An adaptive hp-refinement strategy with inexact solvers and computable guaranteed bound on the error reduction factors

◀ WONAPDE 2019: 6th Chilean Workshop on Numerical Analysis of PDEs ▶

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École des Ponts ParisTech

European Research Council

References

- W. DÖRFLER, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. (1996)
- P. MORIN, R. H. NOCHETTO, AND K. G. SIEBERT, *Data oscillation and convergence of adaptive FEM*, SIAM J. Numer. Anal. (2000)

- P. BINEV, W. DAHMEN, AND R. DEVORE, *Adaptive FEM with convergence rates*, Numer. Math. (2004)
- R. P. STEVENSON, *The uniform saturation property for a singularly perturbed reaction-diffusion equation*, Numer. Math. (2005)

  R. STEVENSON, *Optimality of a standard adaptive finite element method*, F. Comp. Math., (2007).
- M. ARIOLI, E. H. GEORGOULIS, AND D. LOGHIN, *Stopping criteria for adaptive finite element solvers*, SIAM J. Sci. Comput. (2013)

- C. CARSTENSEN, M. FEISCHL, M. PAGE, AND D. PRAETORIUS, *Axioms of adaptivity*, CAMWA (2014)
- G. GANTNER, A. HABERL, D. PRAETORIUS, AND B. STIFTNER, *Rate optimal adaptive FEM with inexact solver for nonlinear operators*, IMA J. Numer. Anal., (2017)
- C. CANUTO, R. H. NOCHETTO, R. STEVENSON, AND M. VERANI, *Convergence and optimality of* $\mathbf{hp}$-**AFEM**, Numer. Math. (2017) & *On p-robust saturation for hp-AFEM*, CAMWA (2017)

Outline

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An adaptive hp-refinement strategy **with exact solver**

An adaptive hp-refinement strategy **with inexact solver**

Convergence of adaptive hp-refinement strategies

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- P. Daniel, A. Ern, I. Smears, and M. Vohralík, *An adaptive hp-refinement strategy with computable guaranteed bound on the error reduction factor*, Comput. Math. Appl., (2018)
- P. Daniel, A. Ern, and M. Vohralík, *An adaptive hp-refinement strategy with inexact solvers and computable guaranteed bound on the error reduction factor.* HAL preprint 01931448, 2018
- P. Daniel, and M. Vohralík, *Convergence of adaptive hp-refinement strategies with computable guaranteed bound on the error reduction factor*, In Preparation

SOLVE → ESTIMATE → MARK → REFINE

Error reduction property: $\left\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\right\| \leq C_{\ell,\mathrm{red}} \left\|\nabla(u - u_{\ell}^{\mathrm{ex}})\right\|$

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$$\texttt{SOLVE} \rightarrow \texttt{ESTIMATE} \rightarrow \texttt{MARK} \rightarrow \texttt{REFINE}$$

$$\|\nabla(u - u_\ell^{\mathrm{ex}})\| \leq C_{\ell,\mathrm{red}} \,\|\nabla(u - u_\ell^{\mathrm{ex}})\| \quad \textbf{and} \quad 0 \leq C_{\ell,\mathrm{red}} \leq C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}} < 1$$

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$$\|\nabla(u-u_{\ell+1})\| \leq C_{\ell,\mathrm{red}} \|\nabla(u-u_\ell)\| \quad \textbf{and} \quad 0 \leq C_{\ell,\mathrm{red}} \leq C_{\theta,\tilde{\gamma}_\ell,d,\kappa_{\mathcal{T}},p_{\max}} < 1$$

Model problem – setting

Poisson equation with (homogeneous) Dirichlet boundary conditions

For $f \in L^2(\Omega)$, seek $u \in H_0^1(\Omega)$

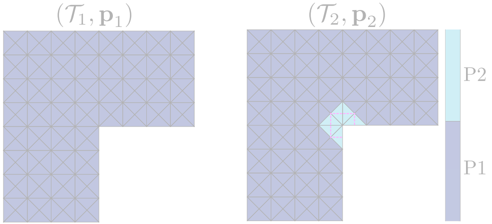
$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Conforming hp–finite element method: (initialize $\ell := 0$)

$$(\nabla u_\ell^{\mathrm{ex}}, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell, \qquad V_\ell := \mathbb{P}_{\mathbf{p}}(\mathcal{T}_\ell) \cap H_0^1(\Omega), \quad \forall \ell \geq 0$$

Built up on the pair $(\mathcal{T}_\ell, \mathbf{p}_\ell)$, $\ell \geq 0$:

- matching simplicial mesh $\mathcal{T}_\ell$
- $\mathbf{p}_\ell := \{p_{\ell,K}\}_{K \in \mathcal{T}_\ell}$
- $p_{\ell,K} \leq p_{\max}$, $\forall K \in \mathcal{T}_\ell$, $\ell \geq 0$



Model problem – setting

Poisson equation with (homogeneous) Dirichlet boundary conditions

For $f \in L^2(\Omega)$, seek $u \in H^1_0(\Omega)$

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_0(\Omega)$$

Conforming hp–finite element method: (initialize $\ell := 0$)

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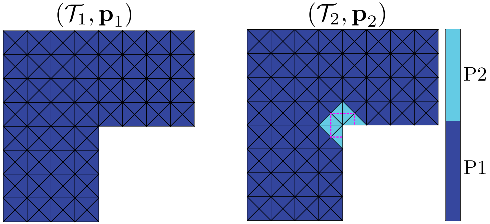

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Module SOLVE

FEM:

$$(\nabla u_\ell^{\mathrm{ex}}, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell \qquad \Longleftrightarrow \qquad \mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$$

- $\mathrm{U}_\ell^{\mathrm{ex}}$ corresponds to the exact FEM solution $u_\ell^{\mathrm{ex}} = \sum_{n=1}^{N_\ell} (\mathrm{U}_\ell^{\mathrm{ex}})_n \, \psi_\ell^n$

