal than 1, the criterion is based on the following family of LMIs: $\exists P_k \in \mathcal{H}^{k^2 n}$, $P_k > 0_{k^2 n}$, $\exists Q_{k,1} \in \mathcal{H}^{k^2 n}$, $\exists Q_{k,2} \in \mathcal{H}^{k(k+1)n}$,

$$\begin{aligned}
&(\hat{J}_k^{2\otimes} \otimes A_0)^H P_k (\hat{J}_k^{2\otimes} \otimes I_n) + (\hat{J}_k \otimes \check{J}_k \otimes A_1)^H P_k (\hat{J}_k^{2\otimes} \otimes I_n) \\
&\quad + \tfrac{1}{2}\Big((\hat{J}_k^{2\otimes} \otimes A_2)^H P_k (\check{J}_k \otimes \hat{J}_k \otimes I_n) + (\check{J}_k \otimes \hat{J}_k \otimes A_2)^H P_k (\hat{J}_k^{2\otimes} \otimes I_n)\Big) \\
&\quad + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k (\hat{J}_k^{2\otimes} \otimes A_0) + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k (\hat{J}_k \otimes \check{J}_k \otimes A_1) \\
&\quad + \tfrac{1}{2}\Big((\check{J}_k \otimes \hat{J}_k \otimes I_n)^T P_k (\hat{J}_k^{2\otimes} \otimes A_2) + (\hat{J}_k^{2\otimes} \otimes I_n)^T P_k (\check{J}_k \otimes \hat{J}_k \otimes A_2)\Big) \\
&\quad + (\hat{J}_k^{2\otimes} \otimes I_n)^T Q_{k,1} (\hat{J}_k^{2\otimes} \otimes I_n) - (\hat{J}_k \otimes \check{J}_k \otimes I_n)^T Q_{k,1} (\hat{J}_k \otimes \check{J}_k \otimes I_n) \\
&+ (\hat{J}_k \otimes I_{(k+1)n})^T Q_{k,2} (\hat{J}_k \otimes I_{(k+1)n}) - (\check{J}_k \otimes I_{(k+1)n})^T Q_{k,2} (\check{J}_k \otimes I_{(k+1)n}) < 0_{(k+1)^2 n} \,.
\end{aligned}$$

The robust stability of a polytope of matrices is now addressed, by reduction to the problems solved in Theorem 4.3. Consider, for $m+1$ fixed matrices $B_1, \dots, B_{m+1} \in \mathbb{C}^{n\times n}$, the class of systems

$$\dot{x} = B(\beta)x, \quad \beta \stackrel{\text{def}}{=} (\beta_0, \beta_1, \dots, \beta_m), \quad B(\beta) \stackrel{\text{def}}{=} \beta_0 B_0 + \beta_1 B_1 + \dots \beta_m B_m \,. \tag{4.4}$$

Define the polytope $\mathbb{P}^{m+1} \stackrel{\text{def}}{=} \{\beta \in \mathbb{R}^{m+1} \ : \ \beta_i \geq 0,\ \beta_0 + \dots + \beta_m = 1\}$.

Corollary 4.4 (Robust stability of real convex polytopic systems). *The following three properties are* equivalent.

(i) *The matrix $B(\beta)$ in (4.4) is Hurwitz for any $\beta \in \mathbb{P}^{m+1}$.*
(ii) *There exists $m+1$ PPDQ functions $x^H P_i(\beta)x$, $i = 0, \dots, m$ such that*

$$\forall \beta \in \mathbb{P}^{m+1}, \qquad \begin{array}{l} P_i(\beta) > 0, i = 0, \dots, m, \\ B(\beta)^H P_{\arg\max \beta_i}(\beta) + P_{\arg\max \beta_i}(\beta) B(\beta) < 0 \ . \end{array}$$

(iii) *For each value of $i = 0, \dots, m$, there exists a positive integer k for which* (LMI$_k$) *with R_k defined in* (3.4) *is solvable, with*

$$A_0 \stackrel{\text{def}}{=} B_i + \frac{1}{2} \sum_{j=0, j\neq i}^{m} B_j, \quad \{A_1, \dots, A_m\} = \left\{ \frac{1}{2} B_j \ : \ j \neq i \right\} \ . \tag{4.5}$$

Proof of Corollary 4.4. Let for example $\max_{0\leq i\leq m} \beta_i = \beta_0 > 0$, write

$$B(\beta) = \beta_0 \left(B_0 + \frac{1}{2} \sum_{i=1}^{m} B_i + \frac{1}{2} \sum_{i=1}^{m} \left(2\frac{\beta_i}{\beta_0} - 1 \right) B_i \right) \ .$$

Remark that the map $[0; +1] \to [-1; +1]$, $u \mapsto 2u - 1$ is one-to-one. For any fixed value of $i = \arg\max \beta_j$ (take any value if the maximum is attained for more than one index), property (i) is thus equivalent to robust stability of $\dot{x} = A(r)x$ with the definition of $A_0, A_1, \dots, A_m$ given in (4.5), and it is possible to apply Theorem 4.3. For any fixed $i = \arg\max \beta_j$, a PPDQ Lyapunov function is found as a function of $r_j \stackrel{\text{def}}{=} -1 + 2\beta_j / \max \beta_i$, $j \neq i$, and may be expressed with respect to β_j after adequate change of the coefficients. □

Application of Theorem 4.3 thus provides for this problem too a family of sufficient conditions for robust stability, whose conservatism vanishes asymptotically: $m+1$ families of LMIs are found, such that robust stability of (4.4) is equivalent to the solvability of at least one LMI in each family. Clearly, this approach amounts to the search for a piecewise PPDQ Lyapunov function, chosen, in $m+1$ quadrants of the parameter space, according to the value of $\max \beta_i$.

5. Numerical example. Consider the following example. Let $n = 3$,

$$A_0 = \begin{pmatrix} -12 & -7 & 7 \\ -11 & -13 & -5 \\ -2 & 9 & -8 \end{pmatrix}, \ A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 1 & 2 & 0 \\ -3 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ .$$

We evaluate the following robustness margins:

$$\alpha_{zz} \stackrel{\text{def}}{=} \sup_{(z_1,z_2)\in\overline{\mathbb{D}}^2} \operatorname{Re}\sigma(A_0 + z_1A_1 + z_2A_2),$$

$$\alpha_{zr} \stackrel{\text{def}}{=} \sup_{(z,r)\in\overline{\mathbb{D}}\times[-1;+1]} \operatorname{Re}\sigma(A_0 + zA_1 + rA_2),$$

$$\alpha_{rz} \stackrel{\text{def}}{=} \sup_{(r,z)\in[-1;+1]\times\overline{\mathbb{D}}} \operatorname{Re}\sigma(A_0 + rA_1 + zA_2),$$

$$\alpha_{z} \stackrel{\text{def}}{=} \sup_{z\in\overline{\mathbb{D}}} \operatorname{Re}\sigma(A_0 + z(A_1 + A_2)),$$

$$\alpha_{rr} \stackrel{\text{def}}{=} \sup_{(r_1,r_2)\in[-1;+1]^2} \operatorname{Re}\sigma(A_0 + r_1A_1 + r_2A_2),$$

$$\alpha_{r} \stackrel{\text{def}}{=} \sup_{r\in[-1;+1]} \operatorname{Re}\sigma(A_0 + r(A_1 + A_2)) \ .$$

Clearly, the latter quantities are linked by the inequalities:

$$\alpha_{zz} \geq \alpha_{zr}, \alpha_{rz} \geq \alpha_{rr} \geq \alpha_r, \quad \text{and } \alpha_{zz} \geq \alpha_z \geq \alpha_r \ . \tag{5.1}$$

We use the previously presented LMIs to find, for each uncertainty structure, the least real number α such that $A(z) - \alpha I_n$ is robustly stable. For each integer k and for each value of α, a convex problem is solved, but the problem is not jointly convex in the four unknowns $P_k, Q_{k,1}, Q_{k,2}$ and α, so a bisection process is achieved.

The computations presented here have been performed using the package `lmitool` of the free software SCILAB. The successive (upper) estimates of the robustness margins, according to the value of k, are given in Table 5.1. Between parentheses is given the CPU time necessary for the solution of the LMIs (for the corresponding values of α and k), measured on a computer equipped with a Pentium III 800MHz.

	α_{zz}	α_{zr}	α_{rz}
$k=1$	-2.42 (0.2s)	-3.17 (0.2s)	-2.42 (0.19s)
$k=2$	-3.87 (7.63s)	-4.57 (9.73s)	-4.46 (7.84s)
	α_z	α_{rr}	α_r
$k=1$	-3.24 (0.06s)	-3.17 (0.2s)	-3.24 (0.07s)
$k=2$	-4.14 (0.23s)	-5.24 (10.1s)	-5.39 (0.26s)
$k=3$			-5.41 (0.65s)

TABLE 5.1
Succesive estimates of the margins and corresponding CPU times.

The values are compared to those obtained by checking directly the robust stability by means of gridding of the parameter space, which are presented in Table 5.2. Due to the small size of the problem, small computation times are required.

One verifies that, for each margin, the successive estimates are nonincreasing functions of k, and that the inequalities corresponding to (5.1) are fulfilled for any value of k. In the present case, the tests achieved for $k = 2$ provide these true values up to three digits, except for α_r ($k = 3$).

