Control of delay systems with relay

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Abstract

We study the periodic oscillations of a 1st order delayed linear system with relay output and proportional + integral feedback and describe the behavior of the general solutions of the closed loop.

For the system under study, we first exhibit a countable set of periodic limit cycles. We show that in the particular case where only proportional control is used, any solution tends in finite time towards one of the limit cycles (whose determination depends on the initial conditions). All the cycles are orbitally unstable except one of them, the only slowly oscillating one. We provide exact computations of their period and amplitude.

We then show how these results may be used to identify the parameters of the plant and to tune the control-law parameters in order to control the amplitude and the period of the slowly oscillating limit cycle.

Finally, we provide some well-posedness and ultimate boundedness results for a time-varying perturbed version of the system under study. The given estimates show that the proportional + integral feedback law permits to reject various parametric perturbations.

Keywords: control of oscillations, fuel-air ratio regulation, delay differential equations, slowly oscillating solutions, super-high-frequency oscillations, relay nonlinearity.

Introduction

We consider periodic oscillations of relay systems. In Automatic Control applications, the relay nonlinearity is used to describe a sensor or actuator behavior. Such a nonlinearity combined with stabilizing feedback loops often leads to limit cycles for the closed loop system.

We study here the behavior of a first order system controlled by a Proportional+Integral control law on the delayed output of a relay sensor. The system under study is the following:

$$\left\{ \begin{array}{l} \dot{x}(t) = -\frac{1}{\tau}\left(k_I y(t) + k_P \mathrm{sgn} x(t-h) + x(t)\right) \ , \\ \dot{y}(t) = \mathrm{sgn} x(t-h) \ , \end{array} \right. \tag{1}$$

where $\tau > 0$ is the time constant of the plant, $h > 0$ the delay, and k_I, k_P the PI controller parameters. It is indeed obtained by closing the open-loop system (where u is the control and Y the output) $\tau\dot{x} + x = u$, $Y = \mathrm{sgn} x(t-h)$ with the P.I. control law $u = -k_P Y - k_I \int Y$, in an attempt to steer x near zero in a short time. Equation (1) writes $\tau\dot{x} + x + k_P \mathrm{sgn} x(t-h) = 0$ (resp. $\tau\ddot{x} + \dot{x} + k_I \mathrm{sgn} x(t-h) = 0$) when $k_I = 0$ (resp. $k_P = 0$). When $k_P = k_I\tau$ and $x(0) = -k_I y(0)$, one has $\dot{x} = -k_I \mathrm{sgn} x(t-h)$, giving saw-tooth evolution for x.

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The motivation for considering such a control system comes from some automotive control problem, namely the fuel-air ratio regulation problem for spark ignition engine [7, 5]. Another point of interest is the identification of the linear plant of the system on the basis of the limit cycle characteristics.

In (1), the interpretation of sgn deserves some attention. In order to have uniqueness of the solution of the Cauchy problem, sgn cannot be chosen as the usual multivalued map with $\text{sgn}0 = [-1, 1]$. Instead, sgn0 must be defined as a single value. The case $\text{sgn}0 = 0$ leads to an *unstable* equilibrium, so we take $\text{sgn}0 \neq 0$ to cancel it. In the context of control, it seems realistic to take $\text{sgn}0 \in \{-1, +1\}$: the sign usually models a binary sensor or actuator. Let us define finally the function sgn as follows. Let $z \in L^\infty(0, +\infty)$ with $|z(t)| = 1$ a.e., and consider the Lebesgue measurable function $t \in (0, +\infty) \mapsto \text{sgn}(t, .)$, such that

$$\text{sgn}(t, x) = \begin{cases} 1 & \text{if } x > 0 \ , \\ -1 & \text{if } x < 0 \ , \\ z(t) & \text{if } x = 0 \ , \end{cases} \tag{2}$$

holds for almost every t. This permits to model various policies: e.g. $\text{sgn}(t, 0) = 1$, or $\text{sgn}(t, x(t-h))$ switches as late as possible .... We then replace Equation (1) by

$$\begin{cases} \dot{x}(t) = -\frac{1}{\tau}\left(k_I y(t) + k_P \text{sgn}(t, x(t-h)) + x(t)\right) \ , \\ \dot{y}(t) = \text{sgn}(t, x(t-h)) \ . \end{cases} \tag{3}$$

The Cauchy problem associated to (3,2) then admits a unique solution (see Theorem 9), and an important property of the solutions x is that the set $\{t \geq 0 \, : \, x(t) = 0\}$ has zero measure (see Theorem 10). The choice of sgn0 is thus a posteriori indifferent.

Let us give some definitions, useful to describe the periodic solutions properties.

Definition 1 (Slowly Oscillating (SO) functions[1]). A continuous function x defined on $[t_0, +\infty)$ is called slowly oscillating (with respect to h) if $x(t) = x(t') = 0$ for $t, t' > t_0$, $t \neq t'$ implies $|t - t'| > h$.

Definition 2 (2-Phase Periodic (2PP) functions [14]). A T-periodic ($T > 0$) continuous function x defined on $[t_0, +\infty)$ is called 2-phase periodic if there exists $t \geq t_0$ such that $x|_{(t,t+T)}$ changes sign (strictly) exactly once.

Definition 3 (Symmetric Periodic functions). A T-periodic ($T > 0$) continuous function x defined on $[t_0, +\infty)$ is called symmetric if $x(t + \frac{T}{2}) = -x(t)$ for $t \geq t_0$.

In the present paper, we study System (1) (or more precisely, System (3,2)) under one of the following assumptions:

$$k_I = 0, \quad k_P > 0 \quad (\text{Proportional control}) \tag{P}$$

$$k_I > 0 \quad (\text{Proportional+Integral control}) \tag{PI}$$

In Section 1, we present the case where $\tau = 0$. Here the periods of the different periodic solutions may be determined by elementary geometrical constructions. Their number is infinite countable, one of them only being SO.

In Section 2, the case where $\tau \neq 0$ is studied and proved to be qualitatively the same. Under any of the hypotheses (P) or (PI), there exists a SO periodic solution of (3,2). This solution is unique in the class of SO periodic solutions (up to time translation). It is asymptotically orbitally stable, symmetric and 2PP.

In the case (P), we give, in Theorem 2, an exhaustive description of the asymptotic behavior: first, the non-SO 2PP solutions are unstable, second, any solution is 2PP symmetric after a finite time. This permits to show that the proportional control law ensures that the mean-value of the solution goes asymptotically to

[1]See [12] for a survey on this notion in the context of periodic solutions of autonomous differential equations with delay.

zero (Corollary 3). One ingredient of Theorem 2 is the property of disappearance of super-high-frequencies [15] for Equation (3,2) (that is solutions having infinite number of zeros on any interval of $[0,+\infty)$ of length h do not exist). This property is proved in [2] for more general systems (see also [15]). The remaining proof of Theorem 2 is postponed to Appendix A.

In Section 3, we show how one may control the characteristics of the SO limit cycle (period, amplitude) by an adequate choice of the control law parameters; also, it is shown how one may identify the linear plant by measuring the limit cycle characteristics.

In Section 4, one introduces perturbations of the nominal system (3,2): the parameters h, τ of the plant are now functions of time, and time-dependent errors ζ and ξ are added respectively to the command and to the sign input (see system (15)). One shows that the Cauchy problem associated with this new system admits a unique solution, and provides asymptotic bounds. These estimates show the possibility to reject the influence of the perturbations (Theorem 11 and Corollary 12).

For a general overview on periodic solutions of autonomous delayed equations, we refer the reader to [6, Chapters XV and XVI].

1 The particular case $\tau = 0$

When $\tau = 0$ and (PI) holds, that is $k_I > 0$, (3) writes

$$x = -k_P \dot{y} - k_I y, \quad \dot{y} = \mathrm{sgn}(t, x(t-h)) \ .$$

If the initial condition $x|_{[-h,0]}$ is continuous and has a finite number of zeros, then $x|_{[0,+\infty)}$ is locally piecewise affine, with slopes $\pm k_I$, and undergo jumps of magnitude $\pm 2k_P$. Moreover, a change of mode at time t implies a change of sign of x at time $t-h$.

We construct 2PP symmetric solutions with period T_n^* satisfying

$$nT_n^* \leq h < (n+1)T_n^* \ .$$

Here, $n = \lfloor \frac{h}{T_n^*} \rfloor$: in other words, n is the maximal number of periods included in an interval of length h. Geometric constructions shown in Figure 1 permit to determine the period T_0^*.

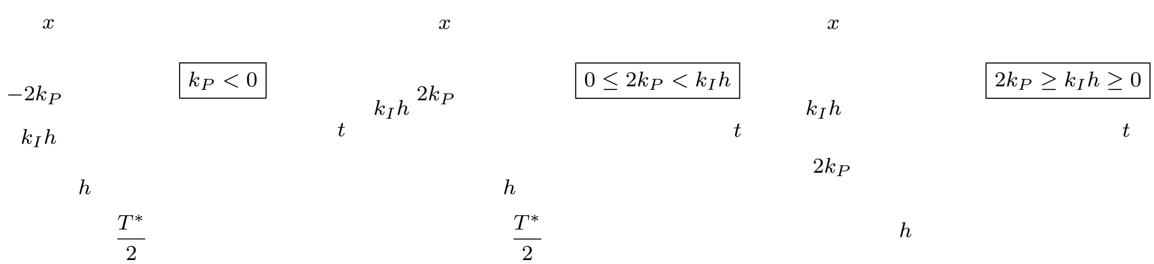

Figure 1: Computation of the period T^* of the 2PP symmetric limit cycles when $\tau = 0$. Case $T^* \geq 2h$

When $2k_P < k_I h$ (resp. $2k_P \geq k_I h$), one has

$$\frac{T_0^*}{2} = h + \frac{k_I h - 2k_P}{k_I} \quad (\text{resp. } \frac{T_0^*}{2} = h) \ . \tag{4}$$

So one deduces the value of the period T_0^*:

$$T_0^* = \max\left\{2h, 4\left(h - \frac{k_P}{k_I}\right)\right\} \ .$$

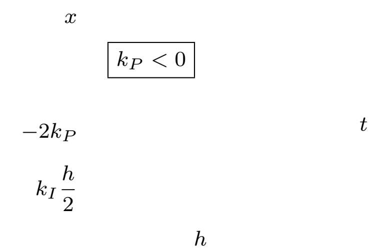

Figure 2: Computation of the period T^* of the 2PP symmetric limit cycles when $\tau = 0$. Case $h \leq T^* < 2h$

The corresponding solution is SO if and only if $h > 2\frac{k_P}{k_I}$.