Galerkin orthogonality (**exact setting**)

$$\left\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\right\|^2 = \left\|\nabla(u - u_\ell^{\mathrm{ex}})\right\|^2 - \left\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_\ell^{\mathrm{ex}})\right\|^2$$

Module SOLVE

FEM:

$$(\nabla u_\ell^{\mathrm{ex}}, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell \qquad \Longleftrightarrow \qquad \mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$$

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Module $\boxed{\texttt{ESTIMATE}}$: Guaranteed upper bound on the energy error

$$\|\nabla(u - u_\ell^{\mathrm{ex}})\| \le \eta(u_\ell^{\mathrm{ex}}, \mathcal{T}_\ell) := \left\{ \sum_{K\in\mathcal{T}_\ell} \eta_K^2 \right\}^{\frac{1}{2}}, \ \eta_K := \|\nabla u_\ell^{\mathrm{ex}} + \boldsymbol{\sigma}_\ell\|_K + \frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_\ell\|_K.$$

Equilibrated flux reconstruction $\boldsymbol{\sigma}_\ell := \sum_{\mathbf{a}\in\mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

For each vertex $\mathbf{a} \in \mathcal{V}_\ell$, we solve a small minimization problem

$$\boldsymbol{\sigma}_\ell^{\mathbf{a}} := \arg \min_{\mathbf{v}_\ell\in\mathbf{V}_\ell^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_\ell = \Pi_{Q_\ell^{\mathbf{a}}}(f\psi_\ell^{\mathbf{a}} - \nabla u_\ell^{\mathrm{ex}}\cdot\nabla\psi_\ell^{\mathbf{a}})} \|\psi_\ell^{\mathbf{a}}\nabla u_\ell^{\mathrm{ex}} + \mathbf{v}_\ell\|_\omega^{\mathbf{a}}.$$

with properly chosen local *Raviart–Thomas–Nédélec* mixed finite element spaces $\mathbf{V}_\ell^{\mathbf{a}} \times Q_\ell^{\mathbf{a}}$ of order $p_{\mathbf{a}} := \max_{K\in\mathcal{T}_\ell^{\mathbf{a}}} p_K$.

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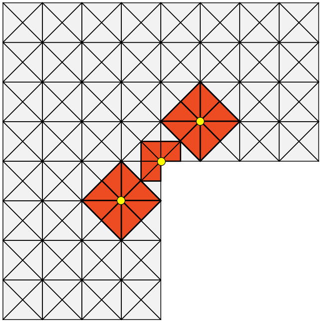
Module MARK

A bulk chasing criterion for marking vertices

For a fixed threshold parameter $\theta \in (0,1]$, the set of **marked vertices** $\widetilde{\mathcal{V}}_\ell^\theta \subset \mathcal{V}_\ell$ is selected in such a way that

$$\underbrace{\eta\Big(u_\ell^{\mathrm{ex}}, \bigcup_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}}\Big)}_{\text{estimated error in marked region}} \geq \underbrace{\theta\, \eta(u_\ell^{\mathrm{ex}}, \mathcal{T}_\ell)}_{\text{bulk of the estimated total error}}$$



- ● $\rightarrow$ **marked vertices** $\widetilde{\mathcal{V}}_\ell^\theta$
- ▲ $\rightarrow$ marked elements $\mathcal{M}_\ell^\theta := \bigcup_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}}$

Set $\omega_\ell := \cup_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \omega_\ell^{\mathbf{a}}$, the open subdomain corresponding to $\mathcal{M}_\ell^\theta$

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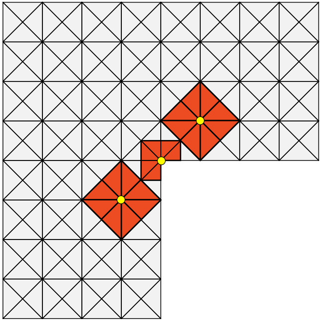

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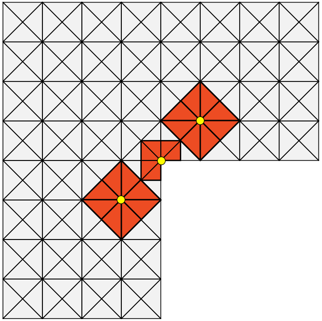

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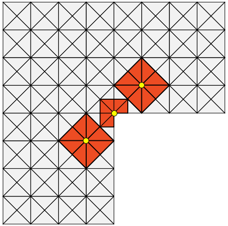
REFINE: residual liftings & local hp-decision criterion $\to (\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$

Two local FE problems on each patch $\mathcal{T}_\ell^{\mathbf{a}}$ attached to a **marked vertex $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell^\theta$**



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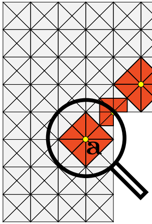

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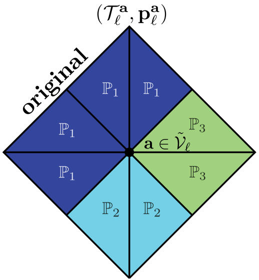

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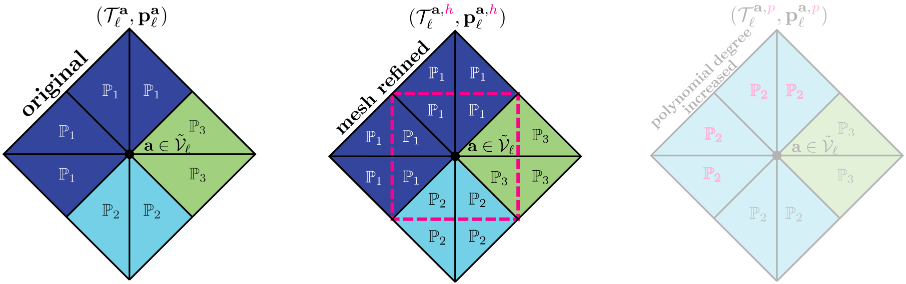

$$(\nabla r^{\mathbf{a},h}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} \quad \forall\, v^{\mathbf{a},h} \in \boxed{V_\ell^{\mathbf{a},h} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},h}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H_0^1(\omega_\ell^{\mathbf{a}})}$$

