An adaptive hp-refinement strategy with inexact solvers and computable guaranteed error reduction factors

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Introduction

In this work we extend our recently proposed hp-**adaptive refinement strategy with computable guaranteed bound on the error reduction factor** [1] for conforming finite element approximations of elliptic problems by taking into account an **inexact algebraic solver**.

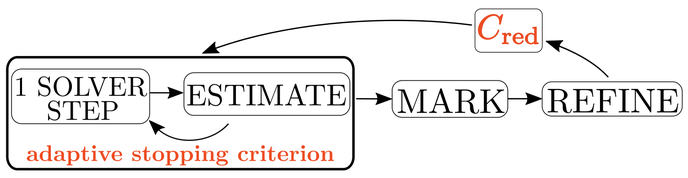


Figure 1: Paradigm of an adaptive loop employing an inexact algebraic solver.

We consider as a model problem the Poisson equation with (homogeneous) Dirichlet boundary conditions: Seek $u \in H^1_0(\Omega)$, such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H^1_0(\Omega). \tag{1}$$

We are particularly interested in recovering the computable guaranteed bound on the error reduction factor introduced in [1] also in the inexact setting described in Figure 1. It turns out that while using a properly designed **stopping criterion for the algebraic solver**, at the end of each iteration of the inexact adaptive loop given in Figure 1 it is still possible to compute a real number $C_{\rm red} \in [0,1]$ such that

$$\|\nabla(u - u_{\ell+1})\| \le C_{\rm red}\, \|\nabla(u - u_\ell)\|\,, \tag{2}$$

where we employ the following notation:

- u - unknown exact solution
- $u_{\ell+1}$ - unknown discrete inexact solution on step $\ell+1$
- u_ℓ - available discrete inexact solution on step ℓ.

The adaptive procedure

The module 1 SOLVER STEP

We employ the conforming hp - finite element method to discretize the model problem (1) on a matching (no hanging nodes) simplicial mesh $\mathcal{T}_\ell$ with varying local polynomial degree $\mathbf{p}_\ell := \{p_{\ell,K}\}_{K\in\mathcal{T}_\ell}$

$$(\nabla u_\ell^{\rm ex}, \nabla v_\ell) = (f, v_\ell)\ \ \forall v_\ell \in V_\ell, \quad V_\ell := \mathbb{P}_{\mathbf{p}}(\mathcal{T}_\ell) \cap H^1_0(\Omega), \ \ \forall \ell \ge 0.$$

We apply only one (or a certain small number of) iteration of the algebraic iterative solver to solve the resulting algebraic problem

$$\mathbb{A}_\ell \mathrm{U}_\ell^{\rm ex} = \mathrm{F}_\ell \quad \Longleftrightarrow \quad (\nabla u_\ell^{\rm ex}, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell,$$

where the vector $\mathrm{U}_\ell^{\rm ex}$ corresponds to the exact FEM solution $u_\ell^{\rm ex} = \sum_{n=1}^{N_\ell} (\mathrm{U}_\ell^{\rm ex})_n\, \psi_\ell^n$. The module outputs an **inexact approximation** u_ℓ ($\approx u_\ell^{\rm ex}$.)

The module ESTIMATE

The obtained inexact solution $u_\ell \in V_\ell$ is then analyzed within the ESTIMATE module which considers the **algebraic error** a part of the **total error**

Guaranteed total energy error upper bound

$$\underbrace{\|\nabla(u - u_\ell)\|}_{\text{total error}} \le \eta(u_\ell, \mathcal{T}_\ell) := \left\{ \sum_{K\in\mathcal{T}_\ell} \eta_K^2(u_\ell) \right\}^{\frac{1}{2}} \text{ with}$$

$$\eta_K(u_\ell) := \overbrace{\underbrace{\|\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\rm dis}\|_K}_{\eta_{{\rm dis},K}(u_\ell)} + \underbrace{\frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_{\ell,\rm tot}\|_K}_{\eta_{{\rm osc},K}(u_\ell)}}^{\sim \text{discretization error } \|\nabla(u - u_\ell^{\rm ex})\|} + \overbrace{\underbrace{\|\boldsymbol{\sigma}_{\ell,\rm alg}\|_K}_{\eta_{{\rm alg},K}(u_\ell)}}^{\ge \text{algebraic error } \|\nabla(u_\ell^{\rm ex} - u_\ell)\|}\,, \ \forall K \in \mathcal{T}_\ell$$

