An adaptive hp-refinement strategy with computable guaranteed error reduction factors

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informatics mathematics

École des Ponts ParisTech

European Research Council

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Motivation

Setting

Reduction factors

hp-strategy

Conclusion

Motivation

References

D. BRAESS, J. SCHÖBERL
Equilibrated residual error estimator for edge elements, Math. Comp. (2008)

W. DÖRFLER
A convergent adaptive algorithm for Poisson's equation, SIAM J. Numer. Anal.,(1996)

C. CANUTO, R. H. NOCHETTO, R. STEVENSON, AND M. VERANI
Convergence and optimality of hp-AFEM, Numer. Math. (2016).

J. M. CASCÓN AND R. H. NOCHETTO
Quasioptimal cardinality of AFEM driven by nonresidual estimators, IMA J. Numer. Anal., (2012)

A. ERN AND M. VOHRALÍK
Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, SIAM J. Numer. Anal.(2015).

General adaptive loop

SOLVE $\rightarrow$ **ESTIMATE** $\rightarrow$ **MARK** $\rightarrow$ **REFINE**

Outline

SOLVE

Laplace model problem

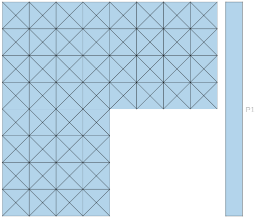
For $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

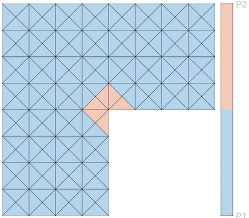
Discretization

- $\{\mathcal{T}_\ell\}_{\ell \geq 0}$ a sequence of nested matching simplicial meshes
- Each element $K \in \mathcal{T}_\ell$ is assigned with a polynomial degree via vector $\mathbf{p}_\ell := \{p_K \geq 1, K \in \mathcal{T}_\ell\}$, $\mathbb{P}_{p_K}(K)$ s.t. $\mathbf{p}_{\ell+1} \geq \mathbf{p}_\ell$

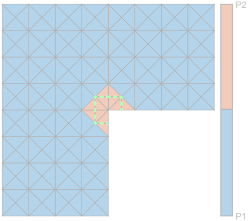
$(\mathcal{T}_1, \mathbf{p}_1)$



$(\mathcal{T}_2, \mathbf{p}_2)$



$(\mathcal{T}_3, \mathbf{p}_3)$



SOLVE

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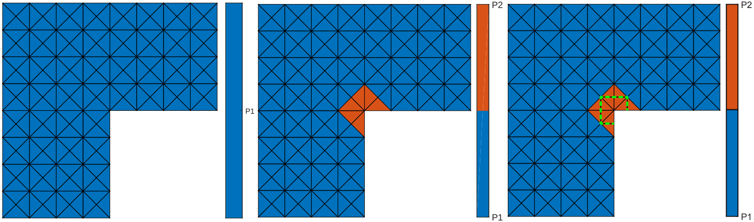
$(\mathcal{T}_1, \mathbf{p}_1)$ $(\mathcal{T}_2, \mathbf{p}_2)$ $(\mathcal{T}_3, \mathbf{p}_3)$



SOLVE

Laplace model problem – FEM

Define the test space $V_\ell := \mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell) \cap H_0^1(\Omega)$. Find $u_\ell \in V_\ell$ s.t.

$$(\nabla u_\ell, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell$$

Due to the nestedness of the spaces $V_\ell \subset V_{\ell+1}$, $\ell \geq 0$:

Galerkin orthogonality

$$\|\nabla(u - u_{\ell+1})\|^2 = \|\nabla(u - u_\ell)\|^2 - \|\nabla(u_{\ell+1} - u_\ell)\|^2$$

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A posteriori error ESTIMATE

Guaranteed upper bound on the energy error $\|\nabla(u - u_\ell)\|$

- for each $\ell \geq 0$ and for each patch $\mathcal{T}_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{T}_\ell$, select $p_{\mathbf{a}} := \max_{K \in \mathcal{T}_{\mathbf{a}}} p_K$

