# Lyapunov equation for the stability of linear delay systems of retarded and neutral type

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**Abstract**

In this note, the delay-independent stability of delay systems is studied. It is shown that the strong delay-independent stability is equivalent to the feasibility of certain linear matrix inequality, that is to the existence of a quadratic Lyapunov-Krasovskii functional, independent of the (nonnegative) value of the delay. This constitutes the analogue of some well-known properties of finite-dimensional systems. This result is then applied to study delay-independent stability of systems with polytopic uncertainties.

**Keywords:** linear delay systems, delay-independent stability, quadratic Lyapunov-Krasovskii functionals, linear matrix inequalities, polytopic uncertainties.

## 1 Introduction

The stability of linear delay systems of retarded or neutral type is a field of intense research [14, 19]. A major difficulty lies in the fact that the delays are usually not perfectly known. A way to ensure stability robustness with respect to this uncertainty, is to employ stability criteria valid for any nonnegative value of the delays, that is *delay-independent results.* This assumption that no information on the value of the delay is known, is often coarse in practice: in general some estimates are available, and it is more appropriate (and sometimes unavoidable) to consider the stability of the systems obtained for the different values of the delays in the corresponding product of bounded intervals. However, the design of stability tests adapted to this task, both *numerically tractable* and *nonconservative*, seems to be more complicated[1]. This is the reason why the delay-independent results are of interest.

Criteria for delay-independent stability (or for *strong* delay-independent stability [18], see below), expressed by conditions on the zeros of a polynomial with two variables in the frequency domain, have been exhibited for retarded type systems [11, 12, 13], and more recently for neutral type systems [9, 16, 20]. Chen *et al.* [7] have shown that checking this property amounts to verify conditions involving structured singular values with respect to complex uncertainties, formally similar to small gain conditions appearing in robust stability analysis. This approach, however, does not give rise to easy-to-check conditions.

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[1] A recent, promising, attempt, may be found in [26].

On the other hand, various easy-to-check stability conditions have appeared in the literature, based on time-domain techniques, see e.g. [10, 22, 21] and the references therein. Approaches by quadratic Lyapunov-Krasovskii functionals are intensively used, leading in particular to conditions expressed under the form of linear matrix inequalities (see [8, 24] for retarded type systems, [21] for neutral type), a class of problems for which widespread powerful numerical algorithms exist. However, these stability criteria are only sufficient. The analysis by Zhang *et al.* [25] shows that they are based on conservative estimation of the structured singular values involved.

In this paper is proposed a LMI condition *equivalent* to the (strong) delay-independent stability of neutral or retarded type delay differential systems. More precisely, one displays a family of LMIs of increasing size, each of them sufficient for delay-independent stability. The key result is that, reciprocally, the strong delay-independent stability implies that the LMIs are solvable beyond a certain size.

The main idea, based on an improvement of the existing time-domain methods, consists in using, instead of the usual state variable, say $\{x(t+\tau)\ :\ -h \leq \tau \leq 0\}$, the augmented, nonminimal, state $\{x(t+\tau)\ :\ -kh \leq \tau \leq 0\}$, for some positive integer $k$. A similar idea has been used in [2] and [8, 132–133] to derive sufficient stability conditions. Of course, when $k > 1$, not any function in the new state space, even sufficiently smooth, can be part of a trajectory of the system under study. The supplementary compatibility constraints are reintroduced when evaluating the derivative of the candidate Lyapunov-Krasovskii functional along the trajectories of the system, considered now in the augmented state space. It turns out that the LMIs found by this method constitute *necessary and sufficient* conditions for strong delay-independent stability for large enough values of $k$. Incidentally, this furnishes a method for checking scaled small gain conditions as the ones depicted by Chen *et al.* [7], by use of LMIs rather than griding. Work is in progress to apply the same ideas to related problems, stability of systems with noncommensurate delays, analysis of $\mathcal{H}_\infty$ performance and other, see [5].

The background is recalled in Section 2. In Section 3, is provided the main result, characterizing the strong delay-independent stability of delay differential systems with one delay (Theorem 1). Numerical examples are presented in Section 4. Application to delay-independent stability of systems with polytopic uncertainties is provided in Section 5 (Theorem 2). Complete proof of Theorem 1 is given in Section 6.

The notations are standard. $I_n$, $0_{m\times n}$ stand respectively for the identity matrix of size $n$ and the null matrix of size $m \times n$ (simply abbreviated $0_n$ when $m = n$). The Kronecker product is denoted $\otimes$. The set of positive integers is denoted $\mathbb{N}$, and the integer part is written int. The set $\overline{\mathbb{R}}$ is defined to be $\mathbb{R} \cup \{\infty\}$. By $\overline{\mathbb{C}^+}$ is meant the set of complex numbers with nonnegative real part, and by $\overline{\mathbb{D}}$ the closed unit disk $\{z \in \mathbb{C}\ :\ |z| \leq 1\}$. The spectral radius and maximal singular value of a matrix $M$ are respectively denoted $\rho(M)$ and $\|M\|$. The spectrum of a square matrix $M$ is written $\sigma(M)$, and applying the operation Re to this set, one defines $\operatorname{Re}\sigma(M) \stackrel{\text{def}}{=} \{\operatorname{Re} s\ :\ s \in \sigma(M)\}$, so $\operatorname{Re}\sigma(M) < 0$ means that $M$ is Hurwitz. Also, the conjugate and transconjugate of $M$, are denoted $M^T$ and $M^*$. Last, for systems with a delay $h \geq 0$, $x_t$ designates the function $x(t + \cdot)$, defined on $[-h, 0]$.

# 2 Delay-independent stability

We consider the following delay differential system:

$$\dot{x}(t) - E\dot{x}(t-h) = Ax(t) + Bx(t-h)\ , \tag{1}$$

$A, B, E \in \mathbb{R}^{n\times n}$. This is a delay differential equation of neutral type when $E \neq 0$, of retarded type when $E = 0$. The asymptotic stability of system (1) is equivalent to [8]

$$\rho(E) < 1 \text{ and } \forall s \in \overline{\mathbb{C}^+},\ \det(s(I_n - e^{-sh}E) - A - e^{-sh}B) \neq 0\ .$$

The notion of *delay-independent stability* has been introduced (see [11, 12, 13] for retarded systems, [9, 16, 20] for neutral systems): by definition, system (1) is (weakly) delay-independently stable if

$$\rho(E) < 1 \text{ and } \forall h \geq 0, \forall s \in \overline{\mathbb{C}^+},\ \det(s(I_n - e^{-sh}E) - A - e^{-sh}B) \neq 0\ .$$