For the other 2PP symmetric solutions, the changes of mode are not consequence of the last change of sign of x, but of the last change situated before an entire number of half-periods. When $0 < h - nT_n^* \leq \frac{T_n^*}{2}$, then we obtain the corresponding period T_n^* by replacing h by $h - nT_n^*$ in (4), under the constraint that $h - nT_n^* > 0$. We get

$$T_n^* = \max\left\{\frac{1}{n+\frac{1}{2}}\,h, \frac{1}{n+\frac{1}{4}}\left(h - \frac{k_P}{k_I}\right)\right\},\ T_n^* < \frac{h}{n},\ n \in \mathbb{N}\ ,$$

which corresponds to the portion of the curve given in Figure 3 for $\frac{k_P}{k_I} > -\frac{h}{4n}$. There is no solution with $\frac{T_n^*}{2} < h - nT_n^* < T_n^*$.

When $h - nT_n^* = 0$, then we get solutions of the type shown in Figure 2, leading to periods

$$T_n^* = \frac{h}{n},\ \frac{k_P}{k_I} \leq -\frac{h}{4n},\ n \in \mathbb{N} - \{0\}\ ,$$

which correspond to the remainder of the curve given in Figure 3. The curve in this Figure is hence depicted by the following formula:

$$\forall n \in \mathbb{N},\ T_n^* = \min\left\{\frac{h}{n}\ ,\ \max\left\{\frac{1}{n+\frac{1}{2}}\ h\ ,\frac{1}{n+\frac{1}{4}}\left(h - \frac{k_P}{k_I}\right)\right\}\right\}\ .$$

We obtain a countable number of branchs of solutions. For any value of $n \in \mathbb{N}$, T_n^* is uniquely defined, and is a continuous nonincreasing function of $\frac{k_P}{k_I}$. The branch corresponding to $n = 0$ is unbounded for $k_P/k_I \to -\infty$. The periods obtained for $n \neq 0$ are smaller than $2h$, so they give rise to non-slowly oscillating periodic solutions.

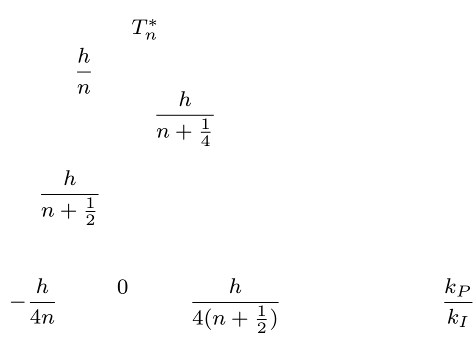

Figure 3: Periods of the nth branch of cycles when $\tau = 0$ (the decreasing part is infinite on the left if $n = 0$)

One may also compute graphically the amplitude of the cycles. The result is given in Figure 4. The curve is given analytically by

$$\forall n \in \mathbb{N}, \quad \|x_n^*\|_\infty = \begin{cases} -k_P + \frac{1}{4n} k_I h & \text{when } \frac{k_P}{k_I} \leq -\frac{h}{4n} \ , \\ -\frac{2(2n+1)}{4n+1} k_P + \frac{1}{4n+1} k_I h & \text{when } -\frac{h}{4n} \leq \frac{k_P}{k_I} \leq 0 \ , \\ \frac{4n}{4n+1} k_P + \frac{1}{4n+1} k_I h & \text{when } 0 \leq \frac{k_P}{k_I} \leq \frac{h}{4(n+\frac{1}{2})} \ , \\ k_P + \frac{1}{2(2n+1)} k_I h & \text{when } \frac{h}{4(n+\frac{1}{2})} \leq \frac{k_P}{k_I} \ . \end{cases}$$

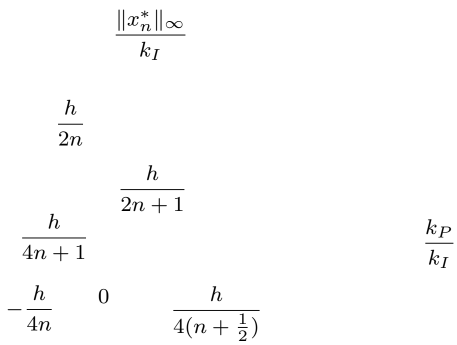

Figure 4: Amplitudes of the nth branch of cycles when $\tau = 0$ (from left to right, the slopes are successively -1, $-1 - \frac{1}{4n+1}$, $1 - \frac{1}{4n+1}$, 1)

When (P) holds, that is $k_I = 0$, $k_P > 0$, the same considerations show that the possible periods are $\frac{h}{n+\frac{1}{2}}$, $n \in \mathbb{N}$: they are obtained in Figure 3 by taking the value T_n^* corresponding to $\frac{k_P}{k_I} \to +\infty$. The amplitudes are obtained in the same manner.

2 The flow and its attractors in the general case $\tau \neq 0$

In the sequel, we denote by a bar over a symbol the normalization by τ:

$$\bar{h} \stackrel{\Delta}{=} \frac{h}{\tau}, \ \bar{T} \stackrel{\Delta}{=} \frac{T}{\tau} \ .$$

Theorem 1 (Number, period and amplitude of the 2PP solutions). *If* (P) *or* (PI) *holds, then the periodic solutions* (x, y) *of system* (3,2) *such that* x *is 2PP, are symmetric and form (up to time-translation) an infinite countable set, denoted* $\{(x_n^*, y_n^*)\}_{n\in\mathbb{N}}$.

For $k_P/k_I \in (-\infty, +\infty]$ *(with the convention* $k_P/k_I = +\infty$ *in the case* (P)*), the period* $T_n^* = T_n^*(k_P/k_I)$ *of* (x_n^*, y_n^*) *is defined uniquely by the following conditions:*

$$k_I\tau \left((n + \frac{1}{4}) \bar{T}_n^* - \bar{h} \right) = (k_I\tau - k_P) \left(1 - \frac{2}{1 + e^{-\frac{\bar{T}_n^*}{2}}} e^{-(n+\frac{1}{2})\bar{T}_n^* + \bar{h}} \right), \quad \frac{\bar{h}}{n + \frac{1}{2}} < \bar{T}_n^*(+\infty) \leq \bar{T}_n^* < \frac{\bar{h}}{n} \ . \tag{5}$$

The function T_n^* *of* k_P/k_I *is strictly decreasing and continuous.*

The solution (x_n^*, y_n^*) *is defined by the symmetry property and, for* $0 \leq t < \frac{T_n^*}{2}$:

$$x_n^*(t) = k_I\tau \left(\frac{\bar{T}_n^*}{4} - \frac{t}{\tau} \right) + (k_I\tau - k_P) \left(1 - \frac{2}{1 + e^{-\frac{\bar{T}_n^*}{2}}} \ e^{-\frac{t}{\tau}} \right), \ \ y_n^*(t) = t - \frac{T_n^*}{4} \ . \tag{6}$$

Moreover,

$$\|x_n^*\|_\infty = \begin{cases} -k_P + k_I\tau \ln\left(\frac{\cosh \frac{\bar{T}_n^*}{4}}{1-\frac{k_P}{k_I\tau}}\right) & \text{if } \frac{k_P}{k_I\tau} < \frac{1-e^{-\frac{\bar{T}_n^*}{2}}}{2} \\ k_I \frac{T_n^*}{4} - (k_I\tau - k_P)\tanh\frac{\bar{T}_n^*}{4} & \text{otherwise} \end{cases} \tag{7}$$

The situation when $\tau \neq 0$ is hence qualitatively the same as in Section 1; the curves, analogous to those shown in Figures 1 to 4, are smoother.

For any $n \in \mathbb{N}$, (5) defines a branch of solutions $\bar{T}_n^*$, which is continuous and decreasing as a function of $\frac{k_P}{k_I} \in (-\infty, +\infty]$. When $\frac{k_P}{k_I} = \tau$, one has $T_n^* = \frac{h}{n+\frac{1}{4}}$, as in the case $\tau = 0$ (see Figure 3).

Remark that the value of the period may be deduced from Tsypkin's results [17], but only "one out of every two" periods obtained by this method really lead to a 2PP solution.

Proof of Theorem 1. Let (x, y) be a periodic solution of (3,2) with x 2PP. One deduces from the periodicity of y that $\mathrm{sgn}(t, x(t-h))$ is symmetric. Suppose that (x_n^*, y_n^*) is such a solution, with $x_n^*(t) \geq 0$ for $t \in [-h, \frac{T_n^*}{2} - h]$, $x_n^*(t) \leq 0$ for $t \in [\frac{T_n^*}{2} - h, T_n^* - h]$, and $nT_n^* \leq h < (n+1)T_n^*$. Solving the equations piecewise and expressing the continuity and periodicity conditions for x_n^* and y_n^* leads to the symmetry of x_n^* and y_n^*, and some necessary conditions on T_n^*. Expressing compatibility conditions on $\mathrm{sgn}(t, x_n^*(t-h))$, we get (5), which is also shown to be sufficient. This method also provides the explicit form (6) of the corresponding periodic solutions. From these explicit formulas is deduced the value of the amplitude, using the fact that $x_n^*(0) > 0 > x_n^*(\frac{T_n^*}{2}) = -x_n^*(0)$ and the monotony or concavity property of x_n^* on $[0, \frac{T_n^*}{2}]$. Indeed, when $k_P/(k_I\tau) \geq (1 - e^{-\frac{\bar{T}_n^*}{2}})/2$, then x_n^* is nonincreasing on $[0, \frac{T_n^*}{2}]$ and $0 < x_n^*(0) = \|x_n^*\|_\infty$; otherwise, x_n^* is concave on $[0, \frac{T_n^*}{2}]$ and its maximum is attained for some $t \in (0, \frac{T_n^*}{2})$ such that $\dot{x}_n^*(t) = 0$. $\square$

We now turn to the question of asymptotic behavior of system (3,2). We first examinate the case (P).

