p-robust equivalence of global continuous and local discontinuous approximation, a p-stable local projector, and optimal elementwise hp approximation estimates in H^1

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Abstract

Let a polygon or polyhedron Ω, a function v in the Sobolev space $H^1(\Omega)$, and a simplicial mesh of Ω be given. We prove the equivalence of two piecewise polynomial best approximations of v: 1) globally on the whole computational domain Ω, with the (trace) continuity requirement; 2) locally on each mesh element, without any interelement continuity requirement. The former (global-best continuous piecewise polynomial approximation) arises in numerical methods for partial differential equations related to the $H^1(\Omega)$ space, whereas the latter (local-best discontinuous piecewise polynomial approximation) is a key quantity in approximation theory. Crucially, we establish p-robustness in that the equivalence constant only depends on the mesh shape regularity and the spatial dimension. This improves the recent results of [Found. Comput. Math. **16** (2016), 723–750] and [Numer. Math. **135** (2017), 1073–1119], where the equivalence constant was possibly dependent (algebraically or logarithmically) on the underlying polynomial degree. Consequently, we obtain fully h- and p- (mesh-size- and polynomial-degree-) optimal approximation estimates under the minimal Sobolev regularity only requested separately on each mesh element, where we also cover locally variable polynomial degrees. These two results immediately follow by our construction of an operator from the infinite-dimensional Sobolev space $H^1(\Omega)$ to its finite-dimensional piecewise polynomial subspace that has the following properties: 1) it is defined over the entire $H^1(\Omega)$ and preserves boundary conditions imposed on a part of the boundary of Ω; 2) it is defined locally in a neighborhood of each mesh element; 3) it is based on elementwise H^1-orthogonal polynomial projections; 4) it is a projector, i.e., it leaves intact objects that are already continuous piecewise polynomials; 5) it is locally and p-robustly stable in the $H^1(\Omega)$-seminorm; 6) its approximation property is locally and p-robustly equivalent to that of the local discontinuous (elementwise H^1-orthogonal) projection.

Key words: Sobolev space H^1, best approximation, continuous approximation, discontinuous approximation, piecewise polynomial, local–global equivalence, minimal regularity, elementwise regularity, projector, hp finite elements, error bound, polynomial-degree robustness

1 Introduction

Let Ω be a polygon or polyhedron and let $H^1(\Omega)$ be the Sobolev space of functions square-integrable together with their weak gradients, cf. Adams [1], Ciarlet [16], or Ern and Guermond [23]. Let $\mathcal{T}_h$ be a simplicial mesh of Ω.

1.1 h and hp approximation in H^1

The question of error arising in approximation of $v \in H^1(\Omega)$ by continuous piecewise polynomials over $\mathcal{T}_h$ is central in numerical approximation and namely in finite element analysis [16, 23]. Numerous (quasi-)interpolation operators were proposed and studied in the past in Clément [17], Dupont and Scott [21], Scott and Zhang [37], Bernardi and Girault [9], Falk and Winther [27, 28], Bank and Ovall [8], Ern and Guermond [22], Arnold and Guzmán [4], Gawlik *et al.* [30], and the references therein. Often, the approximation error is studied with respect to the mesh size h, for a bounded (and uniform) polynomial

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degree p. In the hp context, the approximation error with respect to both the principal parameters, the mesh size h and the polynomial degree p, needs to be examined, see Babuška and Guo [7], Babuška and Suri [6], Schwab [36], Ainsworth and Kay [3], Demkowicz and Buffa [18], Melenk [33], Karkulik and Melenk [31], and the references therein.

1.2 Equivalence of global continuous and local discontinuous approximation

A central result has been recently established in Veeser [38] see also the predecessor results in Carstensen *et al.* [13, Theorem 2.1 and inequalities (3.2), (3.5), and (3.6)] and Aurada *et al.* [5, Proposition 3.1]. It shows the equivalence of the global-best approximation of v by continuous piecewise polynomials over $\mathcal{T}_h$ with the local-best approximation of v by discontinuous piecewise polynomials over $\mathcal{T}_h$. This seems surprising at a first sight, since the discontinuous piecewise polynomial space over $\mathcal{T}_h$ is (much) bigger than the continuous one, not imposing any (trace) continuity constraint. Congruently, the best continuous piecewise polynomial approximation arises from a global minimization over the whole Ω, whereas the best discontinuous piecewise polynomial approximation is obtained by a minimization separately over all mesh elements $K \in \mathcal{T}_h$. In [38], this result is established for a bounded (and uniform) polynomial degree p, where the hidden equivalence constant degrades (algebraically) with increasing the polynomial degree. Improvement of this dependence (to logarithmic) in two space dimensions has been later developed in Canuto *et al.* [12, Theorem 4] in the context of study of convergence and optimality hp finite elements, relying on constructive approximation of Binev [10].

1.3 Main results of this manuscript

Our main result is a construction of an interpolation operator from $H^1_{0,\mathrm{D}}(\Omega)$ to continuous piecewise polynomials over $\mathcal{T}_h$, $P_{hp} : H^1_{0,\mathrm{D}}(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$ (details on notation are given in Section 2 below) with the following properties: 1) it is defined over the entire $H^1_{0,\mathrm{D}}(\Omega)$, preserving the homogeneous boundary conditions imposed on a part Γ_D of the boundary of Ω; 2) it is defined locally in a neighborhood of each mesh element $K \in \mathcal{T}_h$; 3) it is based on elementwise H^1-orthogonal polynomial projections; 4) it is a projector, i.e., it leaves intact objects that are already continuous piecewise polynomials,

$$P_{hp}(v) = v \qquad \forall v \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega); \tag{1.1}$$

(projection)

5) it is locally and p-robustly stable in the $H^1(\Omega)$-seminorm, i.e.,

$$\|\nabla P_{hp}(v)\|_K^2 \lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \|\nabla v\|_L^2 \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega), \tag{1.2}$$

(H^1-stability)

where $\widetilde{\mathcal{T}}_K$ is an extended element patch consisting of two layers of vertex neighbors of $K \in \mathcal{T}_h$; 6) its approximation property is locally and p-robustly equivalent to that of the local discontinuous (elementwise H^1-orthogonal) projection:

$$\|\nabla(v - P_{hp}(v))\|_K^2 \quad \lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \min_{v_p\in\mathcal{P}_{\underline{p}_K}(L)} \|\nabla(v - v_p)\|_L^2 \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega), \tag{1.3}$$

(approximation p-robustly equivalent to elementwise H^1-orthogonal projector)

where $\underline{p}_K$ is the minimal polynomial degree over $\widetilde{\mathcal{T}}_K$, see (2.7). The generic constants hidden in (1.2) and (1.3) only depend on the mesh shape-regularity κ_h given by (2.1) below, the polynomial-variation parameter κ_p given by (2.8) below, and the space dimension d, which improves the results in [8, 22, 4, 30, 18, 33, 31] discussed above in Section 1.1. Details are given in Definition 3.5 and Theorem 3.7. The properties of P_{hp} immediately lead to two important consequences. Let $v \in H^1_{0,\mathrm{D}}(\Omega)$.

First, for a uniform polynomial degree $p \geq 1$ for simplicity,

$$\min_{v_{hp}\in\mathcal{P}_p(\mathcal{T}_h)\cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(v - v_{hp})\|^2 \approx \sum_{K\in\mathcal{T}_h} \min_{v_p\in\mathcal{P}_p(K)} \|\nabla(v - v_p)\|_K^2, \tag{1.4}$$

(p-robust global continuous – local discontinuous equivalence)

where the hidden constant only depends on κ_h and d. The full version of this result, also considering variable polynomial degree, is stated in Theorem 3.3. This improves [38, Theorem 2] and [12, Theorem 4] discussed in Section 1.2, where the equivalence constant was unfavorably (algebraically or logarithmically) dependent on the polynomial degree p.

Second, (1.3) immediately implies the optimal hp approximation bound

$$\|\nabla(v - P_{hp}v)\|_K^2 \lesssim \sum_{L\in\widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(s_L-1,\underline{p}_K)}}{\underline{p}_K^{s_L-1}}\|v\|_{H^{s_L}(L)}\right)^2 \qquad \forall K \in \mathcal{T}_h \tag{1.5}$$

(optimal local hp approximation estimate)

whenever the approximated function $v \in H^1_{0,\mathrm{D}}(\Omega)$ additionally has, separately on each mesh element $K \in \mathcal{T}_h$, the Sobolev regularity

$$v|_K \in H^{s_K}(K) \text{ for a Sobolev regularity exponent } s_K \geq 1. \tag{1.6}$$

The constant hidden in $\lesssim$ in (1.5) only depends on κ_h, κ_p, d, and the regularity exponents s_K. In the context of the discussion in Sections 1.1 and 1.2, this extends [38, equation (3)] to the hp-setting while removing the possible dependence of the generic constant on the polynomial degree p. In comparison with hp approximation estimates such as [31, Corollary 3.5], the regularity of the approximated function v is only requested elementwise. Note that in (1.5)–(1.6), neither any global and minimal regularity over the entire computational domain Ω such as $v \in H^s(\Omega)$ with $s > 1$ is requested, nor $v|_\omega \in H^s(\omega)$ for some patch subdomains $\omega \subset \Omega$ and $s > 1$ is needed. Details form the content of Theorem 3.4.

1.4 Crucial tools: polynomial extension operators and stable decompositions

There are two crucial tools used to obtain the above results. First, these are polynomial extension operators. After the seminal contributions in Gagliardo [29] and Muñoz-Sola [34], these have been obtained in Ainsworth and Demkowicz [2] in two space dimensions and in Demkowicz *et al.* [19] in three space dimensions, see also the references therein. We more precisely employ their broken extensions on patches of elements, obtained in Ern and Vohralík [26, Theorems 2.2 and 2.4, Corollaries 3.1 and 3.7], following Braess *et al.* [11] (cf. also Chaumont-Frelet and Vohralík [14, Corollaries 3.4 and 4.4]). Second, these are p-robust stable decompositions. We will namely use that of Schöberl *et al.* [35] for a uniform polynomial degree in both two and three space dimensions and that of Karkulik *et al.* [32] for a variable polynomial degree in two space dimensions

1.5 Organization of this manuscript

We set up the notation in Section 2. We then present our main results in full details in Section 3, with the more involved proofs collected in Sections 4 and 5. We also state four independent results in the Appendix. We first formulate the stable decomposition results from [35, 32] in a form suitable for us in Appendix A. We then study patch enumerations in respectively two and three space dimensions in Appendices B and C. We finally generalize the results from [26, 14] to larger (extended) patches and no trace boundary conditions in Appendix D.

This contribution only concerns the H^1 case. Extensions to the $\boldsymbol{H}(\mathrm{div})$ context are addressed in Demkowicz and Vohralík [20], whereas the $\boldsymbol{H}(\mathrm{curl})$ case is studied in Vohralík [39].

2 Setting and notation

In this section, we collect the notation.

