Absolute stability criteria with prescribed decay rate for finite-dimensional and delay systems

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Abstract

We provide here an extension of Popov criterion, permitting to check exponential stability with prescribed decay rate (otherwise called α-stability) of nonlinear delay systems with sector-bounded nonlinearities. As for the celebrated result, the main hypothesis is expressed under a frequency form. For the delay-free case, the latter is equivalent to a linear matrix inequality, whose solution may be found by widespread algorithms.

Key words: α-stability, absolute stability, Popov criterion, delay analysis, frequency domains, linear matrix inequality, Lyapunov function.

1 Introduction

Asymptotic stability of the controlled systems is usually not sufficient, one also requires a minimal prescribed decay rate. Results in this direction have been proposed for linear delay systems, see [4,11,15] and the references therein. Also, a result has been obtained for nonlinear delay systems [14]. In the present paper, we provide such a result for nonlinear finite-dimensional or delay systems with sector restricted nonlinearities.

We consider more precisely multivariable nonlinear control systems given by one of the following differential and functional differential equations:

$$\begin{cases} \dot{x} = Ax + Bu, \quad x(0) = \phi \in \mathbb{R}^n \ , \\ u = -\psi(y), \quad y = Cx \ , \end{cases} \tag{1a}$$

$$\begin{cases} \dot{x} = \sum_{l=0}^{L} A_l x(t-h_l) + Bu, \quad x|_{[-h,0]} = \phi \in \mathcal{C}([-h,0]) \ , \\ u = -\psi(y), \quad y = \sum_{l=0}^{L} C_l x(t-h_l) \ , \end{cases} \tag{1b}$$

where $n, p, L \in \mathbb{N}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $A, A_l \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C, C_l \in \mathbb{R}^{p\times n}$, $0 \leq h_0 < \dots < h_L$, $h \stackrel{\text{def}}{=} \max\{h_l\}$. One denotes by H the matrix transfer function corresponding to the system under study, namely:

$$H(s) = C(sI - A)^{-1}B \ , \tag{2a}$$

$$H(s) = \left(\sum_{l=0}^{L} C_l e^{-h_l s}\right)\left(sI - \sum_{l=0}^{L} A_l e^{-h_l s}\right)^{-1} B \ . \tag{2b}$$

Let the nonlinearity $\psi : \mathbb{R}^+ \times \mathbb{R}^p \to \mathbb{R}^p$ be time-independent, *decentralized* [8] ($\forall i \in \{1,\dots,p\}$, $\psi_i(y) = \psi_i(y_i)$), and fulfill the following *sector condition*:

$$\forall y \in \mathbb{R}^p, \ \psi(y)^T(\psi(y) - Ky) \leq 0 \ , \tag{3}$$

for a certain diagonal matrix $K \geq 0$ (the inequality is hence also valid componentwise). By analogy with the concepts of absolute and of α-stability [11,14], we define :

Definition 1 *Let α be a nonnegative scalar. System* (1a) *(resp.* (1b)*) is called* absolutely stable with decay rate α *(or* absolutely α-stable*) if, for any global solution x,*

$$\lim_{t\to+\infty} \frac{e^{\alpha t}x(t)}{\|\phi\|} = 0 \ ,$$

where the convergence is uniform wrt the initial condition $\phi \neq 0$ and to the nonlinearity ψ fulfilling (3). *Here, $\|.\|$ denotes the euclidian norm in $\mathbb{R}^n$ (resp. the uniform convergence norm in $\mathcal{C}([-h,0];\mathbb{R}^n)$).*

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$$R \stackrel{\text{def}}{=} R(P,\eta) = \begin{pmatrix} A^TP+PA+2\alpha P+2\alpha C^TK|\eta|_+KC & -PB+C^TK+A^TC^TK\eta \\ -B^TP+KC+\eta KCA & -2I-\eta KCB-B^TC^TK\eta \end{pmatrix} . \tag{4}$$