In principle, the previous numbers may also be determined using the fact that

$$\alpha_{zz} = \inf \{\alpha \in \mathbb{R} \ : \ \forall \omega \in \mathbb{R}, \forall \alpha' \in (\alpha; +\infty), \ \mu_\Delta(G(j\omega + \alpha')) < 1\} \ ,$$

Number of nodes in parameter space	α_{zz}	α_{zr}	α_{rz}
10×10	-3.93 (0.02s)	-4.63 (0.02s)	-4.48 (0.02s)
100×100	-3.88 (1.67s)	-4.57 (1.68s)	-4.47 (1.66s)
	α_z	α_{rr}	α_r
10×10	-4.15 (0.01s)	-5.24 (0.01s)	-5.42 (0.01s)
100×100	-4.15 (0.02s)	-5.24 (0.84s)	-5.42 (0.01s)

TABLE 5.2
Succesive estimates of the margins by gridding and corresponding CPU times.

where $G(s) \stackrel{\text{def}}{=} \begin{pmatrix} I_n \\ I_n \end{pmatrix} \left(sI_n - A_0\right)^{-1} \begin{pmatrix} A_1 & A_2 \end{pmatrix}$, and for the uncertainty structure $\Delta = \{\text{diag}\{z_1 I_n; z_2 I_n\} \ : \ z_i \in \mathbb{C}\}$. Similar formulas hold for the other margins. Define the constants

$$\overline{\alpha} \stackrel{\text{def}}{=} \inf\{\alpha \in \mathbb{R} \ : \ \forall \omega \in \mathbb{R}, \forall \alpha' \in (\alpha; +\infty), \ \overline{\sigma}(G(j\omega + \alpha')) < 1\},$$
$$\overline{\alpha}_{zz} \stackrel{\text{def}}{=} \inf\{\alpha \in \mathbb{R} \ : \ \forall \omega \in \mathbb{R}, \forall \alpha' \in (\alpha; +\infty), \ \nu_\Delta(G(j\omega + \alpha')) < 1\} \ ,$$

where Δ is the same set than above, and $\overline{\sigma}$ and ν_Δ denote respectively the largest singular value and the usual upper bound of μ_Δ [16]. Based on the properties of ν_Δ [16], one has $\alpha_{zz} \leq \overline{\alpha}_{zz} \leq \overline{\alpha}$. The results of the estimation of the previous constants, based on the underlying LMI problems, are summarized in Table 5.3. As before, between parentheses is indicated the CPU time necessary to check that, for a fixed value of α, $\overline{\sigma}(G(j\omega + \alpha)) < 1$ (respectively $\nu_\Delta(G(j\omega + \alpha)) < 1$) for a discretized sample of frequencies ω on the real axis. Tighter discretization, not reproduced here, shows that the values obtained for 1000 gridding points are the true values of the extrema $\overline{\alpha}$ and $\overline{\alpha}_{zz}$. Recall that the estimate α_{zz} obtained by use of Theorem 4.1 for

Number of nodes in frequency domain	$\overline{\alpha}$	$\overline{\alpha}_{zz}$
100	-0.241 (0.43s)	-1.30 (0.84s)
1000	-0.151 (5.22s)	-1.21 (7.96s)

TABLE 5.3
Succesive estimates of the margins upper bounds and corresponding CPU times.

$k = 2$ is exact (up to the precision considered), while $\overline{\alpha}$ and $\overline{\alpha}_{zz}$ provide conservative robust stability margins. For this simple example, the gain in precision is clear, for comparable computation time.

6. Comments on the results. The results stated in §4 permit a systematic approach to the study of parameter-dependent quadratic Lyapunov functions for robust stability: a class of candidate Lyapunov functions is exhibited (given in Definition 3.1), rich enough to *characterize* robust stability, but structured enough to permit the use of LMI tests. In our opinion, this offers a useful insight into the powerfulness of quadratic Lyapunov functions for stability analysis. Similarly, it provides information on the kind of problems solvable by LMIs: the issue of robust stability analysis is located "on the boundary" of these problems, as it may be relaxed with arbitrary precision into a standard LMI, obtained explicitly. In this sense, the results given here constitute an attempt to investigate in more detail the abilities of the LMIs, which have become, in the last decade, a unifying framework for expressing and solving many problems in control theory.

We believe that, beyond their theoretical interest, the proposed results may offer attractive numerical alternatives for robust stability analysis, at least for problems of low order. The construction of the LMIs involved is reasonably simple, using only elementary algebraic operations. For a given value of k, their complexity is polynomial with respect to the dimension n of the matrices, and exponential with respect to the number m of scalar parameters. More precisely, the total number of scalar elements of the unknowns $P_k, Q_{k,1}, \dots, Q_{k,m}$ in (LMI_k) is $\frac{1}{2}[k^m n(k^m+1) + \sum_{1\leq i\leq m} k^{m-i+1}(k+1)^{i-1}(k^{m-i+1}(k+1)^{i-1}+1)]$, which is equivalent to $\frac{m+1}{2}k^{2m}n^2$ when $k \to +\infty$, while the number of rows of the inequalities involved is $[k^m + (k+1)^m]n$, which is of the order of $2k^m n$ when $k \to +\infty$.

A quantitative evaluation of the relationship between the size of k and the precision of the criteria, should be considered as a natural next step in forthcoming research. In the general case, however, when no special matrix structure exists for the system under study, the growth of the value of k needed to check robust stability of a system cannot be polynomial in the worst case. The effective use of large values of k hinges upon the possibility of intensive computation and use of large memory.

The method for robust stability analysis proposed here may be compared to the one consisting in checking stability in every node of a grid of the parameter space. Both methods are able to provide less and less conservative criteria — when the discretization step goes to zero, or when the degree, with respect to the parameters, of the underlying parameter-dependent Lyapunov function goes to infinity. Both methods are exact, in the sense that they provide asymptotically arbitrarily precise estimates of the true stability margins. The gridding method, however, offers successive (less and less) optimistic estimates, whereas the other one provides (less and less) pessimistic indications, usually more useful in practice.

Also, both methods are, in the present state of knowledge, computationally *undecidable*: no information is known on the size of the least k, if any, for which the LMIs are solvable (in other words, of the largest k which is necessary to test numerically to decide whether the system is robustly stable or not). This is an important question, both from theoretical and practical point of view. Some numerical experiments (see §5) indicate that small values of k often yield correct answers.

Finally, recall that ensuring robust stability analysis is equivalent to checking that a certain structured singular value is less than 1. In consequence, the results presented above may be as well seen as providing a family of more and more precise *upper bounds* for these special-structured singular values. In a future work, the extension of this to the general case of structured singular values with repeated scalar blocks will be investigated. Indeed, it may be reasonable to employ these new upper bounds in a branch and bound algorithm [3], in place of the usual ones.

Alternatively, in the case of real parameters, a possible way to consider problems of larger size rests in the combination of decomposition of the parameter domain and resolution on each subdomain by use of low-order test. This should permit to find a compromise between the number of independent LMIs to be solved (equal to the number of subdivisions) and the computational complexity (due to high degree of the underlying parameter-dependent Lyapunov functions). When coupled with decomposition in the parameter space, one may expect a better fit of the proposed method than the direct use of Theorems 4.1 and 4.3, at least for problems of medium size.

7. Proofs. The following decomposition of the matrix $\hat{J}_k$ defined in §2 will be used

$$\hat{J}_k = \begin{pmatrix} I_k & 0_{k\times 1} \end{pmatrix} = \begin{pmatrix} f_k & F_k \end{pmatrix} \ ,$$

where

$$f_k \overset{\text{def}}{=} \begin{pmatrix} 1 \\ 0_{(k-1)\times 1} \end{pmatrix}, \ F_k \overset{\text{def}}{=} \begin{pmatrix} 0_{1\times(k-1)} & 0 \\ I_{k-1} & 0_{(k-1)\times 1} \end{pmatrix} \ . \tag{7.1}$$

The size of the previous matrices is $f_k : k \times 1$, $F_k : k \times k$, and the spectrum of F_k is $\{0\}$. Simple computation shows that, for any $z \in \mathbb{C}$, $(I_k - zF_k)z^{[k]} = f_k$, that is

$$(I_k - zF_k)^{-1} f_k = z^{[k]} \ . \tag{7.2}$$

Another useful property is the fact that, for any $i \in \mathbb{N}$, $0 \leq i \leq k-1$,

$$F_k^{iT} z^{[k]} = \begin{pmatrix} 0_{(k-i)\times i} & I_{k-i} \\ 0_i & 0_{i\times(k-i)} \end{pmatrix} \begin{pmatrix} z^{[i]} \\ z^i z^{[k-i]} \end{pmatrix} = z^i \begin{pmatrix} z^{[k-i]} \\ 0_{i\times 1} \end{pmatrix} \ . \tag{7.3}$$

7.1. Proof of Theorem 4.1. We now prove Theorem 4.1. The proof of the equivalence between the properties (i), (ii) and (iii) consists of three main stages, that we now present, and which are detailed below in §§7.1.1 to 7.1.3.