Guaranteed algebraic error upper bound

$$\|\nabla(u_\ell^{\rm ex} - u_\ell)\| \le \eta_{\rm alg}(u_\ell, \mathcal{T}_\ell) := \left\{ \sum_{K\in\mathcal{T}_\ell} \eta_{{\rm alg},K}^2(u_\ell) \right\}^{\frac{1}{2}}$$

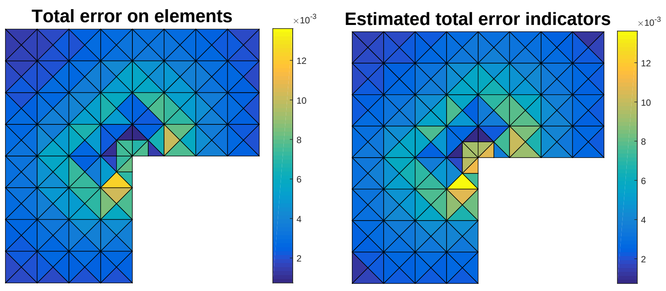


Figure 2: [Model problem (1) posed on the L-shaped domain in 2D] Elementwise distribution of the total energy error $\|\nabla(u - u_7)\|$ (*left*) and the total error indicators $\eta_K(u_7)$ (*right*).

The above a posteriori error estimates crucially rely on the $\mathbf{H}(\mathrm{div},\Omega)$-conforming flux reconstructions

$$\underbrace{\boldsymbol{\sigma}_{\ell,\rm tot}}_{\substack{\text{total flux}\\ \text{reconstruction}}} := \underbrace{\boldsymbol{\sigma}_{\ell,\rm dis}}_{\substack{\text{discretization}\\ \text{flux reconstruction}}} + \underbrace{\boldsymbol{\sigma}_{\ell,\rm alg}}_{\substack{\text{algebraic error}\\ \text{flux reconstruction}}}\,,$$

- $\boldsymbol{\sigma}_{\ell,\rm dis} := \sum_{\mathbf{a}\in\mathcal{V}_\ell} \boldsymbol{\sigma}_{\ell,\rm dis}^{\mathbf{a}}$ ← obtained via solving local mixed-FE problems posed on the finest patches $\mathcal{T}_\ell^{\mathbf{a}}$, $\mathbf{a} \in \mathcal{V}_\ell$
- $\boldsymbol{\sigma}_{\ell,\rm alg} := \sum_{j=1}^{\ell} \sum_{\mathbf{a}\in\mathcal{V}_{j-1}} \boldsymbol{\sigma}_{j,\rm alg}^{\mathbf{a}}$ ← **multilevel approach** ← local problems posed on refined patches $\mathcal{T}_{j,j-1}^{\mathbf{a}}$, $1 \le j \le \ell$

In addition we solve one more set of local, this time primal, finite element problems in order to obtain the *total residual lifting* $\rho_{\ell,\rm tot} := \sum_{\mathbf{a}\in\mathcal{V}_\ell} \psi_\ell^{\mathbf{a}} \rho_{\ell,\rm tot}^{\mathbf{a}} \in H^1_0(\Omega)$ leading to

Guaranteed total energy error lower bound

$$\|\nabla(u - u_\ell)\| \ge \frac{\sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\nabla \rho_{\ell,\rm tot}^{\mathbf{a}}\|^2_{\omega_\ell^{\mathbf{a}}}}{\|\nabla \rho_{\ell,\rm tot}\|} =: \mu(u_\ell).$$