In the present note, some simple absolute α-stability criteria are proposed. The presentation is unified for both finite-dimensional and delay systems, of type (1a) and (1b) respectively. Our main contribution is the following. • A criterion for finite-dimensional systems (1a) is provided (Theorem 2). It is expressed equivalently as a Linear Matrix Inequality (LMI), a standard class of problems for which sound numerical methods have been developed [5], or under frequency domain form. • The previous frequency domain form of the criterion is shown to be valid for delay systems (1b) too (Theorem 5). The results generalize Popov criterion, which is found when taking $\alpha = 0$. Notice that one may easily adapt the results proposed herein to absolute α-stability for systems with *time-varying* nonlinearities, or to local stability results, see [3]. More generally, the results stated here could be applied to more general systems (e.g. systems with distributed delays, integral systems), as it is indeed the case for Popov criterion, see [7, §4.6.] and [6].

The case of finite-dimensional systems is treated in Section 2. The frequency domain criterion for delay systems stability is stated in Section 3. Section 4 is a conclusion. The issues of existence and uniqueness of the solutions are not considered, as they have been extensively studied. In all the sequel, it is assumed that there exist *global solutions of* (1a) (resp. (1b)), that is, *by definition*: for all $\phi \in \mathbb{R}^n$ (resp. for all $\phi \in \mathcal{C}([-h,0];\mathbb{R}^n)$), there exists a continuous function x defined on $[0,+\infty)$ (resp. on $[-h,+\infty)$), absolutely continuous (AC) on $[0,+\infty)$, such that $x(0)=\phi$ (resp. $x|_{[-h,0]}=\phi$) and (1a) (resp. (1b)) is fulfilled almost everywhere on $[0,+\infty)$.

Notations The abbreviation SPR is used for "strictly positive real". For $z \in \mathbb{R}$, one denotes by sgn$z$ the sign of z (sgn$0 = 1$ or -1 indifferently), and $|z|_+ \stackrel{\text{def}}{=} \sup\{z,0\}$, $|z|_- \stackrel{\text{def}}{=} \sup\{-z,0\}$. By convention, one extends the action of any map acting on scalar or scalar-valued functions to an operator acting on matrices or matrix-valued functions, obtained by componentwise application. As an example, for any diagonal matrix η, $|\eta|_\pm = \sup\{\pm\eta,0\} = \text{diag}\{\sup\{\pm\eta_i,0\}\} = \text{diag}\{|\eta_i|_\pm\}$.

2 Finite-dimensional systems

Theorem 2 *Assume that*

(H) *The nonlinearity ψ is measurable, decentralized and there exists a diagonal matrix $K \stackrel{\text{def}}{=} \text{diag}\{K_i\} \geq 0$ such that* (3) *holds.*

Let $\alpha \geq 0$, and η be a diagonal matrix in $\mathbb{R}^{p\times p}$. The following properties are equivalent, and imply absolute α-stability of system (1a).

• *There exists a symmetric definite positive matrix $P \in \mathbb{R}^{n\times n}$ such that $R(P,\eta)$ defined in* (4) *is definite negative.*

• *The roots of* $\det(sI-A)$ *have real part smaller than $-\alpha$, and for H defined in* (2a), *the transfer*

$$\begin{aligned} I + (I+\eta(s-\alpha)I))KH(s-\alpha) \\ -\alpha H^H(s-\alpha)K|\eta|_+KH(s-\alpha) \text{ is SPR} . \end{aligned} \tag{5}$$

Theorem 2 extends, for finite-dimensional systems, the usual form of the Popov criterion, obtained for $\alpha = 0$. The result of α-stability obtained for null Popov slope η constitutes an extension of the circle criterion, and may be found in [10]. Given the sign of the components of the diagonal matrix η, (4) is a linear matrix inequality in the two unknowns P, η.

Proof of Theorem 2 The following lemma plays central role in the demonstration of Theorems 2 and 5.