Define the set of points where the solution x of (3,2) vanishes and changes sign, and the cardinal of the number of zeros with changes of sign [2]:

$$Z \stackrel{\Delta}{=} \{t \geq 0 : x(t) = 0 \text{ and } \forall\varepsilon > 0,\ \exists t' \in [t-\varepsilon, t),\ t'' \in (t, t+\varepsilon],\ x(t')x(t'') < 0\}\ , \tag{8}$$

$$V(t) = \mathrm{card}\, Z \cap [t'-h, t') \text{ where } t' = \inf\ [t, +\infty) \cap Z\ . \tag{9}$$

One has obviously for any $t \geq 0$: $V(t) \in \mathbb{N} \cup \{+\infty\}$. In [9, 15], similar constructions are made. The set Z is unbounded [9, 3] and closed [2].

Remark that $V(t) = 0$ for t large enough if and only if x is slowly oscillating on an interval of the form $[t_0, +\infty)$.

Theorem 2 (Behavior of the solutions in the case (P)**).** *Suppose* (P) *holds. Then, for any solution of the Cauchy problem associated with* (3,2)*,* $V(t)$ *takes on nonincreasing even values, finite after a finite time.*

Denote $2n = \lim_{t\to+\infty} V(t)$*, and let* $\underline{t}$ *be the smallest time* t *for which* $V(t) = 2n$ *(so that* $V(t) = 2n$ *for* $t \geq \underline{t}$*). Then, there exists* $\varphi \geq 0$ *such that*

$$t \geq \underline{t} \Rightarrow x(t) = x_n^*(t+\varphi)\ .$$

The SO cycle is asymptotically orbitally stable, the non-SO cycles are unstable.

In particular, Theorem 2 implies that, in the case (P), the only periodic solutions are the ones exhibited in Theorem 1. Moreover, any trajectory of (3,2) sufficiently close to the SO cycle ultimately coincides with it.

The case $\tau = 0$ appears as degenerated, as the slowest periodic solution is not SO, since its period is $2h$ (as $k_I = 0$).

Corollary 3. *If* (P) *holds, then, for any trajectory of* (3,2),

$$\limsup_{t'\to+\infty} \sup\{t-t' \ : \ t\geq t',\ x\geq 0 \text{ (resp. } \leq 0) \text{ on } [t',t]\} \leq \frac{T_0^*}{2}\ ,$$

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t x(s)\ ds = \lim_{t\to+\infty} \frac{1}{t}\int_0^t \operatorname{sgn}(s+h,x(s))\ ds = 0\ .$$

Corollary 3 shows that in the case (P), the mean value of x is controlled to zero. Remark that, more generally, when (P) holds, the asymptotic properties of the solutions need only to be checked for the periodic ones. As an example,

$$\limsup_{t\to+\infty} |x(t)| \leq \|x_0^*\|_\infty = k_P(1-e^{-\bar{h}})\ .$$

One ingredient of Theorem 2 is the fact that $V(t)$ is finite after a certain time (disappearance of the so-called *super-high-frequency oscillations* [15]). This property follows from [2, Corollary 3], which generalizes [15, Theorem 0.5] with the same type of technique (the result of [15] can be applied here for a small delay only). The remaining proof is presented in Appendix A, where we demonstrate the results related to solutions with finite number of zeros, with techniques analogous to what is done in [8].

A less precise and weaker result on the asymptotics of system (3,2) is given now for the case (PI).

Theorem 4 (Uniqueness and stability of the SO periodic solution in the case (PI)**).** *The solution* (x_0^*,y_0^*) *is the unique periodic solution such that x is SO. It is (locally) orbitally asymptotically stable with exponential asymptotic phase:*

$$\exists c>0, \exists\varepsilon>0, \forall\varphi\in\mathbb{R}, \forall(x_0,y_0)\in\mathcal{C}([-h,0])\times\mathbb{R}\ ,\ \|(x_0,y_0)-(x_0^*(\cdot+\varphi)|_{[-h,0]},y_0^*(\varphi))\|<\varepsilon$$
$$\Longrightarrow \exists\varphi'\in\mathbb{R}\ ,\ \limsup_{t\to+\infty} \frac{\ln\|(x(\cdot+t)|_{[-h,0]},y(t))-(x_0^*(\cdot+t+\varphi')|_{[-h,0]},y_0^*(t+\varphi'))\|}{t} < -c\ ,$$

where $\|\cdot\|$ *is the norm on* $\mathcal{C}([-h,0])\times\mathbb{R}$ *given by* $\|(x,y)\| = \|x\|_\infty + |y|$, *and* (x,y) *is the solution of* (3,2) *with initial condition* $x|_{[-h,0]}=x_0,\ y(0)=y_0$.

Remark that when (PI) holds, the function V does not decrease anymore. Indeed, it is possible to show, as in [16, 10] that the integer part of $\frac{V+1}{2}$ decreases.

Proof of Theorem 4. • Let (x^{**},y^{**}) be a SO periodic solution. The set of zeros of x^{**} is unbounded (on the right). Otherwise, $x^{**}(t)>0$ on $[t_0,+\infty)$ for instance, thus $\dot{y}(t+h)=1$ and $x(t)$ tends to $-\infty$ when t goes to infinity, and we get a contradiction. Denote $-h=t_0<\dots<t_k<\dots$ the zeros of x^{**}, and suppose that $x^{**}(t)>0$ on (t_0,t_1). Since $t_k-t_{k-1}>h$, then y^{**} and x^{**} are $\mathcal{C}^1$ in a neighbourhood of t_k and it is easy to show that $(-1)^k\dot{x}^{**}(t_k)>0$ and $(-1)^k x^{**}(t)>0$ on (t_k,t_{k+1}). Denote $z_k \stackrel{\Delta}{=} \frac{(-1)^{k-1}}{\tau}y^{**}(t_k)$. The previous properties imply $z_k>-k_P/(k_I\tau)$. Now, denote Φ the map such that $z_{k+1}=\Phi(z_k)$, and z^* the (unique) fixed point of Φ; indeed $z^*=\frac{1}{\tau}y_0^*(\frac{T_0^*}{2}-h)$, where y_0^* is defined in (6). Φ is defined implicitly on $(-k_P/(k_I\tau),+\infty)$ by the relation

$$e^{2\bar{h}+z+\Phi(z)}\left(\Phi(z)-1+\frac{k_P}{k_I\tau}\right)+2\left(1-\frac{k_P}{k_I\tau}\right)e^{\bar{h}}+z-1+\frac{k_P}{k_I\tau}=0 \tag{10}$$

From this, it is easy to prove that $\frac{d\Phi}{dz}>-1$, and that $z<z_*\Leftrightarrow z<\Phi(z)$. We deduce from these two facts that $|z_{k+1}-z^*|<|z_k-z^*|$ for any k such that $z_k\neq z^*$. Due to the assumed periodicity, the sequence z_k is cyclic, hence $z_k=z^*$ for any $k\in\mathbb{N}$, that is $(x^{**},y^{**})=(x_0^*,y_0^*)$; the SO periodic solution is hence unique.

• The proof of the orbital stability is based on two auxiliary lemmas. Recall that $x_0^*|_{(-h,\frac{T_0^*}{2}-h)}>0$, $x_0^*|_{(\frac{T_0^*}{2}-h,T_0^*-h)}<0$. The following lemma comes from the fact that $\operatorname{sgn}:\mathcal{C}([-h,0])\to L^1([-h,0])$, $x\mapsto(t\mapsto \operatorname{sgn}(t+h,x(t)))$ is continuous in $x_0^*(\cdot+\varphi)$ for all $\varphi\in\mathbb{R}$ [Hint: use the fact that meas ${x_0^*}^{-1}(\{0\})=0$ and the inequality $\int_{-h}^0|\operatorname{sgn}(x_0^*+x)-\operatorname{sgn}x_0^*|\leq 2\,\text{meas}\,\{{x_0^*}^{-1}([-\|x\|_\infty,\|x\|_\infty])\cap[-h,0]\}$].

Lemma 5 (Continuity of the flow in the neighbourhood of the SO periodic solutions).

$$\forall \varepsilon, T > 0\ \exists \eta > 0\ \forall \varphi \in \mathbb{R}\ \forall (x_0, y_0) \in \mathcal{C}([-h,0]) \times \mathbb{R}\,,\ \|(x_0, y_0) - (x_0^*(\cdot + \varphi)|_{[-h,0]}, y_0^*(\varphi))\| < \eta$$
$$\implies \forall t \in [0,T]\,, \|(x(\cdot + t)|_{[-h,0]}, y(t)) - (x_0^*(\cdot + t + \varphi)|_{[-h,0]}, y_0^*(t+\varphi))\| < \varepsilon\ ,$$

where $\|.\|$ and (x, y) are defined as in the statement of Theorem 4.

Applying Lemma 5 to $T > T_0^*$ and ε small enough, we obtain that there exists $t_0 > 0$ such that $x(t_0) = 0$, $x(\cdot + t_0)|_{[-h,0)} > 0$ and $|y(t_0) - y_0^*(\frac{T_0^*}{2} - h)| < \varepsilon$. But the positive homogeneity of sgn implies that for an initial value (x_0, y_0) such that $x_0 > 0$ (or $x_0 < 0$) and $x_0(0) = 0$, the evolution depends only on y_0. This makes it possible to use a finite-dimensional Poincaré map to study the stability of the SO periodic solution (a similar finite-dimensional technique will be used in the proof of Theorem 2, see Appendix A). This map is related to the map Φ defined above.

