An adaptive hp-refinement strategy with computable guaranteed error reduction factors

The 15th European Finite Element Fair

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Outline

Motivation

References

D. BRAESS, J. SCHÖBERL
Equilibrated residual error estimator for edge elements, Math. Comp. (2008)

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A convergent adaptive algorithm for Poisson's equation, SIAM J. Numer. Anal.,(1996)

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Convergence and optimality of hp-AFEM, Numer. Math. (2016).

J. M. CASCÓN AND R. H. NOCHETTO
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A. ERN AND M. VOHRALÍK
Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, SIAM J. Numer. Anal.(2015).

General adaptive loop

SOLVE $\rightarrow$ **ESTIMATE** $\rightarrow$ **MARK** $\rightarrow$ **REFINE**

Outline

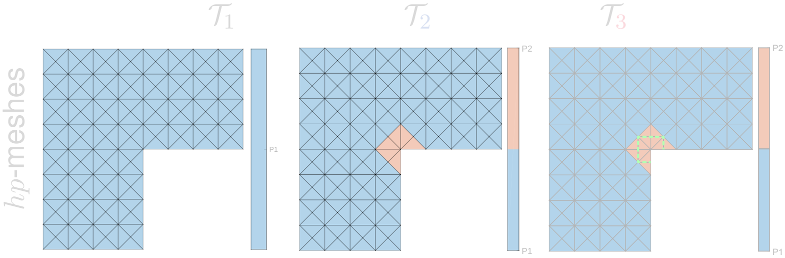
SOLVE

Laplace model problem

For $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Let $\{\mathcal{T}_\ell\}_{\ell \geq 0}$ be a sequence of matching simplicial meshes



Each element $K \in \mathcal{T}_\ell$ is assigned with a polynomial degree via vector $\mathbf{p}_\ell := \{p_K \geq 1, K \in \mathcal{T}_\ell\}$, $\mathbb{P}_{p_K}(K)$

SOLVE

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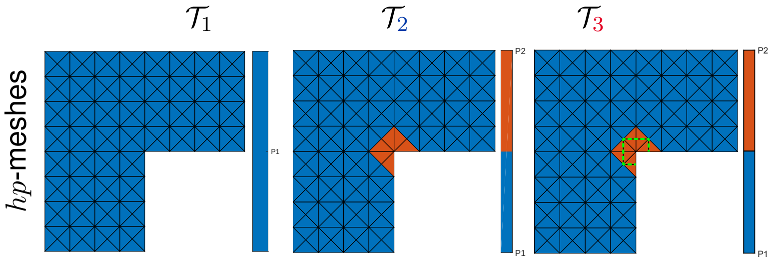

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SOLVE

Laplace model problem – FEM

Define the test space $V_\ell := \mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell) \cap H_0^1(\Omega)$. Find $u_\ell \in V_\ell$ s.t.

$$(\nabla u_\ell, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell$$

Due to the nestedness of the spaces $V_\ell \subset V_{\ell+1}$, $\ell \geq 0$:

Galerkin orthogonality

$$\|\nabla(u - u_{\ell+1})\|^2 = \|\nabla(u - u_\ell)\|^2 - \|\nabla(u_{\ell+1} - u_\ell)\|^2$$

SOLVE

Laplace model problem – FEM

Define the test space $V_\ell := \mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell) \cap H^1_0(\Omega)$. Find $u_\ell \in V_\ell$ s.t.

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A posteriori error ESTIMATE

Guaranteed upper bound on the energy error $\|\nabla(u-u_\ell)\|$

- for each $\ell \geq 0$ and for each patch $\mathcal{T}_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{T}_\ell$, select $p_{\mathbf{a}} := \max_{K\in\mathcal{T}_{\mathbf{a}}} p_K$

Equilibrated flux reconstruction $\boldsymbol{\sigma}_\ell := \sum_{\mathbf{a}\in\mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\mathbf{a}}$

For each vertex $\mathbf{a} \in \mathcal{V}_\ell$, we solve a small minimization problem

$$\boldsymbol{\sigma}_\ell^{\mathbf{a}} := \arg \min_{\mathbf{v}_\ell\in\mathbf{V}_\ell^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_\ell=\Pi_{Q_\ell^{\mathbf{a}}}(f\psi_{\mathbf{a}}-\nabla u_\ell\cdot\nabla\psi_{\mathbf{a}})} \|\psi_{\mathbf{a}}\nabla u_\ell + \mathbf{v}_\ell\|_{\omega_{\mathbf{a}}}.$$

with properly chosen local *Raviart–Thomas–Nédélec* mixed finite element spaces $\mathbf{V}_\ell^{\mathbf{a}} \times Q_\ell^{\mathbf{a}}$ of order $p_{\mathbf{a}}$.