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Local hp-decision criterion is then based on $\|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^{\mathbf{a}}}$, $\|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^{\mathbf{a}}}$, $\forall \mathbf{a} \in \widetilde{\mathcal{V}}_\ell^\theta$

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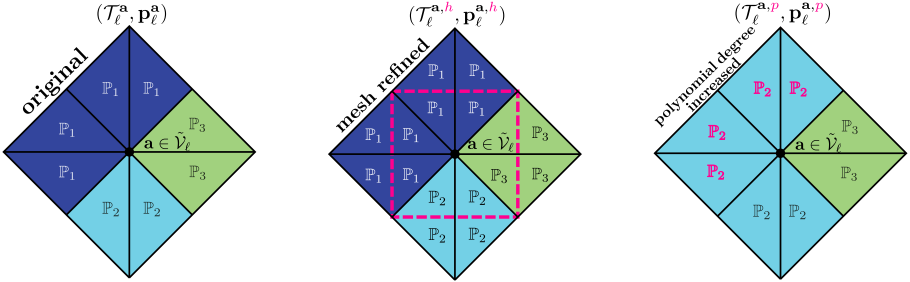

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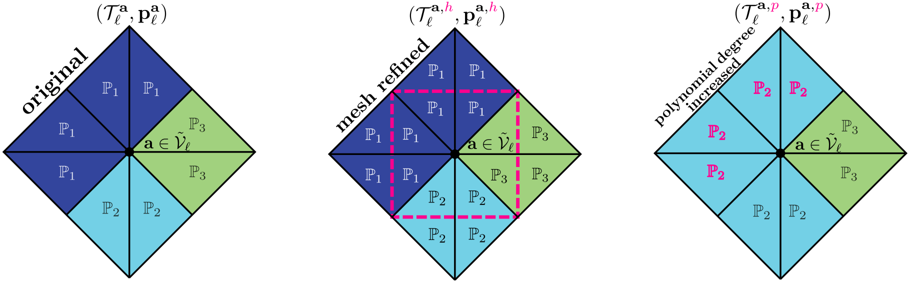

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Error reduction factor I.

SOLVE → ESTIMATE → MARK → REFINE → SOLVE

Goal: $\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\| \leq C_{\ell,\mathbf{red}} \|\nabla(u - u_{\ell}^{\mathrm{ex}})\|$

Error reduction factor I.

$$\begin{tikzcd} \texttt{SOLVE} \arrow[r] & \texttt{ESTIMATE} \arrow[r] & \texttt{MARK} \arrow[r] & \texttt{REFINE} \arrow[lll, bend right, "\boxed{C_{\ell,\mathrm{red}}}"'] \end{tikzcd}$$

Goal: $\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\| \leq C_{\ell,\mathrm{red}} \|\nabla(u - u_\ell^{\mathrm{ex}})\|$

Error reduction factor I.

$\boxed{C_{\ell,\mathrm{red}}}$

SOLVE → ESTIMATE → MARK → REFINE

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Error reduction factor I.

$$
\begin{tikzcd}
\texttt{SOLVE} \arrow[r] & \texttt{ESTIMATE} \arrow[r] & \texttt{MARK} \arrow[r] & \texttt{REFINE} \arrow[lll, bend right, "\boxed{C_{\ell,\mathrm{red}}}"']
\end{tikzcd}
$$

Goal: $\|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\ell+1}^{\mathbf{ex}})\| \le \underbrace{C_{\ell,\mathbf{red}}}_{\textit{fully computable}} \|\nabla(\boldsymbol{u} - u_\ell^{\mathrm{ex}})\|$

Error reduction factor I.

$$\boxed{C_{\ell,\mathrm{red}}}$$

SOLVE → ESTIMATE → MARK → REFINE

Goal: $\|\nabla(\underbrace{u - u_{\ell+1}^{\mathrm{ex}}}_{\textit{both unknown}})\| \leq \underbrace{C_{\ell,\mathrm{red}}}_{\textit{fully computable}} \|\nabla(u - u_{\ell}^{\mathrm{ex}})\|$

Error reduction factor I.

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Goal: $$\|\nabla(\underbrace{\boldsymbol{u} - \boldsymbol{u}_{\ell+1}^{\mathbf{ex}}}_{\textit{both unknown}})\| \leq \underbrace{C_{\ell,\mathbf{red}}}_{\textit{fully computable}} \underbrace{\|\nabla(\boldsymbol{u} - u_\ell^{\mathrm{ex}})\|}_{\textit{only } u_\ell^{\mathrm{ex}} \text{ is known}}$$

Residual liftings & guaranteed lower bound on $\left\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_\ell^{\mathrm{ex}})\right\|_{\omega_\ell}$

Recall: $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$ and space $V_{\ell+1}$ **at our disposal**!