Adaptive stopping criterion for the algebraic solver

$$\eta_{\rm alg}(u_\ell, \mathcal{T}_\ell) \le \gamma_\ell\, \mu(u_\ell), \quad 0 < \gamma_\ell < 1 \quad \text{(typically } \gamma_\ell \approx 0.1) \tag{3}$$

Ensuring the desired balance:

$$\underbrace{\|\nabla(u_\ell^{\rm ex} - u_\ell)\|}_{\text{algebraic error}} \le \gamma_\ell \underbrace{\|\nabla(u - u_\ell)\|}_{\text{total error}}$$

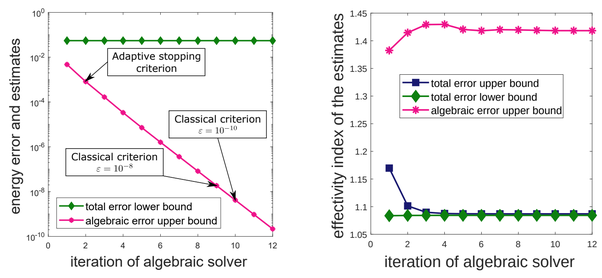


Figure 3: [Model problem (1) posed on the L-shaped domain in 2D] Total error lower bound and algebraic error upper bound (*left*) and the effectivity indices of the total error upper bound, total error lower bound, and the algebraic error upper bound (*right*) throughout the iterations of the algebraic solver, refinement level $\ell = 7$.

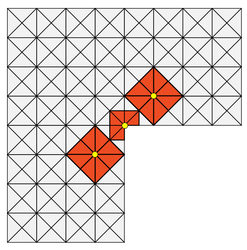
The module MARK

The module MARK takes as input the local error estimators computed within the ESTIMATE module, and outputs a set of **marked vertices** $\widetilde{\mathcal{V}}_\ell^\theta \subset \mathcal{V}_\ell$ using a bulk-chasing criterion. For a fixed threshold parameter $\theta \in (0,1]$, the set of marked vertices $\widetilde{\mathcal{V}}_\ell^\theta$ is selected in such a way that

$$\eta\Big(u_\ell, \bigcup_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}}\Big) \ge \theta\, \eta(u_\ell, \mathcal{T}_\ell).$$

- ● → **marked vertices** $\widetilde{\mathcal{V}}_\ell^\theta$
- ▲ → marked elements $\mathcal{M}_\ell^\theta := \bigcup_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}}$

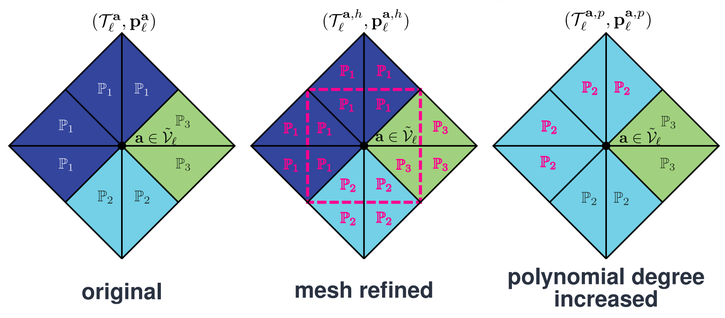
Set $\omega_\ell := \cup_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \omega^{\mathbf{a}}$, the open subdomain corresponding to $\mathcal{M}_\ell^\theta$



The module REFINE

The module REFINE takes as input the set of marked vertices $\widetilde{\mathcal{V}}_\ell^\theta$ and outputs the mesh $\mathcal{T}_{\ell+1}$ and the polynomial-degree distribution $\mathbf{p}_{\ell+1}$ to be used at the next iteration of the adaptive loop from Figure 1.