1st stage. We detail here the ideas given in the Sketch of proof of formula (4.1). We show that the computations proposed there permit to establish that solvability of (LMI$_k$) implies solvability of (1.2) for all $z \in (\partial\mathbb{D})^m$. It remains to show that this implies however Hurwitzness of $A(z)$ for any z *in the whole set* $\overline{\mathbb{D}}^m$. This gives the implication (iii) $\Rightarrow$ (i).

2nd stage. The 2nd step establishes that the robust stability property (i) implies that the parameter-dependent Lyapunov inequality (1.2) admits a solution $P(z)$ of the form (3.1), for a certain $k \in \mathbb{N}$, with P_k *positive definite.* For this, one shows essentially that the associated Lyapunov *equation* $A(z)^H P(z) + P(z)A(z) = -I_n$ admits as a solution an infinite sum of powers of $z, \bar{z}$, converging uniformly in $\overline{\mathbb{D}}^m$. It then suffices to truncate this expansion to obtain a polynomial solution (of unknown degree) to *inequality* (1.2). The corresponding coefficient matrix P_k is positive *semidefinite* by construction, and some more work is necessary to obtain an expression with a *positive* definite matrix. This gap is filled in in Lemma 7.1. As a by-product, the implication (i) $\Rightarrow$ (ii) is obtained here.

3rd stage. At this point, (i) has been shown to imply existence, for large enough k, of a certain $P_k > 0$ such that $R(z)$ given by (3.2), (3.3) is negative definite for any $z \in \overline{\mathbb{D}}^m$. The next step (Lemma 7.2) is the key part of the necessity proof. It consists in showing that $R(z) < 0$ for all $z \in (\partial\mathbb{D})^m$ *if and only if* the second inequality in (LMI$_k$) holds. This is done by applying recursively D-scaling with respect to each of the parameter z_i. At each step, a new matrix, depending upon the remaining parameters $z_{i+1}, \dots, z_m$, is introduced. The latter may be assumed polynomial in the previous parameters and their conjugates (this is deduced from a general result on existence of polynomial solutions to parameter-dependent LMIs, Theorem 7.3), with coefficients defined by a constant matrix, which is precisely the variable $Q_{k,i}$ of (LMI$_k$). The transformation carried out by this procedure is not restrictive, as the scaling technique (the Kalman-Yakubovich-Popov lemma, recalled in Appendix

A) is lossless for one complex parameter. This yields the implication (ii) $\Rightarrow$ (iii). Incidentally, we prove at this stage that the solvability of (LMI$_k$) implies the same property for largest indices.

When the coefficients are real, then the polynomial solutions exhibited above are easily proved to be real too, basically due to the remark on realness given after the version of Kalman-Yakubovich-Popov lemma recalled in Appendix A.

7.1.1. First stage. Suppose (LMI$_k$) holds. As suggested in the Sketch of proof of formula (4.1), left- and right-multiplication of its second inequality by $(z_m^{[k+1]} \otimes \cdots \otimes z_1^{[k+1]} \otimes I_n)$ and its transconjugate yields $R(z) + \sum_{i=1}^m (1-|z_i|^2)(z_m^{[k]} \otimes \cdots \otimes z_i^{[k]} \otimes z_{i-1}^{[k+1]} \otimes \dots z_1^{[k+1]} \otimes I_n)^H Q_{k,i}(z_m^{[k]} \otimes \cdots \otimes z_i^{[k]} \otimes z_{i-1}^{[k+1]} \otimes \dots z_1^{[k+1]} \otimes I_n) < 0_n$. Indeed, this comes directly from the fact that, due to (2.2), for any $i = 1, \dots, m$,

$$\left(\hat{J}_k^{(m-i+1)\otimes} \otimes I_{(k+1)^{i-1}n}\right)(z_m^{[k+1]} \otimes \cdots \otimes z_1^{[k+1]} \otimes I_n) = (z_m^{[k]} \otimes \cdots \otimes z_i^{[k]} \otimes z_{i-1}^{[k+1]} \otimes \dots z_1^{[k+1]} \otimes I_n),$$

$$\left(\hat{J}_k^{(m-i)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{i-1}n}\right)(z_m^{[k+1]} \otimes \cdots \otimes z_1^{[k+1]} \otimes I_n) = z_i(z_m^{[k]} \otimes \cdots \otimes z_i^{[k]} \otimes z_{i-1}^{[k+1]} \otimes \dots z_1^{[k+1]} \otimes I_n).$$

Thus, $R(z) < 0_n$ if $|z_1| = \cdots = |z_m| = 1$, so the matrix $A(z)$ is Hurwitz for all $z \in (\partial\mathbb{D})^m$.

The remaining argument is based on a subharmonicity and continuity argument. Using the fact that the map $\overline{\mathbb{C}^+} \cup \{\infty\} \to \overline{\mathbb{D}}, s_z \mapsto (1-s_z)/(1+s_z)$ is one-to-one, one proves [7] that

$$\begin{aligned}\max_{z\in\overline{\mathbb{D}}^m} \rho(e^{A(z)}) &= \sup_{s\in\overline{\mathbb{C}^+}^m} \rho(e^{A_0+(1-s_1)/(1+s_1)A_1+\cdots+(1-s_m)/(1+s_m)A_m}) \\ &= \sup_{s\in(j\mathbb{R})^m} \rho(e^{A_0+(1-s_1)/(1+s_1)A_1+\cdots+(1-s_m)/(1+s_m)A_m}) \\ &= \max_{z\in(\partial\mathbb{D})^m} \rho(e^{A(z)}).\end{aligned}$$

As a consequence, if all the matrices $A(z)$ are Hurwitz for $z \in (\partial\mathbb{D})^m$, then the previous expression is less than 1, and the same property holds on the whole $\overline{\mathbb{D}}^m$. This shows that (iii) implies (i).

7.1.2. Second stage. Property (i) implies solvability of (1.2) for each $z \in \overline{\mathbb{D}}^m$, which, as is well-known, is equivalent [30] to the solvability of the (Lyapunov) *equation*

$$P(z) > 0_n,\ A(z)^H P(z) + P(z)A(z) = -I_n\ . \tag{7.4}$$

Now, when (i) holds, the latter has a solution *analytic in z, $\bar{z}$ in* $\overline{\mathbb{D}}^m$. Indeed, when $A(z)$ is Hurwitz, the explicit form of the solution of (7.4) is given by

$$P(z) = \int_0^{+\infty} e^{A(z)^H t} e^{A(z)t}\, dt\ .$$

When $A(z)$ is Hurwitz for any z in the compact set $\overline{\mathbb{D}}^m$, the convergence of this integral in $t = +\infty$ is uniform with respect to z, so there exists $T > 0$ independent of

z, such that $P(z)$ defined now by

$$P(z) = \int_0^T e^{A(z)^H t} e^{A(z)t}\, dt \tag{7.5}$$

is positive definite and solves inequality (1.2) in $\overline{\mathbb{D}}^m$.

Expanding the integrand in powers of the $z_i, \bar{z}_i$, $1 \leq i \leq m$, and interverting the sum and the integral, one exhibits an expansion of $P(z)$ in powers of $z, \bar{z}$, converging uniformly for $(z,t) \in \overline{\mathbb{D}}^m \times [0;T]$. More precisely, let $M_k : [0;T] \to \mathbb{C}^{n \times k^m n}$ be such that $M_k(t)(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)$ represents the terms of degree less than k in each of the z_i in the expansion of $e^{A(z)t}$. Then,

$$\int_0^T e^{A(z)^H t} e^{A(z)t}\, dt = \lim_{k \to +\infty} (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H \tilde{P}_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) \tag{7.6}$$

with uniform convergence in $\overline{\mathbb{D}}^m$, where $\tilde{P}_k \in \mathcal{H}^{k^m n}$ is defined by

$$\tilde{P}_k \stackrel{\text{def}}{=} \int_0^T M_k(t)^H M_k(t)\, dt \geq 0\ . \tag{7.7}$$

Now, this implies that, for large enough k, $(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H \tilde{P}_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)$ solves inequality (1.2) in $\overline{\mathbb{D}}^m$. This provides a PPDQ Lyapunov function for (1.1), but not the desired one, as the matrices $\tilde{P}_k$ are only positive *semidefinite* (except $\tilde{P}_1 = P(0)$, which is positive definite).