Lemma 6 (A contracting Poincaré map). *Let $z^* = y_0^*(\frac{T_0^*}{2} - h)$, and for $\varepsilon > 0$, define $\Omega_\varepsilon \overset{\Delta}{=} \{(x_0, y_0) \in \mathcal{C}([-h,0]) \times \mathbb{R} : x_0(t) > 0\ \forall t \in [-h, 0)\,,\ x_0(0) = 0$ and $|y_0 - z^*| < \varepsilon\}$. For $(x_0, y_0) \in \Omega_\varepsilon$, let $\delta = \delta(x_0, y_0) > 0$ be the first time such that $(-x(.+\delta)|_{[-h,0]}, -y(\delta)) \in \Omega_\varepsilon$, and denote $P(x_0, y_0)$ this point.*

Then, for sufficiently small ε, the functions δ and P are well-defined on Ω_ε, $w^* = (x_0^*(.+\frac{T_0^*}{2} - h)|_{[-h,0]}, z^*)$ is a fixed-point of P and $\delta(w^*) = \frac{T_0^*}{2}$. Moreover, P and δ satisfy

$$P(x_0, y_0) = (x_1, y_1) \quad \text{with } y_1 = \tau \Phi(\frac{y_0}{\tau})\,,$$
$$\delta(x_0, y_0) = \frac{T_0^*}{2} + y_0 + y_1 - 2z^*\,,$$

where Φ is defined in (10)*, and there exist $c > 0$ and $\gamma \in (0, 1)$ such that:*

$$|y_1 - z^*| \le \gamma |y_0 - z^*|\,,$$
$$\|P(x_0, y_0) - w^*\| \le c|y_0 - z^*|\,.$$

The proof of Lemma 6 follows by the same arguments as for the uniqueness of the SO solution: the distance between two zeros of x is at least h and an explicit computation of the solution leads to the formulas of P and δ. From $|\frac{d\Phi}{dy}(z^*)| < 1$, one deduces the contraction of $y_1 - z^*$.

The formula of δ proves that the distance δ_n between the first and the $2n$-th zero is such that $nT_0^* - h - \delta_n$ tends to a limit, which is the phase φ' in the conclusion of Theorem 4. □

3 Identification of the plant and control of the oscillations

We now consider the issue of controlling the oscillations, more precisely the period and amplitude of the (orbitally asymptotically stable) SO cycle, by adequate choice of k_P, k_I. Remark that the period T_0^* of the latter verifies

$$2\tau \ln(2e^{\bar{h}} - 1) \le T_0^* < +\infty\ ,$$

where $2h < 2\tau \ln(2e^{\bar{h}} - 1) < 4h$, whatever the values of k_P, k_I fulfilling (P) or (PI): periods smaller than this bound are not realizable directly. It turns out that this is the only restriction, and that the amplitude may be chosen freely.

Theorem 7 (Control algorithm). *For $\tau, h, T, \rho > 0$, there exist k_P, k_I satisfying* (P) *or* (PI) *such that* (3,2) *has a SO cycle with period T and amplitude ρ iff*

$$\bar{T} \ge 2\ln(2e^{\bar{h}} - 1)\,, \quad \text{where } \bar{h} \overset{\Delta}{=} \frac{h}{\tau},\ \bar{T} \overset{\Delta}{=} \frac{T}{\tau}.$$

In this case, the solution is unique and given by the following algorithm:

1. *If* $\bar{T} = 2\ln(2e^{\bar{h}} - 1)$, *then set*

$$k_P = \frac{\rho}{1 - e^{-\bar{h}}}, \quad k_I = 0 \ .$$

2. *Otherwise, compute*

$$\bar{\rho} = \frac{\rho}{\tau}, \quad \alpha = 1 - \frac{1 + e^{-\frac{\bar{T}}{2}}}{(1 + e^{-\frac{\bar{T}}{2}} - 2e^{-\frac{\bar{T}}{2}+\bar{h}})}(\frac{\bar{T}}{4} - \bar{h}) \ ,$$

and set

$$\begin{aligned} k_I &= \begin{cases} \bar{\rho}\left(-\alpha + \ln\left(\frac{\cosh\frac{\bar{T}}{4}}{1-\alpha}\right)\right)^{-1} & \textit{if } \alpha < \frac{1-e^{-\frac{\bar{T}}{2}}}{2} \\ \bar{\rho}\left(\frac{\bar{T}}{4} - (1-\alpha)\tanh\frac{\bar{T}}{4}\right)^{-1} & \textit{otherwise} \end{cases} \ , \\ k_P &= \alpha k_I \tau \ . \end{aligned}$$

Theorem 7 proceeds directly from the previous results. When $T > 2\tau\ln(2e^{\bar{h}} - 1)$, then $k_I > 0$ by construction. Periods T such that $2h < T < 2\tau\ln(2e^{\bar{h}} - 1)$ may be realized by prefiltering: **1.** Choose $\tau' > 0$ sufficiently small, in order that $\frac{T}{\tau'} \geq 2\ln(2e^{\frac{h}{\tau'}} - 1)$, **2.** Compute k_P, k_I as in the preceding algorithm, replacing τ by τ'. Then the control law given by use of $\frac{\tau s+1}{\tau' s+1}(k_P + \frac{k_I}{s})$, instead of $k_P + \frac{k_I}{s}$, achieves the desired goal.

Another interesting issue is the possibility to use the previous results to identify the parameters τ and h of the plant, from the measure of the period and amplitude of the SO cycle.

Theorem 8 (Identification algorithm). *For* $T, \rho > 0$ *and* k_P, k_I *satisfying* (P) *or* (PI), *there exist* $\tau, h > 0$ *such that* (3,2) *has a SO cycle with period* T *and amplitude* ρ *iff*

$$\begin{cases} \rho < k_P + k_I\frac{T}{4} & \textit{if } k_P \geq 0 \, , \\ 0 < \rho_0 < \rho < -k_P + k_I\frac{T}{4} & \textit{if } k_P < 0 \, , \end{cases} \tag{11}$$

where $\rho_0 = f(\tau_0)$ *with* τ_0 *the only* $\tau_0 > 0$ *such that* $g(\tau_0) = 0$ *and the functions* f *and* g *are defined by:*

$$\begin{aligned} f(\tau) &= -k_P + k_I\tau\ln\left(\frac{\cosh\frac{T}{4\tau}}{1 - \frac{k_P}{k_I\tau}}\right) \ , \\ g(\tau) &= k_I\frac{T}{4} - (k_I\tau - k_P)\tanh\frac{T}{4\tau} \ . \end{aligned}$$

In this case, the solution is unique and given by the following algorithm if (P) *holds:*

$$\tau = \frac{T}{2}\frac{1}{\ln(\frac{k_P+\rho}{k_P-\rho})}, \quad h = \frac{T}{2}\frac{\ln(\frac{k_P}{k_P-\rho})}{\ln(\frac{k_P+\rho}{k_P-\rho})} \ , \tag{12}$$

and if (PI) *holds by:*

$$\tau \textit{ is the only } \tau > 0 \textit{ such that } \begin{cases} f(\tau) = \rho \textit{ and } g(\tau) > 0 & \textit{if } \rho < -k_P + k_I\frac{T}{4}, \\ g(\tau) = \rho & \textit{otherwise,} \end{cases} \tag{13}$$

$$\bar{h} \triangleq \frac{h}{\tau} \textit{ is the only } 0 < \bar{h} < \ln\frac{1 + e^{\frac{T}{2\tau}}}{2} \textit{ such that } \frac{T}{4\tau} - \bar{h} - (1 - \frac{k_P}{k_I\tau})\left(1 - 2\frac{e^{\bar{h}}}{1 + e^{\frac{T}{2\tau}}}\right) = 0 \ . \tag{14}$$

Proof. Formulas (12) and (14) are deduced directly from Theorem 1. From (7), we get $\rho = f(\tau)$ if $F(\tau) \stackrel{\Delta}{=} 2(1-\frac{k_P}{k_I\tau})/(1-e^{-\frac{T}{2\tau}}) > 1$ and $\rho = g(\tau)$ otherwise. Denoting $G(\rho) = \rho - k_I\frac{T}{4} + k_P$, we obtain that $\rho = f(\tau)$ if and only if $F(\tau) = \exp -\frac{G(\tau)}{k_I\tau}$ and that $\rho = g(\tau)$ if and only if $F(\tau) = 1 - \frac{G(\tau)}{k_I\tau}$. So in both cases, $F(\tau) > 1$ is equivalent to $G(\rho) < 0$. In Theorem 1, the solution x_0^* satisfies $x_0^*(0) > 0$ which means $g(\tau) > 0$ or equivalently $F(\tau) < 1 + \frac{k_I\frac{T}{4}-k_P}{k_I\tau}$. Using, the properties of F, one deduces that System (13) has a solution if and only if ρ satisfies (11), and that in this case the solution τ is unique and decreases with respect to ρ. Then, $\frac{T}{4\tau} - (1-\frac{k_P}{k_I\tau})\tanh\frac{T}{4\tau} = \frac{g(\tau)}{k_I\tau} > 0$, so the left-hand side of (14) is positive for $\bar{h} = 0$; being negative for $\bar{h} = \ln\frac{1+e^{\frac{T}{2\tau}}}{2}$ and either decreasing, or decreasing-increasing, it possesses a unique zero on $(0, \ln\frac{1+e^{\frac{T}{2\tau}}}{2})$. $\square$

4 Estimates of the solutions of the Cauchy problem robust wrt perturbations of the parameters

We now consider a perturbed version of (3,2), namely the system:

$$\begin{cases} \dot{x}(t) = -\frac{1}{\tau(t)}\left(k_I y(t) + k_P \mathrm{sgn}(t, x(t-h(t)) + \xi(t-h(t))) + x(t) + \zeta(t)\right) , \\ \dot{y}(t) = \mathrm{sgn}(t, x(t-h(t)) + \xi(t-h(t))) \ . \end{cases} \tag{15}$$

Theorem 9 (Existence and uniqueness for solutions of (15)**).** *Suppose that* (P) *or* (PI) *holds, that* $\mathrm{Id} - h$ *is nondecreasing with* $h(t^+) > 0$ *for any* $t \geq 0$*, that* τ *is nonnegative and such that* $\frac{1}{\tau} \in L^1_{\mathrm{loc}}((0,+\infty))$*, and that* $\zeta, \xi \in L^1_{\mathrm{loc}}((0,+\infty))$*. Then, for any* $(x_0, x_{00}, y_0) \in L^1((-h(0+),0))\times\mathbb{R}\times\mathbb{R}$*, there exists a unique pair* $(x,y) \in L^1_{\mathrm{loc}}((-h(0+),+\infty))\times W^{1,\infty}_{\mathrm{loc}}([0,+\infty))$*, such that* $x \in W^{1,\infty}_{\mathrm{loc}}([0,+\infty))$*,* $x|_{(-h(0+),0)} = x_0$*,* $x(0) = x_{00}$*,* $y(0) = y_0$*, and* (x,y) *verifies Equation* (15,2) *for almost every* $t \in \mathbb{R}^+$*.*