2.1 Domain Ω, simplicial mesh $\mathcal{T}_h$, and patch subdomains ω

Let the computational domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open, bounded, and connected Lipschitz polygon or polyhedron. We reserve the notation $\omega \subset \mathbb{R}^d$, possibly with subscripts, for open, bounded, Lipschitz, and polygonal or polyhedral subdomains of Ω corresponding to a face-connected set of mesh elements from $\mathcal{T}_h$; moreover, we suppose $\overline{\omega}$ contractible (homotopic to a ball). Here $\mathcal{T}_h$ is a simplicial mesh of Ω, i.e., a collection of nontrivial closed triangles or tetrahedra K covering $\overline{\Omega}$, where the intersection of two different simplices is either empty or their common vertex, edge, or face. The shape-regularity parameter κ_h is given by

$$\kappa_h := \max_{K\in\mathcal{T}_h} \frac{h_K}{\rho_K}, \tag{2.1}$$

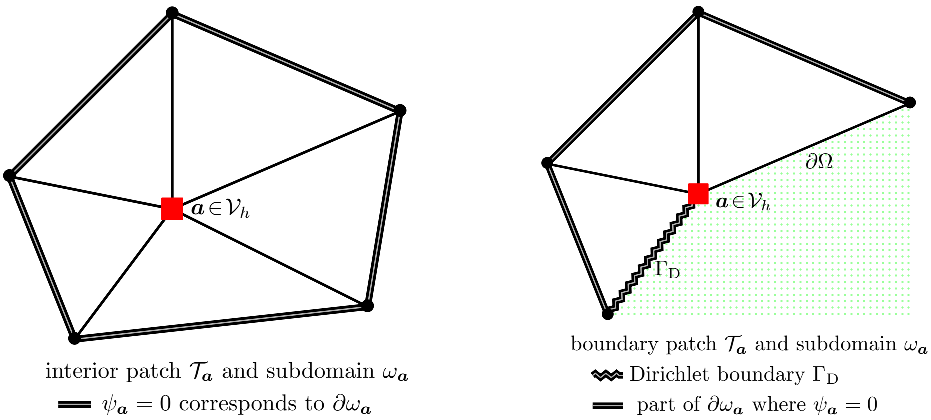

Figure 1: Vertex patch $\mathcal{T}_{\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathcal{V}_h$ in the interior of Ω (left) and on the boundary of Ω (right)

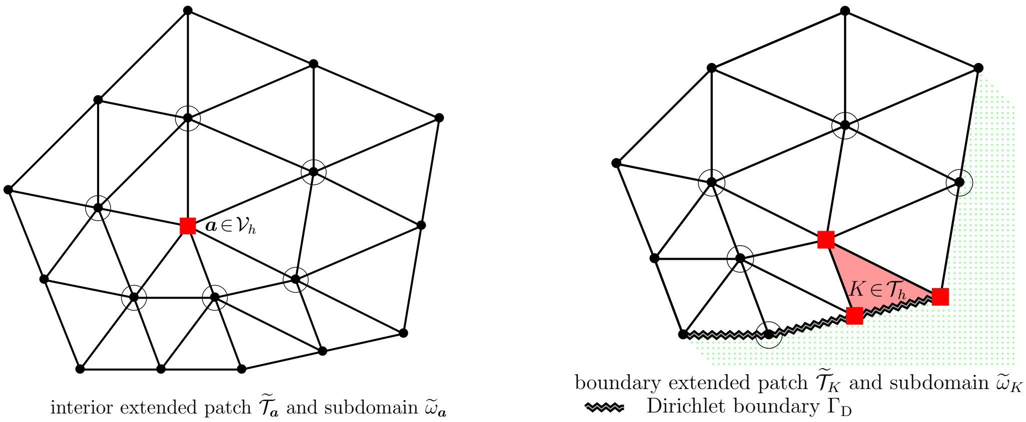

Figure 2: Extended vertex patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathcal{V}_h$ in the interior of Ω (generated by the marked vertex patches $\mathcal{T}_{\boldsymbol{b}}$) (left) and extended element patch $\widetilde{\mathcal{T}}_K$ for an element $K \in \mathcal{T}_h$ on the boundary of Ω (generated by the marked vertex patches $\mathcal{T}_{\boldsymbol{b}}$) (right)

where h_K is the diameter of the simplex K and ρ_K that of the largest ball contained in K. Uniformly bounded κ_h allows for families of strongly graded meshes with local refinements, but not for anisotropic elements.

2.2 Vertices, edges, faces, and patches of mesh elements

For a simplex $K \in \mathcal{T}_h$, denote by $\mathcal{F}_K$ the set of its $(d-1)$-dimensional faces and by $\mathcal{V}_K$ the set of its vertices; in three space dimensions, we additionally use the set $\mathcal{E}_K$ of the edges of K. Let $\mathcal{V}_h$ collect all mesh vertices. For a vertex $\boldsymbol{a} \in \mathcal{V}_h$, denote by $\mathcal{T}_{\boldsymbol{a}}$ the patch of the elements of $\mathcal{T}_h$ that share $\boldsymbol{a}$ and by $\omega_{\boldsymbol{a}}$ the corresponding subdomain, see Figure 1. We will also need the extended vertex patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and the corresponding subdomain $\widetilde{\omega}_{\boldsymbol{a}}$; this includes $\mathcal{T}_{\boldsymbol{a}}$ and all elements $L \in \mathcal{T}_h$ sharing a vertex with $K \in \mathcal{T}_{\boldsymbol{a}}$, see Figure 2 (left) for an illustration. For a simplex $K \in \mathcal{T}_h$, let $\widetilde{\mathcal{T}}_K$ be the extended element patch given by the extended vertex patches $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ for all vertices of $\boldsymbol{a}$ of K, see Figure 2 (right) for an illustration. $\widetilde{\mathcal{T}}_K$ comprises K and all L sharing a vertex with K or its vertex neighbor; the corresponding subdomain is $\widetilde{\omega}_K$. We collect the vertices from respectively $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and $\widetilde{\mathcal{T}}_K$ in the sets $\widetilde{\mathcal{V}}_{\boldsymbol{a}}$ and $\widetilde{\mathcal{V}}_K$. Diameters of respectively $\omega_{\boldsymbol{a}}$, $\widetilde{\omega}_{\boldsymbol{a}}$, and $\widetilde{\omega}_K$ are denoted by $h_{\omega_{\boldsymbol{a}}}$, $h_{\widetilde{\omega}_{\boldsymbol{a}}}$, and $h_{\widetilde{\omega}_K}$.

2.3 Hat functions and the partition of unity

Let the continuous, piecewise first-order polynomial (affine) "hat" function $\psi^{\boldsymbol{a}}$ take value 1 at the vertex $\boldsymbol{a}$ and zero at all the other vertices. We note that $\omega_{\boldsymbol{a}}$ corresponds to the support of $\psi^{\boldsymbol{a}}$ and that these functions form the partition of unity

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi^{\boldsymbol{a}} = 1. \tag{2.2}$$

2.4 Boundary subsets $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$

Let $\Gamma_{\rm D}$, $\Gamma_{\rm N}$ be two disjoint, relatively open, and possibly empty subsets of the boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_{\rm D}} \cup \overline{\Gamma_{\rm N}}$. We also require that $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$ have polygonal Lipschitz boundaries and we assume that each boundary face of the mesh $\mathcal{T}_h$ lies entirely either in $\overline{\Gamma_{\rm D}}$ or in $\overline{\Gamma_{\rm N}}$.

2.5 The space H^1 on the entire computational domain and its subdomains

Let $\omega \subseteq \Omega$. We let $L^2(\omega)$ be the space of scalar-valued square-integrable functions defined on ω and we use the notation $\boldsymbol{L}^2(\omega) := [L^2(\omega)]^d$ for vector-valued functions with each component in $L^2(\omega)$. We denote by $\|\cdot\|_\omega$ the $L^2(\omega)$ or $\boldsymbol{L}^2(\omega)$ norm and by $(\cdot,\cdot)_\omega$ the corresponding scalar product; we drop the index when $\omega = \Omega$. Then, $H^1(\omega) = \{v \in L^2(\omega);\, \nabla v \in \boldsymbol{L}^2(\omega)\}$. Moreover, if $\partial\omega \cap \Gamma_{\rm D}$ is of nonzero measure ($\partial\omega$ contains at least one face from the Dirichlet boundary $\overline{\Gamma_{\rm D}}$), then we use

$$H^1_{0,{\rm D}}(\omega) := \{v \in H^1(\omega);\, v = 0 \text{ on } (\partial\omega \cap \Gamma_{\rm D})^\circ\}, \tag{2.3}$$

Let $\boldsymbol{a} \in \mathcal{V}_h$ be a mesh vertex. For the vertex patch subdomain $\omega_{\boldsymbol{a}}$, cf. Figure 1, we will employ the specific notation $H^1_{0,{\rm D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$ for the subspace of $H^1(\omega_{\boldsymbol{a}})$ with zero trace on that faces in $\partial\omega_{\boldsymbol{a}}$ where the hat function $\psi^{\boldsymbol{a}}$ vanishes (all $\partial\omega_{\boldsymbol{a}}$ for interior vertices) or which lie in the Dirichlet boundary $\overline{\Gamma_{\rm D}}$,

$$H^1_{0,{\rm D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}}) := \{v \in H^1(\omega_{\boldsymbol{a}});\, v = 0 \text{ on } \partial\omega_{\boldsymbol{a}} \cap \{\psi^{\boldsymbol{a}} = 0\} \text{ and } (\partial\omega_{\boldsymbol{a}} \cap \Gamma_{\rm D})^\circ\}. \tag{2.4}$$

In Figure 1, this respectively corresponds to the double line (for interior patches $\mathcal{T}_{\boldsymbol{a}}$, left) or to the double and zigzag lines (for boundary patches $\mathcal{T}_{\boldsymbol{a}}$, right). Congruently, for an arbitrary patch subdomain ω and a vertex $\boldsymbol{a}$ therein, $H^1_{0,{\rm D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}} \cap \omega)$ stands for the subspace of $H^1(\omega_{\boldsymbol{a}} \cap \omega)$ with zero trace on that faces in $\partial(\omega_{\boldsymbol{a}} \cap \omega)$ where the hat function $\psi^{\boldsymbol{a}}$ vanishes or which lie in the Dirichlet boundary $\overline{\Gamma_{\rm D}}$,

$$\begin{aligned} H^1_{0,{\rm D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}} \cap \omega) := \{v \in H^1(\omega_{\boldsymbol{a}} \cap \omega);\, v = 0 \\ \text{on } \partial(\omega_{\boldsymbol{a}} \cap \omega) \cap \{\psi^{\boldsymbol{a}} = 0\} \text{ and } (\partial(\omega_{\boldsymbol{a}} \cap \omega) \cap \Gamma_{\rm D})^\circ\}. \end{aligned} \tag{2.5}$$

This is as above in (2.4), with the exception of vertices $\boldsymbol{a}$ on the boundary of ω: functions from $H^1_{0,{\rm D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}} \cap \omega)$ do not vanish on $\partial(\omega_{\boldsymbol{a}} \cap \omega)$ unless this is a part of $\overline{\Gamma_{\rm D}}$.