Let instead, for the matrix F_k defined in (7.1),

$$P_k \stackrel{\text{def}}{=} \sum_{i_1,\dots,i_m=0}^{k-1} (F_k^{i_m} \otimes \cdots \otimes F_k^{i_1} \otimes I_n) \tilde{P}_k (F_k^{i_m} \otimes \cdots \otimes F_k^{i_1} \otimes I_n)^T\ . \tag{7.8}$$

Lemma 7.1. *The matrix $P_k \in \mathcal{H}^{k^m n}$ defined in* (7.8) *is positive definite and, for large enough $k \in \mathbb{N}$, $(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H P_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)$ solves* (1.2) *in $\overline{\mathbb{D}}^m$.*

Proof. • We begin by the positivity property. Note first that $P_k \geq 0$, because $\tilde{P}_k \geq 0$. Let $u \in \mathbb{C}^{k^m n}$ such that $u^H P_k u = 0$, let us establish that this implies $u = 0$. In view of (7.8), and thanks to the fact that $\tilde{P}_k \geq 0$, this implies that: $\forall\, 0 \leq i_1, \dots, i_m \leq k-1$,

$$u^H (F_k^{i_m} \otimes \cdots \otimes F_k^{i_1} \otimes I_n) \tilde{P}_k (F_k^{i_m} \otimes \cdots \otimes F_k^{i_1} \otimes I_n)^T u = 0. \tag{7.9}$$

First, notice that for any integer i, k, $i \leq k$, all the terms of degree less than $k-i$ in $e^{A(z)t}$, whose total sum is $M_{k-i}(t)(z_m^{[k-i]} \otimes \cdots \otimes z_1^{[k-i]} \otimes I_n)$ by definition, are also present in $M_k(t)(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)$. At the level of the matrices $\tilde{P}_k$, this property reads as

$$\tilde{P}_{k-i} = \left(\begin{pmatrix} I_{k-i} \\ 0_{i \times (k-i)} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} I_{k-i} \\ 0_{i \times (k-i)} \end{pmatrix} \otimes I_n \right)^T \tilde{P}_k \left(\begin{pmatrix} I_{k-i} \\ 0_{i \times (k-i)} \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} I_{k-i} \\ 0_{i \times (k-i)} \end{pmatrix} \otimes I_n \right) . \tag{7.10}$$

Indeed, for $z \in \mathbb{C}$, $\begin{pmatrix} I_{k-i} \\ 0_{i \times (k-i)} \end{pmatrix} z^{[k-i]}$ is equal to $z^{[k]}$, except for the terms of degree larger than $k-i-1$, which are replaced by zero. Remark also that, for $i' \leq i \leq k$,

$$F_k^{iT} = \begin{pmatrix} 0_{(k-i)\times i} & I_{k-i} \\ 0_i & 0_{i \times (k-i)} \end{pmatrix} = \begin{pmatrix} I_{k-i'} \\ 0_{i' \times (k-i')} \end{pmatrix} \begin{pmatrix} 0_{(k-i)\times i} & I_{k-i} \\ 0_{(i-i')\times i} & 0_{(i-i')\times(k-i)} \end{pmatrix}\ . \tag{7.11}$$

Putting now $i_1 = \cdots = i_m = k-1$ in (7.9) and using identity (7.11) with $i = i' = k-1$ and (7.10), one deduces first that

$$\left\| \tilde{P}_1^{1/2} \left(\begin{pmatrix} 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes I_n \right) u \right\|^2 = 0 \ ,$$

i.e., as $\tilde{P}_1 > 0$,

$$\left(\begin{pmatrix} 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes I_n \right) u = 0 \ . \tag{7.12}$$

Taking then $i_1 = k-2, i_2 = \cdots = i_m = k-1$ in (7.9), using (7.11) with $i = i' = k-2$ and $i' = k-2$, $i = k-1$, and then (7.10), yields

$$\left\| \tilde{P}_2^{1/2} \left(\begin{pmatrix} 0_{2\times(k-2)} & I_2 \end{pmatrix} \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes I_n \right) u \right\|^2 = 0 \ .$$

Now,

$$\begin{aligned}
&\tilde{P}_2^{1/2} \left(\begin{pmatrix} 0_{2\times(k-2)} & I_2 \end{pmatrix} \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes I_n \right) u \\
&= \tilde{P}_2^{1/2} \left(\begin{pmatrix} 0_{1\times(k-2)} & 1 & 0 \\ 0_{1\times(k-2)} & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes I_n \right) u \\
&\quad + \tilde{P}_2^{1/2} \left(\begin{pmatrix} 0_{1\times(k-1)} & 0 \\ 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes I_n \right) u \\
&\qquad \text{(by linearity)} \\
&= \tilde{P}_2^{1/2} \left(\begin{pmatrix} 0_{1\times(k-2)} & 1 & 0 \\ 0_{1\times(k-2)} & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes I_n \right) u \\
&\qquad \text{(due to (7.12))} \ ,
\end{aligned}$$

and, thanks to (7.10),

$$\begin{aligned}
&\left\| \tilde{P}_2^{1/2} \left(\begin{pmatrix} 0_{1\times(k-2)} & 1 & 0 \\ 0_{1\times(k-2)} & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \\ 0_{1\times(k-1)} & 0 \end{pmatrix} \otimes I_n \right) u \right\| \\
&\quad = \left\| \tilde{P}_1^{1/2} \left(\begin{pmatrix} 0_{1\times(k-2)} & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes I_n \right) u \right\| \ .
\end{aligned}$$

The last term is thus null, due as before to definiteness of $\tilde{P}_1$, so one concludes that

$$\left(\begin{pmatrix} 0_{1\times(k-2)} & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0_{1\times(k-1)} & 1 \end{pmatrix} \otimes I_n \right) u = 0 \ .$$

Carrying on in this way, one shows that all the components of u, taken n by n, are null. Thus, $u^H P_k u = 0$ implies $u = 0$, so $P_k > 0$, for any $k \in \mathbb{N}$.

• We now show that, for large enough values of k, P_k defined in (7.8) generates a PPDQ function fulfilling the requirement of (ii). For this, let us first establish that

$$\begin{aligned}
\lim_{k\to+\infty} \frac{1}{\|z_1^{[k]}\|^2 \ldots \|z_m^{[k]}\|^2} &(z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H P_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) \\
&= \lim_{k\to+\infty} (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n)^H \tilde{P}_k (z_m^{[k]} \otimes \cdots \otimes z_1^{[k]} \otimes I_n) \ ,
\end{aligned} \tag{7.13}$$

where both limits are uniform in $\overline{\mathbb{D}}^m$. The second limit is already known to exist, and to be equal to $P(z)$ in (7.5).

From identity (7.8) one deduces, thanks to (7.3), that

$$
\begin{aligned}
&(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)^H P_k (z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)\\
=&\sum_{i_1,\dots,i_m=0}^{k-1} |z_1|^{2i_1}\dots|z_m|^{2i_m}\left(\begin{pmatrix} z_m^{[k-i_m]}\\ 0_{i_1\times 1}\end{pmatrix}\otimes\dots\otimes\begin{pmatrix} z_1^{[k-i_1]}\\ 0_{i_m\times 1}\end{pmatrix}\otimes I_n\right)^H \tilde{P}_k\left(\begin{pmatrix} z_m^{[k-i_m]}\\ 0_{i_m\times 1}\end{pmatrix}\otimes\dots\otimes\begin{pmatrix} z_1^{[k-i_1]}\\ 0_{i_1\times 1}\end{pmatrix}\otimes I_n\right).
\end{aligned}
$$

As

$$
\sum_{i_1,\dots,i_m=0}^{k-1} |z_1|^{2i_1}\dots|z_m|^{2i_m} = \|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2 ,
$$

we get

$$
\begin{aligned}
&\|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2 P(z) - (z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)^H P_k (z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)\\
=&\sum_{i_1,\dots,i_m=0}^{k-1} |z_1|^{2i_1}\dots|z_m|^{2i_m}\Bigg[P(z)\\
&-\left(\begin{pmatrix} z_m^{[k-i_m]}\\ 0_{i_m\times 1}\end{pmatrix}\otimes\dots\otimes\begin{pmatrix} z_1^{[k-i_1]}\\ 0_{i_1\times 1}\end{pmatrix}\otimes I_n\right)^H \tilde{P}_k\left(\begin{pmatrix} z_m^{[k-i_m]}\\ 0_{i_m\times 1}\end{pmatrix}\otimes\dots\otimes\begin{pmatrix} z_1^{[k-i_1]}\\ 0_{i_1\times 1}\end{pmatrix}\otimes I_n\right)\Bigg] \qquad (7.14)
\end{aligned}
$$

Now, uniform convergence of the right-hand side of (7.13) yields: for any $\varepsilon>0$, there exists k_ε such that, for any $k>k_\varepsilon$, for any $z\in\overline{\mathbb{D}}^m$,

$$
\|P(z) - (z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)^H \tilde{P}_k (z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)\| < \varepsilon .
$$

Distinguishing between the terms for which $\max\{i_1,\dots,i_m\}<k-k_\varepsilon$ and $\max\{i_1,\dots,i_m\}\geq k-k_\varepsilon$, allows to show that the norm of the left-hand side of (7.14) is bounded from above by

$$
\varepsilon\ \|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2 + 2c \sum_{\substack{i_1,\dots,i_m=0\\ \max\{i_1,\dots,i_m\}\geq k-k_\varepsilon}}^{k-1} |z_1|^{2i_1}\dots|z_m|^{2i_m} .
$$

In the previous expression, c is defined as

$$
c \stackrel{\text{def}}{=} \max\left\{\left\|\left(\begin{pmatrix} z_m^{[k-i_m]}\\ 0_{i_m\times 1}\end{pmatrix}\otimes\dots\otimes\begin{pmatrix} z_1^{[k-i_1]}\\ 0_{i_1\times 1}\end{pmatrix}\otimes I_n\right)^H \tilde{P}_k\left(\begin{pmatrix} z_m^{[k-i_m]}\\ 0_{i_m\times 1}\end{pmatrix}\otimes\dots\otimes\begin{pmatrix} z_1^{[k-i_1]}\\ 0_{i_1\times 1}\end{pmatrix}\otimes I_n\right)\right\| : z\in\overline{\mathbb{D}}^m, i_1,\dots,i_m\leq k, k\in\mathbb{N}\right\} .
$$