Proof. The fact that $\mathrm{Id} - h$ is nondecreasing implies that h has left- and right-limits on any point t, with $h(t^+) \leq h(t) \leq h(t^-)$. This property and the fact that $h(t^+) > 0$ implies that we cannot have $t_n \to t$ with $h(t_n^+) \to 0$. Therefore, $h(t^+)$, and then $h(t)$, is bounded from below on any compact of $[0,+\infty)$. Existence and uniqueness uses the fact that h is locally bounded from below: the integration is performed on intervals of length $h(t)$, using the local integrability of $\frac{1}{\tau}$. $\square$

Theorem 10 (Zeros of the solutions of (15,2)**).** *Suppose that the hypotheses of Theorem* 9 *are fulfilled and that* τ *is finite a.e. on* $\mathbb{R}^+$*.*

- *If* $\mathrm{meas}\,\{t \geq 0 :\ |\zeta(t)| = k_P\} = 0$ *in the case* (P)*, or if* ζ *is differentiable a.e. with* $\mathrm{meas}\,\{t \geq 0 :\ |\frac{d}{dt}\zeta(t)| = k_I\} = 0$ *in the case* (PI)*, then, for all* $(x_0, x_{00}, y_0) \in L^1((-h(0+),0))\times\mathbb{R}\times\mathbb{R}$*,*

$$\mathrm{meas}\,\{t \geq 0 :\ x(t) = 0\} = 0 \ . \tag{16}$$

- *If* ξ *is absolutely continuous, if* $\mathrm{meas}\,\{t \geq 0 :\ |\tau(t)\dot{\xi}(t) + \xi(t) + \zeta(t)| = k_P\} = 0$ *in the case* (P)*, or if* $\tau\dot{\xi} + \xi + \zeta$ *is differentiable a.e. with* $\mathrm{meas}\,\{t \geq 0 :\ |\frac{d}{dt}(\tau\dot{\xi} + \xi + \zeta)| = k_I\} = 0$ *in the case* (PI)*, then, for all* $(x_0, x_{00}, y_0) \in L^1((-h(0+),0))\times\mathbb{R}\times\mathbb{R}$*,*

$$\mathrm{meas}\,\{t \geq 0 :\ x(t) + \xi(t) = 0\} = 0 \ . \tag{17}$$

When (17) is fulfilled, then the solutions of (15) do not depend upon the definition of sgn0. Conversely, one may show that if on a certain time-interval, $\tau\dot{\xi} + \xi + \zeta = k_P$ a.e. in the case (P), or $\frac{d}{dt}(\tau\dot{\xi} + \xi + \zeta) = k_I$ a.e. in the case (PI), then, some trajectories depend upon the choice of sgn0.

Proof of Theorem 10. To prove property (16), suppose first that $k_I = 0$. Denote $N \subset \mathbb{R}^+$ a set of measure zero such that outside N, x is differentiable, (15) is fulfilled, $\tau \neq +\infty$, $|\zeta| \neq k_P$ and $|z| = 1$, where z is the function defining the sign in (2). For any $\varepsilon > 0$, there exists an open set U such that $N \subset U$ and meas $U < \varepsilon$. Denote $S \stackrel{\Delta}{=} \{t \geq 0 \ : \ x(t) = 0\}$ and $N^{\mathrm{c}} \stackrel{\Delta}{=} \{t \geq 0 \ : \ t \notin N\}$, $U^{\mathrm{c}} \stackrel{\Delta}{=} \{t \geq 0 \ : \ t \notin U\}$. It is clear that $S \cap N^{\mathrm{c}}$ has no accumulation point, otherwise on such a point t, one would have $x(t) = 0$, $\dot{x}(t) = 0$, so $k_P \dot{y}(t) + \zeta(t) = 0$, together with $|\dot{y}(t)| = 1$. As $S \cap U^{\mathrm{c}} \subset S \cap N^{\mathrm{c}}$, the set $S \cap U^{\mathrm{c}}$ has no point of accumulation. Being closed (by the continuity of x), it is then finite or countable. Hence, meas $(S \cap U^{\mathrm{c}}) = 0$, so meas $S =$ meas $(S \cap U) < \varepsilon$ for any $\varepsilon > 0$.

Suppose now $k_I \neq 0$. We define the set N with the same properties as before, with the condition $|\zeta| \neq k_P$ replaced by: ζ differentiable and $|\dot{\zeta}| \neq k_I$. The set U is then defined as before. First, the set $\{t \in N^{\mathrm{c}} \ : \ k_P \dot{y}(t) + k_I y(t) + \zeta(t) = 0\}$ has no accumulation point (because on such a point, we would have $k_I \dot{y}(t) + \dot{\zeta}(t) = 0$). As the later set contains the set of accumulation points of $S \cap N^{\mathrm{c}}$, and hence of the closed set $S \cap U^{\mathrm{c}}$, there exists an open set V containing the accumulation points of $S \cap U^{\mathrm{c}}$ and such that meas $V < \varepsilon$. Denote $V^{\mathrm{c}} \stackrel{\Delta}{=} \{t \geq 0 \ : \ t \notin V\}$. Because the set $S \cap U^{\mathrm{c}} \cap V^{\mathrm{c}}$ has no accumulation point, we have meas $(S \cap U^{\mathrm{c}} \cap V^{\mathrm{c}}) = 0$, so meas $S \leq$ meas $(S \cap U) +$ meas $(S \cap V) \leq 2\varepsilon$ for any $\varepsilon > 0$.

The proof of property (17) is conducted similarly. The continuity of ξ is necessary to claim that the set $\{t \geq 0 \ : \ x(t) + \xi(t) = 0\}$ is closed. □

Since we consider only the behavior at infinity of the solutions, we define the following seminorms on $L^\infty((0, +\infty))$ (the subscript a is for *asymptotic* and sup, inf mean sup ess, inf ess):

$$\|x\|_a \stackrel{\Delta}{=} \limsup_{t \to +\infty} |x(t)| \ , \quad |x|_a \stackrel{\Delta}{=} \frac{1}{2} \left(\limsup_{t \to +\infty} x(t) - \liminf_{t \to +\infty} x(t) \right) \ .$$

In the following results, we use the notations $z^+ \stackrel{\Delta}{=} \max\{z, 0\}$, $z^- \stackrel{\Delta}{=} \max\{-z, 0\}$.

Theorem 11 (Estimates for solutions of (15)**).** *Suppose that the hypotheses of Theorem* 9 *are fulfilled and that* $\zeta, \xi, h, \tau \in L^\infty((0, +\infty))$*, with*

$$|\zeta|_a \leq \delta_\zeta \ , \ |\xi|_a \leq \delta_\xi \ , \ \|\zeta\|_a \leq \Delta_\zeta \ , \ \|\xi\|_a \leq \Delta_\xi \ ,$$

for some constants $\delta_\zeta, \delta_\xi, \Delta_\zeta, \Delta_\xi > 0$*. Let* $(x_0, x_{00}, y_0) \in L^1((-h(0+), 0)) \times \mathbb{R} \times \mathbb{R}$*.*

• *In the case* (P)*, then*

$$|x|_a \leq k_P + \delta_\zeta - e^{-\frac{\|h\|_a}{\liminf \tau}} \min\left\{ 2k_P, \ (k_P + \delta_\zeta - \delta_\xi)^+, \ \left(\frac{3}{2} k_P + \delta_\zeta - \frac{\Delta_\zeta}{2} - \frac{\Delta_\xi}{2} \right)^+ \right\} \ ,$$

$$\|x\|_a \leq k_P + \Delta_\zeta - e^{-\frac{\|h\|_a}{\liminf \tau}} \min\{2k_P, \ (k_P + \Delta_\zeta - \Delta_\xi)^+\} \ .$$

• *In the case* (PI)*, then*

$$|x|_a \leq \delta_\xi + 2\Phi(\max\{\hat{T}, \tilde{T}, 2\|h\|_a\}) \ , \quad \|x\|_a \leq \Delta_\xi + 2\Phi(\max\{\hat{T}, \tilde{T}, 2\|h\|_a\}) \ ,$$

where

$$\Phi(T) \stackrel{\Delta}{=} -k_I\left(\frac{T}{4} - \|h\|_a\right) + K_2 + K_3^+ e^{-\frac{\frac{T}{2} - \|h\|_a}{\limsup \tau}} - K_3^- e^{-\frac{\frac{T}{2} - \|h\|_a}{\liminf \tau}} \ ,$$

$$K_1 \stackrel{\Delta}{=} \frac{1}{\liminf \tau} \left(\|k_I \tau - k_P\|_a + |k_P| + 2\delta_\zeta \right) \ ,$$

$$K_2 \stackrel{\Delta}{=} \frac{1}{2} \left(\|k_I \tau - k_P\|_a + \limsup(k_I \tau - k_P) \right) + \delta_\zeta \ ,$$

$$K_3 \stackrel{\Delta}{=} \frac{1}{2} \left(\|k_I \tau - k_P\|_a - \limsup(k_I \tau - k_P) \right) - e^{-\frac{\|h\|_a}{\liminf \tau}} \left(\|k_I \tau - k_P\|_a + \delta_\zeta \right) \ ,$$

and $\hat{T}, \tilde{T}$ are defined by

$$\Phi(\hat{T}) = K_1 \frac{\hat{T}}{4}, \quad \hat{T} > 0 \ ,$$
$$\frac{\tilde{T}}{2} \overset{\Delta}{=} \|h\|_a + \liminf \tau \ \ln\left(\frac{2K_3^-}{k_I \liminf \tau}\right) \ .$$

A proof of Theorem 11 is given in Appendix B. Estimates for y may be deduced from estimates on $x + k_I y$.

The results of Theorem 11 are to be compared with the open-loop system ($k_P = 0, k_I = 0$), for which the worst-case estimate are $|x|_a = |\zeta|_a$, $\|x\|_a = \|\zeta\|_a$. The Proportional control permits in certain cases to reject the perturbation. However, it is not possible to render $\|x\|_a$ small when $|\zeta|_a = 0$ only. This can be achieved by the Proportional-Integral control, as shown in the following result.