Equilibrated flux reconstruction $\boldsymbol{\sigma}_\ell := \sum_{\mathbf{a} \in \mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\mathbf{a}} \in \mathbf{H}(\mathrm{div}, \Omega)$

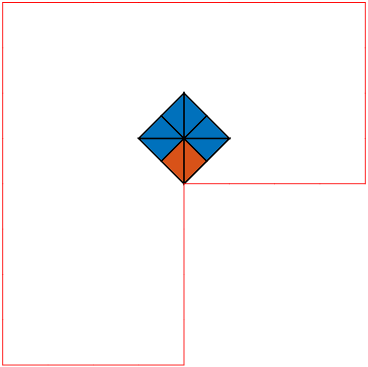
For each vertex $\mathbf{a} \in \mathcal{V}_\ell$, we solve a small minimization problem

$$\boldsymbol{\sigma}_\ell^{\mathbf{a}} := \arg \min_{\mathbf{v}_\ell \in \mathbf{V}_\ell^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_\ell = \Pi_{Q_\ell^{\mathbf{a}}}(f\psi_{\mathbf{a}} - \nabla u_\ell \cdot \nabla \psi_{\mathbf{a}})} \|\psi_{\mathbf{a}} \nabla u_\ell + \mathbf{v}_\ell\|_{\omega_{\mathbf{a}}}.$$

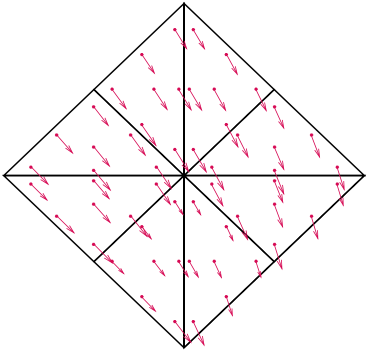
with properly chosen local *Raviart–Thomas–Nédélec* mixed finite element spaces $\mathbf{V}_\ell^{\mathbf{a}} \times Q_\ell^{\mathbf{a}}$ of order $p_{\mathbf{a}}$.

A posteriori error ESTIMATE

Flux reconstruction: illustration on a single patch $\omega_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{T}_2$



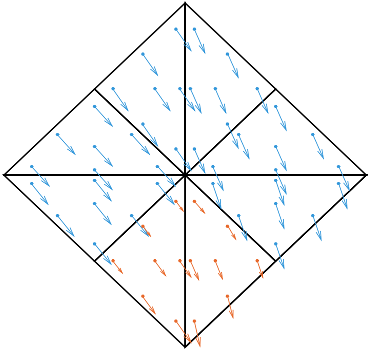
Global position of the patch $\omega_{\mathbf{a}}$



The exact flux
$-\nabla u \in \mathbf{H}(\mathrm{div}, \Omega)$

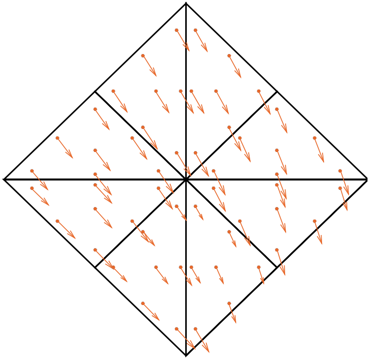
A posteriori error ESTIMATE

Flux reconstruction: illustration on a single patch $\omega_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{T}_2$



Approximate flux

$-\nabla u_\ell \notin \mathbf{H}(\mathrm{div}, \Omega)$



Flux reconstruction

$\boldsymbol{\sigma}_\ell \in \mathbf{V}_\ell \subset \mathbf{H}(\mathrm{div}, \Omega)$