One more local problem per marked vertex $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell^\theta$ (residual lifting)

Define a local space $V_\ell^{\mathbf{a},hp} := V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}} \cap H_0^1(\omega_\ell^{\mathbf{a}})$ and solve

$$(\nabla r^{\mathbf{a},hp}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}.$$

Then, if $\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} r^{\mathbf{a},hp} \neq 0$, we have the **guaranteed lower bound**:

$$\|\nabla(\underbrace{u_{\ell+1}^{\mathrm{ex}} - u_\ell^{\mathrm{ex}}}_{\text{only } u_\ell^{\mathrm{ex}} \text{ is known}})\|_{\omega_\ell} \geq \frac{\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \left\|\nabla r^{\mathbf{a},hp}\right\|^2_{\omega_\ell^{\mathbf{a}}}}{\left\|\nabla\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} r^{\mathbf{a},hp}\right)\right\|_{\omega_\ell}} =: \eta_{\mathcal{M}_\ell^\theta}$$

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Guaranteed lower bound on $\left\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_{\ell}^{\mathrm{ex}})\right\|_{\omega_\ell}$ (proof)

Discrete lower bound $\underline{\eta}_{\mathcal{M}_\ell^\theta}$

We use crucially

$$\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_{\ell}^{\mathrm{ex}})\| \geq \|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_{\ell}^{\mathrm{ex}})\|_{\omega_\ell} \geq \frac{\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \left\|\nabla r^{\mathbf{a},hp}\right\|_{\omega_\ell^{\mathbf{a}}}^2}{\left\|\nabla\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} r^{\mathbf{a},hp}\right)\right\|_{\omega_\ell}} =: \underline{\eta}_{\mathcal{M}_\ell^\theta}$$

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To finish take $\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} r_{\mathbf{a}}^{hp}\right)$ as test function $v_{\ell+1}$

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Error contraction property

Guaranteed bound on the error reduction factor

The new (*unknown*) numerical solution $u_{\ell+1}^{\mathrm{ex}} \in V_{\ell+1}$ satisfies:

$$\left\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\right\| \leq C_{\ell,\mathrm{red}} \left\|\nabla(u - u_{\ell}^{\mathrm{ex}})\right\| \text{ with } 0 < C_{\ell,\mathrm{red}} := \sqrt{1 - \theta^2 \frac{\underline{\eta}^2_{\mathcal{M}_\ell^\theta}}{\eta^2(u_\ell^{\mathrm{ex}}, \mathcal{M}_\ell^\theta)}} \leq 1$$

Proof (sketch)

1. Garlerking orthogonality
$$\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\|^2 = \|\nabla(u - u_{\ell}^{\mathrm{ex}})\|^2 - \underbrace{\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_{\ell}^{\mathrm{ex}})\|^2}_{\geq = \frac{\underline{\eta}^2_{\mathcal{M}_\ell^\theta}}{\eta^2(u_\ell^{\mathrm{ex}}, \mathcal{M}_\ell^\theta)} \eta^2(u_\ell^{\mathrm{ex}}, \mathcal{M}_\ell^\theta)}$$
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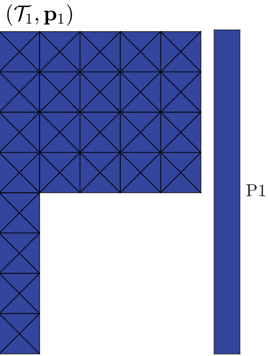
Numerics I.

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

- singular exact solution (in polar coordinates): $u(r,\varphi) = r^{\frac{2}{3}} \sin \frac{2\varphi}{3}$



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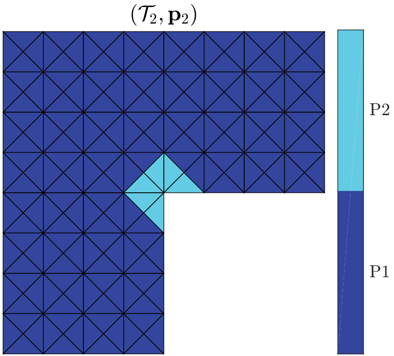

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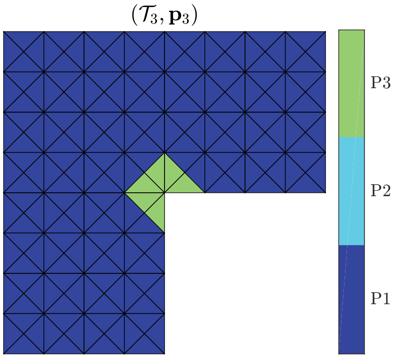

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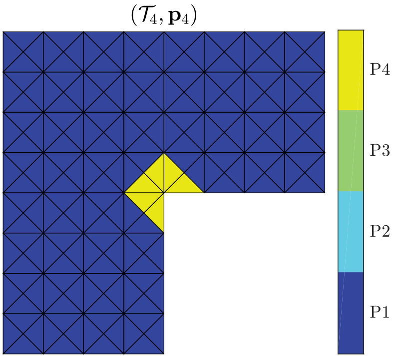

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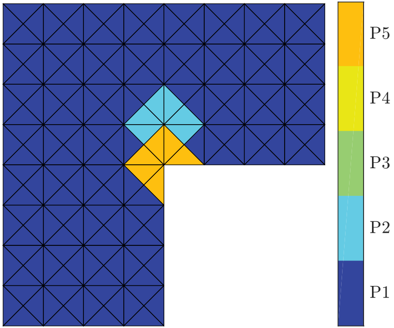

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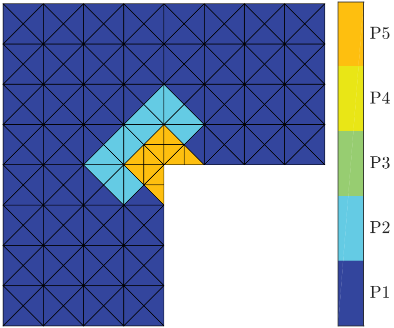

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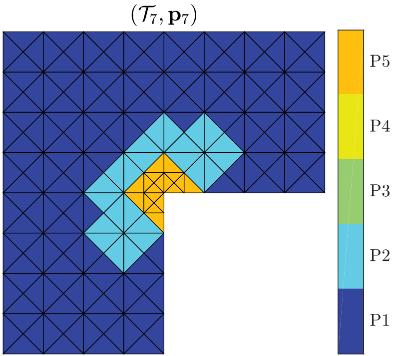
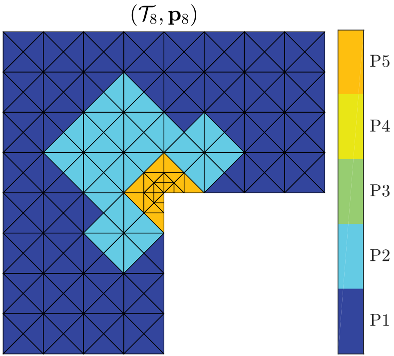


Numerics I.

