The obtained inexact solution \( u \in V \) is then analyzed within the ESTIMATE module which considers the algebraic error as a part of the total error.

**Guaranteed total energy error upper bound**

\[
\| \mathbf{V} u - \mathbf{V} u^e \|_{\mathbf{A}} \leq \sum_{i=1}^{N} q_{\text{dis},i}^e + q_{\text{alg},i}^e, \quad \forall \mathbf{V} \in T.
\]

\[
\begin{align*}
q_{\text{dis},i}^e &= \frac{2}{3} \| \mathbf{V} \nabla u - \mathbf{V} \nabla u^e \|_{L^2}, \\
q_{\text{alg},i}^e &= \sum_{e=1}^{M} \| \mathbf{V} (u - u^e) \|_{L^2}.
\end{align*}
\]

**Guaranteed algebraic error upper bound**

\[
\| \mathbf{V} (u - u^e) \|_{L^2} \leq \sum_{i=1}^{N} q_{\text{alg},i}^e, \quad \forall \mathbf{V} \in T.
\]

The module **REFIN**

The module REFIN takes as input the set of marked vertices \( V^f \) and outputs the mesh \( V_{r+1} \) and the polynomial-degree distribution \( \mu_{r+1} \), to be used at the next iteration of the adaptive loop from Figure 1.

We solve 2 more local problems, but this time only on the patches \( T^h \) attached to a marked vertex \( \mathbf{V} \in V^f \), with either the mesh refined, or the polynomial degree increased.

**Guaranteed bound on the error reduction in presence of inexact solver**

After the end of module REFIN at iteration \( r \), the new pair \( (V_{r+1}, u_{r+1}) \) and thus also the space \( V_{r+1} \) are available. For each marked vertex \( \mathbf{V} \in V^f \) we define a local space \( \mathbf{V}_{r+1} = V_{r+1} \cap \mathbf{V}(\mathbf{V}) \).

At the modest price of solving \( \mathbf{V}_{r+1} \)-adapters \( (\mathbf{V}_{r+1}, u_{r+1}) \) for each \( \mathbf{V} \in V^f \) and \( \mu_{r+1} \neq \mu_r \), we have the following guaranteed lower bound on the incremental error on marked simplices

\[
\| \mathbf{V}_r u - \mathbf{V}_{r+1} u_{r+1} \|_{L^2} \geq (\sum_{\mathbf{V} \in V^f} \| \mathbf{V} (u - u^e) \|_{L^2}) - 2M_{\text{tot}}.
\]

Recall that both \( w_r \) and \( w_{r+1} \) are inexact solutions and only \( \mu_r \) is known. Following [1], we first show that the (unknown) exact next level solution \( w_{r+2} = V_{r+2} u_{r+2} \) satisfies

\[
\| \mathbf{V}_r u - \mathbf{V}_{r+2} u_{r+2} \|_{L^2} \leq C_{\text{est}} \| \mathbf{V}_r u - \mathbf{V}_{r+1} u_{r+1} \|_{L^2}.
\]

This is achieved by using the global stopping criterion \( \| \mathbf{V}_r u - \mathbf{V}_{r+2} u_{r+2} \|_{L^2} \leq C_{\text{est}} \| \mathbf{V}_r u - \mathbf{V}_{r+1} u_{r+1} \|_{L^2} \) fully computable. Then, we show that while using the global stopping criterion \( \| \mathbf{V}_r u - \mathbf{V}_{r+2} u_{r+2} \|_{L^2} \leq C_{\text{est}} \| \mathbf{V}_r u - \mathbf{V}_{r+1} u_{r+1} \|_{L^2} \), the resulting error reduction between the inexact solution \( u_r \) from the current iteration \( r \) and the next inexact approximation \( u_{r+1} \in V_{r+1} \) still to be computed verifies

\[
\| \mathbf{V}_r u - u_{r+1} \|_{L^2} \leq C_{\text{est}} \| \mathbf{V}_r u - \mathbf{V}_{r+1} u_{r+1} \|_{L^2}.
\]

Finally, in a series of numerical experiments, we investigate the practicality of the proposed adaptive solver and the accuracy of our bound on the reduction factor.

**Numerical experiments**

In the following, we summarize the main findings of the experiments. We first test the modules on the L-shaped domain in 2D. In each case, we compare the effectivities of our estimate for the total error, upper and lower bounds, with the total error upper bound (left) and the lower bound from (6) (right).

![Numerical experiments](image)

**References**