Delay-independent stability may be proved to be equivalent to [9, 20]

$$\rho(E) < 1 \text{ and } \forall (s,z) \in \overline{\mathbb{C}^+} \times \overline{\mathbb{D}}, s \neq 0 \text{ or } s = 0, z = 1 \Rightarrow \det(s(I_n - zE) - A - zB) \neq 0 \ .$$

A slightly stronger property may be introduced, as in [18] for retarded type systems: system (1) is called *strongly* delay-independently stable if

$$\rho(E) < 1 \text{ and } \forall (s,z) \in \overline{\mathbb{C}^+} \times \overline{\mathbb{D}},\ \det(s(I_n - zE) - A - zB) \neq 0 \ . \tag{2}$$

We may show easily as in [3] for systems of retarded type, that the strong delay-independent stability is a property robust with respect to perturbations of the matrices $A, B, E$, whereas the delay-independent stability is not. Indeed, infinitely close (in the sense of the distance induced by the maximal singular value on the product space of the triplets $(A, B, E)$) to any system fulfilling the weak property but not the strong one, there exist unstable systems. More precisely, the set of the triplets corresponding to strongly delay-independently stable systems is the *interior* of the set of the triplets corresponding to (weakly) delay-independently stable systems.

Generalizations of the Lyapunov method to delay differential equations have been proposed. In particular, a class of quadratic Lyapunov-Krasovskii functionals [15, 8] has been used early for this purpose, afterwards generalized to neutral type systems [21] under the form

$$V(x_t) = (x(t) - Ex(t-h))^T P(x(t) - Ex(t-h)) + \int_{t-h}^{t} x^T(\tau) Q x(\tau)\ d\tau \ , \tag{3}$$

for positive definite matrices $P, Q \in \mathbb{R}^{n\times n}$. Along the trajectories of (1),

$$\frac{d[V(x_t)]}{dt} = \begin{pmatrix} x(t) - Ex(t-h) \\ x(t-h) \end{pmatrix}^T \left( \begin{pmatrix} A^T P + PA & P(AE+B) \\ (AE+B)^T P & 0_n \end{pmatrix} \right.$$
$$\left. + \begin{pmatrix} Q & QE \\ E^T Q & E^T QE - Q \end{pmatrix} \right) \begin{pmatrix} x(t) - Ex(t-h) \\ x(t-h) \end{pmatrix} \ , \tag{4}$$

and feasibility of the following linear matric inequality

$$P = P^T > 0, Q = Q^T > 0, \begin{pmatrix} A^T P + PA & P(AE+B) \\ (AE+B)^T P & 0_n \end{pmatrix} + \begin{pmatrix} Q & QE \\ E^T Q & E^T QE - Q \end{pmatrix} < 0 \tag{5}$$

is indeed sufficient to have asymptotic stability (by use of the results in [8, §12.7]). However, this is *not* in general a necessary condition. As a matter of fact, it may be deduced from [1, 25], that solvability of (5) is equivalent to:

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \min_{M \text{ invertible}} \sup_{s \in \overline{\mathbb{C}^+}} \| M((sI_n - A)^{-1}(sE + B)M^{-1} \| < 1 \ ,$$

whereas (2) is equivalent to

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \sup_{s \in \overline{\mathbb{C}^+}} \min_{M \text{ invertible}} \| M((sI_n - A)^{-1}(sE + B)M^{-1} \| < 1 \ .$$

# 3 Principle of the method and main result

Our method is based on an improvement of the previous one. Consider, for $k \in \mathbb{N}$,

$$\mathcal{X}_k(t) \stackrel{\text{def}}{=} \begin{pmatrix} x(t) \\ x(t-h) \\ \vdots \\ x(t-(k-1)h) \end{pmatrix} \ . \tag{6}$$

The vector $\mathcal{X}_k(t)$, element of $\mathbb{R}^{kn}$, is an augmented state variable, which contains more information than needed to compute future evolution. From (1) we deduce that

$$\dot{\mathcal{X}}_k(t) - (I_k \otimes E)\dot{\mathcal{X}}_k(t-h) = (I_k \otimes A)\mathcal{X}_k(t) + (I_k \otimes B)\mathcal{X}_k(t-h)\ . \tag{7}$$

Take definite positive matrices $P_k, Q_k \in \mathbb{R}^{kn\times kn}$, and define

$$V_k(\mathcal{X}_{k,t}) \stackrel{\text{def}}{=} (\mathcal{X}_k(t) - (I_k \otimes E)\mathcal{X}_k(t-h))^T P_k(\mathcal{X}_k(t) - (I_k \otimes E)\mathcal{X}_k(t-h)) + \int_{t-h}^{t} \mathcal{X}_k(\tau)^T Q_k \mathcal{X}_k(\tau)\ d\tau\ . \tag{8}$$

Verify that (compare with (4))

$$\begin{aligned}\frac{d[V_k(\mathcal{X}_{k,t})]}{dt} \quad = \quad & \begin{pmatrix} \mathcal{X}_k(t) - (I_k \otimes E)\mathcal{X}_k(t-h) \\ \mathcal{X}_k(t)\end{pmatrix}^T \left[\begin{pmatrix} (I_k \otimes A)^T P_k + P_k(I_k \otimes A) & P_k(I_k \otimes (AE+B)) \\ (I_k \otimes (AE+B))^T P_k & 0_{kn}\end{pmatrix}\right. \\ & \left. + \begin{pmatrix} Q_k & Q_k(I_k \otimes E) \\ (I_k \otimes E)^T Q_k & (I_k \otimes E)^T Q_k (I_k \otimes E) - Q_k\end{pmatrix}\right] \begin{pmatrix} \mathcal{X}_k(t) - (I_k \otimes E)\mathcal{X}_k(t-h) \\ \mathcal{X}_k(t)\end{pmatrix}\ .\end{aligned}$$

Now, the key point of the method consists in remarking that the components of $\mathcal{X}_k(t) - (I_k \otimes E)\mathcal{X}_k(t-h)$ and $\mathcal{X}_k(t)$ are not all independent. More precisely,

$$\mathcal{X}_k(t) = F_k(\mathcal{X}_k(t) - (I_k \otimes E)\mathcal{X}_k(t-h)) + f_k x(t-kh)\ ,$$

where $F_k \in \mathbb{R}^{kn\times kn}$, $f_k \in \mathbb{R}^{kn\times n}$ are defined by induction by

$$f_1 \stackrel{\text{def}}{=} I_n,\ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}E \\ I_n\end{pmatrix}, F_1 \stackrel{\text{def}}{=} 0_n,\ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1} \\ 0_{n\times(k-1)n} & 0_n\end{pmatrix}\ .$$