A posteriori error ESTIMATE

Guaranteed upper bound on the energy error

$$\|\nabla(u - u_\ell)\| \le \eta(\mathcal{T}_\ell) := \left\{ \sum_{K\in\mathcal{T}_\ell} \eta_K^2 \right\}^{\frac{1}{2}}$$

$$\eta_K := \|\nabla u_\ell + \boldsymbol{\sigma}_\ell\|_K + \frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_\ell\|_K.$$

References:

- D. Braess, J. Schöberl, *Equilibrated residual error estimator for edge elements*, Math. Comp. (2008)
- P. Destuynder, B. Métivet, *Explicit error bounds in a conforming finite element method*, Math. Comp. (1999)
- V. Dolejší, A. Ern, and M. Vohralík, *hp-adaptation driven by polynomial-degree-robust a posteriori error estimates for elliptic problems*, SIAM J. Sci. Comput. (2016)

MARK

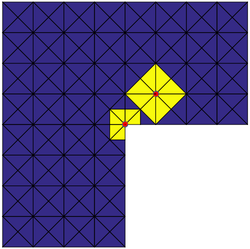
The goal is to mark a set of elements $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ **to be refined**

Classical bulk chasing (Dörfler's marking strategy)

For a *fixed* parameter $\theta \in (0, 1]$ choose (the smallest) set of elements $\mathcal{M}_\ell$ s.t.:

$$\eta(\mathcal{M}_\ell) \geq \theta\, \eta(\mathcal{T}_\ell)$$

- *Notation*: $\eta(\mathcal{M}_\ell) := \left\{ \sum_{K \in \mathcal{M}_\ell} \eta_K^2 \right\}^{\frac{1}{2}}$
- *Remark*: we select the elements **patch-wise**, hence we define the set of **marked vertices** $\widetilde{\mathcal{V}}_\ell$ (●), and ω_ℓ (▲) – **the domain** of the marked elements $\mathcal{M}_\ell$



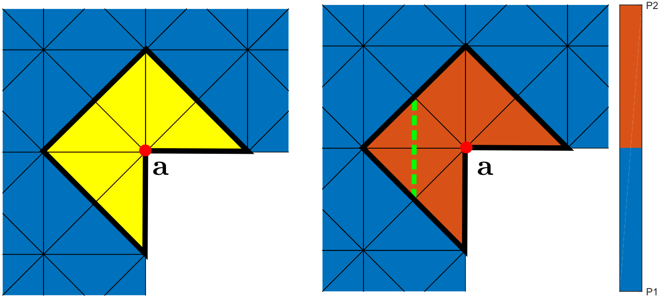
Outline

Residual liftings I

Assumption: the next-level $\mathcal{T}_{\ell+1}$ and $\mathbf{p}_{\ell+1}$ have been determined

Notation: for each marked vertex $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$ (•) and the associated patch $\omega_{\mathbf{a}}$ we define

- the local submesh refinement $\mathcal{T}_{\mathbf{a}}^{hp} = \mathcal{T}_{\ell+1}|_{\omega_{\mathbf{a}}}$
- the local polynomial degrees $\mathbf{p}_{\mathbf{a}}^{hp} = \mathbf{p}_{\ell+1}|_{\mathcal{T}_{\ell+1}}$



$\omega_{\mathbf{a}}|_{\mathcal{T}_\ell}$ $\omega_{\mathbf{a}}|_{\mathcal{T}_{\ell+1}}$

Residual liftings II

Residual liftings' local problems ($\ell \geq 0$)

For each marked vertex $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$, we define the local patch-based space

$$V_{\mathbf{a}}^{hp} := \mathbb{P}_{\mathbf{p}_{\mathbf{a}}^{hp}}(\mathcal{T}_{\mathbf{a}}^{hp}) \cap H_0^1(\omega_{\mathbf{a}}) .$$