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

- singular exact solution (in polar coordinates): $u(r,\varphi) = r^{\frac{2}{3}} \sin \frac{2\varphi}{3}$



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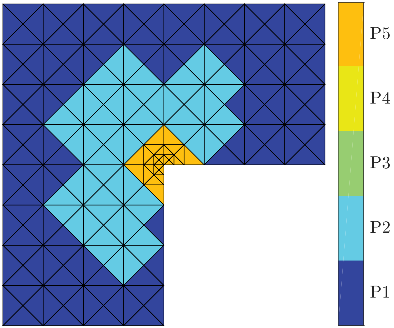
$(\mathcal{T}_9, \mathbf{p}_9)$



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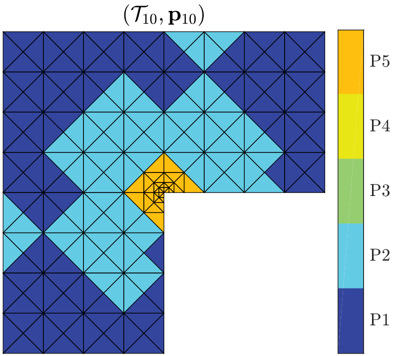

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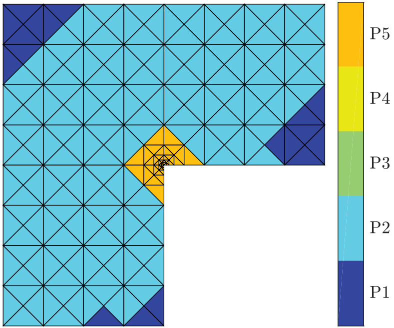
$(\mathcal{T}_{15}, \mathbf{p}_{15})$



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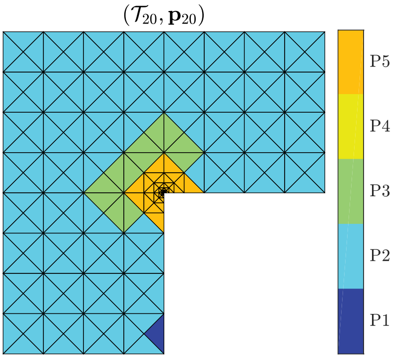

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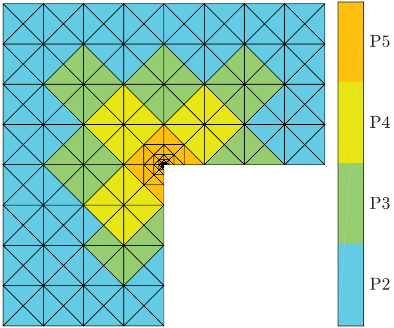
$(\mathcal{T}_{30}, \mathbf{p}_{30})$



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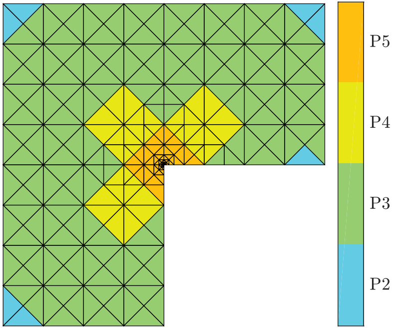
$(\mathcal{T}_{40}, \mathbf{p}_{40})$



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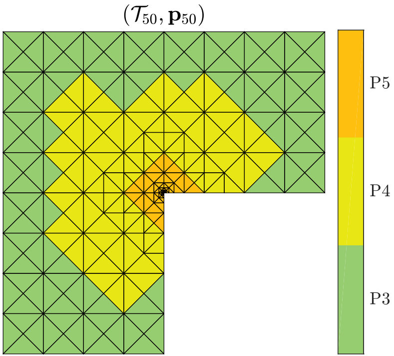

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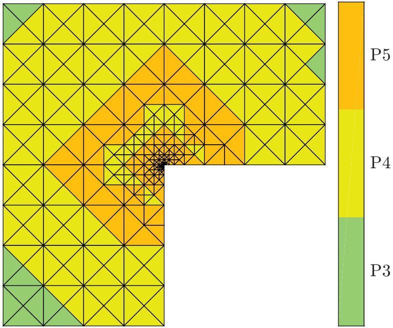
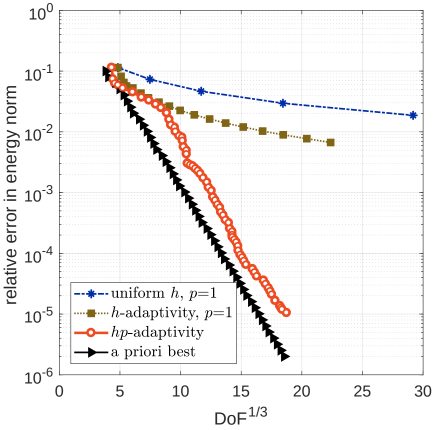
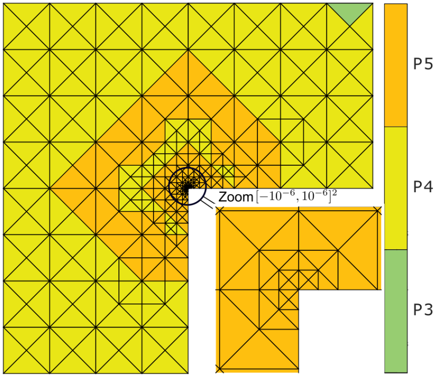
$(\mathcal{T}_{60}, \mathbf{p}_{60})$