ESTIMATE

Guaranteed upper bound on the error

$$\|\nabla(u - u_\ell)\| \leq \eta(\mathcal{T}_\ell) := \left\{ \sum_{K \in \mathcal{T}_\ell} \eta_K^2 \right\}^{\frac{1}{2}}$$

$$\eta_K := \|\nabla u_\ell + \boldsymbol{\sigma}_\ell\|_K + \frac{h_K}{\pi} \|f - \nabla{\cdot}\boldsymbol{\sigma}_\ell\|_K.$$

References:

- D. Braess, J. Schöberl, *Equilibrated residual error estimator for edge elements*, Math. Comp. (2008)
- P. Destuynder, B. Métivet, *Explicit error bounds in a conforming finite element method*, Math. Comp. (1999)
- V. Dolejší, A. Ern, and M. Vohralík, *hp-adaptation driven by polynomial-degree-robust a posteriori error estimates for elliptic problems*, SIAM J. Sci. Comput. (2016)

MARK

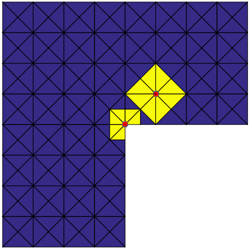
The goal is to mark a set of elements $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ **to be refined**

Classical bulk chasing (Dörfler's marking strategy)

For a *fixed* parameter $\theta \in (0, 1]$ choose (the smallest) set of elements $\mathcal{M}_\ell$ s.t.:

$$\eta(\mathcal{M}_\ell) \geq \theta\, \eta(\mathcal{T}_\ell)$$

- *Notation*: $\eta(\mathcal{M}_\ell) := \left\{ \sum_{K \in \mathcal{M}_\ell} \eta_K^2 \right\}^{\frac{1}{2}}$
- *Remark*: we select the elements **patch-wise**, hence we define the set of **marked vertices** $\widetilde{\mathcal{V}}_\ell$ (●), and ω_ℓ (▲) – **the domain** of the marked elements $\mathcal{M}_\ell$



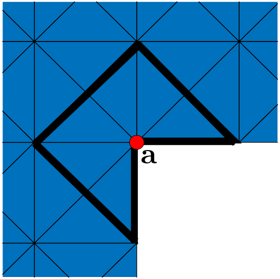
Outline

Residual liftings I

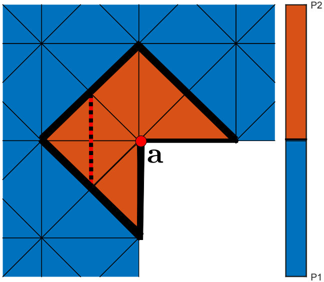
Assumption: the next-level $\mathcal{T}_{\ell+1}$ and $\mathbf{p}_\ell$ have been determined

Notation: for each marked vertex $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$ (●) and the associated patch $\omega_\mathbf{a}$ we define

- the local submesh refinement $\mathcal{T}_\mathbf{a}^{hp} = \mathcal{T}_{\ell+1}|_{\omega_\mathbf{a}}$
- the local polynomial degrees $\mathbf{p}_\mathbf{a}^{hp} = \mathbf{p}_{\ell+1}|_{\mathcal{T}_{\ell+1}}$



$\omega_\mathbf{a}|\mathcal{T}_\ell$



$\omega_\mathbf{a}|\mathcal{T}_{\ell+1}$

Residual liftings II

Residual liftings' local problems ($\ell \geq 0$)

For each marked vertex $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$, we define the local patch-based space

$$V_{\mathbf{a}}^{hp} := \mathbb{P}_{\mathbf{p}_{\mathbf{a}}^{hp}}(\mathcal{T}_{\mathbf{a}}^{hp}) \cap H_0^1(\omega_{\mathbf{a}}) .$$

We define **the local residual lifting** $r_{\mathbf{a}}^{hp}$ as the solution of

$$(\nabla r_{\mathbf{a}}^{hp}, \nabla v_{\mathbf{a}}^{hp})_{\omega_{\mathbf{a}}} = (f, v_{\mathbf{a}}^{hp})_{\omega_{\mathbf{a}}} - (\nabla u_\ell, \nabla v_{\mathbf{a}}^{hp})_{\omega_{\mathbf{a}}} \quad \forall\, v_{\mathbf{a}}^{hp} \in V_{\mathbf{a}}^{hp}.$$