Lemma 3 *Let $i \in \{1,\dots,p\}$, $\mathcal{Y}_i : [0,+\infty) \to \mathbb{R}$ be differentiable a.e. Then, for almost any $T \geq 0$,*

$$\frac{d}{dT}\left[e^{2\alpha T}\int_0^{e^{-\alpha T}\mathcal{Y}_i(T)} \psi_i(z)\, dz\right]$$
$$\leq \alpha K_i \mathcal{Y}_i^2(T) + e^{\alpha T}(\dot{\mathcal{Y}}_i(T)-\alpha\mathcal{Y}_i(T))\psi_i(e^{-\alpha T}\mathcal{Y}_i(T)) .$$

Proof The left-hand side being equal to $2\alpha e^{2\alpha T}\int_0^{e^{-\alpha T}\mathcal{Y}_i(T)} \psi_i(z)\, dz + e^{\alpha T}(\dot{\mathcal{Y}}_i(T)-\alpha\mathcal{Y}_i(T))\psi_i(e^{-\alpha T}\mathcal{Y}_i(T))$, bound the integral term, using the fact that $\forall z \in \mathbb{R}$, $\int_0^z \psi_i(z')\, dz' \leq \frac{1}{2}K_i z^2$, due to sector condition. □

• Assume first that $\eta \geq 0$. With the change of variables $\mathcal{X}(t) \stackrel{\text{def}}{=} e^{\alpha t}x(t)$, $\mathcal{Y}(t) \stackrel{\text{def}}{=} e^{\alpha t}y(t)$, $\varphi(t,\mathcal{Y}) \stackrel{\text{def}}{=} e^{\alpha t}\psi(e^{-\alpha t}\mathcal{Y})$, (1a) is changed into the *nonautonomous* system

$$\dot{\mathcal{X}} = (\alpha I + A)\mathcal{X}(t) - B\varphi(t,\mathcal{Y}(t)), \quad \mathcal{Y} = C\mathcal{X} . \tag{6}$$

From Hypothesis (H), one deduces that

$$\forall \mathcal{Y} \in \mathbb{R}^p, \forall t \geq 0, \quad \varphi(t,\mathcal{Y})^T(\varphi(t,\mathcal{Y}) - K\mathcal{Y}) \leq 0 . \tag{7}$$

Absolute α-stability of (1a) is equivalent to : $\lim_{t\to+\infty} \mathcal{X}(t) = 0$, uniformly wrt ϕ in any compact and to φ verifying (7).

Fora the (positive definite) function $V(t,\mathcal{X}) \stackrel{\text{def}}{=} \mathcal{X}^T P\mathcal{X} + 2e^{2\alpha t}\sum_{i=1}^p \eta_i K_i \int_0^{e^{-\alpha t}(C\mathcal{X})_i} \psi_i(z)\,dz$, $\dot V \stackrel{\text{def}}{=} \frac{d}{dt}[V(t,\mathcal{X}(t))]$ is less than $\begin{pmatrix} \mathcal{X}(t) \\ \varphi(t,\mathcal{Y}(t)) \end{pmatrix}^T R \begin{pmatrix} \mathcal{X}(t) \\ \varphi(t,\mathcal{Y}(t)) \end{pmatrix}$ a.e. on $(0,+\infty)$ along the trajectories of (6), for R defined by (4). To get this, use (H) and Lemma 3, and add the nonnegative term $2\varphi(t,\mathcal{Y}(t))^T(K\mathcal{Y}(t) - \varphi(t,\mathcal{Y}(t)))$. Now, for any $(t,\mathcal{X}) \in \mathbb{R}^+ \times \mathbb{R}^n$, $\mathcal{X}^T P\mathcal{X} \le V(t,\mathcal{X}) \le \mathcal{X}^T(P + C^T K\eta KC)\mathcal{X}$. Thus, $\exists\varepsilon > 0$, independent from ϕ,ψ, such that $\dot V + \varepsilon V \le 0$ a.e. on $(0,+\infty)$. The regularity hypothesis on x implies that $t \mapsto V(t,\mathcal{X}(t))$ too is AC, as $\forall i \in \{1,\dots,p\}, \forall z_1, z_2 \in \mathbb{R}$, $\left|\int_0^{z_1}\psi_i(z)\,dz - \int_0^{z_2}\psi_i(z)\,dz\right| \le K_i \max\{|z_1|,|z_2|\}\ |z_1 - z_2|$. Thus $V(t,\mathcal{X}(t))\ e^{\varepsilon t} - V(0,\mathcal{X}(0)) = \int_0^t(\dot V + \varepsilon V)\ e^{\varepsilon\tau}\ d\tau \le 0$, and, whenever LMI (4) is feasible and $\eta \ge 0$, then system (6) is globally exponentially stable with the prescribed uniform convergence property.