We define **the local residual lifting** $r_{\mathbf{a}}^{hp}$ as the solution of

$$(\nabla r_{\mathbf{a}}^{hp}, \nabla v_{\mathbf{a}}^{hp})_{\omega_{\mathbf{a}}} = (f, v_{\mathbf{a}}^{hp})_{\omega_{\mathbf{a}}} - (\nabla u_\ell, \nabla v_{\mathbf{a}}^{hp})_{\omega_{\mathbf{a}}} \quad \forall\, v_{\mathbf{a}}^{hp} \in V_{\mathbf{a}}^{hp}.$$

A. Ern and M. Vohralík
Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, SIAM (2015)

I. Babuška and T. Strouboulis
The finite element method and its reliability, The Clarendon Press Oxford University Press (2001)

Guaranteed lower bound on $\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell}$

Discrete lower bound $\underline{\eta}_{\mathcal{M}_\ell}$

Let the meshes $\mathcal{T}_\ell$, $\mathcal{T}_{\ell+1}$ and the associated residual liftings $r_{\mathbf{a}}^{hp}$ for each $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$ be given. Then we have

$$\|\nabla(u_{\ell+1} - u_\ell)\| \geq \|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell} \geq \frac{\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell} \|\nabla r_{\mathbf{a}}^{hp}\|_{\omega_{\mathbf{a}}}^2}{\|\nabla\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell} r_{\mathbf{a}}^{hp}\right)\|_{\omega_\ell}} =: \underline{\eta}_{\mathcal{M}_\ell}$$

Proof:

$$\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell} = \sup_{v_{\ell+1}\in V_{\ell+1}(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}$$

To finish take $\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell} r_{\mathbf{a}}^{hp}\right)$ as test function $v_{\ell+1}$

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Proof:

$$\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell} \geq \sup_{v_{\ell+1}\in V_{\ell+1}^0(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}$$

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To finish take $\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell} r_{\mathbf{a}}^{hp}\right)$ as test function $v_{\ell+1}$

Error reduction factor $C_{\mathrm{red}} \in [0,1)$

Guaranteed error contraction property

For given:

- $\mathcal{T}_\ell$, $\mathcal{T}_{\ell+1}$ (s.t. $\mathcal{T}_\ell \subset \mathcal{T}_{\ell+1}$)
- the associated residual liftings $r_{\mathbf{a}}^{hp}$ for each $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$
- $u_\ell \in \mathcal{V}_\ell$ be the FEM solution and $\{\eta_K\}_{K \in \mathcal{T}_\ell}$

The new (*unknown*) numerical solution $u_{\ell+1} \in V_{\ell+1}$ satisfies:

$$\|\nabla(u - u_{\ell+1})\| \leq C_{\mathrm{red}} \, \|\nabla(u - u_\ell)\| \;\text{ with }\; C_{\mathrm{red}} := \sqrt{1 - \theta^2 \frac{\underline{\eta}^2_{\mathcal{M}_\ell}}{\eta^2(\mathcal{M}_\ell)}}$$

Guaranteed error reduction factor - proof sketch

Contraction property

$$\|\nabla(u - u_{\ell+1})\| \leq C_{\mathrm{red}} \|\nabla(u - u_\ell)\| \text{ with } C_{\mathrm{red}} := \sqrt{1 - \theta^2 \frac{\underline{\eta}^2_{\mathcal{M}_\ell}}{\eta^2(\mathcal{M}_\ell)}}$$

Proof

1. Garlerking orthogonality
$$\|\nabla(u - u_{\ell+1})\|^2 = \|\nabla(u - u_\ell)\|^2 - \quad \|\nabla(u_{\ell+1} - u_\ell)\|^2$$
2. Employ the discrete lower bound $\underline{\eta}_{\mathcal{M}_\ell}$
3. Use the Dörfler marking property $\eta^2(\mathcal{M}_\ell) \geq \theta^2 \, \eta^2(\mathcal{T}_\ell)$
4. Employ the error estimate $\eta^2(\mathcal{T}_\ell) \geq \|\nabla(u - u_\ell)\|^2$
5. Factorize & take square root