The constant c is finite, because, when $i_j\to+\infty$, $1\leq j\leq m$, the expression inside the norm converges uniformly in $\overline{\mathbb{D}}^m$ towards $P(z)$, which, being continuous, is itself bounded. On the other hand, for any $z \in \mathbb{C}^m$,

$$
\begin{aligned}
&\sum_{\substack{i_1,\dots,i_m=0\\ \max\{i_1,\dots,i_m\}\geq k-k_\varepsilon}}^{k-1} |z_1|^{2i_1}\dots|z_m|^{2i_m}\\
&= \sum_{i_1,\dots,i_m=0}^{k-1} |z_1|^{2i_1}\dots|z_m|^{2i_m} - \sum_{i_1,\dots,i_m=0}^{k-k_\varepsilon-1} |z_1|^{2i_1}\dots|z_m|^{2i_m}\\
&= \|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2 - \|z_1^{[k-k_\varepsilon]}\|^2\dots\|z_m^{[k-k_\varepsilon]}\|^2\\
&= \left(\|z_1^{[k]}\|^2 - \|z_1^{[k-k_\varepsilon]}\|^2\right)\|z_2^{[k]}\|^2\dots\|z_m^{[k]}\|^2\\
&\quad + \dots + \|z_1^{[k-k_\varepsilon]}\|^2\dots\|z_{m-1}^{[k-k_\varepsilon]}\|^2\left(\|z_m^{[k]}\|^2 - \|z_m^{[k-k_\varepsilon]}\|^2\right)\\
&\leq m\|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2 \max_{i=1,\dots,m}\frac{\|z_i^{[k]}\|^2 - \|z_i^{[k-k_\varepsilon]}\|^2}{\|z_i^{[k]}\|^2}\\
&= m\|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2 \max_{i=1,\dots,m}|z_i|^{2(k-k_\varepsilon)}\frac{1+|z_i|^2+\dots+|z_i|^{2(k_\varepsilon-1)}}{1+|z_i|^2+\dots+|z_i|^{2(k-1)}}\ .
\end{aligned}
$$

It turns out that, uniformly in $\overline{\mathbb{D}}^m$, the following estimate holds:

$$
\begin{aligned}
&\left\|P(z) - \frac{1}{\|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2}(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)^H P_k(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)\right\|\\
&\qquad\qquad\qquad\qquad \leq \varepsilon + 2mc \sup_{r\in[0;1)} r^{k-k_\varepsilon}\frac{1-r^{k_\varepsilon}}{1-r^k}\ ,
\end{aligned}
$$

provided that $k > k_\varepsilon$. Notice that, for any fixed k_ε, the quantity

$$
\sup_{r\in[0;1)} r^{k-k_\varepsilon}\frac{1-r^{k_\varepsilon}}{1-r^k}
$$

vanishes when k goes to infinity. Thus, for large enough k, it is smaller than $\varepsilon/2mc$ and then, for any $z \in \overline{\mathbb{D}}^m$,

$$
\left\|P(z) - \frac{1}{\|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2}(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)^H P_k(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)\right\| < 2\varepsilon\ .
$$

This achieves the proof of the announced convergence property (7.13).

As a consequence of (7.13), the truncated expression $\frac{1}{\|z_1^{[k]}\|^2\dots\|z_m^{[k]}\|^2}(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)^H P_k(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)$ solves (1.2) for large enough k, and $x^H(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)^H P_k(z_m^{[k]}\otimes\dots\otimes z_1^{[k]}\otimes I_n)x$ with $P_k > 0$ is also a PPDQ Lyapunov function for (1.1). This achieves the proof of Lemma 7.1. $\square$

As a conclusion of this second stage of the proof of Theorem 4.1, we have shown until now that property (i) is equivalent to (ii), in which moreover the hermitian P_k defining $P(z)$ may be supposed *positive definite* without loss of generality.

7.1.3. Third stage. This part is achieved by induction. Consider for any $i = 0, \dots, m$, the following

Property $(\mathcal{P}_i)$: $\exists k \in \mathbb{N}$, $\exists P_k \in \mathcal{H}^{k^m n}$, $P_k > 0$, $\exists Q_{k,j} \in \mathcal{H}^{k^{m-j+1}(k+1)^{j-1}n}$, $j = 1, \dots, i$, $\forall (z_{i+1}, \dots, z_m) \in (\partial\mathbb{D})^{m-i}$,

$$\left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+1}^{[k+1]} \otimes I_{(k+1)^i n}\right)^H \left[R_k + \sum_{j=1}^{i} \left(\hat{J}_k^{(m-j+1)\otimes} \otimes I_{(k+1)^{j-1}n}\right)^T Q_{k,j} \left(\hat{J}_k^{(m-j+1)\otimes} \otimes I_{(k+1)^{j-1}n}\right) - \sum_{j=1}^{i} \left(\hat{J}_k^{(m-j)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{j-1}n}\right)^T Q_{k,j} \left(\hat{J}_k^{(m-j)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{j-1}n}\right)\right] \left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+1}^{[k+1]} \otimes I_{(k+1)^i n}\right) < 0_{(k+1)^i n} .$$

In the previous expression, the matrix $R_k = R_k(P_k)$ is defined in (3.3). One verifies easily that $(\mathcal{P}_0)$ may be as well expressed as: there exists $P(z)$ as in (3.1) such that $P_k > 0$ and $R(z)$ defined in (3.2), (3.3) is negative definite for all $z \in (\partial\mathbb{D})^m$. Property $(\mathcal{P}_0)$ is thus a consequence of (ii) (see §7.1.2), while in parallel $(\mathcal{P}_m)$ writes simply: there exists $k \in \mathbb{N}$ such that (LMI$_k$) holds, that is (iii).

In order to prove that $(\mathcal{P}_0)$ implies $(\mathcal{P}_m)$, we establish the slightly stronger following result.

LEMMA 7.2. *For all* $i = 1, \dots, m-1$, $(\mathcal{P}_i) \Leftrightarrow (\mathcal{P}_{i+1})$.

Proof of Lemma 7.2. First, remark that,

$$\begin{aligned}
&\left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+1}^{[k+1]} \otimes I_{(k+1)^i n}\right) \\
&= \left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+2}^{[k+1]} \otimes I_{(k+1)^{i+1} n}\right) \left(z_{i+1}^{[k+1]} \otimes I_{(k+1)^i n}\right) \\
&= \left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+2}^{[k+1]} \otimes I_{(k+1)^{i+1} n}\right) \begin{pmatrix} I_{(k+1)^i n} \\ z_{i+1}\left(I_{k(k+1)^i n} - z_{i+1}(F_k \otimes I_{(k+1)^i n})\right)^{-1} (f_k \otimes I_{(k+1)^i n}) \end{pmatrix} ,
\end{aligned}$$

the last identity being obtained after writing $z_{i+1}^{[k+1]} = \begin{pmatrix} 1 \\ z_{i+1} z_{i+1}^{[k]} \end{pmatrix}$ and using (7.2).

Applying Kalman-Yakubovich-Popov lemma as recalled in Appendix A, with $p = k(k+1)^i n$, $q = (k+1)^i n$, $A = F_k \otimes I_{(k+1)^i n}$, $B = f_k \otimes I_{(k+1)^i n}$, and remarking that the following identities hold:

$$\begin{pmatrix} B & A \end{pmatrix} = \hat{J}_k \otimes I_{(k+1)^i n}, \ \begin{pmatrix} 0_{p \times q} & I_p \end{pmatrix} = \check{J}_k \otimes I_{(k+1)^i n} ,$$

property $(\mathcal{P}_i)$ is proved to be equivalent to: $\exists k \in \mathbb{N}$, $\exists P_k \in \mathcal{H}^{k^m n}$, $P_k > 0$, $\exists Q_{k,j} \in \mathcal{H}^{k^{m-j+1}(k+1)^{j-1}n}$, $j = 1, \dots, i$, $\forall (z_{i+2}, \dots, z_m) \in (\partial\mathbb{D})^{m-i-1}$, $\exists \tilde{Q}_{k,i+1}(z_{i+2}, \dots, z_m) \in$

$\mathcal{H}^{k(k+1)^i n}$,

$$\left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+2}^{[k+1]} \otimes I_{(k+1)^i n}\right)^H \left[R_k + \sum_{j=1}^{i} \left(\hat{J}_k^{(m-j+1)\otimes} \otimes I_{(k+1)^{j-1} n}\right)^T Q_{k,j} \left(\hat{J}_k^{(m-j+1)\otimes} \otimes I_{(k+1)^{j-1} n}\right) - \sum_{j=1}^{i} \left(\hat{J}_k^{(m-j)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{j-1} n}\right)^T Q_{k,j} \left(\hat{J}_k^{(m-j)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{j-1} n}\right) \right] \left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+2}^{[k+1]} \otimes I_{(k+1)^i n}\right)$$
$$+ \left(\hat{J}_k \otimes I_{(k+1)^i n}\right)^T \tilde{Q}_{k,i+1} \left(\hat{J}_k \otimes I_{(k+1)^i n}\right) - \left(\check{J}_k \otimes I_{(k+1)^i n}\right)^T \tilde{Q}_{k,i+1} \left(\check{J}_k \otimes I_{(k+1)^i n}\right) < 0_{(k+1)^{i+1} n} . \quad \blacksquare$$

The next step consists in assigning polynomial form to $\tilde{Q}_{k,i+1}$. This is done with the help of the following general result, proved in Appendix B, and which, up to our knowledge, is original.