Last, check that

$$\begin{pmatrix} -((s-\alpha)I - A)^{-1}B \\ I \end{pmatrix}^H R \begin{pmatrix} -((s-\alpha)I - A)^{-1}B \\ I \end{pmatrix}$$

is equal to $-[I + (I + \eta(s-\alpha)I))KH(s-\alpha) - \alpha H^H(s-\alpha)K\eta KH(s-\alpha)] - [I + (I + \eta(s-\alpha)I))KH(s-\alpha) - \alpha H^H(s-\alpha)K\eta KH(s-\alpha)]^H + (s+\bar s)B^T((s-\alpha)I - A)^{-H}P((s-\alpha)I - A)^{-1}B$. Based on this identity, equivalence between feasibility of (4) and condition (5) is proved, by use of Kalman-Yakubovich-Popov lemma. This achieves the proof of Theorem 2 in the case $\eta \ge 0$.

• To remove the constraints on the sign of η [1], consider instead of $\psi(y)$ the inputs $\hat\psi(y) \stackrel{\text{def}}{=} \text{sgn}\eta\psi(y) + \frac{1}{2}(I - \text{sgn}\eta)Ky$, and apply the previous result to the transformed system [13]. Direct calculation shows that this amounts to consider the system obtained when replacing A (resp. B, resp. η) by $\hat A \stackrel{\text{def}}{=} A - B\frac{I-\text{sgn}\eta}{2}KC$ (resp. $\hat B \stackrel{\text{def}}{=} B\text{sgn}\eta$, resp. $\hat\eta \stackrel{\text{def}}{=} |\eta| \ge 0$). To verify this [2,3], check that $Ax - B\psi(y) = \hat Ax - \hat B\hat\psi(y)$ for $y = Cx$, and that $\psi(y)^T(\psi(y) - Ky) = \hat\psi(y)^T(\hat\psi(y) - Ky)$, so $\hat\psi$ fulfills same sector condition than ψ. Denote $\hat H(s) = C(sI - \hat A)\hat B$ the transfer obtained when replacing ψ by $\hat\psi$. We prove that (5) is equivalent to: $I + (I + \hat\eta(s-\alpha)I))K\hat H(s-\alpha) - \alpha\hat H^H(s-\alpha)K\hat\eta K\hat H(s-\alpha)$ is SPR.

Lemma 4 *Let us denote* $\hat I \stackrel{\text{def}}{=} \text{sgn}\ \eta$, $\hat J \stackrel{\text{def}}{=} (I - \hat I)/2$. *Then,* $\hat H(s) = (I + H(s)\hat JK)^{-1}H(s)\hat I = H(s)(I + \hat JKH(s))^{-1}\hat I$.