Thus,

$$f_2 = \begin{pmatrix} E \\ I_n\end{pmatrix},\ f_3 = \begin{pmatrix} E^2 \\ E \\ I_n\end{pmatrix},\ F_2 = \begin{pmatrix} 0_n & I_n \\ 0_n & 0_n\end{pmatrix},\ F_3 = \begin{pmatrix} 0_n & I_n & E \\ 0_n & 0_n & I_n \\ 0_n & 0_n & 0_n\end{pmatrix}\dots$$

We deduce that, along the trajectories of (1),

$$\frac{d[V_k(\mathcal{X}_{k,t})]}{dt} = \begin{pmatrix} \mathcal{X}_k(t) - (I_k \otimes E)\mathcal{X}_k(t-h) \\ x(t-kh)\end{pmatrix}^T R_k \begin{pmatrix} \mathcal{X}_k(t) - (I_k \otimes E)\mathcal{X}_k(t-h) \\ x(t-kh)\end{pmatrix}\ , \tag{9}$$

where

$$\begin{aligned} R_k = R_k(P_k, Q_k) \stackrel{\text{def}}{=} & \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k\end{pmatrix}^T \left[\begin{pmatrix} (I_k \otimes A)^T P_k + P_k(I_k \otimes A) & P_k(I_k \otimes (AE+B)) \\ (I_k \otimes (AE+B))^T P_k & 0_{kn}\end{pmatrix}\right. \\ & \left. + \begin{pmatrix} Q_k & Q_k(I_k \otimes E) \\ (I_k \otimes E)^T Q_k & (I_k \otimes E)^T Q_k (I_k \otimes E) - Q_k\end{pmatrix}\right] \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k\end{pmatrix}\ .\end{aligned}$$

We are hence naturally led to study the solvability of the following LMIs, defined for any $k \in \mathbb{N}$.

$$P_k, Q_k \in \mathbb{R}^{kn\times kn}, P_k = P_k^T > 0, Q_k = Q_k^T > 0,\ R_k < 0\ .$$

The case $k = 1$ reduces to (5).

Our central result tells that the solvability of the previous linear matrix inequality becomes *equivalent* to the asymptotic stability of (1) when $k$ goes to infinity.

**Theorem 1 (LMI characterization of the strong delay-independent stability).** *The strong delay-independent stability of system* (1) *is equivalent to any of the following properties.*

1. $\rho(E) < 1$ *and* $\forall (s,z) \in \overline{\mathbb{C}^+} \times \overline{\mathbb{D}}$, $\det(s(I_n - zE) - A - zB) \neq 0$.
2. *There exists* $k \in \mathbb{N}$ *such that the following LMI is feasible:*

$$P_k \in \mathbb{R}^{kn\times kn}, Q_k \in \mathbb{R}^{kn\times kn}, P_k = P_k^T > 0, Q_k = Q_k^T > 0,\ R_k < 0, \tag{$10_k$}$$

*where* $R_k \in \mathbb{R}^{(k+1)n\times(k+1)n}$ *is defined by*

$$\begin{aligned}
R_k = R_k(P_k, Q_k) &\stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix}^T \left[ \begin{pmatrix} (I_k\otimes A)^T P_k + P_k(I_k \otimes A) & P_k(I_k\otimes (AE+B)) \\ (I_k \otimes (AE+B))^T P_k & 0_{kn} \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} Q_k & Q_k(I_k\otimes E) \\ (I_k\otimes E)^T Q_k & (I_k\otimes E)^T Q_k (I_k \otimes E) - Q_k \end{pmatrix} \right] \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix}, \\
f_1 &\stackrel{\text{def}}{=} I_n,\ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}E \\ I_n \end{pmatrix} \in \mathbb{R}^{kn\times n}, \\
F_1 &\stackrel{\text{def}}{=} 0_n,\ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1} \\ 0_{n\times(k-1)n} & 0_n \end{pmatrix} \in \mathbb{R}^{kn\times kn}\ .
\end{aligned}$$

3. *There exists* $k^* \in \mathbb{N}$ *such that, for any* $k \geq k^*$, $(10_k)$ *is feasible.*

■

Recalling the analysis given previously, we see that system (1) is strongly delay-independently stable *if and only if* it possesses, for a certain $k \in \mathbb{N}$, a quadratic Lyapunov-Krasovskii functional, valid for *any nonnegative value of the delay* $h \geq 0$, of the form (8), where $P_k, Q_k$ are positive definite matrices from $\mathbb{R}^{kn\times kn}$ (compare with (3)). The derivative of this functional along the trajectories of (1) is given in (9). Also, if system (1) is (weakly) delay-independently stable, but does not possess a Lyapunov-Krasovskii functional of the previous type for a certain $k \in \mathbb{N}$, then infinitesimal parametric perturbations make it unstable for some $h \geq 0$.

Theorem 1 gives a formal analogue to the equivalence between spectral characterization of the asymptotic stability of $\dot{x} = Ax$, $A \in \mathbb{R}^{n\times n}$:

$$\forall s \in \overline{\mathbb{C}^+},\ \det(sI_n - A) \neq 0\ ,$$

and solvability of the Lyapunov inequation

$$\exists P \in \mathbb{R}^{n\times n}, P = P^T > 0, A^T P + PA < 0\ .$$

Theorem 1 furnishes a family of LMI criteria, of arbitrary precision. Concerning the way the precision changes with $k$, the following remark may be made. It may be checked from the proof of Theorem 1, that a sufficient condition for solvability of $(15_k)$ is

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \sup_{s\in\overline{\mathbb{C}^+}} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| < 1\ ,$$

whereas stability appears to be equivalent to

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \lim_{k\to+\infty} \sup_{s\in\overline{\mathbb{C}^+}} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| = 0\ .$$

For systems of retarded type, $E = 0$, so $\begin{pmatrix} F_k & f_k \end{pmatrix} = \begin{pmatrix} 0_{kn\times n} & I_{kn} \end{pmatrix}$. The feasibility of $(10_k)$ then implies the feasibility of $(16_{k+1})$, see [4].