We solve 2 more local problems, but this time only on the patches $\mathcal{T}_\ell^{\mathbf{a}}$ attached to a **marked vertex** $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell^\theta$, with either the **mesh refined**, or the **polynomial degree increased**.



$$(\nabla r^{\mathbf{a},h}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},h} \in V_\ell^{\mathbf{a},h} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},h}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H^1_0(\omega_\ell^{\mathbf{a}})$$

$$(\nabla r^{\mathbf{a},p}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},p} \in V_\ell^{\mathbf{a},p} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a},p}) \cap H^1_0(\omega_\ell^{\mathbf{a}})$$

Local hp-decision criterion is then based on the norms $\|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^{\mathbf{a}}}$ and $\|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^{\mathbf{a}}}$ for each marked vertex $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell^\theta$.

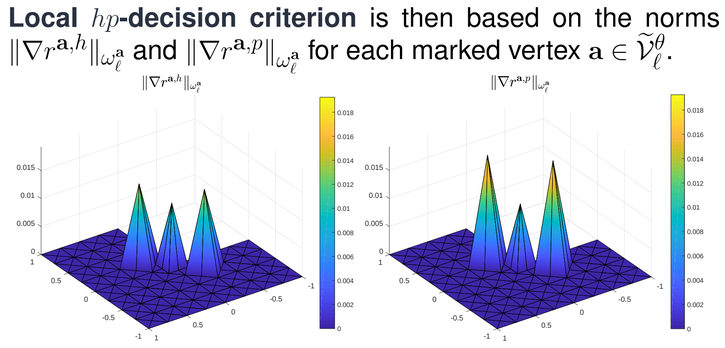


Figure 4: Norms $\|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^{\mathbf{a}}}$ and $\|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^{\mathbf{a}}}$ for three marked vertices.

Guaranteed bound on the error reduction in presence of inexact solver

After the end of module REFINE at iteration ℓ, the new pair $(\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$ and thus also the space $V_{\ell+1}$ are available. For each marked vertex $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell^\theta$, we define a local space

$$V_\ell^{\mathbf{a},hp} := V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}} \cap H^1_0(\omega_\ell^{\mathbf{a}}).$$

At the modest price of solving

$$(\nabla r^{\mathbf{a},hp}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} \quad \forall v^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}$$

for each $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell^\theta$, and if $\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} r^{\mathbf{a},hp} \ne 0$, we have the following guaranteed lower bound on the incremental error on marked simplices

$$\|\nabla(u_{\ell+1}^{\rm ex} - u_\ell)\|_{\omega_\ell} \ge \frac{\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} \|\nabla r^{\mathbf{a},hp}\|^2_{\omega_\ell^{\mathbf{a}}}}{\left\|\nabla\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell^\theta} r^{\mathbf{a},hp}\right)\right\|_{\omega_\ell}} =: \underline{\eta}_{\mathcal{M}_\ell^\theta}. \tag{4}$$

Recall that both $u_{\ell+1}$ and u_ℓ are inexact solutions and only u_ℓ is known. Following [1], we first show that the (unknown) exact next level solution $u_{\ell+1}^{\rm ex} \in V_{\ell+1}$ satisfies

$$\|\nabla(u - u_{\ell+1}^{\rm ex})\| \le C_{\rm red}^*\, \|\nabla(u - u_\ell)\|\,, \quad 0 \le C_{\rm red}^* \le 1, \tag{5}$$

with $C_{\rm red}^* := \sqrt{1 - \left(\theta \frac{\underline{\eta}_{\mathcal{M}_\ell^\theta}}{\eta(u_\ell, \mathcal{M}_\ell^\theta)}\right)^2}$ fully computable. Then, we show that while using the global stopping criterion (3) at the level $\ell+1$ with $0 < \gamma_{\ell+1} \le (1 - C_{\rm red}^*)$, the resulting error reduction between the inexact solution $u_\ell \in V_\ell$ from the current iteration ℓ and the next inexact approximation $u_{\ell+1} \in \mathcal{V}_{\ell+1}$ still to be computed verifies

$$\|\nabla(u - u_{\ell+1})\| \le C_{\rm red}\, \|\nabla(u - u_\ell)\|\,, \tag{6}$$

with a **computable guaranteed bound on the error reduction factor**

$$0 \le C_{\rm red} := \frac{C_{\rm red}^*}{(1 - \gamma_{\ell+1})} \le 1. \tag{7}$$

Numerical experiments

Finally, in a series of numerical experiments, we investigate the practicality of the proposed adaptive solver and the accuracy of our bound on the reduction factor.