Proof the new input $\hat\psi$ being defined as in the proof of Theorem 2, one has: $y = -H\psi = -H(\hat J\psi + (I-\hat J)\psi) = -H(\hat J(Ky - \varphi) + (I-\hat J)\psi) = -H(\hat JKy - \hat J\hat\psi + (I - \hat J)\hat\psi) = -H(\hat I\hat\psi + \hat JKy)$, and finally: $(I + H\hat JK)y = -H\hat I\hat\psi$, which gives the 1st equality. Deduction of the 2nd equality is straightforward. $\square$

Applying Lemma 4 and using the identities $\hat J^2 = \hat J = -\hat I\hat J = -\hat J\hat I$, $2\hat J + \hat I = \hat I^2 = I_p$, $\hat\eta\hat I = \eta$, $(\hat J\hat I + I)\hat\eta = |\eta|_+$, one gets $2I + (I + \hat\eta(s-\alpha))K\hat H(s-\alpha) + \hat H^H(s-\alpha)K(I + \hat\eta(\bar s - \alpha)) - 2\alpha\hat H^H(s-\alpha)K\hat\eta K\hat H(s-\alpha) = \hat I(I + H^H(s-\alpha)K\hat J)^{-1}\hat G(s)(I + \hat JKH(s-\alpha))^{-1}\hat I$, where $\hat G(s) = 2(I + H^H(s-\alpha)K\hat J)(I + \hat JKH(s-\alpha)) + (I + H^H(s-\alpha)K\hat J)\hat I(I + \hat\eta(s-\alpha))KH(s-\alpha) + H^H(s-\alpha)K(I + \hat\eta(\bar s - \alpha))\hat I(I + \hat JKH(s-\alpha)) - 2\alpha H^H(s-\alpha)K\hat\eta H(s-\alpha) = 2I + [2\hat J + \hat I + \hat\eta\hat I(s-\alpha)]KH(s-\alpha) + H^H(s-\alpha)K[2\hat J + \hat I + \hat\eta\hat I(\bar s - \alpha)] - H^H(s-\alpha)K[2\hat J^2 + 2\hat J\hat I - 2\alpha(\hat J\hat I + I)\hat\eta + \hat J\hat I(s+\bar s)\hat\eta]KH(s-\alpha) = 2I + (I + \eta(s-\alpha))KH(s-\alpha) + H^H(s-\alpha)K(I + \eta(\bar s - \alpha)) - 2\alpha H^H(s-\alpha)K|\eta|_+ KH(s-\alpha) - (s+\bar s)H^H(s-\alpha)K\hat J\hat I\hat\eta KH(s-\alpha)$. Putting $\text{Re}\, s = 0$, the last term vanishes, so G is SPR if and only if $\hat G$ is SPR. This achieves the proof of Theorem 2 for general systems. $\square$

3 Delay systems

We hereafter show that the frequency domain form of Theorem 2 is true for delay systems too.

Theorem 5 *Assume that Hypothesis* (H) *holds. Let* $\alpha \ge 0$, *and suppose that the roots of* $\det(sI - A - \sum_{l=0}^L A_l e^{-h_l s})$ *have real part smaller than* $-\alpha$, *and that there exists a diagonal matrix* η *such that* (5) *holds, for* H *defined in* (2b). *Then, the delay system* (1b) *is absolutely* α*-stable.*

Again, for $\alpha = 0$, Theorem 5 reduces to the frequency domain form of the Popov criterion [12,9].

Proof of Theorem 5 The proof of Theorem 5 is inspired from [12,9,7], with adequate improvements. Suppose $\eta \ge 0$, the other cases are treated as in 2nd part of proof of Theorem 2, by considering $\hat H(s) \stackrel{\text{def}}{=} (\sum_{l=0}^L C_l e^{-h_l s})(sI - \sum_{l=0}^L \hat A_l e^{-h_l s})^{-1}\hat B$, $\hat A_l \stackrel{\text{def}}{=} A_l - B\frac{I-\text{sgn}\eta}{2}KC_l$.

