The diameter of the minimum spanning tree of a complete graph

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(joint work with L. Addario and B. Reed)

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Erdős-Rényi random graphs

The random graph $G_{n,m}$

- $n$ vertices,
- random permutation of the $\binom{n}{2}$ possible edges,
The random graph $G_{n,m}$

- $n$ vertices,
- random permutation of the $\binom{n}{2}$ possible edges,
- Then, $G_{n,m}$ consists of the $m$ first edges.
A related process

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Connection between $G_{n,m}$ and MST

The diameter of the minimum spanning tree of a complete graph
The diameter of the MST of a complete graph

$D_n$: maximum number of edges on a path
The diameter of the MST of a complete graph

\( D_n \): maximum number of edges on a path
The diameter of the MST of a complete graph

$D_n$: maximum number of edges on a path

**Theorem**

$ED_n = \Theta(n^{1/3})$
Erdős-Rényi random graphs... again

Random graphs: $G_{n,p}$

Random forest: $F_{n,p}$

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Random graphs: $G_{n,p}$
- each edge present with probability $p$
- essentially, $G_{n,p} \sim G_{n,m}$ if $p = m/\binom{n}{2}$
- easier to deal with (independence)

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Random forest: $F_{n,p}$
- increasing sequence of forests
- diameter $D_n(p)$ is increasing
- tracking the diameter up to $D_n(1)$
Evolution of the $G_{n,p}$ graph process (Erdős & Rényi, 1960)
$G_{n,p}$, what does this look like?

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- $p = \frac{\log n}{n}$: the graph is connected
From $G_{n,p}$ to $F_{n,p}$

The diameter of the minimum spanning tree of a complete graph $7/12$
The diameter of the minimum spanning tree of a complete graph

$E D_n(p)$

The diagram shows the expected diameter of the minimum spanning tree $E D_n(p)$ as a function of $p$, where $p$ ranges from 0 to 1.
From $G_{n,p}$ to $F_{n,p}$

The diameter of the minimum spanning tree of a complete graph.
The diameter of the minimum spanning tree of a complete graph \( G_{n,p} \) to \( F_{n,p} \).
The diameter of the minimum spanning tree of a complete graph
From $G_{n,p}$ to $F_{n,p}$

The diameter of the minimum spanning tree of a complete graph

$ED_n(p)$

- $O(\log n)$
- $\Theta(n^{1/3})$
- $? - O(\log^2 n)$
- $? - \Theta(n^{1/3})$

$p$

$0 \quad \frac{1-\epsilon}{n} \quad \frac{1}{n} \quad \frac{1+\epsilon}{n} \quad \frac{\log n}{n} \quad 1$
From $G_{n,p}$ to $F_{n,p}$

The diameter of the minimum spanning tree of a complete graph
From $G_{n,p}$ to $F_{n,p}$

The diameter of the minimum spanning tree of a complete graph $n^{1/3}$
The diameter of the minimum spanning tree of a complete graph

\[ \mathbb{E}D_n(p) \]

\[ \Theta(n^{1/3}) \]

\[ \frac{1}{n} \]

\[ \frac{1 + \epsilon}{n} \]
Thou shalt not jump too far

The diameter of the minimum spanning tree of a complete graph $d(n) = \Theta(n^{1/3})$.
Thou shalt not jump too far

\[ ED_n(p_k) = ED_n(p_0) + \sum_{i=0}^{k-1} ED_n(p_i+1) - ED_n(p_i) \]

\[ \Theta(n^{1/3}) \]

The diameter of the minimum spanning tree of a complete graph
Thou shalt not jump too far

\[ \text{ED}_n(p_k) = \begin{cases} \text{ED}_n(p_0) \\ + \sum_{i=0}^{k-1} \text{ED}_n(p_{i+1}) - \text{ED}_n(p_i) \end{cases} \]
Thou shalt not jump too far

\[ ED_n(p_k) = \begin{cases} ED_n(p_0) \\ + \sum_{i=0}^{k-1} ED_n(p_{i+1}) - ED_n(p_i) \end{cases} \]

The diagram illustrates the behavior of \( ED_n(p_k) \) as \( p \) increases from \( p_0 = \frac{1}{n} \) to \( p_k = \frac{1+\epsilon}{n} \). The function \( \Theta(n^{1/3}) \) is shown to bound the growth of \( ED_n(p) \) as \( p \) increases.
Bounding $\mathbf{ED}_n(p_{i+1}) - \mathbf{ED}_n(p_i)$

Assume $G_i \subset G_{i+1}$, with diameters $D_n(p_i)$ and $D_n(p_{i+1})$
Bounding $ED_n(p_{i+1}) - ED_n(p_i)$

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Claim

\[ D_n(p_{i+1}) \leq D_n(p_i) \]
Assume $G_i \subset G_{i+1}$, with diameters $D_n(p_i)$ and $D_n(p_{i+1})$.

Claim

$$D_n(p_{i+1}) \leq D_n(p_i) + 2L_n(G_{i+1} \setminus G_i) + 2$$
How to choose $p_i$?
Using the right lens

How to choose $p_i$?

- need a non-trivial scaling
Using the right lens

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- need a non-trivial scaling
- self-similarity
Using the right lens

How to choose $p_i$?

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- self-similarity
- bounding $E L_n(G_{n,p_{i+1}} \setminus C_1(G_{n,p_i}))$
How to choose $p_i$?
- need a non-trivial scaling
- self-similarity
- bounding $E \log_n(G_{n,p_{i+1}} \setminus C_1(G_{n,p_i}))$

**Theorem (Łuczak, 1990)**

Let $p = 1/n + fn^{-4/3}$, with $f \to \infty$. Then, with probability going to $1$,
- largest component $H_n(p)$ has size $\Omega(fn^{2/3})$
- all other have size $O(n^{2/3}/f)$
- $H_n(p) \subset H_n(p + \delta)$ for all $\delta \geq 0$
Putting all together

Choose \( p_i = \frac{1}{n} + f_i n^{-4/3} \) with \( f_i = c^i \).
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- improved Łuczak’s results in the critical window
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$$\Rightarrow \mathbb{E}D_n(p_{i+1}) - \mathbb{E}D_n(p_i) = O(n^{1/3}/f_{1/6})$$
Putting all together

Choose \( p_i = 1/n + f_i n^{-4/3} \) with \( f_i = c^i \).

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\[
\Rightarrow E D_n(p_{i+1}) - E D_n(p_i) = O(n^{1/3}/f^{1/6})
\]

Then,

\[
E D_n\left(\frac{1+\epsilon}{n}\right) = \Theta(n^{1/3})
\]
And now?

Further developments:

- ∃c : $ED_n \sim cn^{1/3}$? Which $c$?
- what about sparser ground graphs? Hypercube?
Further developments:

- $\exists c : ED_n \sim cn^{1/3}$? Which $c$?
- what about sparser ground graphs? Hypercube?

Thank you!