THEOREM 7.3. *Suppose $G_0, G_1, \ldots, G_p$ are continuous mappings defined in a* compact *subset K of $\mathbb{R}^m$, and taking values in the set of symmetric matrices of $\mathbb{R}^{n\times n}$. If, for any $\delta \in K$, there exists a solution $x(\delta) \in \mathbb{R}^p$ to the parameter-dependent LMI*

$$\exists x \in \mathbb{R}^p,\ G(x,\delta) \stackrel{\text{def}}{=} G_0(\delta) + x_1 G_1(\delta) + \cdots + x_p G_p(\delta) > 0 \ , \tag{7.15}$$

then there exists a polynomial *function $x^* : K \to \mathbb{R}^p$, such that, for any $\delta \in K$, $G(x^*(\delta), \delta) > 0$.*

REMARK 7.4. *Incidentally, one may wonder why Theorem* 7.3 *was not used in* §7.1.2*, in order to get a polynomial expansion of $P(z)$, see formula* (7.6) *above. The reason is that semidefiniteness of the matrices $\tilde{P}_k$ as given by* (7.7)*, which cannot be obtained by Theorem* 7.3*, was a crucial point to carry on the second stage.*

Notice that any LMI depending upon a finite number of scalar parameters may be put under the form (7.15).

By use of the previous result, $\tilde{Q}_{k,i+1}(z_{i+2}, \ldots, z_m)$, being solution of a LMI continuous with respect to the parameters $(z_{i+2}, \ldots, z_m)$ in $(\partial\mathbb{D})^{m-i-1}$ (seen as a compact set in $\mathbb{R}^{2(m-i-1)}$), may be chosen polynomial in the real and imaginary parts of the z_i, or as well in the $z_i, \bar{z}_i$, that is

$$\tilde{Q}_{k,i+1} = \left(z_m^{[\tilde{k}]} \otimes \cdots \otimes z_{i+2}^{[\tilde{k}]} \otimes I_{k(k+1)^i n}\right)^H Q_{\tilde{k},i+1} \left(z_m^{[\tilde{k}]} \otimes \cdots \otimes z_{i+2}^{[\tilde{k}]} \otimes I_{k(k+1)^i n}\right) , \tag{7.16}$$

for certain degree $\tilde{k}-1$ and coefficient matrix $Q_{\tilde{k},i+1} \in \mathcal{H}^{k\tilde{k}^{m-i-1}(k+1)^i n}$.

A priori, the integers k and $\tilde{k}$ are different. If $\tilde{k} < k$, one may as well suppose that $\tilde{k} = k$, enlarging the coefficient matrix $Q_{\tilde{k},i+1}$ by addition of zeros. If $\tilde{k} > k$, one shows now that k may be as well replaced by $k+1$. For this, define

$$P_{k+1} \stackrel{\text{def}}{=} \sum_{M_i \in \{\hat{J}_k, \check{J}_k\},\ i=1,\ldots,m} (M_m \otimes \cdots \otimes M_1 \otimes I_n)^T P_k (M_m \otimes \cdots \otimes M_1 \otimes I_n),$$

and, for $j = 1, \dots, i$,

$$Q_{k+1,j} \stackrel{\text{def}}{=} \sum_{\substack{M_l \in \{\hat{J}_{k+1}, \check{J}_{k+1}\},\ l=1,\dots,j-1,\\ M_l \in \{\hat{J}_k, \check{J}_k\},\ l=j,\dots,m}} (M_m \otimes \cdots \otimes M_1 \otimes I_n)^T Q_{k,j} (M_m \otimes \cdots \otimes M_1 \otimes I_n),$$

$$\tilde{Q}_{k+1,i+1} \stackrel{\text{def}}{=} \sum_{\substack{M_j \in \{\hat{J}_{k+1}, \check{J}_{k+1}\},\ j=1,\dots,i,\\ M_j \in \{\hat{J}_k, \check{J}_k\},\ j=i+1,\dots,m}} (M_m \otimes \cdots \otimes M_1 \otimes I_n)^T \tilde{Q}_{k,i+1} (M_m \otimes \cdots \otimes M_1 \otimes I_n).$$

One first shows that the positivity of P_k implies positivity of P_{k+1}: for any $u \in \mathbb{C}^{(k+1)^m n}$ such that $u^H P_{k+1} u = 0$, one has $P_k^{1/2}(M_m \otimes \cdots \otimes M_1 \otimes I_n)u = 0$ for any $M_i \in \{\hat{J}_k, \check{J}_k\}, i = 1, \dots, m$, and this implies that $u = 0$, whence the positivity of P_{k+1}. One then shows that the matrix R_{k+1} obtained from P_{k+1} by formula (3.3) verifies:

$$R_{k+1} \stackrel{\text{def}}{=} \sum_{M_i \in \{\hat{J}_{k+1}, \check{J}_{k+1}\},\ i=1,\dots,m} (M_m \otimes \cdots \otimes M_1 \otimes I_n)^T R_k (M_m \otimes \cdots \otimes M_1 \otimes I_n)\ .$$

This requires cumbersome but straightforward calculations, using property (2.3). A new set of matrices verifying property $(\mathcal{P}_i)$ has thus been generated, with index $k+1$ instead of k. Remark that otherwise, the degree $\tilde{k} - 1$ in the unknowns $z_{i+2}, \dots, z_m$ of the new matrix $\tilde{Q}_{k+1,i+1}$ is the same than for $\tilde{Q}_{k,i+1}$. It thus suffices to repeat this operation to obtain a solution with $k = \tilde{k}$. Finally, *up to a possible increase of* k, one may always suppose that $k = \tilde{k}$ in the decomposition (7.16) of $\tilde{Q}_{k,i+1}$.

REMARK 7.5. *Applying the previous argument to* $(\mathcal{P}_m)$ *proves that solvability of* (LMI$_k$) *implies the same property for the larger values of the index, as announced in the Sketch of proof of formula* (4.1).

It now remains to achieve some matrix manipulations. Using the following formula, obtained by use of (2.4),

$$\begin{aligned}
&\left(z_m^{[k]} \otimes \cdots \otimes z_{i+2}^{[k]} \otimes I_{k(k+1)^i n}\right)\left(\hat{J}_k \otimes I_{(k+1)^i n}\right)\\
&= \left(I_{k^{m-i-1}} \otimes \hat{J}_k \otimes I_{(k+1)^i n}\right)\left(z_m^{[k]} \otimes \cdots \otimes z_{i+2}^{[k]} \otimes I_{(k+1)^{i+1} n}\right)\\
&= \left(\hat{J}_k^{(m-i)\otimes} \otimes I_{(k+1)^i n}\right)\left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+2}^{[k+1]} \otimes I_{(k+1)^{i+1} n}\right)\ ,
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left(z_m^{[k]} \otimes \cdots \otimes z_{i+2}^{[k]} \otimes I_{k(k+1)^i n}\right)\left(\check{J}_k \otimes I_{(k+1)^i n}\right)\\
&\qquad = \left(\hat{J}_k^{(m-i-1)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^i n}\right)\left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+2}^{[k+1]} \otimes I_{(k+1)^{i+1} n}\right)\ ,
\end{aligned}$$

one finally proves that $(\mathcal{P}_i)$ is equivalent to:

$\exists k \in \mathbb{N}, \exists P_k \in \mathcal{H}^{k^m n}, P_k > 0, \exists Q_{k,j} \in \mathcal{H}^{k^{m-j+1}(k+1)^{j-1} n},\ j = 1, \dots, i+1,$

$\forall (z_{i+2}, \ldots, z_m) \in (\partial\mathbb{D})^{m-i-1}$,

$$\left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+2}^{[k+1]} \otimes I_{(k+1)^{i+1}n}\right)^H$$
$$\left[R_k + \sum_{j=1}^{i+1} \left(\hat{J}_k^{(m-j+1)\otimes} \otimes I_{(k+1)^{j-1}}\right)^T Q_{k,j} \left(\hat{J}_k^{(m-j+1)\otimes} \otimes I_{(k+1)^{j-1}}\right)\right.$$
$$\left. - \sum_{j=1}^{i+1} \left(\hat{J}_k^{(m-j)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{j-1}}\right)^T Q_{k,j} \left(\hat{J}_k^{(m-j)\otimes} \otimes \check{J}_k \otimes I_{(k+1)^{j-1}}\right)\right]$$
$$\left(z_m^{[k+1]} \otimes \cdots \otimes z_{i+2}^{[k+1]} \otimes I_{(k+1)^{i+1}n}\right) < 0_{(k+1)^{i+1}n} \ .$$

One recognizes property $(\mathcal{P}_{i+1})$. Hence, $(\mathcal{P}_i) \Leftrightarrow (\mathcal{P}_{i+1})$, and Lemma 7.2 is proved. $\square$

The equivalence between $(\mathcal{P}_0)$ and $(\mathcal{P}_m)$ shows in particular that (ii) implies (iii). This achieves the proof of Theorem 4.1.