A. Ern and M. Vohralík
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Guaranteed lower bound on $\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell}$

Discrete lower bound $\underline{\eta}_{\mathcal{M}_\ell}$

Let the meshes $\mathcal{T}_\ell$, $\mathcal{T}_{\ell+1}$ and the associated residual liftings $r_{\mathbf{a}}^{hp}$ for each $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$ be given. Then we have

$$\|\nabla(u_{\ell+1} - u_\ell)\| \geq \|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell} \geq \frac{\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell} \|\nabla r_{\mathbf{a}}^{hp}\|^2_{\omega_{\mathbf{a}}}}{\|\nabla\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell} r_{\mathbf{a}}^{hp}\right)\|_{\omega_\ell}} =: \underline{\eta}_{\mathcal{M}_\ell}$$

Proof:

$$\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell} = \sup_{v_{\ell+1}\in V_{\ell+1}(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}$$

To finish take $\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell} r_{\mathbf{a}}^{hp}\right)$ as test function $v_{\ell+1}$

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$$\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell} \geq \sup_{v_{\ell+1}\in V_{\ell+1}^0(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}$$

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To finish take $\left(\sum_{\mathbf{a}\in\widetilde{\mathcal{V}}_\ell} r_{\mathbf{a}}^{hp}\right)$ as test function $v_{\ell+1}$

Error reduction factor $C_{\text{red}} \in [0, 1)$

Guaranteed contraction property

For given:

- $\mathcal{T}_\ell$, $\mathcal{T}_{\ell+1}$ (s.t. $\mathcal{T}_\ell \subset \mathcal{T}_{\ell+1}$)
- the associated residual liftings $r_{\mathbf{a}}^{hp}$ for each $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$
- $u_\ell \in \mathcal{V}_\ell$ be the FEM solution and $\{\eta_K\}_{K \in \mathcal{T}_\ell}$

The new (*unknown*) numerical solution $u_{\ell+1} \in V_{\ell+1}$ satisfies:

$$\|\nabla(u - u_{\ell+1})\| \leq C_{\text{red}} \|\nabla(u - u_\ell)\| \text{ with } C_{\text{red}} := \sqrt{1 - \theta^2 \frac{\eta^2_{\mathcal{M}_\ell}}{\eta^2(\mathcal{M}_\ell)}}$$

Guaranteed error reduction factor - proof sketch

Contraction property

$$\|\nabla(u-u_{\ell+1})\| \le C_{\mathrm{red}} \|\nabla(u-u_\ell)\| \text{ with } C_{\mathrm{red}} := \sqrt{1-\theta^2 \frac{\underline{\eta}^2_{\mathcal{M}_\ell}}{\eta^2(\mathcal{M}_\ell)}}$$

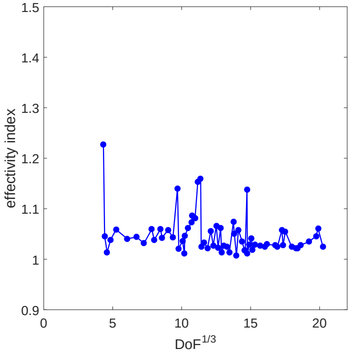
Proof

1. Galerkin orthogonality
$$\|\nabla(u-u_{\ell+1})\|^2 = \underbrace{\|\nabla(u-u_\ell)\|^2}_{\le \eta^2(\mathcal{T}_\ell)} - \underbrace{\|\nabla(u_{\ell+1}-u_\ell)\|^2}_{\ge \underline{\eta}^2_{\mathcal{M}_\ell} = \frac{\underline{\eta}^2_{\mathcal{M}_\ell}}{\eta^2(\mathcal{M}_\ell)} \eta^2(\mathcal{M}_\ell)}$$
2. Employ the discrete lower bound $\underline{\eta}_{\mathcal{M}_\ell}$
3. Employ the error estimate $\eta(\mathcal{T}_\ell)$
4. Use the Dörfler marking property $\eta(\mathcal{M}_\ell) \ge \theta\, \eta(\mathcal{T}_\ell)$
5. Factorize & take square root