# 4 Numerical examples

Let us give two simple numerical examples of utilization of Theorem 1. Consider the matrices

$$A = \begin{pmatrix} -4 & 10 & 5.7 & -6.5 & -2 \\ -0.2 & 1.2 & 1.5 & -1.9 & -0.6 \\ 15 & -16 & -7 & -3.6 & -6.4 \\ 8.7 & 5.5 & 6.1 & -5.9 & -7.7 \\ 7.6 & -0.9 & 1.9 & -7.2 & -7.5 \end{pmatrix}, \ B = \begin{pmatrix} 9.1 & 0 & 4.1 & 6.9 & 5.1 \\ -2.6 & -1.8 & 1.6 & 1.1 & 2.7 \\ -0.5 & -9.9 & -0.4 & 7.5 & 3.2 \\ -6 & 3.5 & 7.8 & 0.8 & 6.4 \\ 1.1 & -12 & 4.4 & -7.5 & 6 \end{pmatrix} .$$

We first study the strong delay-independent stability of the following system, of retarded type

$$\dot{x}(t) = Ax(t) + \alpha Bx(t-h) \ , \tag{11}$$

for various values of the real parameter $\alpha$. We check that $\operatorname{Re}\sigma(A) < 0$, so system (11) is asymptotically stable for $\alpha = 0$. Also, $\sup\{\alpha \in \mathbb{R} \ : \ \sigma(A + \alpha B) \cap \overline{\mathbb{C}^+} = \emptyset\} = 0.1726$ : for this value, system (11) is unstable when $h = 0$. We wish to estimate the largest $\alpha$ such that (11) is strongly delay-independently stable. This value, denoted $\alpha_\infty$, is computed by frequency sweeping as [7]

$$\alpha_\infty = \sup\{\alpha \in \mathbb{R} \ : \ \sup_{s \in \overline{\mathbb{C}^+}} \rho(\alpha(sI_5 - A)^{-1}B) < 1\} = \left( \sup_{\omega \in \mathbb{R}} \rho((j\omega I_5 - A)^{-1}B) \right)^{-1} \simeq 0.1647 \ .$$

Denote by $\alpha_k$ the supremum over the set of all $\alpha$ such that: $\exists P_k, Q_k, P_k = P_k^T > 0, Q_k = Q_k^T > 0, R_k < 0$, where

$$R_k \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ 0_{kn\times n} & I_{kn} \end{pmatrix}^T \begin{pmatrix} (I_k \otimes A)^T P_k + P_k(I_k \otimes A) + Q_k & \alpha P_k(I_k \otimes B) \\ \alpha(I_k \otimes B)^T P_k & -Q_k \end{pmatrix} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ 0_{kn\times n} & I_{kn} \end{pmatrix} .$$

The previous LMI is just ($10_k$), with $E = 0$ and $B$ replaced by $\alpha B$.

The computations are achieved by the SCILAB package `lmitool`[2]. We find

$$\alpha_1 = 0.1488, \ \alpha_2 = \alpha_3 = 0.1647 \ .$$

The criterion obtained with $k = 2$ is hence exact in the present case up to four digits.

Consider now the system of neutral type given by

$$\dot{x}(t) - E\dot{x}(t-h) = Ax(t) + \alpha Bx(t-h) \ , \tag{12}$$

where $A$ and $B$ are chosen as above, and

$$E = \begin{pmatrix} 0.11 & 0.32 & -0.29 & 0.12 & 0.15 \\ -0.38 & 0.42 & -0.33 & 0.12 & 0.47 \\ 0 & 0.35 & -0.36 & 0.11 & -0.11 \\ 0.17 & 0.43 & 0.11 & 0.43 & 0.15 \\ 0.33 & 0.03 & -0.27 & 0.33 & 0.18 \end{pmatrix} .$$

We check that $\rho(E) \simeq 0.8407 < 1$. The supremum over all $\alpha$ such that (12) is strongly delay-independently stable is here given by

$$\alpha_\infty = \sup\{\alpha \in \mathbb{R} \ : \ \sup_{\omega \in \mathbb{R}} \rho((j\omega I_5 - A)^{-1}(\alpha B + j\omega E)) < 1\} \simeq 0.1211 \ .$$

Denoting as before by $\alpha_k$ the supremum over all $\alpha$ such that the LMI ($10_k$) corresponding to the system under study is feasible ($B$ has to be replaced by $\alpha B$), gives:

$$\alpha_1 = -\infty = \sup \ \emptyset, \ \alpha_2 = \alpha_3 = 0.1211 \ .$$

Remark that, for both examples, $\alpha_\infty$ is also the supremum over all $\alpha$ such that system (11), resp. (12), is *weakly* delay-independently stable.

[2] SCILAB is a free software, distributed with its source code, see the homepage at `http://www-rocq.inria.fr/scilab/`

## 5 Delay-independent stability of systems with polytopic uncertainties

We consider here, for fixed $A_l, B_l \in \mathbb{R}^{n\times n}$, $l=\overline{1,L}$, the convex class of non-stationary systems

$$\dot{x}(t) - E\dot{x}(t-h) = A(t)x(t) + B(t)x(t-h) \ , \tag{13}$$

with $A(t), B(t), E \in \mathbb{R}^{n\times n}$, such that

$$\forall t \geq 0, (A(t), B(t)) = \sum_{l=1}^{L} \lambda_l(t)(A_l, B_l) \text{ for some functions } 0 \leq \lambda_l \leq 1, \sum_{l=1}^{L} \lambda_l = 1 \ . \tag{14}$$

Of course, the previous setting may be used to model some nonlinear systems. As for the finite-dimensional systems, a way to ensure stability is to exhibit a Lyapunov-Krasovskii functional common to the $L$ stationary systems obtained with $\lambda_l \equiv 1$, $l=\overline{1,L}$. This leads to the following result. Its residual conservatism is due only to the method of simultaneous stability itself (as is the case for the finite-dimensional systems).

**Theorem 2.** *Suppose there exist $k\in\mathbb{N}$ and positive matrices $P_k, Q_k \in \mathbb{R}^{kn\times kn}$ such that, for all $l=\overline{1,L}$,*

$$\begin{aligned} R_{l,k} = R_{l,k}(P_k,Q_k) \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix}^T \Bigg[ & \begin{pmatrix} (I_k\otimes A_l)^T P_k + P_k(I_k\otimes A_l) & P_k(I_k\otimes(A_lE+B_l)) \\ (I_k\otimes(A_lE+B_l))^T P_k & 0_{kn} \end{pmatrix} \\ & + \begin{pmatrix} Q_k & Q_k(I_k\otimes E) \\ (I_k\otimes E)^T Q_k & (I_k\otimes E)^T Q_k (I_k\otimes E) - Q_k \end{pmatrix} \Bigg] \begin{pmatrix} I_{kn} & 0_{kn\times n} \\ F_k & f_k \end{pmatrix} < 0 \ , \end{aligned}$$

*where*

$$\begin{aligned} & f_1 \stackrel{\text{def}}{=} I_n, \ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}E \\ I_n \end{pmatrix} \in \mathbb{R}^{kn\times n}, \\ & F_1 \stackrel{\text{def}}{=} 0_n, \ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1} \\ 0_{n\times(k-1)n} & 0_n \end{pmatrix} \in \mathbb{R}^{kn\times kn} \ . \end{aligned}$$

*Then, the system* (13) *with constraint* (14) *is asymptotically stable for any value of $h \geq 0$.* ■