7.2. Proof of Theorem 4.3. The proof proceeds by using the change of variables $r = (z + \bar{z})/2$, $z \in \overline{\mathbb{D}}^m$, already introduced to get Proposition 3.3, and by achieving the slight necessary adaptations of the proof of Theorem 4.1, using R_k defined in (3.4) and not in (3.3). The argument used in the third stage is here trivial, as the sets $\{A((z + \bar{z})/2) \ : \ z \in \overline{\mathbb{D}}^m\}$ and $\{A((z + \bar{z})/2) \ : \ z \in (\partial\mathbb{D})^m\}$ are identical.

REMARK 7.6. *Notice that the change of variables which is used leads to D-scaling as in the complex parameter case, and not DG-scaling, although the parameters involved here are real.*

8. Conclusion. Robust stability of linear systems with several scalar (complex or real) parameters has been studied. For each problem, a family of LMIs, indexed by a positive integer k, is provided. Their solvability is sufficient for robust stability, and the corresponding conditions are becoming less conservative with increasing k. Conversely, if robust stability holds, then the corresponding LMI problems are solvable from a certain k and beyond. The method involves search for a quadratic Lyapunov function depending polynomially on the parameters and their conjugates.

The LMIs are obtained in a constructive and systematic way, resulting from a limited set of elementary algebraic matrix operations. In consequence, the derived algorithms are immediatly implementable in a MATLAB/SCILAB-like environment. In practice, the accuracy of the approximation is only limited by computation time and available memory size.

Further research includes the following aspects.

1. Determination of the degree of accuracy needed to test the robust stability of any specific system; that is, of an a priori (upper) estimate on the least k, if any, for which the LMIs are solvable. More generally, the complexity and numerical aspects have to be analyzed.

2. Extension of the results to robust input/output performance evaluation for systems with scalar parameters, and to systems with polynomial and LFT dependency (see first results in [6] and [5, 4] respectively). Application to μ-analysis.

Appendix A. Discrete-time version of Kalman-Yakubovich-Popov lemma.

Initially appearing in [45], the result has been first published under its discrete-time form by Szegö and Kalman [37]. We use the statement as expressed e.g. in [35].

A proof of the result in the complex case (and for the continuous-time case) may be found in [31, Theorem 1.11.1 and Remark 1.11.1].

Let $A \in \mathbb{C}^{p\times p}, B \in \mathbb{C}^{p\times q}, M \in \mathcal{H}^{p+q}$.

LEMMA A.1. *If* $\det(I_n - zA) \neq 0$ *for any* $z \in \partial\mathbb{D}$*, then the following two statements are equivalent.*

(i) *There exists* $Q \in \mathcal{H}^p$ *such that*

$$0_{p+q} > \begin{pmatrix} B & A \end{pmatrix}^H Q \begin{pmatrix} B & A \end{pmatrix} - \begin{pmatrix} 0_{p\times q} & I_p \end{pmatrix}^H Q \begin{pmatrix} 0_{p\times q} & I_p \end{pmatrix} + M \ .$$

(ii) *For any* $z \in \partial\mathbb{D}$,

$$\begin{pmatrix} I_p \\ z(I_p - zA)^{-1}B \end{pmatrix}^H M \begin{pmatrix} I_p \\ z(I_p - zA)^{-1}B \end{pmatrix} < 0_p \ .$$

When in the statements the matrices A, B, M are real, then Q is real, symmetric.

Appendix B. Proof of Theorem 7.3.

Under the hypothesis of solvability of (7.15) for any $\delta \in K$, there exists, by continuity and compactness, a real number $\alpha > 0$ such that

$$\forall \delta \in K,\ \{x \in \mathbb{R}^p \ :\ G_0(\delta) + x_1G_1(\delta) + \cdots + x_pG_p(\delta) \geq 2\alpha I_n\} \neq \emptyset \ .$$

Define

$$\begin{aligned} F \ &:\ K \to 2^{\mathbb{R}^p}, \\ &\delta \mapsto F(\delta) = \{x \in \mathbb{R}^p \ :\ G_0(\delta) + x_1G_1(\delta) + \cdots + x_pG_p(\delta) \geq \alpha I_n\}. \end{aligned} \tag{B.1}$$

The set-valued map F maps K into the non-void closed convex subsets of $\mathbb{R}^p$.

Let us first establish that F fulfils the following property of *lower semicontinuity*, see e.g. [2].

DEFINITION B.1. *Let X be a topological space, Y a metric space. A set-valued map F from X to Y is said* lower semicontinuous *at* $x^0 \in X$ *if for any* $y^0 \in F(x^0)$ *and any neighborhood* $N(y^0)$ *of* y^0*, there exists a neighborhood* $N(x^0)$ *such that*

$$\forall x \in N(x^0),\ F(x) \cap N(y^0) \neq 0 \ .$$

F is said lower semicontinuous *if it is lower semicontinuous at every point* $x^0 \in X$.

Let $\delta^0 \in K$, $x^0 \in F(\delta^0)$, $\varepsilon > 0$. To prove lower semicontinuity of F at δ^0, we exhibit $\eta > 0$ such that for any $\delta \in K$ with $\|\delta - \delta^0\|_m < \eta$, there exists $x \in F(\delta)$, $\|x - x^0\|_p < \varepsilon$.

Indeed, by assumption, there exists $x^{\delta^0} \in \mathbb{R}^p$ such that $G(x^{\delta^0}, \delta^0) \geq 2\alpha I_n$. For $\lambda \in [0, 1]$ to be defined afterwards, let $x \overset{\text{def}}{=} (1-\lambda)x^0 + \lambda x^{\delta^0}$. Then, the fact that G is affine with respect to x implies for any $\eta > 0$, any $\delta \in K$ such that $\|\delta - \delta^0\|_m < \eta$:

$$\begin{aligned} G(x, \delta) &= (1-\lambda)G(x^0, \delta) + \lambda G(x^{\delta^0}, \delta) \\ &= (1-\lambda)G(x^0, \delta^0) + \lambda G(x^{\delta^0}, \delta^0) \\ &\quad + (1-\lambda)\left(G(x^0, \delta) - G(x^0, \delta^0)\right) + \lambda\left(G(x^{\delta^0}, \delta) - G(x^{\delta^0}, \delta^0)\right) \\ &\geq \alpha(1+\lambda)I_n \\ &\quad - \left(\sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^0, \delta) - G(x^0, \delta^0)\|_n + \sup_{\|\delta-\delta^0\|_m<\eta} \|G(x^{\delta^0}, \delta) - G(x^{\delta^0}, \delta^0)\|_n\right) I_n \ . \end{aligned}$$

On the other hand,

$$\|x - x^0\|_p = \lambda \|x^{\delta^0} - x^0\|_p \ .$$

So, take $\lambda \in [0, 1]$ such that

$$\lambda \leq \frac{\varepsilon}{2\|x^{\delta^0} - x^0\|_p} \ ,$$

and choose $\eta > 0$ such that

$$\sup_{\|\delta - \delta^0\|_m < \eta} \|G(x^0, \delta) - G(x^0, \delta^0)\|_n + \sup_{\|\delta - \delta^0\|_m < \eta} \|G(x^{\delta^0}, \delta) - G(x^{\delta^0}, \delta^0)\|_n \leq \alpha\lambda \ .$$

With these choices, one has $\|x - x^0\|_p \leq \varepsilon/2 < \varepsilon$, and $G(x, \delta) \geq \alpha(1 + \lambda)I_n - \alpha\lambda I_n = \alpha I_n$, so $x \in F(\delta)$, provided that $\delta \in K$ and $\|\delta - \delta^0\|_m < \eta$. One concludes that F is lower continuous at δ^0. This achieves the proof of lower semicontinuity of F.

We now apply to F defined in (B.1) Michael's Selection Theorem [32], see also [2].

THEOREM B.2 (Michael's Selection Theorem). *Let X be a metric space, Y a Banach space. Let F from X into the closed convex subsets of Y be lower semicontinuous. Then there exists $f : X \to Y$, a continuous selection from F.*

This yields existence of a continuous selection $f : K \to \mathbb{R}^p$ from F defined in (B.1). This function is such that

$$\forall \delta \in K, \ G(f(\delta), \delta) \geq \alpha I_n \ .$$

It remains to apply to each of the p^2 coefficients of f the following result, see e.g. [14].

THEOREM B.3 (Weierstrass Approximation Theorem). *Every continuous real-valued function defined on a compact subset K of $\mathbb{R}^m$, is the limit of a sequence of polynomials, which converges uniformly in K.*

Thus, the selection f previously exhibited is uniform limit in K of a sequence of (matrix-valued) polynomials in x. In particular, there exists a polynomial function $x^* : K \to \mathbb{R}^p$ such that

$$\forall \delta \in K, \ G(x^*(\delta), \delta) \geq \frac{\alpha}{2} I_n > 0 \ .$$

One concludes that there exists a polynomial solution to the parameter-dependent LMI (7.15), and this achieves the proof of Theorem 7.3.

Acknowledgement. The revised form of this paper has benefited from many comments and encouragements during seminaries given in summer 2002 in St-Petersburg, Russia. The author wishes to express his indebtedness to his colleagues of this community, especially to Prof. V. A. Yakubovich and to Prof. A. L. Fradkov. He also wishes to thank Prof. P. Tsiotras for his attentive reading, and anonymous reviewers whose constructive criticism allowed precious improvements of the manuscript.