2.6 Discontinuous piecewise polynomials

We suppose that with each mesh element $K \in \mathcal{T}_h$, there is an associated polynomial degree $p_K \geq 1$. This is a general variable polynomial degree case. In turn, we call a uniform polynomial degree case the situation where $p_K = p \geq 1$ for all $K \in \mathcal{T}_h$. For each simplex $K \in \mathcal{T}_h$, we denote by $\mathcal{P}_{p_K}(K)$ the space of scalar-valued polynomials on K of total degree at most p_K. On the whole mesh, we then let $\mathcal{P}_p(\mathcal{T}_h)$ be given by the discontinuous (broken) piecewise polynomials elementwise from $\mathcal{P}_{p_K}(K)$,

$$\mathcal{P}_p(\mathcal{T}_h) := \{v_h \in L^2(\Omega);\, v_h|_K \in \mathcal{P}_{p_K}(K) \quad \forall K \in \mathcal{T}_h\}. \tag{2.6}$$

To form the usual continuous (conforming) finite element spaces, we will write $\mathcal{P}_p(\mathcal{T}_h) \cap H^1(\Omega)$ (this could be equivalently written as $\mathcal{P}_p(\mathcal{T}_h) \cap C^0(\overline{\Omega})$) and similarly for the subspaces reflecting the boundary conditions. The same notation will also be used on the patches $\mathcal{T}_{\boldsymbol{a}}$, $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$, and $\widetilde{\mathcal{T}}_K$. For v_{hp} from $\mathcal{P}_p(\mathcal{T}_h)$, we denote by ∇v_{hp} the broken (elementwise) gradient, $(\nabla v_{hp})|_K := \nabla(v_{hp}|_K)$ for all $K \in \mathcal{T}_h$.

For a mesh element $K \in \mathcal{T}_h$, let

$$\underline{p}_K := \min_{L \in \widetilde{\mathcal{T}}_K} \{p_L\} \tag{2.7}$$

be the smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$. We will need below the polynomial-variation parameter κ_p given by

$$\kappa_p := \max_{K \in \mathcal{T}_h} \frac{p_K}{\underline{p}_K}. \tag{2.8}$$

Note that $\kappa_p = 1$ if the polynomial degree p is uniform, $p_K = p$ for all $K \in \mathcal{T}_h$.

2.7 Discontinuous piecewise polynomials with degree lowered according to the neighbors

Let $K \in \mathcal{T}_h$. Let simply $\underline{\mathcal{P}_{p_K}(K)} := \mathcal{P}_{p_K}(K) = \mathcal{P}_p(K)$ if the polynomial degree p is uniform, $p_K = p$ for all $K \in \mathcal{T}_h$. To work with p variable, we let $\underline{\mathcal{P}_{p_K}(K)}$ be the subspace of $\mathcal{P}_{p_K}(K)$ where the trace on each face $F_L \in \mathcal{F}_K$ between the element K and its face neighbor $L \in \mathcal{T}_h$ is a polynomial of degree $\min\{p_K, p_L\}$ and, if $d = 3$, the trace on each edge $e_L \in \mathcal{E}_K$ between the element K and its edge neighbor $L \in \mathcal{T}_h$ is a polynomial of degree $\min\{p_K, p_L\}$. The space $\underline{\mathcal{P}_{p_K}(K)}$ is "aware" of the polynomial degrees of the edge and face neighbors of K. We then define

$$\underline{\mathcal{P}_p(\mathcal{T}_h)} := \{v_h \in L^2(\Omega);\, v_h|_K \in \underline{\mathcal{P}_{p_K}(K)} \quad \forall K \in \mathcal{T}_h\}. \tag{2.9}$$

In the uniform polynomial degree case,

$$\underline{\mathcal{P}_p(\mathcal{T}_h)} = \mathcal{P}_p(\mathcal{T}_h),$$

i.e., $\underline{\mathcal{P}_p(\mathcal{T}_h)}$ from (2.9) is simply the broken piecewise polynomial space $\mathcal{P}_p(\mathcal{T}_h)$ of (2.6). For a general variable polynomial degree, there holds

$$\underline{\mathcal{P}_p(\mathcal{T}_h)} \cap H^1_{0,\mathrm{D}}(\Omega) = \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega), \tag{2.10}$$

since the trace-continuity requirement in $H^1_{0,\mathrm{D}}(\Omega)$ exactly leads to the polynomial order decrease of $\underline{\mathcal{P}_p(\mathcal{T}_h)}$ with respect to $\mathcal{P}_p(\mathcal{T}_h)$. We, however, note that

$$\underline{\mathcal{P}_p(\mathcal{T}_\omega)} \cap H^1(\omega) \neq \mathcal{P}_p(\mathcal{T}_\omega) \cap H^1(\omega), \qquad \underline{\mathcal{P}_p(\mathcal{T}_\omega)} \cap H^1_{0,\mathrm{D}}(\omega) \neq \mathcal{P}_p(\mathcal{T}_\omega) \cap H^1_{0,\mathrm{D}}(\omega) \tag{2.11}$$

for a submesh $\mathcal{T}_\omega$ of $\mathcal{T}_h$ and the corresponding patch subdomain $\omega \subseteq \Omega$. This follows since the polynomial order decrease in $\underline{\mathcal{P}_p(\mathcal{T}_\omega)}$ also applies on the boundary of ω on the above left-hand sides, but not on the above right-hand sides.

2.8 Elementwise Lagrange interpolator

Separately in each mesh element $K \in \mathcal{T}_h$, we will need the Lagrange interpolate operator, cf. Ciarlet [16, Section 2.2] or Ern and Guermond [23, Section 7.4]. We more precisely let

$$I^{\mathcal{L}}_{hp} : \{v \in L^2(\Omega); v|_K \in C^1(K) \quad \forall K \in \mathcal{T}_h\} \to \underline{\mathcal{P}_p(\mathcal{T}_h)}$$

be prescribed, separately on each $K \in \mathcal{T}_h$, by

$$(I^{\mathcal{L}}_{hp}(v)(\boldsymbol{x}))|_K = (v|_K)(\boldsymbol{x}) \qquad \text{in all Lagrange nodes } \boldsymbol{x} \text{ of the space } \underline{\mathcal{P}_{p_K}(K)}. \tag{2.12}$$

We stress that we will only employ $I^{\mathcal{L}}_{hp}$ on (broken) piecewise polynomials which clearly belong to $\{v \in L^2(\Omega); v|_K \in C^1(K) \quad \forall K \in \mathcal{T}_h\}$; recall that $I^{\mathcal{L}}_{hp}$ cannot be applied directly to functions $v \in H^1_{0,\mathrm{D}}(\Omega)$ which may not admit point values.

2.9 Notation $\lesssim$

We will use the notation $a \lesssim b$ when there holds $a \leq Cb$ for a positive constant C and $a \approx b$ when $a \lesssim b$ and $b \lesssim a$ hold simultaneously. All dependencies of C will systematically be given. In any case, all such constants C in this manuscript are independent of the mesh size h and of the polynomial degree p.

3 Main results

We present here our main results.

We will need the following assumption on enumeration of extended patches if $d = 3$ or if $d = 2$ and Γ_{D} is non-empty:

Assumption 3.1 (Enumeration of extended patches). *Let $K \in \mathcal{T}_h$. Consider a patch given by a collection of extended vertex patches $\widehat{\mathcal{T}}_K := \cup_{\boldsymbol{a} \in \widehat{\mathcal{V}}_K} \{\widetilde{\mathcal{T}}_{\boldsymbol{a}}\}$ as per Section 2.2, where $\widehat{\mathcal{V}}_K$ is a (sub)set of vertices $\mathcal{V}_K$. Let $\widehat{\omega}_K$ be the associated open subdomain. If $d = 3$ and $\partial\widehat{\omega}_K$ does not contain any face from $\partial\Omega$, suppose that $\widehat{\mathcal{T}}_K$ can be enumerated as per Definition C.1. If $d = 3$ and if $\partial\widehat{\omega}_K$ contains at least one face from $\partial\Omega$, or if $d = 2$ with Γ_{D} non-empty and if $\partial\widehat{\omega}_K$ contains at least one face $\overline{\Gamma_{\mathrm{D}}}$, suppose that $\widehat{\mathcal{T}}_K$ can be mapped by d symmetries as in [14] for boundary patches into a patch that can be enumerated as per Definition C.1 or Definition B.1.*

Similarly, for the variable polynomial degree case, since Karkulik *et al.* [32, Theorem 2.5] is only stated in two space dimensions, we assume:

Assumption 3.2 (Stable decomposition with variable polynomial degree in three space dimensions)**.** *Suppose that [32, Theorem 2.5] holds for $d = 3$ and variable polynomial degree $p_K \geq 1$ for all $K \in \mathcal{T}_h$.*

3.1 p-robust equivalence of continuous and discontinuous approximation in H^1

The following result improves [38, Theorem 2] and [12, Theorem 4], removing the possible dependence of the equivalence constant on the polynomial degree p. It also addresses the case of variable polynomial degree.

Theorem 3.3 (p-robust equivalence of global-best and local-best approximations in H^1)**.** *Let $v \in H^1_{0,\mathrm{D}}(\Omega)$, a simplicial mesh $\mathcal{T}_h$ of Ω, and a (variable) polynomial degree $p_K \geq 1$ for all $K \in \mathcal{T}_h$ be given. Let Assumptions 3.1 and 3.2 hold. Then*

$$\min_{v_{hp} \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(v - v_{hp})\|^2 \approx \sum_{K \in \mathcal{T}_h} \min_{v_p \in \mathcal{P}_p(K)} \|\nabla(v - v_p)\|_K^2 \tag{3.1a}$$

(p-robust global continuous – local discontinuous equivalence)

if the polynomial degree is uniform, $p_K = p \geq 1$ for all $K \in \mathcal{T}_h$, and

$$\min_{v_{hp} \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(v - v_{hp})\|^2 \lesssim \sum_{K \in \mathcal{T}_h} \sum_{L \in \widetilde{\mathcal{T}}_K} \min_{v_p \in \mathcal{P}_{\underline{p}_K}(L)} \|\nabla(v - v_p)\|_L^2 \tag{3.1b}$$

if the polynomial degree is variable, $p_K \geq 1$ for all $K \in \mathcal{T}_h$. The hidden constant only depends on the mesh shape-regularity parameter κ_h given by (2.1), the polynomial-variation parameter κ_p given by (2.8), and the space dimension d.

Proof. The left-hand sides of (3.1) employ continuous ($H^1_{0,\mathrm{D}}(\Omega)$-conforming) piecewise polynomials from $\mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, whereas the right-hand sides employ (broken) piecewise polynomials without any interelement continuity requirement.

As for (3.1a), for uniform p, $\mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega) \subset \mathcal{P}_p(\mathcal{T}_h)$, so that the inequality $\gtrsim$ follows trivially. For the $\lesssim$ inequality, we bound the minimum by employing the projector P_{hp} from Definition 3.5. The elementwise use of (3.20) from Theorem 3.7 together with a finite overlap argument by the mesh shape regularity yield the claim:

$$\begin{aligned}
\min_{v_{hp} \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)} \|\nabla(v - v_{hp})\|^2 &\leq \|\nabla(v - P_{hp}(v))\|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla(v - P_{hp}(v))\|_K^2 \\
&\overset{(3.20)}{\lesssim} \sum_{K \in \mathcal{T}_h} \sum_{L \in \widetilde{\mathcal{T}}_K} \min_{v_p \in \mathcal{P}_p(L)} \|\nabla(v - v_p)\|_L^2 \lesssim \sum_{K \in \mathcal{T}_h} \min_{v_p \in \mathcal{P}_p(K)} \|\nabla(v - v_p)\|_K^2.
\end{aligned}$$

As for (3.1b), for variable p, the $\lesssim$ inequality is as above, though we are obliged to keep the lowered polynomial degree $\underline{p}_K$ from (2.7) and we cannot combine the double sum into a single one. Congruently, the inequality $\gtrsim$ does not hold in this case, since, recalling (2.11), for a mesh element $K \in \mathcal{T}_h$, $(\mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega))|_{\widetilde{\omega}_K}$ may contain functions that are not lowered-degree polynomials from $\mathcal{P}_{\underline{p}_K}(\widetilde{\mathcal{T}}_K)$. □

3.2 Optimal local hp approximation estimates under minimal elementwise Sobolev regularity in H^1

We now focus on functions with additional regularity only requested locally on each mesh element. For any element $K \in \mathcal{T}_h$, let $H^{s_K}(K)$ be the usual Sobolev space on the element K with a fixed regularity exponent $s_K \geq 1$, cf. [1, 23]. The following is a fully h- and p- (mesh-size- and polynomial-degree-) optimal approximation estimate under the minimal Sobolev regularity only requested separately on each mesh element. It extends [38, equation (3)] to the hp-setting while removing the possible (unfavorable) dependence of the generic constant on the polynomial degree p. In comparison with [31, Corollary 3.5], the regularity of the approximated function v is only requested elementwise and not patchwise, but the dependency region contains an additional layer of neighbors (extended vertex patch in contrast to vertex patch). Recall the lowered polynomial degree $\underline{p}_K$ from (2.7).