Define $\mathcal{X}, \mathcal{Y}, \varphi$ as in proof of Theorem 2, and $\phi_\alpha \stackrel{\text{def}}{=} e^{\alpha t}\phi$. One deduces from (1b) that $\dot{\mathcal{X}} = \alpha\mathcal{X} + \sum_{l=0}^L e^{\alpha h_l}A_l\mathcal{X}(t - h_l) - B\varphi(t,\mathcal{Y})$, $\mathcal{Y} = \sum_{l=0}^L e^{\alpha h_l}C_l\mathcal{X}(t - h_l)$, $\mathcal{X}|_{[-h,0]} = \phi_\alpha$. For $T \ge 0$, fix $\varphi_T(t) = \varphi(t,\mathcal{Y}(t))$ if $0 \le t \le T$,

$= 0$ if $-h \leq t < 0$ or $t > T$; $\dot{\mathcal{X}}_T = \alpha\mathcal{X}_T + \sum_{l=0}^{L} e^{\alpha h_l} A_l \mathcal{X}_T(t-h_l) - B\varphi_T$, $\mathcal{X}_T|_{[-h,0]} = 0$; and $\mathcal{Y}_T = \sum_{l=0}^{L} e^{\alpha h_l} C_l \mathcal{X}_T(t-h_l)$. Then, $\dot{\mathcal{X}} - \dot{\mathcal{X}}_T = \alpha(\mathcal{X}-\mathcal{X}_T) + \sum_{l=0}^{L} e^{\alpha h_l} A_l(\mathcal{X}(t-h_l) - \mathcal{X}_T(t-h_l))$ for $t \in [0,T]$, $(\mathcal{X}-\mathcal{X}_T)|_{[-h,0]} = \phi_\alpha$. From the hypothesis on the location of the roots of the characteristic polynomial, $\exists c_1, \lambda > 0$, independent of ϕ_α and T, such that

$$\forall t \in [-h,T],\ \|\mathcal{X}(t) - \mathcal{X}_T(t)\| \leq c_1 e^{-\lambda t}\|\phi_\alpha\|_{\mathcal{C}([-h,0])}, \quad (8a)$$

$$\forall t \geq T,\ \|\mathcal{X}_T(t)\| \leq c_1 e^{-\lambda(t-T)}\|\mathcal{X}_T(T+\cdot)\|_{\mathcal{C}([-h,0])}, \quad (8b)$$

$$\forall t \in [0,T],\ \|\mathcal{Y}(t)\| \leq c_1 e^{-\lambda t}\|\phi_\alpha\|_{\mathcal{C}([-h,0])} + \|\mathcal{Y}_T(t)\|. \quad (8c)$$

Formula (8c) is deduced from the preceding, by writing $\mathcal{Y}(t) = \sum_{l=0}^{L} e^{\alpha h_l} C_l(\mathcal{X}(t-h_l) - \mathcal{X}_T(t-h_l)) + \mathcal{Y}_T(t)$.

Obviously, $\int_0^T [K_i\mathcal{Y}_i(t) - \varphi_i(t,\mathcal{Y}_i(t))]\varphi_i(t,\mathcal{Y}_i(t))\ dt + e^{2\alpha T}\eta_i K_i \int_0^{e^{-\alpha T}\mathcal{Y}_i(T)} \psi_i(z)\ dz \geq 0$, for any $T \geq 0$, any $i \in \{1,\dots,p\}$. As the solution is AC by assumption, one may integrate the inequality in Lemma 3. Using (8) and the fact that $\alpha, \eta \geq 0$, yields, by summation on i: $\int_0^{+\infty} (\varphi_T^T(t)(K\mathcal{Y}_T(t) - \varphi_T(t)) + \varphi_T^T(t)\eta K\dot{\mathcal{Y}}_T(t) + \alpha\mathcal{Y}_T^T(t)K\eta K\mathcal{Y}_T(t))\ dt \geq -c_2\|\phi_\alpha\|_{\mathcal{C}([-h,0])}\left(\|\phi_\alpha\|_{\mathcal{C}([-h,0])} + \sup_{t\in[0,T]}\|\mathcal{Y}_T(t)\|\right)$. The constant $c_2 \geq 0$ is independent of ϕ_α and T, and the terms with $\varphi_T(t)$ in the previous integral vanish on $[T,+\infty)$. This implies $\mathcal{Y}_T, \dot{\mathcal{Y}}_T \in L^2(0,+\infty)$, due to the fact that $H(s-\alpha)$ is strictly proper.