Numerics: L-shape problem - solution with corner singularity

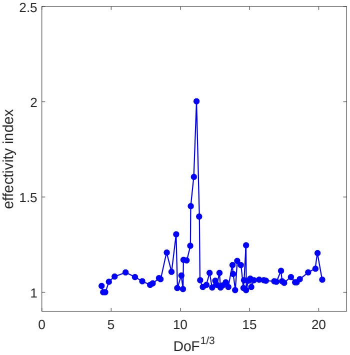
$$u(r,\varphi) = r^{\frac{2}{3}} \sin\left(\frac{2\varphi}{3}\right)$$

$$I_{\text{red}}^{\text{eff}} = \frac{C_{\text{red}}}{\|\nabla(u-u_{\ell+1})\| \,/\, \|\nabla(u-u_\ell)\|}$$



Effectivity index of the reduction factor C_{red}

$$I_{\text{low}}^{\text{eff}} = \frac{\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell}}{\underline{\eta}_{\mathcal{M}_\ell}}$$



Effectivity index of $\underline{\eta}_{\mathcal{M}_\ell}$

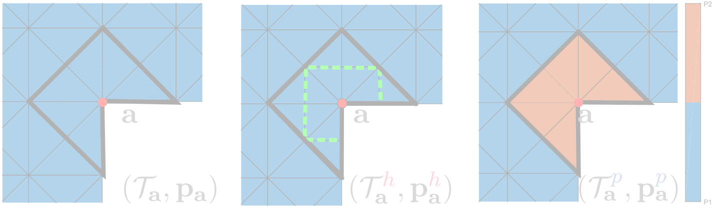
Outline

hp-strategy - the 1st attempt

Goal: to determine the next-level mesh $\mathcal{T}_{\ell+1}$ and degrees $\mathbf{p}_{\ell+1}$

- On each marked patch $\omega_{\mathbf{a}}$, $\mathbf{a} \in \widetilde{\mathcal{V}}_\ell$ calculate the **2 residual liftings** $r_{\mathbf{a}}^h \in V_{\mathbf{a}}^h$ and $r_{\mathbf{a}}^p \in V_{\mathbf{a}}^p$: (ref $\in \{h, p\}$)

$$(\nabla r_{\mathbf{a}}^{\mathrm{ref}}, \nabla v_{\mathbf{a}}^{\mathrm{ref}})_{\omega_{\mathbf{a}}} = (f, v_{\mathbf{a}}^{\mathrm{ref}})_{\omega_{\mathbf{a}}} - (\nabla u_\ell, \nabla v_{\mathbf{a}}^{\mathrm{ref}})_{\omega_{\mathbf{a}}} \quad \forall\, v_{\mathbf{a}}^{\mathrm{ref}} \in V_{\mathbf{a}}^{\mathrm{ref}}.$$



- **Idea:** max $\|\nabla r_{\mathbf{a}}^{\mathrm{ref}}\|_{\omega_{\mathbf{a}}}$ max $\eta_{\mathcal{M}_\ell}$ min $\|\nabla(u - u_{\ell+1})\|$
- If $\|\nabla r_{\mathbf{a}}^h\|_{\omega_{\mathbf{a}}} \geq \|\nabla r_{\mathbf{a}}^p\|_{\omega_{\mathbf{a}}}$, then $\mathbf{a}$ flagged for h-refinement
- If $\|\nabla r_{\mathbf{a}}^h\|_{\omega_{\mathbf{a}}} < \|\nabla r_{\mathbf{a}}^p\|_{\omega_{\mathbf{a}}}$, then $\mathbf{a}$ flagged for p-refinement

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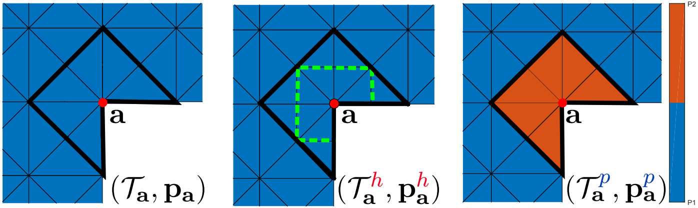

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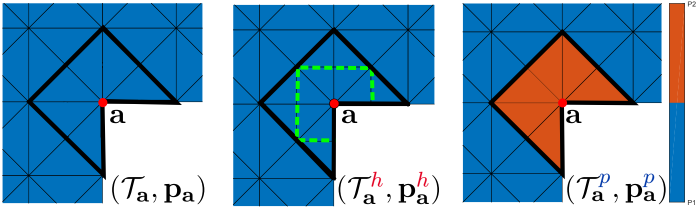