Theorem 3.4 (hp-optimal approximation estimate in H^1 under minimal elementwise Sobolev regularity)**.** *Let a simplicial mesh $\mathcal{T}_h$ of Ω and a (variable) polynomial degree $p_K \geq 1$ for all $K \in \mathcal{T}_h$ be given. Let Assumptions 3.1 and 3.2 hold. There exists a stable local projector $P_{hp} : H^1_{0,\mathrm{D}}(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$ such that, if separately on each mesh element $K \in \mathcal{T}_h$,*

$$v|_K \in H^{s_K}(K) \text{ for a Sobolev regularity exponent } s_K \geq 1, \tag{3.2}$$

then

$$\|\nabla(v - P_{hp}v)\|_K^2 \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \left(\frac{h_L^{\min(s_L-1, \underline{p}_K)}}{\underline{p}_K^{s_L-1}} \|v\|_{H^{s_L}(L)} \right)^2 \qquad \forall K \in \mathcal{T}_h. \tag{3.3}$$

The constant hidden in $\lesssim$ only depends on the mesh shape-regularity parameter κ_h given by (2.1), the polynomial-variation parameter κ_p given by (2.8), the space dimension d, and the regularity exponents s_K.

Proof. We take the projector P_{hp} from Definition 3.5 below. Let $K \in \mathcal{T}_h$ be fixed. The approximation property (3.20) from Theorem 3.7 gives

$$\|\nabla(v - P_{hp}(v))\|_K^2 \quad \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \min_{v_p \in \mathcal{P}_{\underline{p}_K}(L)} \|\nabla(v - v_p)\|_L^2.$$

Thus the claim (3.3) follows from the well-known hp-approximation bounds, see, e.g., [6, Lemma 4.1], with the hidden constant only depending on κ_h, κ_p, d, and s_K. □

3.3 A p-stable local projector in H^1

We finally define our p-stable local projector in H^1 and state its properties. Our construction extends and builds on some ideas from potential reconstruction in a posteriori error estimation, namely [25, 26]. In order to achieve p-robust approximation, broken polynomial extension generalizing [26, 14] and the stable decomposition of [35] or [32] is employed in a correction stage. The construction proceeds in five stages: 1) elementwise L^2-orthogonal projection (local-best approximation); 2) patchwise potential reconstruction and gluing of the patchwise contributions; this stage employs the hat functions $\psi^{\boldsymbol{a}}$ from (2.2) together with the canonical elementwise Lagrange projector $I_{hp}^{\mathcal{L}}$ from (2.12) and builds a projector that would not be p-robust; 3) correction by elementwise bubbles as in [24, equation (3.16)] to achieve an elementwise mean value property; 4) patchwise potential reconstruction of the remainder (relying on the broken polynomial extension from Appendix D) followed by the stable decomposition of Appendix A and gluing of the patchwise contributions into a correction; here, crucially, no hat functions $\psi^{\boldsymbol{a}}$ from (2.2) and no elementwise projector such as $I_{hp}^{\mathcal{L}}$ from (2.12) are used; and 5) combination of the previous steps.

3.3.1 Definition of the projector

Recall the notation from Section 2, namely the definition (2.9) of $\underline{\mathcal{P}_p(\mathcal{T}_h)}$ of piecewise polynomials with degree lowered according to the neighbors.

Definition 3.5 (A stable local projector in H^1)**.** *Let a function $v \in H^1_{0,\mathrm{D}}(\Omega)$, a simplicial mesh $\mathcal{T}_h$ of Ω, and a (variable) polynomial degree $p_K \geq 1$ for all $K \in \mathcal{T}_h$ be given.*

1. *On each mesh element $K \in \mathcal{T}_h$, consider the $H^1(K)$-orthogonal projection of v onto $\underline{\mathcal{P}_{p_K}(K)}$ (without any trace prescription)*

$$\tau_{hp}|_K := \arg \min_{\substack{v_p \in \underline{\mathcal{P}_{p_K}(K)} \\ (v_p,1)_K = (v,1)_K}} \|\nabla(v - v_p)\|_K. \tag{3.4}$$

(elementwise projection τ_{hp})

This gives the broken piecewise polynomial

$$\tau_{hp} \in \underline{\mathcal{P}_p(\mathcal{T}_h)}. \tag{3.5}$$

2. *Starting from τ_{hp}:*
   (a) *On each vertex patch $\mathcal{T}_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{V}_h$, see Figure 1, define the continuous piecewise polynomial $s_p^{\boldsymbol{a}} \in \underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$ via*

$$s_p^{\boldsymbol{a}} := \arg \min_{v_p \in \underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})} \left\| \nabla \left(I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}} \tau_{hp}) - v_p \right) \right\|_{\omega_{\boldsymbol{a}}}; \tag{3.6a}$$

(patchwise "homogeneous" potential reconstruction)

recall from (2.4) that $H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$ *is the subspace of* $H^1(\omega_{\boldsymbol{a}})$ *with zero trace on that faces in* $\partial\omega_{\boldsymbol{a}}$ *where the hat function* $\psi^{\boldsymbol{a}}$ *vanishes or which lie in the Dirichlet boundary* $\overline{\Gamma_{\mathrm{D}}}$. *See Figure 3 (left) for illustration.*

(b) *Extending* $s_p^{\boldsymbol{a}}$ *by zero outside of the patch subdomain* $\omega_{\boldsymbol{a}}$, *define*

$$s_{hp} := \sum_{\boldsymbol{a}\in\mathcal{V}_h} s_p^{\boldsymbol{a}}. \tag{3.6b}$$

(gluing patchwise contributions)

This gives the intermediate continuous piecewise polynomial

$$s_{hp} \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega) \tag{3.7}$$

(standard potential reconstruction s_{hp}*)*

and the broken piecewise polynomial

$$\tau_{hp} - s_{hp} \in \underline{\mathcal{P}_p(\mathcal{T}_h)}. \tag{3.8}$$

(provisional remainder $\tau_{hp} - s_{hp}$*)*

3. *Starting from* $\tau_{hp} - s_{hp}$, *on each mesh element* $K \in \mathcal{T}_h$, *let*

$$\delta_{hp}|_K := 0 \qquad \text{if } p_K < d+1, \tag{3.9a}$$

$$\delta_{hp}|_K := \frac{(\tau_{hp} - s_{hp}, 1)_K}{(b_K, 1)_K} b_K \qquad \text{if } p_K \geq d+1, \tag{3.9b}$$

(elementwise bubble correction if $p_K \geq d+1$*)*

where b_K *is the bubble function on* K, $b_K \in \mathcal{P}_{d+1}(K) \cap H^1_0(K)$, $b_K = \Pi_{\boldsymbol{a}\in\mathcal{V}_K}(\psi^{\boldsymbol{a}}|_K)$. *This gives the intermediate continuous piecewise polynomial*

$$\delta_{hp} \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega) \tag{3.10}$$

and the broken piecewise polynomial

$$\tau_{hp} - s_{hp} - \delta_{hp} \in \underline{\mathcal{P}_p(\mathcal{T}_h)}, \tag{3.11a}$$

$$(\tau_{hp} - s_{hp} - \delta_{hp}, 1)_K = 0 \qquad \forall K \in \mathcal{T}_h \text{ such that } p_K \geq d+1. \tag{3.11b}$$

(remainder $\tau_{hp} - s_{hp} - \delta_{hp}$ *with vanishing lowest-order moments if* $p_K \geq d+1$*)*

4. *Starting from* $\tau_{hp} - s_{hp} - \delta_{hp}$:
   (a) *On each extended vertex patch* $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{V}_h$, *see Figure 2 (left), define*

$$\zeta_p^{\boldsymbol{a}} := 0 \qquad \text{if } p_K < d+1 \text{ for at least one } K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}, \tag{3.12a}$$

or the continuous piecewise polynomial $\zeta_p^{\boldsymbol{a}} \in \underline{\mathcal{P}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_{\boldsymbol{a}})$ *via*

$$\zeta_p^{\boldsymbol{a}} := \arg \min_{\substack{v_p \in \underline{\mathcal{P}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_{\boldsymbol{a}}) \\ (v_p,1)_{\widetilde{\omega}_{\boldsymbol{a}}} = (\tau_{hp} - s_{hp} - \delta_{hp},1)_{\widetilde{\omega}_{\boldsymbol{a}}} = 0 \text{ if } |\partial\widetilde{\omega}_{\boldsymbol{a}} \cap \Gamma_{\mathrm{D}}| = 0}} \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - v_p)\|_{\widetilde{\omega}_{\boldsymbol{a}}} \quad \text{if } p_K \geq d+1 \text{ for all } K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}; \tag{3.12b}$$

(patchwise remainder potential reconstruction)

   recall from (2.3) that $H^1_{0,\mathrm{D}}(\widetilde{\omega}_{\boldsymbol{a}})$ *is the subspace of* $H^1(\widetilde{\omega}_{\boldsymbol{a}})$ *with zero trace on* $\partial\widetilde{\omega}_{\boldsymbol{a}} \cap \Gamma_{\mathrm{D}}$ *when some boundary faces from* $\partial\widetilde{\omega}_{\boldsymbol{a}}$ *lie in* $\overline{\Gamma_{\mathrm{D}}}$. *See Figure 3 (right) for illustration.*

   (b) *On each extended vertex patch* $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{V}_h$, *set*

$$\zeta_p^{\boldsymbol{a},\boldsymbol{a}} := 0 \qquad \text{if } p_K < d+1 \text{ for at least one } K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}, \tag{3.13a}$$

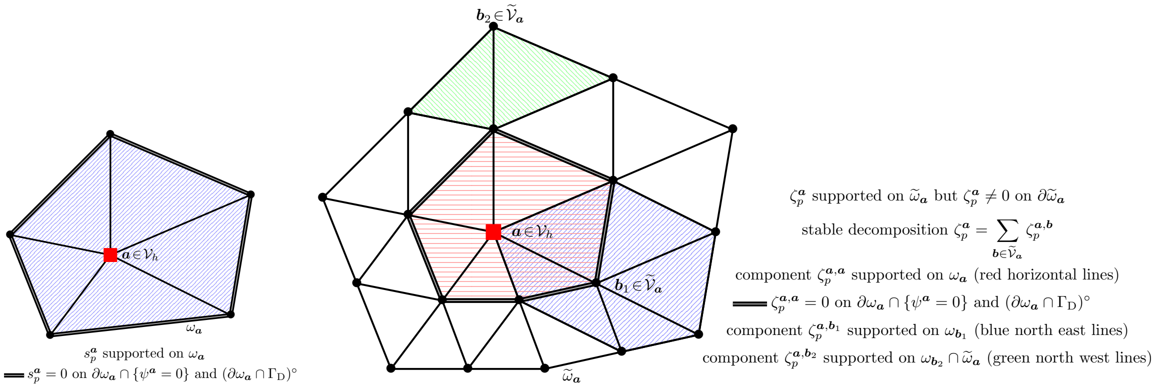

Figure 3: The standard non-p-robust potential reconstruction component $s_p^{\boldsymbol{a}}$ from (3.6a) (left) and the p-robust correction $\zeta_p^{\boldsymbol{a}}$ from (3.12b) together with its stable decomposition (3.13b); only the "interior" component $\zeta_p^{\boldsymbol{a},\boldsymbol{a}}$ is used (right)

or employ to $\zeta_p^{\boldsymbol{a}}$ from (3.12b) the stable decomposition of Corollary A.1 (with $\mathcal{T}_\omega = \widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and $\mathcal{V}_\omega = \widetilde{\mathcal{V}}_{\boldsymbol{a}}$),

$$\zeta_p^{\boldsymbol{a}} = \sum_{\boldsymbol{b} \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \zeta_p^{\boldsymbol{a},\boldsymbol{b}} \quad \text{with in particular} \quad \zeta_p^{\boldsymbol{a},\boldsymbol{a}} \in \underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}}) \\ \text{if } p_K \geq d+1 \text{ for all } K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}. \tag{3.13b}$$

(patchwise p-stable reconstructed remainder decomposition)

See Figure 3 (right) for illustration.