Corollary 12 (Rejection of the perturbations). *Let us use the same notations and assumptions as in Theorem* 11*, and suppose that*

$$\|\tau\|_a > 4|\tau|_a \ .$$

If the control parameters are chosen in such a way that

$$k_P = k_I(\|\tau\|_a - |\tau|_a), \quad k_I \geq \frac{2\delta_\zeta}{\|\tau\|_a - 4|\tau|_a} \ ,$$

then,

$$|x|_a \leq \delta_\xi + 2k_I\|h\|_a, \ \|x\|_a \leq \Delta_\xi + 2k_I\|h\|_a \ .$$

Corollary 12 demonstrates that the perturbation ζ may be rejected when the measurement noise ξ is "small" wrt the model error ζ, the delay h is "small" wrt τ, and τ "does not vary too much". In particular, the integral term permits to reject the constant perturbations ζ.

Proof of Corollary 12*.* With the proposed choice for the control parameters, one gets, using the notations of Theorem 11

$$K_1 = \frac{1}{\liminf \tau}\left(k_I\|\tau\|_a + 2\delta_\zeta\right) \ ,$$
$$K_2 = k_I|\tau|_a + \delta_\zeta \ ,$$
$$K_3 = -e^{-\frac{\|h\|_a}{\liminf \tau}} K_2 \ .$$

We then have

$$\tilde{T} = 2\liminf \tau \ln\left(2\frac{k_I|\tau|_a + \delta_\zeta}{k_I \liminf \tau}\right) \ ,$$

so $\tilde{T} < 0$, due to the bounds on k_I. Writing that

$$K_2 - K_3^- e^{-\frac{\frac{\hat{T}}{2} - \|h\|_a}{\liminf \tau}} = K_2(1 - e^{-\frac{\hat{T}}{2\liminf \tau}}) \leq K_2 \frac{\hat{T}}{2\liminf \tau} \ ,$$

we obtain that $\hat{T} \leq 2\|h\|_a$. Hence, we have

$$|x|_a \leq \delta_\xi + 2\Phi(2\|h\|_a) = \delta_\xi + k_I\|h\|_a + 2(K_2 + K_3) \ ,$$

and a similar formula for $\|x\|_a$. Using again the bounds on k_I, we get $|x|_a \leq \delta_\xi + 2k_I\|h\|_a$, and similarly for $\|x\|_a$. □

A Proof of Theorem 2 when the initial condition has a finite number of zeros

Let x be a solution of (3,2) corresponding to an initial condition with a *finite* number of zeros.

A.1 Evolution of the number of zeros

Since the elements of Z, defined in (8), are the zeros of x where x changes sign, the slope of x changes sign between two consecutive points of Z. Moreover, by Equation (3,2), the sign of the slope of x around $t \in Z$ is opposite to the sign of x around $t-h$. On the one hand, this implies that there is a change of the sign of $x(t-h)$ (and then a point of $Z-h$ if $t \geq h$) between two consecutive points of Z. Hence $x|_{(t-h,t]}$ has a finite number of zeros for $t \geq 0$ and V, defined in (9), is nondecreasing.

On the other hand, the sign of the slope of x around $t \in Z$ is the same as that of the first point of Z following $t-h$. Hence, V is even. Since V is nonincreasing with integer even values, V converges in finite time towards some limit $2n$. From now on, the proof essentially reduces to the study of discrete systems describing the evolution on each level $V \equiv 2n$, $n \in \mathbb{N}$. For advanced studies on related subjects, see [6, 11]. See also [9].

A.2 The levels $V \equiv 2n$, $n \in \mathbb{N} - \{0\}$

We consider the increasing unbounded sequence t_k of elements of Z. Here, we prove that if $V(t_k) = 2n \neq 0$ for any k greater than a certain $k_0 \geq 0$, then x is equal to the periodic solution given in Theorem 1 with period T_n^* for $t \geq t_{k_0}$. To this end, let us define the simplex $\Delta_n \stackrel{\Delta}{\equiv} \{b \in \mathbb{R}^{2n} \;:\; b_j \geq 0,\ \sum_{j=1}^{2n} b_j \leq \bar{h}\}$, and the map $\Phi_n : \Delta_n \to \mathbb{R}^{2n}$ by:

$$b' = \Phi_n(b) \text{ where } b'_j = b_{j+1} \text{ for } j \leq 2n-1,\ b'_{2n} = \ln\left(2e^{\bar{h} - \sum_1^{2n} b_j} - 1\right) .$$

The set Δ_n and the map Φ_n are respectively the state-space and the flow associated with the evolution on the level $V \equiv 2n$. In other words, defining, for $k \geq k_0$:

$$b^{k,n} \stackrel{\Delta}{\equiv} \frac{1}{\tau}(t_{k-2n+j} - t_{k-2n+j-1})_{1 \leq j \leq 2n} ,$$

we have

$$b^{k+1,n} = \Phi_n(b^{k,n}) .$$

The map Φ_n has a unique fixed point $b_n^* \in \Delta_n$, defined by $(b_n^*)_j = \frac{\bar{T}_n^*}{2}$, $j = 1, \ldots, 2n$ and corresponding to the unique periodic solution at level $V \equiv 2n$. We shall show that, for any $b \in \Delta_n$, the sequence[2] $[\Phi_n^{-1}]^k(b)$ exists (Δ_n is Φ_n^{-1}-stable) and tends to b_n^* when $k \to +\infty$. To this end, remark that

$$\Phi_n^{-1}(b) - \Phi_n^{-1}(b') = \left(\int_0^1 \nabla\Phi_n^{-1}(b' + s(b-b')) \cdot ds\right)(b - b') .$$

Now, for any $b \in \Delta_n$, one has $\nabla\Phi_n^{-1}(b) \in \mathcal{M}$, where $\mathcal{M} \stackrel{\Delta}{\equiv} \{M \in \mathbb{R}^{2n \times 2n} \;:\; M_{j+1,j} = 1, M_{1,j} = -1 \text{ for } j = 1, \ldots, 2n-1, -\dfrac{1}{1+e^{-\bar{h}}} \leq M_{1,2n} \leq -\dfrac{1}{2}, M_{j,j'} = 0 \text{ otherwise}\}$. So we deduce, since $\mathcal{M}$ and Δ_n are convex sets:

$$\forall b,\ b' \in \Delta_n, \exists M \in \mathcal{M},\ \Phi_n^{-1}(b) - \Phi_n^{-1}(b') = M(b - b') .$$

We are now led to the demonstration of an absolute stability property for a class of discrete dynamical systems. Defining a norm in $\mathbb{R}^{2n}$ by $\|b\| \stackrel{\Delta}{\equiv} \sum_j |b_j| + |\sum_j b_j|$, one shows the following result:

[2]Here, $[\Phi_n^{-1}]^k = [\Phi_n^{-1}]^{k-1} \circ \Phi_n^{-1}$, $k \in \mathbb{N} - \{0\}$, where $\circ$ denotes the composition.

Lemma 13. *The following claims are true:*

- $\forall M \in \mathcal{M}$, $\forall b \in \mathbb{R}^{2n}$, $\|Mb\| \le \|b\|$.
- $\forall M^{(k)} \in \mathcal{M}$, $k = 1, \ldots, 2n$, $\forall b \in \mathbb{R}^{2n}$, $\left\|\prod_{k=1}^{2n} M^{(k)} b\right\| = \|b\|$ *implies* $b_j b_{j+1} \ge 0$ *for* $j = 1, \ldots, 2n-1$.
- $\forall m = 1, \ldots, 2n-1$, $\forall M^{(k)} \in \mathcal{M}$, $k = 1, \ldots, 2nm$, $\forall b \in \mathbb{R}^{2n}$, $\left\|\prod_{k=1}^{2nm} M^{(k)} b\right\| = \|b\|$ *implies* $b_j b_{j+m} \ge 0$ *for* $j = 1, \ldots, 2n-m$.
- $\forall b \in \mathbb{R}^{2n}$, $\forall M^{(k)} \in \mathcal{M}$, $k = 1, \ldots, 2n(2n-1)+1$, $\left\|\prod_{k=1}^{2n(2n-1)+1} M^{(k)} b\right\| = \|b\| \Rightarrow b = 0$.

Proof. • The first point is deduced by the fact that M has the following form:

$$Mb = \begin{pmatrix} -\sum_{j=1}^{2n-1} b_j + M_{1,2n} b_{2n} \\ b_1 \\ \cdots \\ b_{2n-1} \end{pmatrix},$$

and $-1 \le M_{1,2n} \le 0$.

• Defining for the second point, $M = \prod_{k=1}^{2n} M^{(k)}$ (where the product is taken from left to right), we deduce:

$$Mb = \begin{pmatrix} (1 + M_{1,2n}^{(1)}) b_1 - M_{1,2n}^{(2)} b_2 \\ \cdots \\ (1 + M_{1,2n}^{(2n-1)}) b_{2n-1} - M_{1,2n}^{(2n)} b_{2n} \\ -\sum_{j=1}^{2n-1} b_j + M_{1,2n}^{(2n)} b_{2n} \end{pmatrix}, \tag{18}$$

and (recall that $-1 < M_{1,2n}^{(j)} < 0$):

$$\begin{aligned}
\|Mb\| &= \sum_{j=1}^{2n-1} |(1 + M_{1,2n}^{(j)}) b_j - M_{1,2n}^{(j+1)} b_{j+1}| + \left| \sum_{j=1}^{2n-1} b_j - M_{1,2n}^{(2n)} b_{2n} \right| - M_{1,2n}^{(1)} |b_1| \\
&\le \sum_{j=1}^{2n-1} \Big((1 + M_{1,2n}^{(j)}) |b_j| - M_{1,2n}^{(j+1)} |b_{j+1}| \Big) + \left| \sum_{j=1}^{2n-1} b_j - M_{1,2n}^{(2n)} b_{2n} \right| - M_{1,2n}^{(1)} |b_1| \\
&\qquad \text{(with equality if and only if } b_j b_{j+1} \ge 0,\ j = 1, \ldots, 2n-1) \\
&= \sum_{j=1}^{2n-1} |b_j| - M_{1,2n}^{(2n)} |b_{2n}| + \left| \sum_{j=1}^{2n-1} b_j - M_{1,2n}^{(2n)} b_{2n} \right| \\
&\le \|b\| .
\end{aligned}$$

• The third point is proved by induction on m. For instance, for $m = 2$,

$$\left\| \prod_{k=1}^{4n} M^{(k)} b \right\| \le \left\| \prod_{k=2n+1}^{4n} M^{(k)} b \right\| \le \|b\| ,$$

and, if the extremal expressions are equal, then, by second point,

$$b_j b_{j+1} \geq 0 \quad \text{and} \quad (Mb)_j (Mb)_{j+1} \geq 0, \ \forall j = 1, \dots, 2n-1 \ ,$$

where $Mb = \prod_{k=2n+1}^{4n} M^{(k)} b$. Using the expression of $(Mb)_j$ as in (18), and the fact that $-1 < M^{(j)}_{1,2n} < 0$, we deduce that $b_j b_{j+2} \geq 0$, $j = 1, \dots, 2n-2$.