*Proof.* Suppose $(P_k, Q_k)$ is solution of the LMI given in the statement. Consider the corresponding Lyapunov-Krasovskii functional defined in (8). Proceeding as before to obtain (9), the derivative of $V_k$ along the trajectories of (13) is proved to be equal to

$$\frac{d[V_k(\mathcal{X}_t)]}{dt} = \begin{pmatrix} x(t)-Ex(t-h) \\ \vdots \\ x(t-(k-1)h)-Ex(t-kh) \\ x(t-kh) \end{pmatrix}^T \sum_{l=1}^{L} \lambda_l(t) R_{l,k} \begin{pmatrix} x(t)-Ex(t-h) \\ \vdots \\ x(t-(k-1)h)-Ex(t-kh) \\ x(t-kh) \end{pmatrix} ,$$

which, due to (14), is negative except if $x(t)-Ex(t-h)=\cdots=x(t-(k-1)h)-Ex(t-kh)=x(t-kh)=0$. Asymptotic stability is then deduced as in [8, §12.7]. □

## 6 Proof of Theorem 1

A slightly more general result will be proved. Let $n_1, n_2 \in \mathbb{N}$, and let $A\in\mathbb{R}^{n_1\times n_1}$, $B\in\mathbb{R}^{n_1\times n_2}$, $C\in\mathbb{R}^{n_2\times n_1}$, $D\in\mathbb{R}^{n_2\times n_2}$. The proof of Theorem 1 ensues from the following Lemma, replacing respectively $A,B,C,D$ by $A, AE+B, I_n, E$.

**Lemma 3.** *The four following properties are equivalent.*

1. $\rho(D) < 1$ *and* $\forall z \in \overline{\mathbb{D}},\ \operatorname{Re}\sigma(A + zB(I_{n_2} - zD)^{-1}C) < 0$.

2. $\operatorname{Re}\sigma(A) < 0$ *and* $\sup_{s\in\overline{\mathbb{C}^+}} \rho(C(sI_{n_1} - A)^{-1}B + D) < 1$.

3. *There exists* $k \in \mathbb{N}$ *such that the LMI* $(15_k)$ *is feasible, where*

$$\begin{aligned} P_k \in \mathbb{R}^{kn_1\times kn_1}, Q_k \in \mathbb{R}^{kn_2\times kn_2}, P_k = P_k^T > 0, Q_k = Q_k^T > 0,\ R_k < 0, \\ \text{where } R_k \in \mathbb{R}^{(kn_1+n_2)\times(kn_1+n_2)} \text{is defined by (16), (17)}\ . \end{aligned} \tag{$15_k$}$$

$$f_1 \stackrel{\text{def}}{=} I_{n_2},\ f_k \stackrel{\text{def}}{=} \begin{pmatrix} f_{k-1}D \\ I_{n_2} \end{pmatrix} \in \mathbb{R}^{kn_2\times n_2}, \tag{16a}$$

$$F_1 \stackrel{\text{def}}{=} 0_{n_2\times n_1},\ F_k \stackrel{\text{def}}{=} \begin{pmatrix} F_{k-1} & f_{k-1}C \\ 0_{n_2\times(k-1)n_1} & 0_{n_2\times n_1} \end{pmatrix} \in \mathbb{R}^{kn_2\times kn_1}\ . \tag{16b}$$

$$\begin{aligned} R_k = R_k(P_k, Q_k) \stackrel{\text{def}}{=} \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix}^T \Bigg[ \begin{pmatrix} (I_k\otimes A)^T P_k + P_k(I_k\otimes A) & P_k(I_k\otimes B) \\ (I_k\otimes B)^T P_k & 0_{kn_2} \end{pmatrix} \\ + \begin{pmatrix} (I_k\otimes C)^T Q_k(I_k\otimes C) & (I_k\otimes C)^T Q_k(I_k\otimes D) \\ (I_k\otimes D)^T Q_k(I_k\otimes C) & (I_k\otimes D)^T Q_k(I_k\otimes D) - Q_k \end{pmatrix} \Bigg] \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix}\ . \end{aligned} \tag{17}$$

4. *There exists* $k^* \in \mathbb{N}$ *such that, for any* $k \geq k^*$, $(15_k)$ *is feasible.*

■

The equivalence between 1. and 2. is known [1, 20], and the implication 4. ⇒ 3. is straightforward. We show that 3. implies 1. (Section 6.1), and then that 2. implies 4. (Section 6.2).

## 6.1 Proof of the implication 3. ⇒ 1.

Consider first that feasibility of $(15_k)$ implies

$$0 > \begin{pmatrix} 0_{kn_1\times n_2} \\ I_{n_2} \end{pmatrix}^T R_k \begin{pmatrix} 0_{kn_1\times n_2} \\ I_{n_2} \end{pmatrix} = f_k^T[(I_k\otimes D)^T Q_k(I_k\otimes D) - Q_k]f_k = D^T f_k^T Q_k f_k D - f_k^T Q_k f_k\ .$$

We deduce that, for any nonzero eigenvector $u$ of $D$ associated to an eigenvalue $z$, $(|z|^2 - 1)\|Q_k^{1/2} f_k u\|^2 < 0$, so $\rho(D) < 1$. This is the first part of 1.

Define now, for any $z \in \mathbb{C}$ and for $k \in \mathbb{N}$, the matrices $v_{1,k}(z) \in \mathbb{R}^{kn_1\times n_1}$, $v_{2,k}(z) \in \mathbb{R}^{kn_2\times n_2}$, $w_k(z) \in \mathbb{R}^{(kn_1+n_2)\times n_1}$ by

$$v_{1,k}(z) \stackrel{\text{def}}{=} \begin{pmatrix} I_{n_1} \\ zI_{n_1} \\ \vdots \\ z^{k-1}I_{n_1} \end{pmatrix},\ v_{2,k}(z) \stackrel{\text{def}}{=} \begin{pmatrix} I_{n_2} \\ zI_{n_2} \\ \vdots \\ z^{k-1}I_{n_2} \end{pmatrix},\ w_k(z) \stackrel{\text{def}}{=} \begin{pmatrix} v_{1,k}(z) \\ z^k(I_{n_2} - zD)^{-1}C \end{pmatrix}\ .$$

Then,

$$\begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} w_k(z) = \begin{pmatrix} v_{1,k}(z) \\ F_k v_{1,k}(z) + z^k f_k(I_{n_2} - zD)^{-1}C \end{pmatrix}\ ,$$

and, using (16),

$$F_k v_{1,k}(z) + z^k f_k (I_{n_2} - zD)^{-1} C = \begin{pmatrix} F_{k-1} v_{1,k-1}(z) + z^{k-1} f_{k-1} (I_{n_2} - zD)^{-1} C \\ z^k (I_{n_2} - zD)^{-1} C \end{pmatrix} .$$