By assumption, $\exists\varepsilon > 0$ such that $(1-\varepsilon)I + (I + \eta(s-\alpha)I))KH(s-\alpha) - \alpha H^H(s-\alpha)K\eta KH(s-\alpha)$ is still SPR. Thus, performing Fourier transform and using the identity $\tilde{\mathcal{Y}}_T(\omega) = -H(j\omega-\alpha)\tilde{\varphi}_T(\omega)$ linking Fourier transforms, yields: $\exists\varepsilon > 0$, independent from ϕ, ψ, such that $c_2\|\phi_\alpha\|_{\mathcal{C}([-h,0])}\left(\|\phi_\alpha\|_{\mathcal{C}([-h,0])} + \sup_{t\in[0,T]}\|\mathcal{Y}_T(t)\|\right) \geq \varepsilon\int_0^{+\infty}\|\varphi_T(t)\|^2\ dt \geq \frac{\varepsilon}{\|H(s-\alpha)\|_\infty^2}\int_0^{+\infty}\|\mathcal{Y}_T(t)\|^2\ dt$.

As $H(s-\alpha)$ is strictly proper, one gets similarly $\int_0^T \|\dot{\mathcal{Y}}_T(t)\|^2\ dt \leq c_3\|\phi_\alpha\|_{\mathcal{C}([-h,0])}(\|\phi_\alpha\|_{\mathcal{C}([-h,0])} + \sup_{t\in[0,T]}\|\mathcal{Y}_T(t)\|)$. One infers that, $\forall t \in [0,T]$: $\|\mathcal{Y}_T(t)\|^2 = \|\mathcal{Y}_T(0)\|^2 + 2\int_0^t \mathcal{Y}_T(\tau)^T\dot{\mathcal{Y}}_T(\tau)\ d\tau \leq 2\|H(s-\alpha)\|_\infty\sqrt{c_2c_3/\varepsilon}\ \|\phi_\alpha\|_{\mathcal{C}([-h,0])}(\|\phi_\alpha\|_{\mathcal{C}([-h,0])} + \sup_{t\in[0,T]}\|\mathcal{Y}_T(t)\|)$, using Cauchy-Schwarz inequality and $\mathcal{Y}_T(0) = 0$. Solving the polynomial inequality then leads to $\sup_{t\in[0,T]}\|\mathcal{Y}_T(t)\| \leq c_4\|\phi_\alpha\|_{\mathcal{C}([-h,0])}$, for a certain constant $c_4 > 0$. This in turn implies that, $\forall T \geq 0$, $\int_0^{+\infty}\|\mathcal{Y}_T(t)\|^2\ dt$, $\int_0^{+\infty}\|\mathcal{Y}(t)\|^2\ dt$, $\int_0^T\|\varphi(t,\mathcal{Y}(t))\|^2\ dt \leq c_5\|\phi_\alpha\|_{\mathcal{C}([-h,0])}^2$. In the previous estimates, c_5 is independent of T. Thus, $\mathcal{X}, \dot{\mathcal{X}} \in L^2(0,+\infty)$, and $\mathcal{X}(t) \to 0$ when $t \to +\infty$. Moreover, the uniformity of the estimates implies that the convergence is uniform wrt ϕ in any bounded set and to φ verifying (7). This ends the proof of Theorem 5. □

4 Conclusion

In this note, some extensions of Popov criterion for finite-dimensional systems or delay systems with sector-bounded nonlinearities are given, permitting to integrate constraint on the decay rate of the solutions. For finite-dimensional systems, the assumptions are expressed in terms of solvability of a certain linear matrix inequality (obtained by use of Lyapunov method), or equivalently under a frequency domain condition. This previous form is also shown to be valid for delay systems.

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