(c) Extending the "interior" component $\zeta_p^{\boldsymbol{a},\boldsymbol{a}}$ by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, define

$$\zeta_{hp} := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \zeta_p^{\boldsymbol{a},\boldsymbol{a}}. \tag{3.14}$$

(gluing patchwise correction contributions)

This gives the intermediate continuous piecewise polynomial

$$\zeta_{hp} \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega). \tag{3.15}$$

(p-robust correction ζ_{hp} by treatment of $\tau_{hp} - s_{hp} - \delta_{hp}$ without $\psi^{\boldsymbol{a}}$ and $I^{\mathcal{L}}_{hp}$)

5. Define

$$P_{hp}(v) := s_{hp} + \delta_{hp} + \zeta_{hp}. \tag{3.16}$$

(combining the previous steps)

This gives the final continuous piecewise polynomial

$$P_{hp}(v) \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega). \tag{3.17}$$

Crucially, this definition is correct:

Lemma 3.6 (Well-posedness of P_{hp})**.** *The linear operator $P_{hp} : H^1_{0,\mathrm{D}}(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$ from Definition 3.5 is well defined.*

We will prove Lemma 3.6 in Section 4 below, along with stating the properties of the various objects from Definition 3.5.

3.3.2 Design principles

Let us stress the design principles of Definition 3.5.

1. The construction of τ_{hp} in step 1 sets our local-best discontinuous projection "target". There holds $\tau_{hp} \in \underline{\mathcal{P}_p(\mathcal{T}_h)}$ but in general $\tau_{hp} \notin H^1_{0,\mathrm{D}}(\Omega)$. In the rest of Definition 3.5, we search to stay in $\underline{\mathcal{P}_p(\mathcal{T}_h)}$, as close as possible to τ_{hp}, and keeping its approximation power, but recovering $H^1_{0,\mathrm{D}}(\Omega)$-conformity.

2. The construction of s_{hp} in step 2 is identical to [25, Construction 3.8 and Remark 3.10] and [26, Corollaries 3.1 and 3.7 together with equation (4.3)] from a posteriori error analysis. This is not p-robust since the cut-off by the hat functions $\psi^{\boldsymbol{a}}$ from (2.2) increases the polynomial degree by one and is brought back down to p by the canonical elementwise projector $I_{hp}^{\mathcal{L}}$ from (2.12). The purpose here is to obtain the provisional "remainder" $\tau_{hp} - s_{hp}$ such that if $v \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$, then $\tau_{hp} = s_{hp} = v$ and the provisional remainder vanishes (projection property)(then the remaining steps give $\delta_{hp} = \zeta_{hp} = 0$).
3. The purpose of the construction of δ_{hp} in step 3 is to obtain the (final) remainder $\tau_{hp} - s_{hp} - \delta_{hp}$. This has vanishing lowest-order moments as per (3.11b) on all mesh elements $K \in \mathcal{T}_h$ where the polynomial degree is at least equal to $d+1$ (cf. [24, equation (3.16)]). Note that $\delta_{hp}|_K$, if nonzero, is only a $(d+1)$-degree polynomial.
4. The construction of ζ_{hp} in step 4 is the salient feature for p-robustness (note that it is only nontrivial when $p_K \geq d+1$ for all mesh elements K in the extended vertex patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$). Neither the hat functions $\psi^{\boldsymbol{a}}$ nor the elementwise projector $I_{hp}^{\mathcal{L}}$ are present. First, in (3.12b), we employ a potential reconstruction similar to (3.6a) which however does not impose zero trace on $\partial\widetilde{\omega}_{\boldsymbol{a}}$ (except for $\partial\widetilde{\omega}_{\boldsymbol{a}} \cap \Gamma_{\mathrm{D}}$). Second, in (3.13b), p-stable decomposition is applied (this cannot be applied directly to the remainder $\tau_{hp} - s_{hp} - \delta_{hp}$ which lies in $\underline{\mathcal{P}_p(\widetilde{\mathcal{T}}_{\boldsymbol{a}})}$ but not in $H^1_{0,\mathrm{D}}(\widetilde{\omega}_{\boldsymbol{a}})$). In (3.14), we then merely employ the "interior" component which does have zero trace on $\partial\omega_{\boldsymbol{a}}$ (for interior vertices) or on $\partial\omega_{\boldsymbol{a}} \cap \{\psi^{\boldsymbol{a}} = 0\}$ and $(\partial\omega_{\boldsymbol{a}} \cap \Gamma_{\mathrm{D}})^\circ$ (for boundary vertices) as per the definition of $H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$ in (2.4), cf. Figure 3.
5. In step 5, $P_{hp}(v)$ is defined as $s_{hp} + \delta_{hp}$ corrected by ζ_{hp}.

3.3.3 Properties of the projector

Recall the definition (2.7) of the minimal polynomial degree $\underline{p}_K$. The following theorem summarizes the properties of the projector from Definition 3.5, improving the results in [37, 9, 18, 33, 27, 31, 28, 8, 22, 4, 30].

Theorem 3.7 (Projection, stability, and approximation of P_{hp})**.** *Let a simplicial mesh $\mathcal{T}_h$ of Ω and a (variable) polynomial degree $p_K \geq 1$ for all $K \in \mathcal{T}_h$ be given. Let Assumptions 3.1 and 3.2 hold. The operator $P_{hp} : H^1_{0,\mathrm{D}}(\Omega) \to \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$ from Definition 3.5 satisfies*

$$P_{hp}(v) = v \qquad \forall v \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega). \tag{3.18}$$

(projection)

Moreover, for any function $v \in H^1_{0,\mathrm{D}}(\Omega)$ and any mesh element $K \in \mathcal{T}_h$, there holds

$$\|\nabla P_{hp}(v)\|_K^2 \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \|\nabla v\|_L^2, \tag{3.19}$$

(H^1-stability)

$$\|\nabla(v - P_{hp}(v))\|_K^2 \quad \lesssim \sum_{L \in \widetilde{\mathcal{T}}_K} \min_{v_p \in \mathcal{P}_{\underline{p}_K}(L)} \|\nabla(v - v_p)\|_L^2. \tag{3.20}$$

(approximation p-robustly equivalent to elementwise H^1-orthogonal projector)

The constant hidden in $\lesssim$ only depends on the mesh shape-regularity parameter κ_h given by (2.1), the polynomial-variation parameter κ_p given by (2.8), and the space dimension d.

4 Properties of the intermediate objects from Definition 3.5 and proof of Lemma 3.6

We justify here all steps of Definition 3.5 and summarize the properties of the intermediate objects τ_{hp}, s_{hp}, δ_{hp}, and ζ_{hp} therefrom. Collecting the results from this section in particular proves Lemma 3.6. Let us start by observing that all the minimization problems (3.4), (3.6a), and (3.12b) are well posed.

4.1 Step 1 (construction and properties of the elementwise projection τ_{hp})

We note that by (3.4), there is no interelement continuity in τ_{hp}; τ_{hp} is a broken piecewise polynomial in $\underline{\mathcal{P}_p(\mathcal{T}_h)}$ which typically does not lie in $H^1_{0,\mathrm{D}}(\Omega)$. This confirms:

Lemma 4.1 (Property (3.5))**.** *Property (3.5) holds true.*

4.2 Step 2 (construction and properties of the standard potential reconstruction s_{hp})

Let us now confirm that s_{hp} is also well-defined:

Lemma 4.2 (Properties (3.7), (3.8))**.** *Properties* (3.7) *and* (3.8) *hold true.*

Proof. In view of the definition (2.4), all $s_p^{\boldsymbol{a}}$ from (3.6a) have zero trace on that faces in $\partial\omega_{\boldsymbol{a}}$ where the hat function $\psi^{\boldsymbol{a}}$ vanishes or which lie in the Dirichlet boundary $\overline{\Gamma_{\mathrm{D}}}$. Thus it follows that all $s_p^{\boldsymbol{a}}$ extended by zero outside of the patch subdomains $\omega_{\boldsymbol{a}}$ belong to $\underline{\mathcal{P}_p(\mathcal{T}_h)}\cap H^1_{0,\mathrm{D}}(\Omega) = \mathcal{P}_p(\mathcal{T}_h)\cap H^1_{0,\mathrm{D}}(\Omega)$, recalling (2.10). Consequently, (3.7) follows by (3.6b). Then, (3.8) is a trivial consequence of (3.5) and (3.7). □

4.3 Step 3 (construction and properties of the elementwise bubble correction δ_{hp})

Let us move to δ_{hp}:

Lemma 4.3 (Properties (3.10), (3.11))**.** *Properties* (3.10) *and* (3.11) *hold true.*

Proof. Property (3.10) follows immediately from the definition (3.9), since $\delta_{hp}|_K$ is either zero, or a polynomial of degree $(d+1) \le p_K$; importantly, δ_{hp} vanishes on all mesh faces. Then (3.11a) is an immediate consequence of (3.10) and (3.8). As for (3.11b), let a mesh element $K \in \mathcal{T}_h$ with $p_K \ge d+1$ be fixed. By construction, we see from (3.9b)

$$(\delta_{hp}, 1)_K = \frac{(\tau_{hp} - s_{hp}, 1)_K}{(b_K, 1)_K}(b_K, 1)_K = (\tau_{hp} - s_{hp}, 1)_K.$$

□

4.4 Step 4 (construction and properties of the p-robust correction ζ_{hp})

We continue with ζ_{hp}:

Lemma 4.4 (Construction (3.12b) and decomposition (3.13b))**.** *For each mesh vertex $\boldsymbol{a} \in \mathcal{V}_h$ such that $p_K \ge d+1$ for all $K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}$, the construction* (3.12b) *and the decomposition* (3.13b) *are well defined.*

Proof. We first note that if $p_K \ge d+1$ for all $K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}$, then (3.11b) implies

$$(\tau_{hp} - s_{hp} - \delta_{hp}, 1)_{\widetilde{\omega}_{\boldsymbol{a}}} = 0.$$