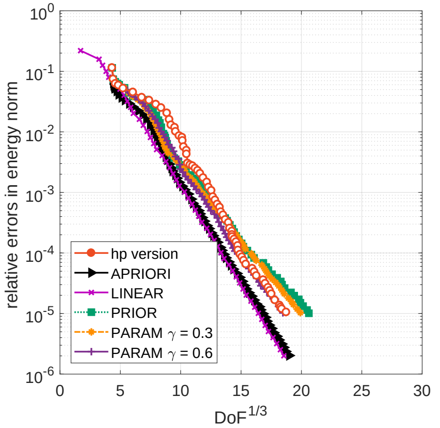
Numerics – exponential convergence

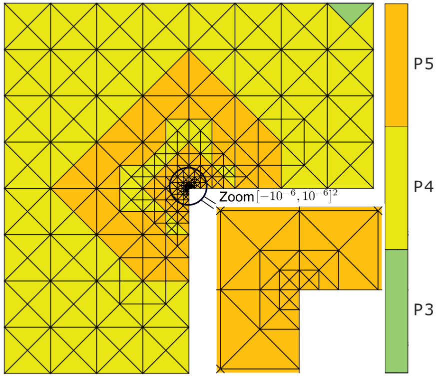




Obtained exponential convergence in **comparison with classical approaches** and the final mesh with polynomial degree distribution $(\mathcal{T}_{65}, \mathbf{p}_{65})$.

Numerics – exponential convergence

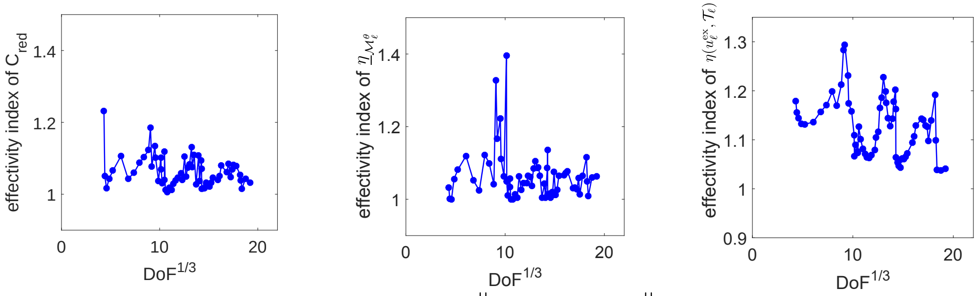




Obtained exponential convergence in **comparison with other hp-adaptive approaches** and the final mesh with polynomial degree distribution $(\mathcal{T}_{65}, \mathbf{p}_{65})$.

Numerics II.

Effectivity indices of the estimated error reduction factor $C_{\ell,\mathrm{red}}$ and $\underline{\eta}_{\mathcal{M}_\ell^\theta}$:



$$I_{\mathrm{red}}^{\mathrm{eff}} = \frac{C_{\ell,\mathrm{red}}}{\left\|\nabla(u-u_{\ell+1}^{\mathrm{ex}})\right\| / \left\|\nabla(u-u_{\ell}^{\mathrm{ex}})\right\|} \qquad I_{\mathrm{LB}}^{\mathrm{eff}} = \frac{\left\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_{\ell}^{\mathrm{ex}})\right\|_{\omega_\ell}}{\underline{\eta}_{\mathcal{M}_\ell^\theta}} \qquad I_{\eta}^{\mathrm{eff}} = \frac{\eta(u_\ell^{\mathrm{ex}},\mathcal{T}_\ell)}{\left\|\nabla(u-u_\ell^{\mathrm{ex}})\right\|}$$

Outline

Introduction

An adaptive hp-refinement strategy **with exact solver**

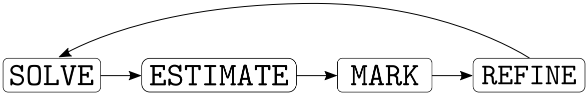
An adaptive hp-refinement strategy **with inexact solver**

Convergence of adaptive hp-refinement strategies

Extension to the **inexact algebraic solver setting**

Goal:

- avoid the *unrealistic* exact solution of $\mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$



$$|\!|\!|\nabla(\boldsymbol{u} - u_{\ell+1}^{\mathrm{ex}})|\!|\!| \le C_{\ell,\mathbf{red}} |\!|\!|\nabla(\boldsymbol{u} - u_{\ell}^{\mathrm{ex}})|\!|\!|, \quad 0 \le C_{\ell,\mathbf{red}} \le 1$$

- only *approximate* solution $\mathbb{A}_\ell \mathrm{U}_\ell \approx \mathrm{F}_\ell$ (corresponding $u_\ell \approx u_\ell^{\mathrm{ex}}$)

- recover the **contraction property** also in the inexact setting

Extension to the **inexact algebraic solver setting**

Goal:

- avoid the *unrealistic* exact solution of $\mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$ ✓

$$\begin{tikzcd} \texttt{SOLVE} \arrow[r] & \texttt{ESTIMATE} \arrow[r] & \texttt{MARK} \arrow[r] & \texttt{REFINE} \arrow[lll, bend right] \end{tikzcd}$$

$$\left\|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\ell+1}^{\mathrm{ex}})\right\| \le \boldsymbol{C}_{\ell,\mathbf{red}} \left\|\nabla(\boldsymbol{u} - u_\ell^{\mathrm{ex}})\right\|, \qquad 0 \le \boldsymbol{C}_{\ell,\mathbf{red}} \le 1$$