We then prove by induction that

$$\begin{pmatrix} I_{kn_1} & 0_{kn_1 \times n_2} \\ F_k & f_k \end{pmatrix} w_k(z) = \begin{pmatrix} I_{kn_1} \\ zI_k \otimes (I_{n_2} - zD)^{-1} C \end{pmatrix} v_{1,k}(z) .$$

From the solvability of $(15_k)$ we hence deduce that, for any $z \in \mathbb{C}$ such that $I_{n_2} - zD$ is invertible,

$$\begin{aligned}
0 &> w_k(z)^* R_k w_k(z) \\
&= \left[ \begin{pmatrix} I_{kn_1} \\ zI_k \otimes (I_{n_2} - zD)^{-1} C \end{pmatrix} v_{1,k}(z) \right]^* \left[ \begin{pmatrix} (I_k \otimes A)^T P_k + P_k (I_k \otimes A) & P_k (I_k \otimes B) \\ (I_k \otimes B)^T P_k & 0_{kn_2} \end{pmatrix} \right. \\
&+ \left. \begin{pmatrix} (I_k \otimes C)^T Q_k (I_k \otimes C) & (I_k \otimes C)^T Q_k (I_k \otimes D) \\ (I_k \otimes D)^T Q_k (I_k \otimes C) & (I_k \otimes D)^T Q_k (I_k \otimes D) - Q_k \end{pmatrix} \right] \begin{pmatrix} I_{kn_1} \\ zI_k \otimes (I_{n_2} - zD)^{-1} C \end{pmatrix} v_{1,k}(z) \\
&= v_{1,k}(z)^* \left[ (I_k \otimes (A + zB(I_{n_2} - zD)^{-1} C))^* P_k + P_k (I_k \otimes (A + zB(I_{n_2} - zD)^{-1} C)) \right] v_{1,k}(z) \\
&+ (1 - |z|^2) \left[ (I_k \otimes (I_{n_2} - zD)^{-1} C) v_{1,k}(z) \right]^* Q_k (I_k \otimes (I_{n_2} - zD)^{-1} C) v_{1,k}(z) \\
&= (A + zB(I_{n_2} - zD)^{-1} C)^* v_{1,k}(z)^* P_k v_{1,k}(z) + v_{1,k}(z)^* P_k v_{1,k}(z) (A + zB(I_{n_2} - zD)^{-1} C) \\
&+ (1 - |z|^2) [v_{2,k}(z) (I_{n_2} - zD)^{-1} C]^* Q_k v_{2,k}(z) (I_{n_2} - zD)^{-1} C .
\end{aligned}$$

In particular, if $z \in \overline{\mathbb{D}}$ (and then $I_{n_2} - zD$ invertible, as $\rho(D) < 1$), this yields

$$(A + zB(I_{n_2} - zD)^{-1} C)^* v_{1,k}(z)^* P_k v_{1,k}(z) + v_{1,k}(z)^* P_k v_{1,k}(z) (A + zB(I_{n_2} - zD)^{-1} C) < 0 .$$

As the matrix $v_{1,k}(z)^* P_k v_{1,k}(z)$ is positive definite, we deduce that for any $z \in \overline{\mathbb{D}}$, the matrix $A + zB(I_{n_2} - zD)^{-1} C$ is solution of a Lyapunov equation, so $\operatorname{Re} \sigma(A + zB(I_{n_2} - zD)^{-1} C) < 0$. This achieves the proof of the implication 3. $\Rightarrow$ 1.

## 6.2 Proof of the implication 2. $\Rightarrow$ 4.

• We first transform condition 2. It is well-known that, for any square matrix $M$,

$$\rho(M) < 1 \Leftrightarrow \lim_{k \to +\infty} \|M^k\| = 0 \Leftrightarrow \limsup_{k \to +\infty} \|M^k\| < 1$$

(where the second equivalence is obtained using the fact that the matrix norm induced by the euclidian norm is submultiplicative). We hence deduce that condition 2. is indeed *equivalent* to

$$\operatorname{Re} \sigma(A) < 0 \text{ and } \sup_{s \in \overline{\mathbb{C}^+}} \limsup_{k \to +\infty} \|[C(sI_{n_1} - A)^{-1} B + D]^k\| < 1 ,$$

or even, using a classical argument of complex analysis [6]: $\operatorname{Re} \sigma(A) < 0$ and

$$\sup_{s \in j\mathbb{R}} \limsup_{k \to +\infty} \|[C(sI_{n_1} - A)^{-1} B + D]^k\| < 1 . \tag{18}$$

• From now on, we assume that condition 2. holds; in particular, $\operatorname{Re} \sigma(A) < 0$. Let us transform $(15_k)$ into a form comparable with (18), but in which $k^*$ has to be chosen *uniformly* wrt $z$ in the unit circle, namely:

$$\limsup_{k \to +\infty} \sup_{s \in j\mathbb{R}} \|[C(sI_{n_1} - A)^{-1} B + D]^k\| < 1 . \tag{19}$$

Developing the first term in $R_k$ given in (17) leads to the identity

$$\begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix}^T \begin{pmatrix} (I_k\otimes A)^T P_k + (I_k\otimes A)P_k & P_k(I_k\otimes B) \\ (I_k\otimes B)^T P_k & 0_{kn_2} \end{pmatrix} \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix}$$
$$= \begin{pmatrix} ((I_k\otimes A)+(I_k\otimes B)F_k)^T P_k + P_k((I_k\otimes A)+(I_k\otimes B)F_k) & P_k(I_k\otimes B)f_k \\ ((I_k\otimes B)f_k)^T P_k & 0_{n_2} \end{pmatrix} .$$

Written under this form, we apply Kalman-Yakubovich-Popov lemma (reproduced in Appendix) to $(15_k)$, taking into account the fact that $\sigma((I_k\otimes A)+(I_k\otimes B)F_k) = \sigma(A) \subset \mathbb{C}\setminus\overline{\mathbb{C}^+}$, as 2. holds. Denoting

$$\mathcal{S}_k = \mathcal{S}_k(s) \stackrel{\text{def}}{=} (sI_{kn_1} - (I_k\otimes A) - (I_k\otimes B)F_k)^{-1} ,$$