We have directly included this equality in the definition (3.12b), where we ask $\zeta_p^{\boldsymbol{a}}$ to either take zero values on the boundary faces from $\partial\widetilde{\omega}_{\boldsymbol{a}}$ lying in $\overline{\Gamma_{\mathrm{D}}}$, or $\zeta_p^{\boldsymbol{a}}$ to have mean value zero. Thus assumption (A.1) below is satisfied with $\mathcal{T}_\omega = \widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and $\omega = \widetilde{\omega}_{\boldsymbol{a}}$. Consequently, (3.13b) follows immediately from (A.2) (with $\mathcal{V}_\omega = \widetilde{\mathcal{V}}_{\boldsymbol{a}}$) and (A.4) in Corollary A.1. Note that we only employ the "interior" component $\zeta_p^{\boldsymbol{a},\boldsymbol{a}}$; this is from (A.3) supported on the vertex patch subdomain $\omega_{\boldsymbol{a}} \cap \widetilde{\omega}_{\boldsymbol{a}}$ which is simply $\omega_{\boldsymbol{a}}$ (no patch truncation happens for the "interior" component, cf. Figure 3 (right)). □

Lemma 4.5 (Property (3.15))**.** *Property* (3.15) *holds true.*

Proof. As in the proof of Lemma 4.2 above, recalling definition (2.4), all $\zeta_p^{\boldsymbol{a}}$ extended by zero outside of the patch subdomains $\omega_{\boldsymbol{a}}$ belong to $H^1_{0,\mathrm{D}}(\Omega)$. Thus the inclusion $\zeta_{hp} \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$ follows immediately by (3.14) and (2.10). □

4.5 Step 5 (combining the previous steps)

We finish by $P_{hp}(v)$:

Lemma 4.6 (Property (3.17))**.** *Property* (3.17) *holds true.*

Proof. This is an immediate consequence of the definition (3.16) and the properties (3.7), (3.10), and (3.15). □

5 Proof of Theorem 3.7

Let the assumptions of Theorem 3.7 be satisfied. We prove the three claims separately.

5.1 Projection

Projection is ensured by Steps 1–2 of Definition 3.5:

Lemma 5.1 (Projection property (3.18))**.** *The projection property* (3.18) *holds true.*

Proof. Let $v \in \mathcal{P}_p(\mathcal{T}_h) \cap H^1_{0,\mathrm{D}}(\Omega)$. Then clearly τ_{hp} from (3.4) satisfies $\tau_{hp} = v$. Moreover, $I^{\mathcal{L}}_{hp}(\psi^{\boldsymbol{a}}\tau_{hp}) \in \underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$ by (2.12). Indeed, this follows since $\tau_{hp} = v$ is continuous, with no face jumps, and since (crucial for p variable) the canonical elementwise Lagrange projection (2.12) employs the space $\underline{\mathcal{P}_{p_K}(K)}$ from Section 2.7 with the polynomial degrees lowered according to the neighbors. Consequently, by (3.6a), $s_p^{\boldsymbol{a}} = I^{\mathcal{L}}_{hp}(\psi^{\boldsymbol{a}}\tau_{hp})$. Then, (3.6b), the partition of unity (2.2), the linearity as well as the projection property of $I^{\mathcal{L}}_{hp}$, and (2.10) give

$$s_{hp} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} s_p^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} I^{\mathcal{L}}_{hp}(\psi^{\boldsymbol{a}}\tau_{hp}) = I^{\mathcal{L}}_{hp}\Big(\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi^{\boldsymbol{a}}\tau_{hp}\Big) = I^{\mathcal{L}}_{hp}(\tau_{hp}) = \tau_{hp}. \tag{5.1}$$

Thus, also $s_{hp} = v$. Next, step 3 of Definition 3.5 builds on $\tau_{hp} - s_{hp}$, so that the bubble correction δ_{hp} is always zero (for any polynomial degree). Finally, step 4 builds on $\tau_{hp} - s_{hp} - \delta_{hp}$, so that all $\zeta_p^{\boldsymbol{a}}$, $\zeta_p^{\boldsymbol{a},\boldsymbol{a}}$, and ζ_{hp} are likewise zero. Then, from (3.16), $P_{hp}(v) = s_{hp} + \delta_{hp} + \zeta_{hp} = s_{hp} = v$. □

5.2 Stability

Stability is a simple consequence of approximation (3.20):

Lemma 5.2 (Stability property (3.19))**.** *The stability property* (3.19) *holds true.*

Proof. This follows by the triangle inequality from the approximation (3.20). Indeed, let $K \in \mathcal{T}_h$ be fixed. Then

$$\|\nabla P_{hp}(v)\|_K^2 \le (\|\nabla v\|_K + \|\nabla(v - P_{hp}(v))\|_K)^2 \overset{(3.20)}{\lesssim} \sum_{L\in\widetilde{\mathcal{T}}_K} \|\nabla v\|_L^2,$$

where we have also used the trivial $H^1(L)$-orthogonal projection stability

$$\min_{v_p\in\mathcal{P}_{\underline{p}_K}(L)} \|\nabla(v - v_p)\|_L \le \|\nabla v\|_L.$$

□

5.3 Approximation

We are left to prove the approximation property (3.20). We establish it first in the case of a uniform polynomial degree such that $p \ge d+1$. Then we generalize it to the variable polynomial degree $p_K \ge 1$ for all $K \in \mathcal{T}_h$.

Lemma 5.3 (Approximation property (3.20), uniform polynomial degree $p \ge d+1$)**.** *Consider a uniform polynomial degree $p_K = p \ge d+1$ for all $K \in \mathcal{T}_h$. Then the approximation property* (3.20) *holds true.*

Proof. Let $K \in \mathcal{T}_h$ be fixed. As $p \ge d+1$, formulas (3.9b), (3.12b), (3.13b) apply.

(i) Like in problem (3.12b), but on the extended element patch $\widetilde{\mathcal{T}}_K$ in place of the extended vertex patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$, see Section 2.2 and Figure 2 (right), define

$$\zeta_p^K := \arg \min_{\substack{v_p\in\mathcal{P}_p(\widetilde{\mathcal{T}}_K)\cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)\\ (v_p,1)_{\widetilde{\omega}_K}=(\tau_{hp}-s_{hp}-\delta_{hp},1)_{\widetilde{\omega}_K}=0 \text{ if } |\partial\widetilde{\omega}_K\cap\Gamma_\mathrm{D}|=0}} \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - v_p)\|_{\widetilde{\omega}_K}. \tag{5.2}$$

To ζ_p^K, we furthermore apply a mean value ajustement by elementwise bubble functions as in (3.9b), i.e., we define $\tilde{\delta}_{hp}^K \in \mathcal{P}_{d+1}(\widetilde{\mathcal{T}}_K)$ by

$$\tilde{\delta}_{hp}^K|_L := \frac{(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K, 1)_L}{(b_L, 1)_L} b_L \qquad \forall L \in \widetilde{\mathcal{T}}_K, \tag{5.3}$$

so that

$$\tilde{\zeta}_p^K := \zeta_p^K + \tilde{\delta}_{hp}^K \tag{5.4}$$

was of elementwise mean value zero,

$$\begin{aligned}(\tilde{\zeta}_p^K, 1)_L &= (\zeta_p^K, 1)_L + \frac{(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K, 1)_L}{(b_L, 1)_L}(b_L, 1)_L \\ &= (\tau_{hp} - s_{hp} - \delta_{hp}, 1)_L \overset{(3.11b)}{=} 0 \qquad \forall L \in \widetilde{\mathcal{T}}_K,\end{aligned} \tag{5.5}$$

where we have crucially used that $\tau_{hp} - s_{hp} - \delta_{hp}$ is of elementwise mean value zero.

(ii) Importantly, the above bubble correction is p-stable in that

$$\|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \tilde{\zeta}_p^K)\|_{\widetilde{\omega}_K} \lesssim \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K} + \|\nabla(v - \tau_{hp})\|_{\widetilde{\omega}_K}, \tag{5.6}$$

where the constant hidden in $\lesssim$ only depends on the shape-regularity of the mesh $\mathcal{T}_h$ and the space dimension d. Indeed, on the one hand,

$$\begin{aligned}\|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \tilde{\zeta}_p^K)\|_{\widetilde{\omega}_K} &= \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K - \tilde{\delta}_{hp}^K)\|_{\widetilde{\omega}_K} \\ &\le \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K} + \|\nabla\tilde{\delta}_{hp}^K\|_{\widetilde{\omega}_K}.\end{aligned}$$

On the other hand, for any fixed $L \in \widetilde{\mathcal{T}}_K$,

$$\|\nabla b_L\|_L \lesssim h_L^{-1}\|b_L\|_L \tag{5.7a}$$

by the inverse inequality and

$$\|b_L\|_L |L|^{1/2} \approx (b_L, 1)_L \tag{5.7b}$$

by equivalence of norms on finite-dimensional spaces and mesh shape regularity; recall that b_L is a low-order $(d+1)$-degree nonnegative polynomial, so that the constants hidden in $\lesssim$ and $\approx$ above are indeed independent of p. Thus, also using the fact that $(\tau_{hp}, 1)_L = (v, 1)_L$ from the constraint in (3.4) and the Cauchy–Schwarz inequality,

$$\begin{aligned}\|\nabla\tilde{\delta}_{hp}^K\|_L &= \frac{|(v - s_{hp} - \delta_{hp} - \zeta_p^K, 1)_L|}{(b_L, 1)_L}\|\nabla b_L\|_L \le \frac{\|v - s_{hp} - \delta_{hp} - \zeta_p^K\|_L |L|^{1/2}}{(b_L, 1)_L}\|\nabla b_L\|_L \\ &\lesssim h_L^{-1}\|v - s_{hp} - \delta_{hp} - \zeta_p^K\|_L \lesssim h_{\widetilde{\omega}_K}^{-1}\|v - s_{hp} - \delta_{hp} - \zeta_p^K\|_{\widetilde{\omega}_K} \\ &\lesssim \|\nabla(v - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K} \le \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K} \\ &\quad + \|\nabla(v - \tau_{hp})\|_{\widetilde{\omega}_K};\end{aligned}$$

in the last but one step, we have applied the Poincaré inequality, since, crucially from (5.2) (recall $(\tau_{hp}, 1)_L = (v, 1)_L$ on L), $(v - s_{hp} - \delta_{hp} - \zeta_p^K) \in H^1(\widetilde{\omega}_K)$ is zero on $\partial\widetilde{\omega}_K \cap \Gamma_{\mathrm{D}}$ or of mean value zero on $\widetilde{\omega}_K$.

(iii) Now, as in (3.13b), we decompose the mean-value adjusted $\tilde{\zeta}_p^K$ from (5.4) using Corollary A.1 (with $\mathcal{T}_\omega = \widetilde{\mathcal{T}}_K$ and $\mathcal{V}_\omega = \widetilde{\mathcal{V}}_K$)

$$\tilde{\zeta}_p^K = \sum_{\boldsymbol{b} \in \widetilde{\mathcal{V}}_K} \tilde{\zeta}_p^{K,\boldsymbol{b}} \ \text{ with } \ \tilde{\zeta}_p^{K,\boldsymbol{b}} \in \underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{b}} \cap \widetilde{\mathcal{T}}_K)} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{b}}}(\omega_{\boldsymbol{b}} \cap \widetilde{\omega}_K). \tag{5.8}$$

Assumptions (A.1) are indeed satisfied by $\tilde{\zeta}_p^K$; in particular, the boundary condition comes from the constraint in (5.2) (the shift from ζ_p^K to $\tilde{\zeta}_p^K$ is by bubbles that are zero on the mesh faces) and the mean value condition comes from (5.5) (mean value zero on all elements $L \in \widetilde{\mathcal{T}}_K$ implies mean value zero on $\widetilde{\omega}_K$). Now, crucially, as in (3.13b), the contributions for the vertices $\boldsymbol{a}$ of the element K, $\boldsymbol{a} \in \mathcal{V}_K$, lie in $\underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$ (as $\mathcal{T}_{\boldsymbol{a}}$ are included in $\widetilde{\mathcal{T}}_K$, $\mathcal{T}_{\boldsymbol{a}} \cap \widetilde{\mathcal{T}}_K = \mathcal{T}_{\boldsymbol{a}}$ and no patch truncation happens).