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$$\begin{tikzcd} \boxed{\texttt{ONE\_SOLVER\_STEP} \to \texttt{ESTIMATE}} \arrow[r] & \texttt{MARK} \arrow[r] & \texttt{REFINE} \arrow[ll, bend right] \end{tikzcd}$$

adaptive stopping criterion

- recover the **contraction property** also in the inexact setting

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Tools:

- a posteriori error bounds on the **total** and **algebraic** errors

Guaranteed total energy error upper bound

$$\|\nabla(u-u_\ell)\| \leq \eta(u_\ell,\mathcal{T}_\ell) := \left\{\sum_{K\in\mathcal{T}_\ell} \eta_K^2(u_\ell)\right\}^{\frac{1}{2}}$$ with

$$\eta_K(u_\ell) := \underbrace{\|\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\mathrm{dis}}\|_K}_{\eta_{\mathrm{dis},K}(u_\ell)} + \underbrace{\frac{h_K}{\pi}\|f-\nabla\cdot\boldsymbol{\sigma}_{\ell,\mathrm{tot}}\|_K}_{\eta_{\mathrm{osc},K}(u_\ell)} + \underbrace{\|\boldsymbol{\sigma}_{\ell,\mathrm{alg}}\|_K}_{\eta_{\mathrm{alg},K}(u_\ell)}, \qquad \forall K\in\mathcal{T}_\ell$$

J. Papež, U. Rüde, M. Vohralík, and B. Wohlmuth, *Sharp algebraic and total a posteriori error bounds for h and p finite elements via a multilevel approach*. HAL preprint 01662944, Dec. 2017.

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- **adaptive stopping criteria** for the algebraic solver

$$\eta_{\mathrm{alg}}(u_\ell,\mathcal{T}_\ell) \le \gamma_\ell\,\mu(u_\ell), \qquad 0<\gamma_\ell<1 \quad \text{(typically } \gamma_\ell\approx 0.1)$$

Ensuring the desired balance: $\underbrace{\|\nabla(u_\ell^{\mathrm{ex}}-u_\ell)\|}_{\text{algebraic error}} \le \gamma_\ell \underbrace{\|\nabla(\boldsymbol{u}-u_\ell)\|}_{\text{total error}}$

Error reduction factor in presence of inexact solver

- **Galerkin orthogonality** relation between $u_{\ell+1}^{\text{ex}}$ and u_ℓ

$$\left\|\nabla(u - u_{\ell+1}^{\text{ex}})\right\|^2 = \|\nabla(u - u_\ell)\|^2 - \left\|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell)\right\|^2$$

- **Intermediate result** $\leftarrow \underline{\eta}_{\mathcal{M}_\ell^\theta}$ & D., Ern, Smears, Vohralík (2018)

Computable guaranteed bound on the reduction factor $C_{\ell,\text{red}}$

Using the adaptive stopping criterion at level $\ell+1$ with $0 < \gamma_{\ell+1} \leq (1 - C_{\ell,\text{red}}^*)$:

$$\|\nabla(\underbrace{u - u_{\ell+1}}_{\text{both unknown}})\| \leq C_{\ell,\text{red}} \|\nabla(\underbrace{u - u_\ell}_{\text{only } u_\ell \text{ known}})\|, \qquad 0 \leq C_{\ell,\text{red}} := \frac{\sqrt{1 - \left(\frac{\underline{\eta}_{\mathcal{M}_\ell^\theta}}{\eta(u_\ell, \mathcal{T}_\ell)}\right)^2}}{(1 - \gamma_{\ell+1})} \leq 1$$

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- singular exact solution (in polar coordinates): $u(r,\varphi) = r^{\frac{2}{3}} \sin \frac{2\varphi}{3}$

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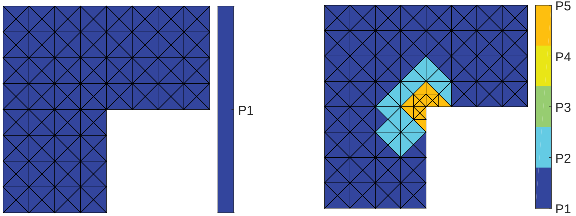
Inexact setting: V-cycle multigrid with Gauss-Seidel as a smoother



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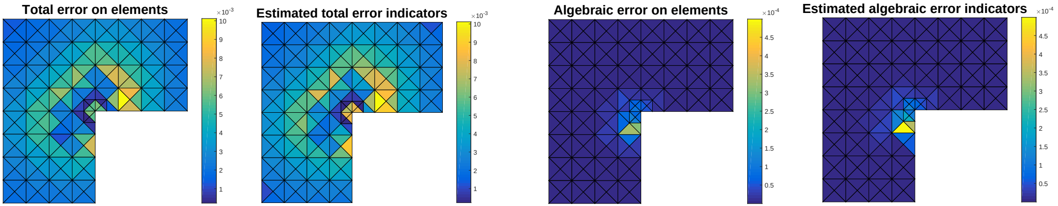

$I_{\rm eff}^{\rm tot} = 1.096$ $\qquad$ $I_{\rm eff}^{\rm alg} = 1.365$

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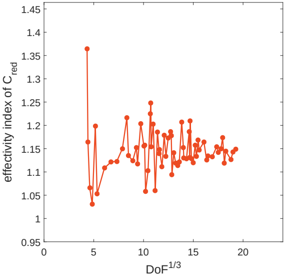
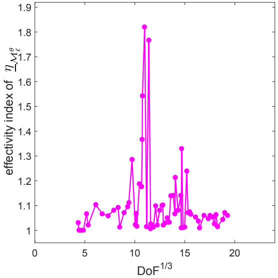
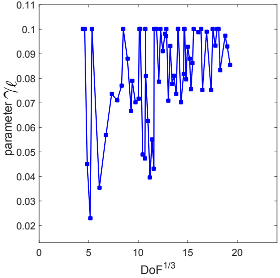
Numerics II.