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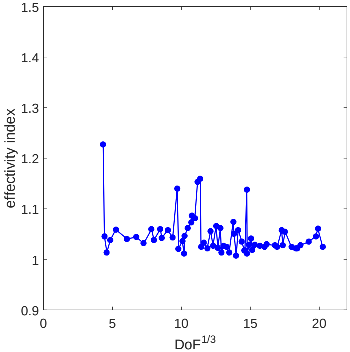
Numerics: L-shape problem - solution with corner singularity

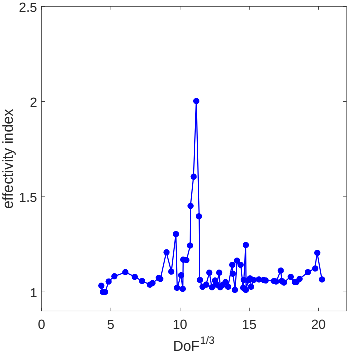
$$u(r,\varphi) = r^{\frac{2}{3}} \sin\left(\frac{2\varphi}{3}\right)$$

$$I_{\mathrm{red}}^{\mathrm{eff}} = \frac{C_{\mathrm{red}}}{\|\nabla(u-u_{\ell+1})\| \,/\, \|\nabla(u-u_\ell)\|}$$



Effectivity index of the reduction factor C_{red}

$$I_{\mathrm{low}}^{\mathrm{eff}} = \frac{\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell}}{\underline{\eta}_{\mathcal{M}_\ell}}$$



Effectivity index of $\underline{\eta}_{\mathcal{M}_\ell}$

Outline

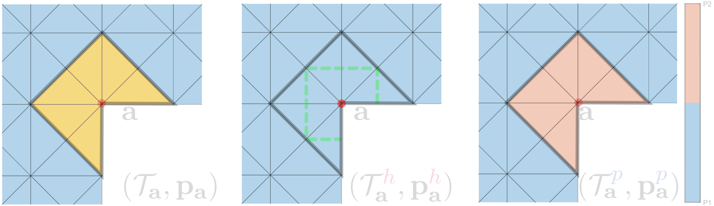
hp-strategy – the 1st attempt

Goal: to determine the next-level mesh $\mathcal{T}_{\ell+1}$ and degrees $\mathbf{p}_{\ell+1}$

- On each marked patch $\omega_{\mathbf{a}}$, $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$ calculate the **2 residual liftings** $r_{\mathbf{a}}^h \in V_{\mathbf{a}}^h$ and $r_{\mathbf{a}}^p \in V_{\mathbf{a}}^p$:

$$(\nabla r_{\mathbf{a}}^h, \nabla v_{\mathbf{a}}^h)_{\omega_{\mathbf{a}}} = (f, v_{\mathbf{a}}^h)_{\omega_{\mathbf{a}}} - (\nabla u_\ell, \nabla v_{\mathbf{a}}^h)_{\omega_{\mathbf{a}}} \quad \forall\, v_{\mathbf{a}}^h \in V_{\mathbf{a}}^h$$

$$(\nabla r_{\mathbf{a}}^p, \nabla v_{\mathbf{a}}^p)_{\omega_{\mathbf{a}}} = (f, v_{\mathbf{a}}^p)_{\omega_{\mathbf{a}}} - (\nabla u_\ell, \nabla v_{\mathbf{a}}^p)_{\omega_{\mathbf{a}}} \quad \forall\, v_{\mathbf{a}}^p \in V_{\mathbf{a}}^p$$