(iv) For each vertex $\boldsymbol{a} \in \mathcal{V}_K$, let us also consider $\tilde{\zeta}_p^K$ from (5.4) restricted to the extended vertex patch $\widetilde{\omega}_{\boldsymbol{a}}$ (recall that $\widetilde{\omega}_{\boldsymbol{a}} \subset \widetilde{\omega}_K$ by definition, see Figure 2, right). We again decompose $\tilde{\zeta}_p^K|_{\widetilde{\omega}_{\boldsymbol{a}}}$ using Corollary A.1 (this time with $\mathcal{T}_\omega = \widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and $\mathcal{V}_\omega = \widetilde{\mathcal{V}}_{\boldsymbol{a}}$)

$$\tilde{\zeta}_p^K|_{\widetilde{\omega}_{\boldsymbol{a}}} = \sum_{\boldsymbol{b} \in \widetilde{\mathcal{V}}_{\boldsymbol{a}}} \tilde{\zeta}_p^{K,\boldsymbol{a},\boldsymbol{b}} \text{ with } \tilde{\zeta}_p^{K,\boldsymbol{a},\boldsymbol{b}} \in \underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{b}} \cap \widetilde{\mathcal{T}}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{b}}}(\omega_{\boldsymbol{b}} \cap \widetilde{\omega}_{\boldsymbol{a}}). \tag{5.9}$$

Again, $\tilde{\zeta}_p^K|_{\widetilde{\omega}_{\boldsymbol{a}}}$ satisfy assumptions (A.1). This would not be the case of ζ_p^K from (5.2), which only satisfies the mean value condition (A.1b) on the extended element patch $\widetilde{\omega}_K$ but not on its subpatches $\widetilde{\omega}_{\boldsymbol{a}}$; for this reason, we pass from ζ_p^K to $\tilde{\zeta}_p^K$.

(v) Now a central observation comes from (A.3): since $\tilde{\zeta}_p^K$ and $\tilde{\zeta}_p^K|_{\widetilde{\omega}_{\boldsymbol{a}}}$ are identical on the extended patches $\widetilde{\omega}_{\boldsymbol{a}}$, the $d+1$ contributions $\tilde{\zeta}_p^{K,\boldsymbol{a}}$ from (5.8) for the vertices $\boldsymbol{a}$ of the element K respectively coincide with the $d+1$ contributions $\tilde{\zeta}_p^{K,\boldsymbol{a},\boldsymbol{a}}$ from (5.9),

$$\tilde{\zeta}_p^{K,\boldsymbol{a}} = \tilde{\zeta}_p^{K,\boldsymbol{a},\boldsymbol{a}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_K. \tag{5.10}$$

Indeed, by (A.3), these contributions have the vertex patches $\mathcal{T}_{\boldsymbol{a}}$ as support and the extended vertex patches $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ as dependency regions. This is actually the reason for the remainder potential reconstruction (3.12b) and the decomposition (3.13b) to be performed on the extended vertex patches $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$; merely the vertex patches $\mathcal{T}_{\boldsymbol{a}}$ would not be sufficient, as the dependency regions are not given by the vertex patches $\mathcal{T}_{\boldsymbol{a}}$ but by the extended vertex patches $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$. Overall, from (5.8)–(5.10), we conclude that

$$\tilde{\zeta}_p^K|_K \overset{(5.8)}{=} \sum_{\boldsymbol{a} \in \mathcal{V}_K} \tilde{\zeta}_p^{K,\boldsymbol{a}}|_K \overset{(5.10)}{=} \sum_{\boldsymbol{a} \in \mathcal{V}_K} \tilde{\zeta}_p^{K,\boldsymbol{a},\boldsymbol{a}}|_K. \tag{5.11}$$

(vi) Recall now the definition of τ_{hp} from (3.4) and that $K \in \mathcal{T}_h$ is fixed. We estimate by the triangle inequality and employing the definitions (3.16) and (3.14) together with the equality (5.11),

$$\begin{aligned}
&\|\nabla(v - P_{hp}(v))\|_K \\
\leq\ & \|\nabla(v - \tau_{hp})\|_K + \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_{hp})\|_K \\
\overset{(3.14)}{\underset{(5.11)}{=}}\ & \|\nabla(v - \tau_{hp})\|_K + \left\|\nabla\left(\tau_{hp} - s_{hp} - \delta_{hp} - \tilde{\zeta}_p^K + \sum_{\boldsymbol{a} \in \mathcal{V}_K} \left(\tilde{\zeta}_p^{K,\boldsymbol{a},\boldsymbol{a}} - \zeta_p^{\boldsymbol{a},\boldsymbol{a}}\right)\right)\right\|_K \\
\overset{(5.6)}{\lesssim}\ & \|\nabla(v - \tau_{hp})\|_{\widetilde{\omega}_K} + \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K} + \sum_{\boldsymbol{a} \in \mathcal{V}_K} \|\nabla(\tilde{\zeta}_p^{K,\boldsymbol{a},\boldsymbol{a}} - \zeta_p^{\boldsymbol{a},\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}};
\end{aligned} \tag{5.12}$$

in the last estimate, we have crucially used (5.6) to go back from the bubble-shifted $\tilde{\zeta}_p^K$ to ζ_p^K. From (3.4), the first term above already has the form requested in (3.20). For the last term, we crucially use the linearity of the decomposition (A.3) and its p-robust stability expressed by (A.5) from Corollary A.1 (note that $h_{\widetilde{\omega}_K} / \min_{K \in \widetilde{\mathcal{T}}_K} h_K$ only depends on κ_h). Thus, for a vertex $\boldsymbol{a} \in \mathcal{V}_K$, recalling (5.9) and (3.13b),

$$\begin{aligned}
&\|\nabla(\tilde{\zeta}_p^{K,\boldsymbol{a},\boldsymbol{a}} - \zeta_p^{\boldsymbol{a},\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}} \\
\overset{(A.5)}{\lesssim}\ & \|\nabla(\tilde{\zeta}_p^K - \zeta_p^{\boldsymbol{a}})\|_{\widetilde{\omega}_{\boldsymbol{a}}} \\
\leq\ & \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \tilde{\zeta}_p^K)\|_{\widetilde{\omega}_K} + \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^{\boldsymbol{a}})\|_{\widetilde{\omega}_{\boldsymbol{a}}} \\
\overset{(5.6)}{\lesssim}\ & \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K} + \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^{\boldsymbol{a}})\|_{\widetilde{\omega}_{\boldsymbol{a}}} + \|\nabla(v - \tau_{hp})\|_{\widetilde{\omega}_K},
\end{aligned} \tag{5.13}$$

where we have added and subtracted $\tau_{hp} - s_{hp} - \delta_{hp}$, used the triangle inequality, extended the integration region, and again used (5.6). We are thus left estimating $\|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^{\boldsymbol{a}})\|_{\widetilde{\omega}_{\boldsymbol{a}}}$ for $\zeta_p^{\boldsymbol{a}}$ given by (3.12b) and $\|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K}$ for ζ_p^K given by (5.2). As these take the same form, we only show details for the latter.

(vii) Let us finally consider $\|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K}$ with ζ_p^K given by (5.2). Such problems (recall that τ_{hp} from (3.4) is merely a broken polynomial from $\underline{\mathcal{P}_p(\mathcal{T}_h)}$ but not from $H^1_{0,\mathrm{D}}(\Omega)$) have recently been analyzed and p-robust stability has been shown in [26, Corollaries 3.1 and 3.7], see also [14, Corollaries 3.4 and 4.4] on: 1) vertex patch subdomains $\omega_{\boldsymbol{a}}$; and 2) with zero Dirichlet boundary conditions on $\partial\omega_{\boldsymbol{a}}$. We now rely on the extension of this result to larger (extended) patch subdomains where zero Dirichlet boundary conditions are not prescribed from Appendix D. As a preliminary step, though, first note that

$$\begin{aligned}\|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - \zeta_p^K)\|_{\widetilde{\omega}_K} &= \min_{v_p \in \underline{\mathcal{P}_p(\widetilde{\mathcal{T}}_K)} \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)} \|\nabla(\tau_{hp} - s_{hp} - \delta_{hp} - v_p)\|_{\widetilde{\omega}_K} \\ &= \min_{v_p \in \underline{\mathcal{P}_p(\widetilde{\mathcal{T}}_K)} \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)} \|\nabla(\tau_{hp} - v_p)\|_{\widetilde{\omega}_K}.\end{aligned} \tag{5.14}$$

Indeed, this follows by the shift by $(s_{hp} + \delta_{hp})|_{\widetilde{\omega}_K}$ since, by (3.7) and (3.10), it lies in $\underline{\mathcal{P}_p(\widetilde{\mathcal{T}}_K)} \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)$; the (trace) continuity of $s_{hp} + \delta_{hp}$ together with $s_{hp} + \delta_{hp} = 0$ on Γ_{D} are crucial here. In this important conceptual step, the non p-robust usual potential reconstruction s_{hp}, together with its mean value correction δ_{hp}, are played out. Please note that the minimum in (5.2) and that in the above first line coincide: the (possible) mean value constraint in (5.2) only fixes ζ_p^K uniquely but has no influence on its gradient.

We now finally apply Corollary D.2 on the extended vertex patch $\widetilde{\mathcal{T}}_K$ (recall that we suppose a uniform polynomial degree p here) to deduce that

$$\min_{v_p \in \underline{\mathcal{P}_p(\widetilde{\mathcal{T}}_K)} \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)} \|\nabla(\tau_{hp} - v_p)\|_{\widetilde{\omega}_K} \lesssim \min_{w \in H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)} \|\nabla(\tau_{hp} - w)\|_{\widetilde{\omega}_K}. \tag{5.15}$$

Finally, we can play in the target function $v \in H^1_{0,\mathrm{D}}(\Omega)$ from the announcement of Theorem 3.7, which satisfies $v|_{\widetilde{\omega}_K} \in H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)$, and obtain

$$\min_{w \in H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)} \|\nabla(\tau_{hp} - w)\|_{\widetilde{\omega}_K} \lesssim \|\nabla(\tau_{hp} - v)\|_{\widetilde{\omega}_K}. \tag{5.16}$$

Combining the above bounds (5.12)–(5.16) together with the definition (3.4) of τ_{hp} gives the assertion (3.20). $\square$

Lemma 5.4 (Approximation property (3.20), variable polynomial degree)**.** *Consider the general case of a variable polynomial degree $p_K \geq 1$ for all $K \in \mathcal{T}_h$. Then the approximation property* (3.20) *holds true.*

Proof. Let $K \in \mathcal{T}_h$ be fixed. We consider several cases.