Effectivity indices of the estimated error reduction factor $C_{\ell,\mathrm{red}}$ and $\underline{\eta}_{\mathcal{M}_\ell^\sharp}$:







$$I^{\mathrm{eff}}_{\mathrm{red}} = \frac{C_{\ell,\mathrm{red}}}{\|\nabla(u-u_{\ell+1})\| \,/\, \|\nabla(u-u_\ell)\|}$$

$$I^{\mathrm{eff}}_{\mathrm{LB}} = \frac{\left\|\nabla(u^{\mathrm{ex}}_{\ell+1} - u_\ell)\right\|_{\omega_\ell}}{\underline{\eta}_{\mathcal{M}_\ell^\theta}}$$

parameter of the adaptive stopping criterion γ_ℓ

Numerics III. – adaptivity for algebraic solver

Adaptive stopping criterion $\boxed{\eta_{\mathrm{alg}}(u_\ell, \mathcal{T}_\ell) \leq \gamma_\ell \, \mu(u_\ell)}$ **in practice:**

Recall: *classical stopping criterion* $\frac{\|\mathrm{F}_\ell - \mathbb{A}_\ell\|}{\|\mathrm{F}_\ell\|} \leq \varepsilon$

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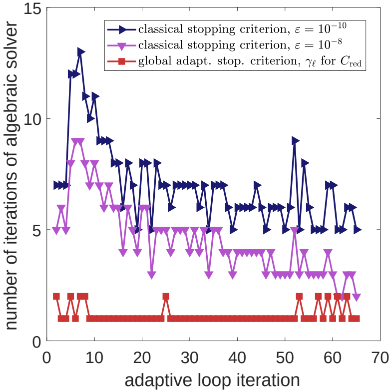
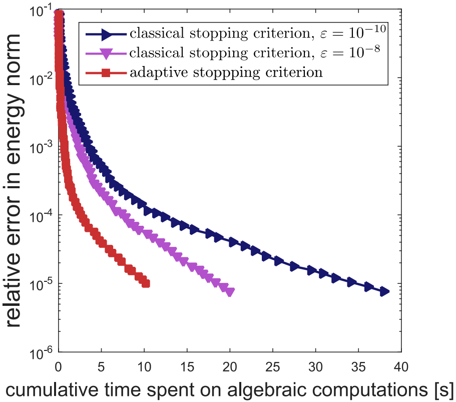
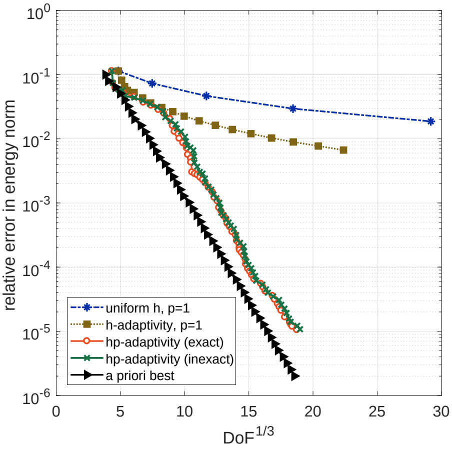




Numerics IV. – exponentail convergence retained



- Obtained exponential convergence in comparison with classical approaches.

Outline

Introduction

An adaptive hp-refinement strategy **with exact solver**

An adaptive hp-refinement strategy **with inexact solver**

Convergence of adaptive hp-refinement strategies

Convergence proofs – *sneak peek*

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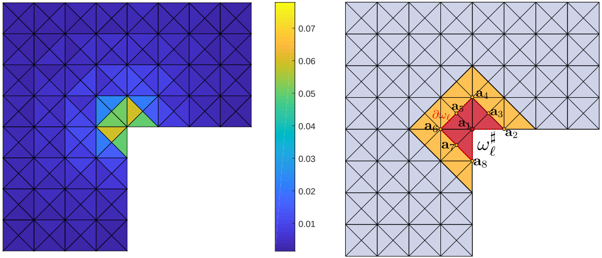

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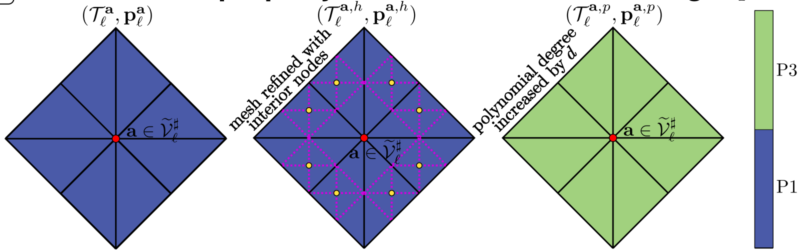
- $\boxed{\texttt{REFINE}}$: **interior node property** for h-refinement and **stronger** p-refinement



Discrete stability (DS) of the local flux equilibration (exact setting)

Let:

- the hat function orthogonality: $(f, \psi_\ell^{\mathbf{a}})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla \psi_\ell^{\mathbf{a}})_{\omega_\ell^{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_\ell^{\mathrm{int}}$,
- the local equilibrated flux $\boldsymbol{\sigma}_\ell^{\mathbf{a}}$ be constructed by the local minimization
- the residual lifting $r^{\mathbf{a},hp}$ be constructed by the local primal FE problem

Then there holds

$$\|\psi_\ell^{\mathbf{a}} \nabla u_\ell^{\mathrm{ex}} + \boldsymbol{\sigma}_\ell^{\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}} \lesssim \left\|\nabla r^{\mathbf{a},hp}\right\|_{\omega_\ell^{\mathbf{a}}} \qquad \forall \mathbf{a} \in \widetilde{\mathcal{V}}_\ell^\sharp$$

Bubble function technique:

R. Verfürth, *A posteriori error estimation techniques for finite element methods*, Num. Math. & Sc. Comp., Oxford, 2013.

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- analysis of the computational (quasi-)optimality

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