• For the last point, we apply the third one to b and Mb, where $M = M^{(2n(2n-1)+1)}$. We then obtain that all the coefficients of b (resp. Mb) have same sign. If $b \neq 0$, then for instance $b_j \geq 0$ for any $j = 1, \dots, 2n$. Hence, $(Mb)_1 < 0$ (as $M_{1,2n} < 0$) and $(Mb)_j = b_{j-1} \leq 0$ for any $j = 2, \dots, 2n$. This implies $b_1 = \dots = b_{2n-1} = 0$, $b_{2n} > 0$, $(Mb)_1 = M_{1,2n} b_{2n} < 0$, and $\|Mb\| < \|b\|$, which contradicts the hypotheses. □

From the last property of Lemma 13 and compactnes of $\mathcal{M}$, we deduce

$$\sup_{\substack{M^{(k)} \in \mathcal{M} \\ k=1,\dots,2n(2n-1)+1}} \left\| \prod_{k=1}^{2n(2n-1)+1} M^{(k)} \right\| < 1 \ .$$

Then, $[\Phi_n^{-1}]^{2n(2n-1)+1}$ is a contraction on Δ_n. Since by hypothesis $b^{k,n} = [\Phi_n]^{k-k_0}(b^{k_0,n}) \in \Delta_n$, $k \geq k_0$, and Δ_n is bounded, then $b^{k_0,n} = b^*_n$. From this, we deduce that $x(t) = x^*_n(t-h-t_{k_0})$ or $x(t) = x^*_n(t-h-t_{k_0}-\frac{T^*_n}{2})$ for $t \geq t_{k_0}$. In any neighborhood of the periodic solution (x^*_n, y^*_n), one may construct a solution (x, y) of (3,2) with $V(0) = 2n$, for which $(b^{1,n}, \dots, b^{2n,n}) \neq (b^*_n, \dots, b^*_n)$. From what precedes, we obtain $\lim_{t\to+\infty} V(t) < 2n$. This yields the unstability of (x^*_n, y^*_n).

A.3 The level 0

Any evolution at level $V \equiv 0$ tends in a finite time towards the SO cycle. Indeed, this is clear, as this happens as soon as $V(t) = 0$: at that instant, x crosses zero and the evolution is the slowly oscillating periodic one, see also [18, 9].

The stability of the SO cycle x^*_0 follows from a continuity result similar to Lemma 5, which permits to consider time intervals $[t-h; t]$ on which x^*_0 has constant sign: in this case, any trajectory sufficiently close to x^*_0 verifies the same property, and satisfies $V(t) = 0$.

B Proof of Theorem 11

In order to prove Theorem 11, we gather some estimates in the following technical result, which gives indeed more informations than what is needed.

Proposition 14. *Under the hypotheses of Theorem* 11*, for all* $(x_0, x_{00}, y_0) \in L^1((-h(0+), 0)) \times \mathbb{R} \times \mathbb{R}$*, the following estimates hold.*

• *In the case* (P)*, we have*

$$\limsup x \leq k_P - \liminf \zeta - e^{-\frac{\|h\|_a}{\liminf \tau}} \min\{2k_P, (k_P - \liminf \zeta + \liminf \xi)^+\} \ . \tag{19}$$

Moreover, if

$$k_P > \limsup \xi - \liminf \zeta \ , \tag{20}$$

then

$$\limsup_{t'\to+\infty} \sup\{t - t' \ : \ t \geq t', \ x + \xi > 0 \text{ a.e. on } (t', t)\}$$
$$\leq (\limsup \tau) \left(\ln \left(\frac{k_P (2e^{\frac{\|h\|_a}{\liminf \tau}} - 1) + \min\{\liminf \zeta - \liminf \xi, k_P\}}{k_P + \liminf \zeta - \limsup \xi} \right) \right)^+ \ . \tag{21}$$

• *In the case* (PI)*, defining*

$$K_1 \triangleq \frac{1}{\liminf \tau}\left(\|k_I\tau - k_P\|_a + |k_P| + 2|\zeta|_a\right) \ ,$$
$$K_2 \triangleq \frac{1}{2}\left(\|k_I\tau - k_P\|_a + \limsup(k_I\tau - k_P)\right) + |\zeta|_a \ ,$$
$$K_3 \triangleq \frac{1}{2}\left(\|k_I\tau - k_P\|_a - \limsup(k_I\tau - k_P)\right) - e^{-\frac{\|h\|_a}{\liminf \tau}}\left(\|k_I\tau - k_P\|_a + |\zeta|_a\right) \ ,$$
$$\Phi(T) \triangleq -k_I(\frac{T}{4} - \|h\|_a) + K_2 + K_3^+ e^{-\frac{\frac{T}{2}-\|h\|_a}{\limsup \tau}} - K_3^- e^{-\frac{\frac{T}{2}-\|h\|_a}{\liminf \tau}} \ ,$$

we have:

$$\limsup x \leq -\liminf \xi + 2\Phi(\max\{\hat{T}, \tilde{T}, 2\|h\|_a\}) \ , \tag{22}$$
$$\limsup_{t'\to+\infty} \sup\{t - t' \ : \ t \geq t', \ x + \xi > 0 \text{ a.e. on } (t', t)\} \leq \frac{T^+}{2} \ ,$$

where $\hat{T}, \tilde{T}, T^+$ *are defined by*

$$\Phi(\hat{T}) = K_1\frac{\hat{T}}{4}, \quad \hat{T} > 0 \ ,$$
$$\frac{\tilde{T}}{2} \triangleq \|h\|_a + \liminf \tau \ \ln\left(\frac{2K_3^-}{k_I \liminf \tau}\right) \ ,$$
$$\Phi(T^+) = -|\xi|_a, \quad T^+ > 0 \ . \tag{23}$$

Proof.

• We begin with the case (PI). We have

$$\tau(t)(\dot{x} + k_I\dot{y}) + (x + k_I y) = (k_I\tau - k_P)\dot{y} - \zeta \ . \tag{24}$$

Hence, using the fact that $\frac{1}{\tau} \notin L^1((0, +\infty))$, we get:

$$-\|k_I\tau - k_P\|_a - \limsup \zeta \leq \liminf(x + k_I y) \leq \limsup(x + k_I y) \leq \|k_I\tau - k_P\|_a - \liminf \zeta \ , \tag{25}$$
$$\limsup \dot{x} \leq \left\|\frac{1}{\tau}\right\|_a (\|k_I\tau - k_P\|_a + |k_P| + 2|\zeta|_a) = K_1 \ . \tag{26}$$

Define $X \triangleq x + \xi$. Firstly, there is no unbounded time-interval on which e.g. $X > 0$ almost everywhere. Indeed, we would have $\dot{y} = 1$ a.e. on an unbounded interval, so $x(t) \to -\infty$ when $t \to +\infty$, due to the boundedness of ζ. The boundedness of ξ then implies that $X(t) \to -\infty$, which contradicts the hypothesis.

Secondly, for any $\varepsilon > 0$, there is a $t_0 > 0$ such that the lim sup and lim inf which will be involved in the sequel are approached up to ε for $t \geq t_0$. As we are interested in the asymptotic behavior only, we shall omit in the following the ε's, for sake of simplicity.

Let t', t be such that $(t_0 \leq)t' < t$ and $X > 0$ almost everywhere on (t', t). We may indeed suppose without loss of generality that $t' = \inf\{s \ : \ X > 0 \text{ a.e. on } (s, t)\}$. This implies, due to the continuity of x, that

$$x(t') \leq -\liminf \xi, \ x(t) \geq -\limsup \xi \ . \tag{27}$$

Let us define

$$t'' \triangleq \sup\{s \in [0, t] \ : \ s - h(s) \leq t'\} \ .$$

One has $t' < t'' \leq t$ and $t'' - h(t''^-) \leq t'$. Then,

$$y(t) - y(t') = \int_{t'}^{t''} \text{sgn}(s, X(s - h(s))) \ ds + t - t'' \geq t - t'' - \|h\|_a \ , \tag{28}$$

using the fact that $X(s-h(s))>0$ a.e. on (t'',t). From (24), we deduce:

$$(x+k_Iy)(t)=e^{-\int_{t'}^t \frac{1}{\tau}}(x+k_Iy)(t')+\int_{t'}^t \frac{1}{\tau(s)}e^{-\int_s^t \frac{1}{\tau}}((k_I\tau-k_P)\mathrm{sgn}(s,X(s-h(s)))-\zeta(s))\ ds\ . \qquad (29)$$

From inequalities (25) and (28), we obtain independently

$$\begin{aligned}
&e^{-\int_{t'}^t \frac{1}{\tau}}(x+k_Iy)(t')-k_Iy(t)-x(t')\\
&= \quad k_I(y(t')-y(t))-(x(t')+k_Iy(t'))\left(1-e^{-\int_{t'}^t \frac{1}{\tau}}\right)\\
&\leq \quad k_I(\|h\|_a-(t-t''))+(\|k_I\tau-k_P\|_a+\limsup\zeta)(1-e^{-\int_{t'}^{t''} \frac{1}{\tau}}\ e^{-\int_{t''}^t \frac{1}{\tau}})\ .
\end{aligned}$$