Kalman-Yakubovich-Popov lemma establishes the equivalence between solvability of (20a) and property (20b). Here, one has put

$$\begin{gathered} P_k\in\mathbb{R}^{kn_1\times kn_1}, Q_k\in\mathbb{R}^{kn_2\times kn_2}, P_k=P_k^T, Q_k=Q_k^T>0,\ R_k<0, \\ \textit{where } R_k\in\mathbb{R}^{(kn_1+n_2)\times(kn_1+n_2)} \textit{is defined by } (16),(17) . \end{gathered} \tag{20a}$$

and

$$\begin{gathered} \exists Q_k=Q_k^T>0, \forall s\in j\overline{\mathbb{R}}, \\ \begin{bmatrix} \cdot \end{bmatrix}^* \begin{pmatrix} (I_k\otimes C)^T Q_k (I_k\otimes C) & (I_k\otimes C)^T Q_k (I_k\otimes D) \\ (I_k\otimes D)^T Q_k (I_k\otimes C) & (I_k\otimes D)^T Q_k (I_k\otimes D) - Q_k \end{pmatrix} \begin{pmatrix} I_{kn_1} & 0_{kn_1\times n_2} \\ F_k & f_k \end{pmatrix} \begin{pmatrix} \mathcal{S}_k(s)(I_k\otimes B)f_k \\ I_{n_2} \end{pmatrix} < 0 , \end{gathered} \tag{20b}$$

where the dot in the brackets has to be replaced by the last two matrices. Remark that (20a) is *identical to* $(15_k)$*, except that* $P_k$ *here has not to be positive.*

Developing the expression in (20b) yields:

$$\begin{gathered} \exists Q_k=Q_k^T>0, \forall s\in j\overline{\mathbb{R}}, \\ [(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k + (I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k)]^* Q_k [(I_k\otimes C)\mathcal{S}_k(I_k\otimes B)f_k + (I_k\otimes D)(F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k)] \\ < [F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k]^* Q_k [F_k\mathcal{S}_k(I_k\otimes B)f_k+f_k] . \end{gathered} \tag{20c}$$

Let us evaluate these expressions. From (16), we get that

$$\begin{aligned} \mathcal{S}_k(s) &= \begin{pmatrix} sI_{(k-1)n_1} - (I_{k-1}\otimes A) - (I_{k-1}\otimes B)F_{k-1} & -(I_{k-1}\otimes B)f_{k-1}C \\ 0_{n_1\times(k-1)n_1} & sI_{n_1}-A \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathcal{S}_{k-1} & \mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}C(sI_{n_1}-A)^{-1} \\ 0_{n_1\times(k-1)n_1} & (sI_{n_1}-A)^{-1} \end{pmatrix}, \\ (I_k\otimes B)f_k &= \begin{pmatrix} (I_{k-1}\otimes B)f_{k-1}D \\ B \end{pmatrix} . \end{aligned}$$

This permits to establish that

$$\mathcal{S}_k(I_k\otimes B)f_k = \begin{pmatrix} \mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}(C(sI_{n_1}-A)^{-1}B+D) \\ (sI_{n_1}-A)^{-1}B \end{pmatrix} ,$$

from which it is deduced that

$$F_k\mathcal{S}_k(I_k\otimes B)f_k + f_k = \begin{pmatrix} (F_{k-1}\mathcal{S}_{k-1}(I_{k-1}\otimes B)f_{k-1}+f_{k-1})(C(sI_{n_1}-A)^{-1}B+D) \\ I_{n_2} \end{pmatrix}$$

and

$$
\begin{aligned}
&(I_k \otimes C)\mathcal{S}_k(I_k \otimes B)f_k + (I_k \otimes D)(F_k\mathcal{S}_k(I_k \otimes B)f_k + f_k)\\
&= \begin{pmatrix} ((I_{k-1} \otimes C)\mathcal{S}_{k-1}(I_{k-1} \otimes B)f_{k-1} + (I_{k-1} \otimes D)(F_{k-1}\mathcal{S}_{k-1}(I_{k-1} \otimes B)f_{k-1} + f_{k-1}))(C(sI_{n_1} - A)^{-1}B + D) \\ C(sI_{n_1} - A)^{-1}B + D \end{pmatrix} .
\end{aligned}
$$

In the left-hand and right-hand sides of the two previous identities, the same expressions appear with adjacent indexes. This permits to prove recursively that

$$
F_k\mathcal{S}_k(I_k \otimes B)f_k + f_k = \begin{pmatrix} [C(sI_{n_1} - A)^{-1}B + D]^{k-1} \\ \vdots \\ I_{n_2} \end{pmatrix} ,
$$

$$
(I_k \otimes C)\mathcal{S}_k(I_k \otimes B)f_k + (I_k \otimes D)(F_k\mathcal{S}_k(I_k \otimes B)f_k + f_k) = \begin{pmatrix} [C(sI_{n_1} - A)^{-1}B + D]^{k} \\ \vdots \\ [C(sI_{n_1} - A)^{-1}B + D] \end{pmatrix} .
$$

The previous formulas show that, when $Q_k = I_{kn_2}$, (20c) just writes:

$$
\forall s \in j\overline{\mathbb{R}},\ \|[C(sI_{n_1} - A)^{-1}B + D]^k\| < 1 . \tag{21}
$$

Condition (21) is hence *sufficient* for existence of $Q_k = Q_k^T > 0$ such that (20c), or equivalently (20a), holds for any $s \in j\overline{\mathbb{R}}$. In this case, (20a) admits a solution $(P_k, Q_k)$ such that $Q_k = I_{kn_2}$.

We now prove that the symmetric matrix $P_k$ previously obtained is indeed *definite positive*. Considering the left-upper block in the decomposition of $R_k$ given by (17), one has

$$
\begin{aligned}
0 >&((I_k \otimes A) + (I_k \otimes B)F_k)^T P_k + P_k((I_k \otimes A) + (I_k \otimes B)F_k)\\
&+ ((I_k \otimes C) + (I_k \otimes D)F_k)^T Q_k((I_k \otimes C) + (I_k \otimes D)F_k) - F_k^T Q_k F_k .
\end{aligned}
$$

When (20a) holds with $Q_k = I_{kn_2}$, then, for any $u_1, \dots, u_k \in \mathbb{R}^{n_1}$,

$$
\begin{aligned}
&\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}^T [((I_k \otimes C) + (I_k \otimes D)F_k)^T Q_k((I_k \otimes C) + (I_k \otimes D)F_k) - F_k^T Q_k F_k] \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{pmatrix}\\
&\qquad = \|Cu_1 + DCu_2 + \dots + D^{k-1}Cu_k\|^2 \geq 0 ,
\end{aligned}
$$

so

$$
((I_k \otimes C) + (I_k \otimes D)F_k)^T Q_k((I_k \otimes C) + (I_k \otimes D)F_k) - F_k^T Q_k F_k \geq 0
$$

and

$$
0 > ((I_k \otimes A) + (I_k \otimes B)F_k)^T P_k + P_k((I_k \otimes A) + (I_k \otimes B)F_k) .
$$

As $\mathrm{Re}\,\sigma((I_k \otimes A) + (I_k \otimes B)F_k) < 0$, the previous inequality implies [17, Theorem 5.3.1] that any symmetric matrix $P_k$ such that $(P_k, I_{kn_2})$ solves (20a) is indeed *definite positive*: this pair also solves ($15_k$). One has thus deduced that, if (20a) holds with $Q_k = I_{kn_2}$, then ($15_k$) admits a solution.