- **Idea:** $\max \|\nabla r_{\mathbf{a}}^{\mathrm{ref}}\|_{\omega_{\mathbf{a}}}$ $\max \underline{\eta}_{\mathcal{M}_\ell}$ $\min \|\nabla(u - u_{\ell+1})\|$
- If $\|\nabla r_{\mathbf{a}}^h\|_{\omega_{\mathbf{a}}} \geq \|\nabla r_{\mathbf{a}}^p\|_{\omega_{\mathbf{a}}}$, then $\mathbf{a}$ flagged for h-refinement
- If $\|\nabla r_{\mathbf{a}}^h\|_{\omega_{\mathbf{a}}} < \|\nabla r_{\mathbf{a}}^p\|_{\omega_{\mathbf{a}}}$, then $\mathbf{a}$ flagged for p-refinement

hp-strategy – the 1st attempt

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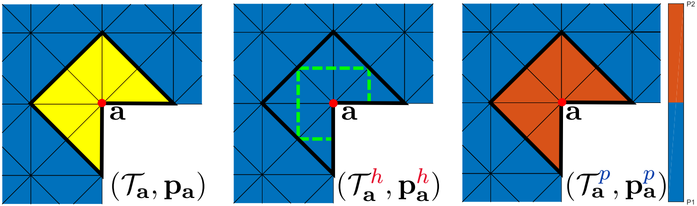

- **Idea:** max $\|\nabla r_{\mathbf{a}}^{\mathrm{ref}}\|_{\omega_{\mathbf{a}}}$ ➞ max $\underline{\eta}_{\mathcal{M}_\ell}$ ➞ min $\|\nabla(u - u_{\ell+1})\|$
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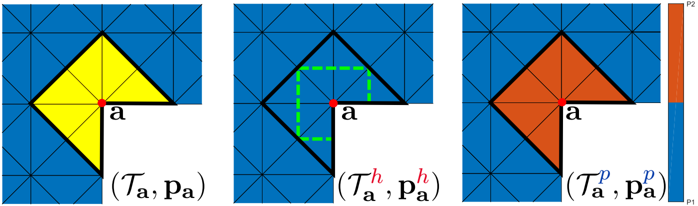

- **Idea:** max $\|\nabla r_{\mathbf{a}}^{\text{ref}}\|_{\omega_{\mathbf{a}}}$ ➞ max $\underline{\eta}_{\mathcal{M}_\ell}$ ➞ min $\|\nabla(u - u_{\ell+1})\|$
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hp-strategy – the 1st attempt

The final element-wise decision

- An element $K \in \mathcal{T}_\ell$ is h-refined if it contains at least one vertex flagged for h-refinement
- An element $K \in \mathcal{T}_\ell$ is p-refined if it contains at least one vertex flagged for p-refinement

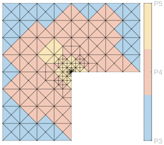
An element K has $(d+1)$ vertices – possible hp-refinement

Our choice of refinement methods

- Newest vertex bisection for h-refinement method
- Locally adjusted p-refinement rules on the patches:

$\mathbf{p}_{\mathbf{a}}^{p} = \{p_K + \triangle_{\mathbf{a}}(K), K \in \mathcal{T}_{\mathbf{a}}\}$, where

$$\triangle_{\mathbf{a}}(K) = \begin{cases} 1 & \text{if} \quad p_K = \min_{K' \in \mathcal{T}_{\mathbf{a}}} p_{K'} \\ 0 & \text{otherwise.} \end{cases}$$



→ *Avoiding the staggered polynomial degree distribution*

hp-strategy – the 1st attempt

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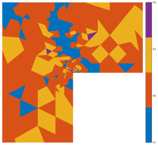
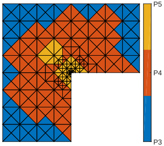