- **Idea:** max $\|\nabla r_{\mathbf{a}}^{\text{ref}}\|_{\omega_{\mathbf{a}}}$ ➞ max $\underline{\eta}_{\mathcal{M}_\ell}$ ➞ min $\|\nabla(u - u_{\ell+1})\|$
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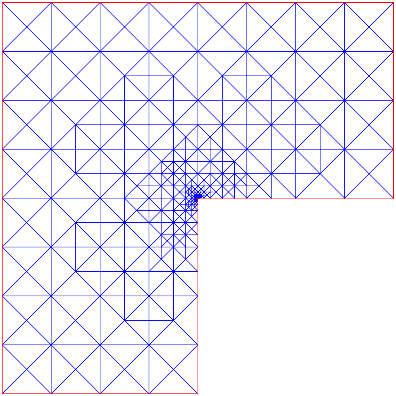
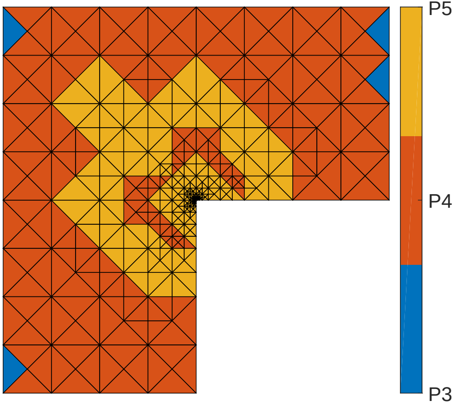
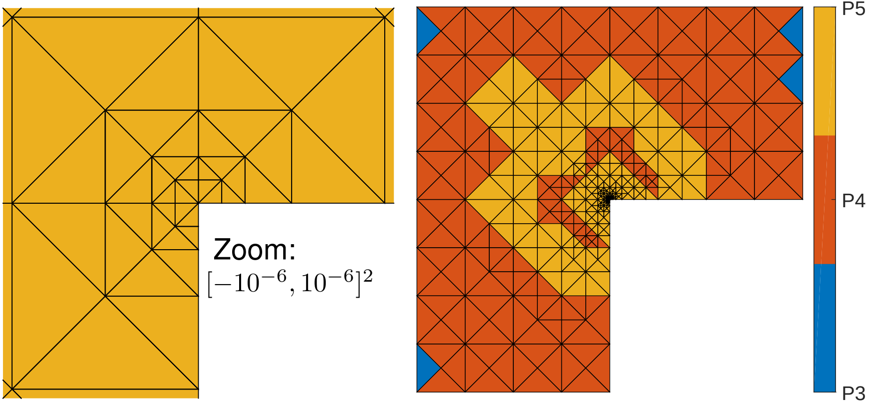
Numerics – hp-strategy





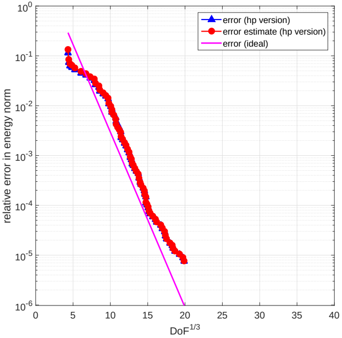
- The initial mesh and the final polynomial degree distribution after 65 iterations of the proposed hp-strategy

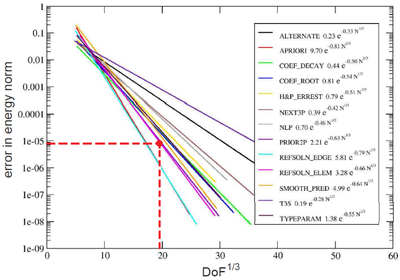
Numerics – hp-strategy



- The final polynomial degree distribution after 65 iterations of the proposed hp-strategy and its detail near the corner (*left*).

Exponential convergence & Assessment of the strategy





W. F. MITCHELL AND M. A. MCCLAIN
A comparison of hp-adaptive strategies for elliptic partial differential equations (2014).

Outline

Motivation

Setting

Reduction factors

hp-strategy

Conclusion

Conclusion

- The effectivity indices of estimates of the reduction factor C_{red} and $\underline{\eta}_{\mathcal{M}_\ell}$ very close to ideal value of $\mathbf{1}$
- the first attempt hp-strategy with exponential order of convergence observed

Future work:

- try to exploit the estimates of C_{red} inside the hp-strategy
- try to prove the convergence of the hp-strategy
- exploiting the multilevel structure in an inexact algebraic solver (multigrid, ...)

Thank you for your attention!

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