Also,

$$\begin{aligned}
&\int_{t'}^t \frac{1}{\tau(s)}e^{-\int_s^t \frac{1}{\tau}}((k_I\tau-k_P)\mathrm{sgn}(s,X(s-h(s)))-\zeta(s))\ ds\\
&\leq \quad -(\liminf\zeta)(1-e^{-\int_{t'}^t \frac{1}{\tau}})+\limsup(k_I\tau-k_P)(1-e^{-\int_{t''}^t \frac{1}{\tau}})\\
&\qquad +\|k_I\tau-k_P\|_a(e^{-\int_{t''}^t \frac{1}{\tau}}-e^{-\int_{t'}^t \frac{1}{\tau}})\\
&= \quad \Big(-\liminf\zeta+\limsup(k_I\tau-k_P)\Big)\\
&\qquad +\Big(\|k_I\tau-k_P\|_a-\limsup(k_I\tau-k_P)-(\|k_I\tau-k_P\|_a-\liminf\zeta)e^{-\int_{t'}^{t''} \frac{1}{\tau}}\Big)\,e^{-\int_{t''}^t \frac{1}{\tau}}\ .
\end{aligned}$$

Adding the two inequalities and using (22), we deduce that:

$$\begin{aligned}
&x(t)-x(t')\\
&\leq \quad k_I(\|h\|_a-(t-t''))+\Big(\|k_I\tau-k_P\|_a+\limsup(k_I\tau-k_P)+2|\zeta|_a\Big)\\
&\qquad +\Big(\|k_I\tau-k_P\|_a-\limsup(k_I\tau-k_P)-2e^{-\int_{t'}^{t''} \frac{1}{\tau}}(\|k_I\tau-k_P\|_a+|\zeta|_a)\Big)\,e^{-\int_{t''}^t \frac{1}{\tau}}\\
&= \quad k_I(\|h\|_a-(t-t''))+2K_2+2K_3e^{-\int_{t''}^t \frac{1}{\tau}}\\
&\leq \quad k_I(\|h\|_a-(t-t''))+2K_2+2K_3^+e^{-\frac{t-t''}{\limsup\tau}}-2K_3^-e^{-\frac{t-t''}{\liminf\tau}}\\
&= \quad 2\Phi(2(t-t''+\|h\|_a))\ ,
\end{aligned}$$

where K_2,K_3,Φ are defined in the statement of the proposition. Now, (27) implies that $x(t)-x(t')\geq -2|\xi|_a$, so $t-t'\leq t-t''+\|h\|_a\leq \frac{T^+}{2}$, with T^+ defined by formula (23).

Using (26), we get $x(t)-x(t')\leq K_1(t-t')\leq K_1(t-t''+\|h\|_a)$. Hence,

$$x(t)-x(t')\leq \sup_{T\geq 2\|h\|_a}\min\{K_1\frac{T}{2},2\Phi(T)\}\ .$$

Using the fact that Φ decreases on $[\tilde{T},+\infty)$, we obtain

$$\sup_{T\geq 2\|h\|_a}\min\{K_1\frac{T}{2},2\Phi(T)\}=2\Phi(\max\{\hat{T},\tilde{T},2\|h\|_a\})\ ,$$

and the statement of the proposition follows from (27).

• Let us now consider the case (P). With the same techniques as before, we estimate (29) and get:

$$x(t) \leq x(t')e^{-\int_{t'}^{t}\frac{1}{\tau}} - (\liminf \zeta + k_P)(1 - e^{-\int_{t'}^{t}\frac{1}{\tau}}) + 2k_P(e^{-\int_{t''}^{t}\frac{1}{\tau}} - e^{-\int_{t'}^{t}\frac{1}{\tau}}) \ .$$

From this, we deduce

$$\begin{aligned}
&\liminf \zeta + k_P - \limsup \xi \\
&\quad \leq \liminf \zeta + k_P + x(t) \\
&\quad \leq \left(x(t') + \liminf \zeta + k_P + 2k_P(e^{\int_{t'}^{t''}\frac{1}{\tau}} - 1)\right) e^{-\int_{t'}^{t}\frac{1}{\tau}} \\
&\quad \leq \left(\min\{-\liminf \xi, k_P - \liminf \zeta\} + \liminf \zeta + k_P(2e^{\int_{t'}^{t''}\frac{1}{\tau}} - 1)\right) e^{-\int_{t'}^{t}\frac{1}{\tau}} \ .
\end{aligned}$$

Under the hypothese (20) and using again $\int_{t'}^{t''}\frac{1}{\tau} \leq \frac{\|h\|_a}{\liminf \tau}$, we get (21).

To get (19), we deduce from the previous inequality:

$$\begin{aligned}
\liminf \zeta + k_P + x(t) &\leq \left(\min\{k_P + \liminf \zeta - \liminf \xi,\ 2k_P\} + 2k_P(e^{\int_{t'}^{t''}\frac{1}{\tau}} - 1)\right) e^{-\int_{t'}^{t}\frac{1}{\tau}} \\
&\leq \min\{(k_P + \liminf \zeta - \liminf \xi)^+,\ 2k_P\}e^{-\int_{t'}^{t''}\frac{1}{\tau}} + 2k_P(1 - e^{-\int_{t'}^{t''}\frac{1}{\tau}}) \ ,
\end{aligned}$$

remarking that the largest value of the right-hand side is attained for $t = t''$. We then use the bound on $\int_{t'}^{t''}\frac{1}{\tau}$. □

We now achieve the proof of Theorem 11.

Results for $\liminf x$ similar to (19) and (22) are obtained by changing x, ζ, ξ into their opposite; one hence gets estimates for $\|x\|_a$ and $|x|_a$. At last, one verifies that the obtained upper bounds are nondecreasing functions of $|\zeta|_a, |\xi|_a, \|\zeta\|_a, \|\xi\|_a$, which permits to prove Theorem 11. This last step is clear for the case (P). For the case (PI), one uses the fact that

$$2\Phi(\max\{\hat{T}, \tilde{T}, 2\|h\|_a\}) = \sup_{T \geq 2\|h\|_a} \min\{K_1\frac{T}{2}, 2\Phi(T)\} \ ,$$

where K_1 and $\Phi(T)$ are nondecreasing functions of $|\zeta|_a$ when $T \geq 2\|h\|_a$.

The form under which are presented the results is close to the one used for the period in the unperturbed case (see formula (5) above). Remark that in the case (P), the estimates are optimal for the unperturbed system: they yield $\limsup x \leq \|x_0^*\|_\infty$ and $\limsup_{t' \to +\infty} \sup\{t - t' \ : \ t \geq t',\ x > 0 \text{ a.e. on } (t', t)\} \leq \frac{T_0^*}{2}$, where x_0^* is the slowly oscillating periodic solution and T_0^* its period (see Corollary 3). In the case (PI), even in the unperturbed case, the estimates are less accurate, as T^+ defined by (23) is greater than T_0^*. Indeed, computations are difficult, due to the fact that there exist slowly oscillating *non periodic* solutions (which tend asymptotically to x_0^*) [3], contrary to the case (P), where the convergence is effective after the first zero.

References

[1] M. Akian, P.-A. Bliman, On super-high-frequencies in discontinuous 1st-order delay-differential equations, *in Proc. of 6th IEEE Mediterranean Conference on Control and Systems*, Alghero (Italy), 9-11 June, 1998.

[2] M. Akian, P.-A. Bliman, On super-high-frequencies in discontinuous 1st-order delay-differential equations, *J. Diff. Equations* **162**, no 2 (2000), 326–358.

[3] K. Aouchiche, P.-A. Bliman, M. Sorine, P.I. control of periodic oscillations of relay systems, *Conference on Control of Oscillations and Chaos*, St-Petersburg, Russia, August 27-29 (1997).

[4] K.J. Åström, T. Hägglund, Automatic tuning of simple regulators, *Proc. of IFAC 9th World congress*, Budapest (1984).

[5] K.J. Åström, B. Wittenmark, *Adaptive control*, Addison-Wesley Publishing Company (1989).

[6] O. Diekmann, S.A. van Gils, S.M. Verduyn Lunel, H.-O. Walther, *Delay equations: functional-, complex- and nonlinear analysis*, Applied Mathematical Sciences 110, Springer-Verlag, New-York, 1995.

[7] G.F. Franklin, J.D. Powell, A. Emami-Naeini, *Feedback control of dynamic systems*, Addison-Wesley Publishing Company (1988).

[8] L. Fridman, E. Fridman, E. Shustin, Steady modes in an autonomous system with break and delay, *Differential equations* (translation of *Differentsial'nye uravneniya*), vol. 29, no 8 (1993), 1161–1166.

[9] L. Fridman, E. Shustin, E. Fridman, Steady modes and sliding modes in the relay control systems with time delay, *Proc. 35th Conf. on Decision and Control*, Kobe, Japan (1996), 4601–4606.

[10] L. Fridman, E. Fridman, E. Shustin, The phenomenon of the second order steady modes in the relay control systems with time delay, *Proc. of the 4th European Control Conference*, Brussels, Belgium (1997).

[11] J. Mallet-Paret, Morse decompositions for delay-differential equations, *J. of Diff. Equations* **72** (1988), 270–315.

[12] R.D. Nussbaum, Periodic solutions of nonlinear autonomous functional differential equations, *Functional differential equations and approximation of fixed points*, Lecture Notes in Mathematics 730, Springer-Verlag (1979), 283–325.

[13] N. Rouche, J. Mawhin, *Equations différentielles ordinaires*, Masson Paris (1973).

[14] T. Seidman, The dynamics of some systems with hysteretic nonlinearities in *Optimization and Nonlinear Analysis*, (Ioffe, Marcus, Reich), Longman, Harlow (1992), 273–285.

[15] E. Shustin, Super-high-frequency oscillations in a discontinuous dynamic system with time delay, *Israel J. of Mathematics*, **90** (1995), 199–219.

[16] E. Shustin, E. Fridman, L. Fridman, Stable oscillations in a discontinuous delay system of the second order, *Proc. Int. Conf. on Differential equations and dynamical systems*, Springfield, Missouri, May 29-June 1 (1996).

[17] Ya.Z. Tsypkin, *Relay control systems*, Cambridge University Press (1984).

[18] X. Xie, Uniqueness and stability of SOP solutions of delay equations with bounded nonlinearity, *Journal of dynamics and differential equations* **3**, n°4 (1991), 515–540.