REFERENCES

[1] T. ASAI, S. HARA, T. IWASAKI, *Simultaneous modeling and synthesis for robust control by LFT scaling*, in Proc. IFAC World Congress, part G (1996), p. 309–314.

[2] J.-P. AUBIN, A. CELLINA, *Differential Inclusions. Set-Valued Maps and Viability Theory*, Springer-Verlag, Berlin Heidelberg New-York Tokyo, 1984.

[3] V. Balakrishnan, S. Boyd, S. Balemi, *Branch and bound algorithm for computing the minimum stability degree of parameter-dependent linear systems*, Int. J. of Robust and Nonlinear Control, 1, no 4 (1992), pp. 295–317.

[4] P.-A. Bliman, *LMI approach to spectral stabilizability of linear delay systems and stabilizability of linear systems with complex parameter*, in Proc. of 40th IEEE CDC, Orlando (Florida), 2001.

[5] ———, *Lyapunov equation for the stability of linear delay systems of retarded and neutral type*, IEEE Trans. Automat. Control, 47, no 2 (2002), pp. 327–335.

[6] ———, *LMIs for delay-independent properties of delay systems and stability/input-output analysis of systems with complex parameter*, in Proc. of 41st IEEE CDC, Las Vegas (Nevada), 2002.

[7] S. Boyd, C. A. Desoer, *Subharmonic functions and performance bounds on linear time-invariant feedback systems*, IMA Journal of Mathematical Control & Information, 2 (1985), pp. 153–170.

[8] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics vol. 15, SIAM, Philadelphia, 1994.

[9] R. P. Braatz, P.M. Young, J.C. Doyle, M. Morari, *Computational complexity of μ calculation*, IEEE Trans. Automat. Control, 39, no 5 (1994), pp. 1000–1002.

[10] J. Chen, H. A. Latchman, *Frequency sweeping tests for stability independent of delay*, IEEE Trans. Automat. Control, 40, no 9 (1995), pp. 1640–1645.

[11] R. R. De Gaston, M. G. Safonov, *Exact calculation of the multiloop stability margin*, IEEE Trans. Automat. Control, 33, no 2 (1988), pp. 156–171.

[12] M. Dettori, C. W. Scherer, *Robust stability analysis for parameter dependent systems using full block S-procedure*, in Proc. of 37th IEEE CDC, Tampa (Florida), 1998, pp. 2798–2799.

[13] ———, *New robust stability and performance conditions based on parameter dependent multipliers*, in Proc. of 39th IEEE CDC, Sydney (Australia), 2000.

[14] J. Dieudonné, *Foundations of modern analysis*, Enlarged and corrected printing. Pure and Applied Mathematics Vol. 10-I, Academic Press, New-York London, 1969.

[15] J. C. Doyle, *Analysis of feedback systems with structured uncertainties*, IEE Proc. Part D, 129, no 6 (1982), pp. 242–250.

[16] M. K. H. Fan, A. L. Tits, J. C. Doyle, *Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics*, IEEE Trans. Automat. Control, 36, no 1 (1991), pp. 25–38.

[17] E. Feron, P. Apkarian, P. Gahinet, *S-procedure for the analysis of control systems with parametric uncertainties via parameter-dependent Lyapunov functions*, in Proc. European Contr. Conf., Roma (Italy), 1995, pp. 141–146.

[18] ———, *Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions*, IEEE Trans. Automat. Control, 41, no 7 (1996), pp. 1041–1046.

[19] M. Fu, *The real structured singular value is hardly approximable*, IEEE Trans. Automat. Control, 42, no 9 (1997), pp. 1286–1288.

[20] M. Fu, N. E. Barabanov, *Improved upper bounds for the mixed structured singular value*, IEEE Trans. Automat. Control, 42, no 10 (1997), pp. 1447–1452.

[21] M. Fu, S. Dasgupta, *Parametric Lyapunov functions for uncertain systems: the multiplier approach*, in Advances in Linear Matrix Inequality Methods in Control, L. El Ghaoui, S.-I. Niculescu, eds., SIAM, Philadelphia, 2000, pp. 95–108.

[22] P. Gahinet, P. Apkarian, M. Chilali, *Affine parameter-dependent Lyapunov functions and real parametric uncertainty*, IEEE Trans. Automat. Control, 41, no 3 (1996), pp. 436–442.

[23] J. C. Geromel, M. C. de Oliveira, L. Hsu, *LMI characterization of structural and robust stability*, Linear Algebra Appl., 285, no 1–3 (1998), pp. 69–80.

[24] J. K. Hale, E. F. Infante, F. S. P. Tsen, *Stability in linear delay equations*, J. Math. Anal. Appl., 115 (1985), pp. 533–555.

[25] T. Iwasaki, *LPV system analysis with quadratic separator*, in Proc. of 37th IEEE CDC, Tampa (Florida), 1998.

[26] ———, *Generalized quadratic Lyapunov functions for nonlinear/uncertain systems analysis*, in Perspectives in Robust Control, S. O. Reza Moheimani, ed., Lecture notes in control and information sciences 268, Springer-Verlag London, 2001, pp. 149–174.

[27] T. Iwasaki, S. Hara, *Well-posedness of feedback systems: insights into exact robustness analysis and approximate computations*, IEEE Trans. Automat. Control, 43, no 5 (1998), pp. 619–630.

[28] E. W. Kamen, *Linear systems with commensurate time delays: stability and stabilization independent of delay*, IEEE Trans. Automat. Control, 27, no 2 (1982), pp. 367–375.

[29] ———, *Correction to "Linear systems with commensurate time delays: stability and stabilization independent of delay"*, IEEE Trans. Automat. Control, 28, no 2 (1983), pp. 248–249.

[30] P. Lancaster, L. Rodman, *Algebraic Riccati Equations*, The Clarendon Press, Oxford University Press, New York, 1995.

[31] G. A. Leonov, D. V. Ponomarenko, V. B. Smirnova, *Frequency-domain Methods for Nonlinear Analysis. Theory and Applications*, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 9. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[32] E. Michael, *Continuous selections I*, Annals of Math., 63 (1956), pp. 361–381.

[33] D. Peaucelle, D. Arzelier, O. Bachelier, J. Bernussou, *A new robust D-stability condition for real convex polytopic uncertainty*, Syst. Contr. Lett., 40, no 1 (2000), pp. 21–30.

[34] D. C. W. Ramos, P. L. D. Peres, *An LMI approach to compute robust stability domains for uncertain linear systems*, in Proc. American Contr. Conf., Arlington (Virginia), 2001, pp. 4073–4078.

[35] A. Rantzer, *On the Kalman-Yakubovich-Popov lemma*, Syst. Contr. Lett., 28, no 1 (1996), pp. 7–10.

[36] A. Rantzer, M. Johansson, *Piecewise linear quadratic optimal control*, IEEE Trans. Automat. Control, 45, no 4 (2000), pp. 629–637.

[37] G. Szegö, R. E. Kalman, *Sur la stabilité absolue d'un système d'équations aux différences finies*, Comp. Rend. Acad. Sci., 257, no 2 (1963), pp. 338–390.

[38] M. Sznaier, P. A. Parrilo, *On the gap between μ and its upper bound for systems with repeated uncertainty blocks*, in Proc. of 38th IEEE CDC, Phoenix (Arizona), 1999, pp. 4511–4516.

[39] O. Toker, H. Özbay, *Complexity issues in robust stability of linear delay-differential systems*, Math. Control Signals Systems, 9, no 4 (1996), pp. 386–400.

[40] ———, *On the complexity of purely complex μ computation and related problems in multidimensional systems*, IEEE Trans. Automat. Control, 43, no 3 (1998), pp. 409–414.

[41] S. Treil, *The gap between complex structured singular value μ and its upper bound is infinite*, Available at: `http://www.math.msu.edu/treil/papers/mu/mu-abs.html`, 1999.

[42] A. Trofino, *Parameter dependent Lyapunov functions for a class of uncertain linear systems: a LMI approach*, in Proc. of 38th IEEE CDC, Phoenix (Arizona), 1999, pp. 2341–2346.

[43] A. Trofino, C. E. de Souza, *Bi-quadratic stability of uncertain linear systems*, in Proc. of 38th IEEE CDC, Phoenix (Arizona), 1999.

[44] L. Xie, S. Shishkin, M. Fu, *Piecewise Lyapunov functions for robust stability of linear time-varying systems*, in Syst. Contr. Lett., 31, no 3 (1997), pp. 165–171.

[45] V. A. Yakubovich, *Solution of certain matrix inequalities in the stability theory of nonlinear control systems*, Dokl. Akad. Nauk. SSSR, 143 (1962), pp. 1304–1307 (English translation in Soviet Math. Dokl., 3 (1962), pp. 620–623).

[46] P. M. Young, M. P. Newlin, J. C. Doyle, *Let's get real*, in Robust Control Theory, Minneapolis, MN, 1992, IMA Vol. Math. Appl., 66, Springer New York (1995), pp. 143–173.

[47] K. Zhou, with J. C. Doyle, K. Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, 1996.