Finally, when 2. holds, then $\mathrm{Re}\,\sigma(A) < 0$, and (21) is *sufficient* for solvability of ($15_k$). Consequently, when 2. holds, (19) is *sufficient* for realization of 4., as announced.

• In view of (18) and (19), it remains, in order to prove that 2. implies 4., to show that one may choose in (18) the index $k$ *uniformly* with respect to $s \in j\mathbb{R}$. The final argument, based on compactness, is itself decomposed into two parts. We show first that, when $\mathrm{Re}\,\sigma(A) < 0$, then (18) implies

$$
\exists k \in \mathbb{N}, \sup_{s \in j\mathbb{R}} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| < 1 , \tag{22}
$$

and then that (22) implies (19).
• For $k \in \mathbb{N}$, let

$$K_k \stackrel{\text{def}}{=} \{s \in j\mathbb{R} \ : \ \|[C(sI_{n_1} - A)^{-1}B + D]^k\| \geq 1\} \ .$$

The matrix $A$ being Hurwitz, $sI_{n_1} - A$ is invertible for $s \in j\mathbb{R}$. By continuity, the sets $K_k$ are closed. Moreover,

$$s \in K_{2k} \Rightarrow 1 \leq \|[C(sI_{n_1} - A)^{-1}B + D]^{2k}\| \leq \|[C(sI_{n_1} - A)^{-1}B + D]^k\|^2 \Rightarrow s \in K_k \ .$$

Hence $K_{2k} \subset K_k$, for any $k \in \mathbb{N}$.

Assume now that (22) does *not* hold. If $\rho(D) \geq 1$, then (18) does not hold either, as necessarily one would have $\limsup_{k\to+\infty} \|D^k\| < 1$. Suppose now that $\rho(D) < 1$. Then, for any $k \in \mathbb{N}$, the sets $K_k$ are nonempty and bounded (as $\rho(D) < 1$). The sequence $K_{2^k}$ is thus a nested sequence of nonempty compact sets. In particular,

$$\exists s_0 \in \bigcap_{k\in\mathbb{N}\cup\{0\}} K_{2^k} \ ,$$

that is

$$\exists s_0 \in j\mathbb{R}, \forall k \in \mathbb{N}\cup\{0\}, \ \|[C(s_0I_{n_1} - A)^{-1}B + D]^{2^k}\| \geq 1 \ .$$

Hence,

$$\forall k^* \in \mathbb{N}, \ \sup_{k\geq k^*} \|[C(s_0I_{n_1} - A)^{-1}B + D]^k\| \geq 1 \ ,$$

and (18) does not hold either. Hence, we have proved by contradiction that, when $\operatorname{Re}\sigma(A) < 0$, then (18) implies (22).
• Let us prove now that, when $\operatorname{Re}\sigma(A) < 0$, then (22) implies (19). Suppose that (22) holds, and let $k^* \in \mathbb{N}$ and $c_1 > 0$ be such that

$$\sup_{s\in j\mathbb{R}} \|[C(sI_{n_1} - A)^{-1}B + D]^{k^*}\| \stackrel{\text{def}}{=} c_1 < 1 \ .$$

Define also

$$c_2 \stackrel{\text{def}}{=} \sup \left\{ \sup_{s\in j\mathbb{R}} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| \ : \ k \in \{0, \ldots, k^* - 1\} \right\} \ .$$

Then $c_2$ is finite.

Now, fix $k^{**} \in \mathbb{N}$ such that

$$k^{**} > \left( -\frac{\log c_2}{\log c_1} + 3 \right) k^* \ ,$$

and let $s \in j\mathbb{R}$ and $k \in \mathbb{N}\cup\{0\}$ such that $k \geq k^{**}$. Denote $q$ and $r$ the quotient and the rest of the euclidian division of $k$ by $k^*$, that is:

$$q \in \mathbb{N}\cup\{0\}, \ r \in \{0, 1, \ldots, k^* - 1\}, \ k = qk^* + r \ .$$

Remark that $k \geq k^{**}$ implies

$$q \geq -\frac{\log c_2}{\log c_1} + 2 > -\frac{\log c_2}{\log c_1} + 1 \ . \tag{23}$$

Then,

$$\|[C(sI_{n_1} - A)^{-1}B + D]^k\| \leq \|[C(sI_{n_1} - A)^{-1}B + D]^{k^*}\|^q \|[C(sI_{n_1} - A)^{-1}B + D]^r\| \leq c_1^q c_2 < c_1 < 1 \ ,$$

due to (23) and the fact that $c_1 < 1$. From (22), we have hence deduced the existence of $k^{**}$ such that

$$\forall k \in \mathbb{N}\cup\{0\}, k \geq k^{**}, \sup_{s\in j\mathbb{R}} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| < c_1 < 1 \ ,$$

so one gets

$$\operatorname{Re}\sigma(A) < 0 \text{ and } \limsup_{k\to+\infty} \sup_{s\in j\mathbb{R}} \|[C(sI_{n_1} - A)^{-1}B + D]^k\| < 1 \ ,$$

which is nothing but (19).

To summarize, it has been shown first that: condition 2. $\Leftrightarrow$ (18); under the assumption $\operatorname{Re}\sigma(A) < 0$ (consequence of 2.), it has been successively shown that: (19) $\Rightarrow$ condition 4., (18) $\Rightarrow$ (22), (22) $\Rightarrow$ (19). This shows finally that condition 2. implies condition 4., and concludes the proof of Lemma 3.


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# A Appendix – Kalman-Yakubovich-Popov lemma

We use the statement as given e.g. in [23]. Let $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, M = M^T \in \mathbb{R}^{(n+p)\times(n+p)}$.

**Lemma 4.** *If $\det(sI_n - A) \neq 0$ for $s \in j\mathbb{R}$, then the following two statements are equivalent.*

1. *For any $s \in j\overline{\mathbb{R}}$,*
$$\begin{pmatrix} (sI_n - A)^{-1}B \\ I_p \end{pmatrix}^* M \begin{pmatrix} (sI_n - A)^{-1}B \\ I_p \end{pmatrix} < 0 \ .$$

2. *There exists $P = P^T \in \mathbb{R}^{n\times n}$ such that*
$$\begin{pmatrix} A^TP + PA & PB \\ B^TP & 0_p \end{pmatrix} + M < 0 \ .$$

■

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