→ *Avoiding the staggered polynomial degree distribution*

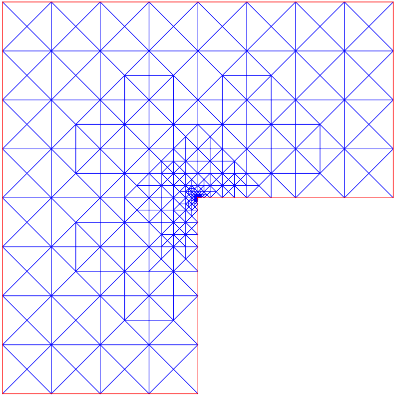
hp-strategy

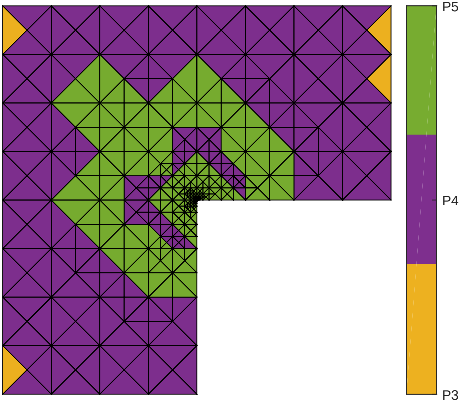
Numerics: L-shape problem - solution with corner singularity

$$u(r,\varphi) = r^{\frac{2}{3}} \sin\left(\frac{2\varphi}{3}\right)$$

Videos illustrating the adaptive process

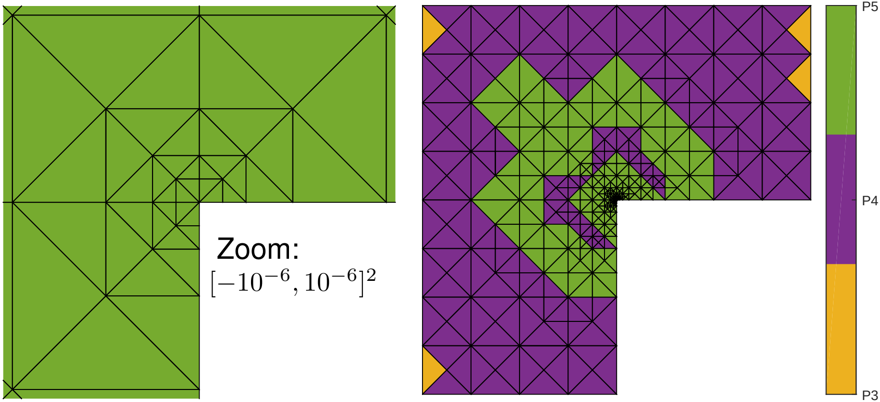
Numerics – hp-strategy





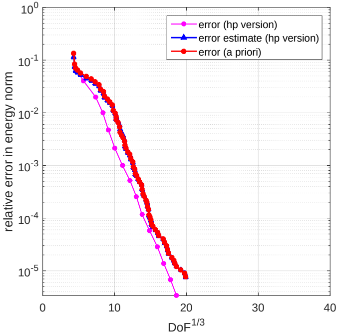
- The final mesh and the final polynomial degree distribution after 65 iterations of the proposed hp-strategy

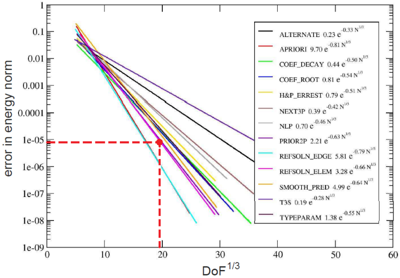
Numerics – hp-strategy



- The final polynomial degree distribution after 65 iterations of the proposed hp-strategy and its detail near the corner (*left*).

Exponential convergence & Assessment of the strategy





W. F. MITCHELL AND M. A. MCCLAIN
A comparison of hp-adaptive strategies for elliptic partial differential equations (2014).

Outline

Conclusion

- The effectivity indices of estimates of the reduction factor C_{red} and $\underline{\eta}_{\mathcal{M}_\ell}$ very close to ideal value of $\mathbf{1}$
- the first attempt hp-strategy with exponential order of convergence observed

Future work:

- try to exploit the estimates of C_{red} inside the hp-strategy
- try to prove the convergence of the hp-strategy
- exploiting the multilevel structure in an inexact algebraic solver (multigrid, ...)

Thank you for your attention!

☺