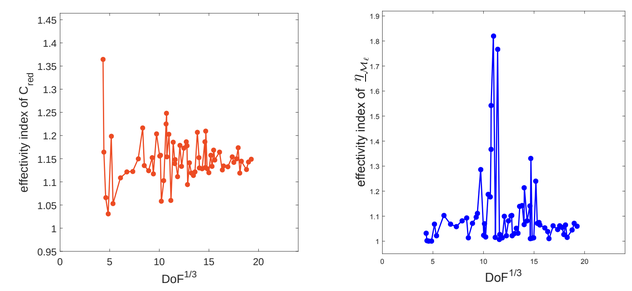


Figure 5: [Model problem (1) posed on the L-shaped domain in 2D] Effectivity indices for the error reduction factor estimate $C_{\rm red}$ of (7) (*left*) and the lower bound $\underline{\eta}_{\mathcal{M}_\ell^\theta}$ from (4) (*right*).

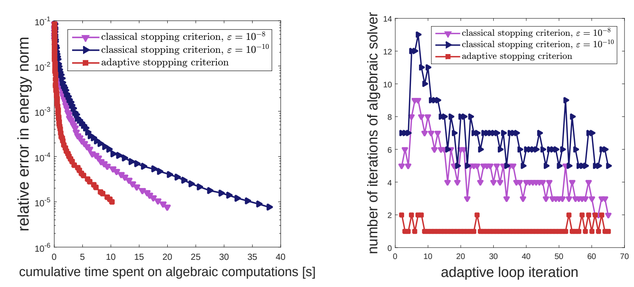


Figure 6: [Model problem (1) posed on the L-shaped domain in 2D] Relative energy error $\|\nabla(u - u_\ell)\| / \|\nabla u\|$ as a function of cumulative time spent on algebraic computations throughout the adaptive process (*left*) and number of algebraic solver iterations per adaptive loop iteration for different stopping criteria.

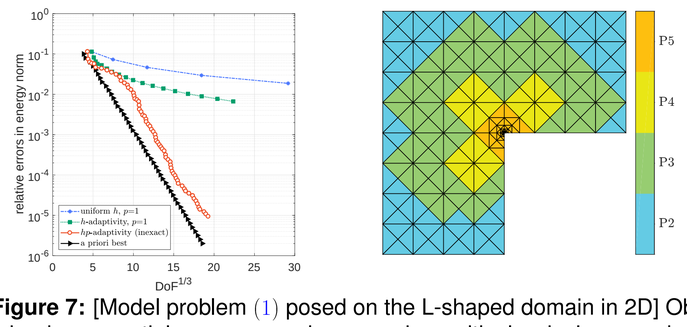


Figure 7: [Model problem (1) posed on the L-shaped domain in 2D] Obtained exponential convergence in comparison with classical approaches and example mesh and polynomial degree distribution $(\mathcal{T}_{33}, \mathbf{p}_{33})$.

References

[1] P. Daniel, A. Ern, I. Smears, and M. Vohralík, An adaptive hp-refinement strategy with computable guaranteed bound on the error reduction factor, Comput. Math. Appl., (2018).

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Since October 2015 I am a member of research team SERENA, where I work on my Ph.D. thesis under the supervision of Martin Vohralík and Alexandre Ern. My work on subject *Adaptive multilevel solvers with a posteriori error control for porous media flows* is carried out within the project ERC Consolidator Grant GATIPOR.