(i) Case $p_K < d+1$. In this low-polynomial-degree case, following (3.9a), there is no bubble correction, $\delta_{hp}|_K = 0$. Moreover, on each extended vertex patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathcal{V}_h$ such that $K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}$, by (3.12a) and (3.13a), there holds $\zeta_p^{\boldsymbol{a}} = \zeta_p^{\boldsymbol{a},\boldsymbol{a}} = 0$, so that, in view of (3.14), there is also no p-robust correction, $\zeta_{hp}|_K = 0$. Thus, by (3.16), $(P_{hp}(v))|_K = s_{hp}|_K$, where s_{hp} is given by (3.6). This is the usual construction of s_{hp} from [25, Construction 3.8 and Remark 3.10] and [26, Corollaries 3.1 and 3.7 together with equation (4.3)]. As discussed in point 2 in Section 3.3.2, it is not p-robust, but here the polynomial degree is low, $p_K < d+1$, so there will be no problem. To see (3.20), we start by the triangle inequality,

$$\|\nabla(v - P_{hp}(v))\|_K = \|\nabla(v - s_{hp})\|_K \leq \|\nabla(v - \tau_{hp})\|_K + \|\nabla(\tau_{hp} - s_{hp})\|_K.$$

Then, similarly to the above references and using the three last equalities from (5.1) together with (3.6b), we see

$$\begin{aligned}\|\nabla(\tau_{hp} - s_{hp})\|_K = \left\|\nabla\Big(\sum_{\boldsymbol{a} \in \mathcal{V}_K} (I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp}) - s_p^{\boldsymbol{a}})\Big)\right\|_K &\leq \sum_{\boldsymbol{a} \in \mathcal{V}_K} \|\nabla(I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp}) - s_p^{\boldsymbol{a}})\|_K \\ &\leq \sum_{\boldsymbol{a} \in \mathcal{V}_K} \|\nabla(I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp}) - s_p^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}}.\end{aligned}$$

If the polynomial degree in the neighbors of K is different from p_K, then the use of the polynomial space $\underline{\mathcal{P}_{p_K}(K)}$ with degree lowered according to neighbors from Section 2.7 in (3.4) is crucial, since only then, cf. (3.5), $\tau_{hp} \in \underline{\mathcal{P}_p(\mathcal{T}_h)}$, and only then $I_{hp}^{\mathcal{L}}(\tau_{hp}) = \tau_{hp}$, which is crucially used in (5.1).

For each vertex of the element K, $\boldsymbol{a} \in \mathcal{V}_K$, let us next introduce the following auxiliary problem:

$$\hat{s}_p^{\boldsymbol{a}} := \arg \min_{v_p \in \mathcal{P}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})} \big\|\nabla\big(\psi^{\boldsymbol{a}}\tau_{hp} - v_p\big)\big\|_{\omega_{\boldsymbol{a}}}.$$

This problem takes a similar form to (3.6a), but increases the polynomial degree by one and does not employ the elementwise Lagrange projector $I_{hp}^{\mathcal{L}}$. We observe that $I_{hp}^{\mathcal{L}}(\hat{s}_p^{\boldsymbol{a}}) \in \underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{a}})} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$. Thus, since $s_p^{\boldsymbol{a}}$ is the minimizer from (3.6a) and since $I_{hp}^{\mathcal{L}}$ is linear, we have

$$\|\nabla(I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp}) - s_p^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}} \leq \|\nabla(I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp} - \hat{s}_p^{\boldsymbol{a}}))\|_{\omega_{\boldsymbol{a}}}. \tag{5.17}$$

Finally, we employ the stability of the Lagrange projector $I_{hp}^{\mathcal{L}}$ (this is not p-robust, but fine here, since $p_K < d+1$ and since the polynomial degrees in the patch $\mathcal{T}_{\boldsymbol{a}}$ are controlled by κ_p of (2.8)). Employing also [26, Corollaries 3.1 and 3.7 together with equation (4.3)], since, for the given function $v \in H^1_{0,\mathrm{D}}(\Omega)$, $\psi^{\boldsymbol{a}} v \in H^1_{0,\mathrm{D},\psi^{\boldsymbol{a}}}(\omega_{\boldsymbol{a}})$, we finally arrive at

$$\|\nabla(I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp} - \hat{s}_p^{\boldsymbol{a}}))\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla(\psi^{\boldsymbol{a}}\tau_{hp} - \hat{s}_p^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla(\psi^{\boldsymbol{a}}(\tau_{hp} - v))\|_{\omega_{\boldsymbol{a}}}.$$

Finally,

$$\begin{aligned}\|\nabla(\psi^{\boldsymbol{a}}(\tau_{hp} - v))\|_{\omega_{\boldsymbol{a}}}^2 &= \sum_{L \in \mathcal{T}_{\boldsymbol{a}}} \|\nabla\psi^{\boldsymbol{a}}(\tau_{hp} - v) + \psi^{\boldsymbol{a}}\nabla(\tau_{hp} - v)\|_L^2 \\ &\leq \sum_{L \in \mathcal{T}_{\boldsymbol{a}}} \{\|\nabla\psi^{\boldsymbol{a}}\|_{\infty,L}\|\tau_{hp} - v\|_L + \|\psi^{\boldsymbol{a}}\|_{\infty,L}\|\nabla(\tau_{hp} - v)\|_L\}^2 \\ &\lesssim \|\nabla(\tau_{hp} - v)\|_{\omega_{\boldsymbol{a}}}^2,\end{aligned}$$

applying $\|\nabla\psi^{\boldsymbol{a}}\|_{\infty,L} \lesssim h_L^{-1}$, $\|\psi^{\boldsymbol{a}}\|_{\infty,L} = 1$, and the Poincaré inequality $\|\tau_{hp} - v\|_L \leq h_L/\pi \|\nabla(\tau_{hp} - v)\|_L$, since $(\tau_{hp}, 1)_L = (v, 1)_L$ for all $L \in \mathcal{T}_{\boldsymbol{a}}$ from the constraint in (3.4), similarly to, e.g., [25, inequality (3.29)]. Thus, by combining the above inequalities,

$$\|\nabla(v - P_{hp}(v))\|_K \lesssim \|\nabla(v - \tau_{hp})\|_{\omega_K},$$

which implies (3.20).

(ii) Case $p_K \geq d+1$ and $\zeta_p^{\boldsymbol{a}} = \zeta_p^{\boldsymbol{a},\boldsymbol{a}} = 0$ for all $\boldsymbol{a} \in \mathcal{V}_h$ such that $K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}$, so that $\zeta_{hp}|_K = 0$, resulting from $p_L < d+1$ for some $L \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}$ and the prescriptions (3.12a), (3.13a). Consequently, in view of (2.8), p_K is still small in that it can be controlled by the space dimension d and the polynomial-variation parameter κ_p from (2.8). Here, by (3.16), $(P_{hp}(v))|_K = s_{hp}|_K + \delta_{hp}|_K$, where s_{hp} is given by (3.6) and δ_{hp} by (3.9b). As in the proof of Lemma 5.3, step (i),

$$\begin{aligned}\|\nabla\delta_{hp}\|_K &\overset{(3.9b)}{=} \frac{|(\tau_{hp} - s_{hp}, 1)_K|}{(b_K, 1)_K}\|\nabla b_K\|_K \leq \frac{\|\tau_{hp} - s_{hp}\|_K |K|^{1/2}}{(b_K, 1)_K}\|\nabla b_K\|_K \\ &\overset{(5.7)}{\lesssim} h_K^{-1}\|\tau_{hp} - s_{hp}\|_K \overset{\substack{(5.1)\\(3.6b)}}{=} h_K^{-1}\Big\|\sum_{\boldsymbol{a} \in \mathcal{V}_K}(I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp}) - s_p^{\boldsymbol{a}})\Big\|_K \\ &\leq \sum_{\boldsymbol{a} \in \mathcal{V}_K}\|\nabla(I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp}) - s_p^{\boldsymbol{a}})\|_K;\end{aligned}$$

note that $I_{hp}^{\mathcal{L}}(\psi^{\boldsymbol{a}}\tau_{hp}) - s_p^{\boldsymbol{a}}$ is zero on at least one face of K, which allows us to use the Poincaré inequality in the last step. Thus, the triangle inequality

$$\|\nabla(v - P_{hp}(v))\|_K = \|\nabla(v - s_{hp} - \delta_{hp})\|_K \leq \|\nabla(v - \tau_{hp})\|_K + \|\nabla(\tau_{hp} - s_{hp})\|_K + \|\nabla\delta_{hp}\|_K \tag{5.18}$$

in combination with step (i) above, branching at (5.17), implies

$$\|\nabla(v - P_{hp}(v))\|_K \lesssim \|\nabla(v - \tau_{hp})\|_{\omega_K}$$

and consequently (3.20) also in this case.

(iii) Case $p_K \geq d+1$ and $\zeta_p^{\boldsymbol{a}}, \zeta_p^{\boldsymbol{a},\boldsymbol{a}}$ non (necessarily) zero for all $\boldsymbol{a} \in \mathcal{V}_h$ such that $K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}$, resulting from the prescriptions (3.12b), (3.13b). Note that consequently $p_L \geq d+1$ for all $L \in \widetilde{\mathcal{T}}_K$, and we are in a high-polynomial-degree case. The proof of Lemma 5.3 applies verbatim until (5.14); in particular, the mean value fixes (5.5) are well defined for all $L \in \widetilde{\mathcal{T}}_K$, since $p_L \geq d+1$, and variable polynomial degree is treated in Corollary A.1. We cannot, unfortunately, proceed as in the last paragraph, namely for (5.15), since Corollary D.2 only addresses uniform polynomial degree. We thus adjust the arguments as follows.

In addition to τ_{hp} from (3.4), let, for all $L \in \widetilde{\mathcal{T}}_K$,

$$\underline{\tau_{hp}^K}|_L := \arg \min_{\substack{v_p \in \mathcal{P}_{\underline{p}_K}(L) \\ (v_p, 1)_L = (v, 1)_L}} \|\nabla(v - v_p)\|_L, \tag{5.19}$$

where we recall that $\underline{p}_K$ is the smallest polynomial degree over the extended element patch $\widetilde{\mathcal{T}}_K$, see (2.7). Here, $\underline{\tau_{hp}^K}$ is a lowered uniform polynomial degree local-best approximate of v, $\underline{\tau_{hp}^K} \in \mathcal{P}_{\underline{p}_K}(\widetilde{\mathcal{T}}_K)$. Let us also denote the argument of the minimization on the left-hand side of (5.15) by ι_p^K,

$$\iota_p^K := \arg \min_{\substack{v_p \in \mathcal{P}_{\underline{p}}(\widetilde{\mathcal{T}}_K) \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_K) \\ (v_p,1)_{\widetilde{\omega}_K} = (s_{hp}+\delta_{hp},1)_{\widetilde{\omega}_K} \text{ if } |\partial\widetilde{\omega}_K \cap \Gamma_{\mathrm{D}}|=0}} \|\nabla(\tau_{hp} - v_p)\|_{\widetilde{\omega}_K}; \tag{5.20}$$

note that from (5.2) and (5.14), $\iota_p^K = \zeta_p^K + (s_{hp} + \delta_{hp})|_{\widetilde{\omega}_K}$. Then, also consider

$$\underline{\iota_p^K} := \arg \min_{\substack{v_p \in \mathcal{P}_{\underline{p}_K}(\widetilde{\mathcal{T}}_K) \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_K) \\ (v_p,1)_{\widetilde{\omega}_K} = (s_{hp}+\delta_{hp},1)_{\widetilde{\omega}_K} \text{ if } |\partial\widetilde{\omega}_K \cap \Gamma_{\mathrm{D}}|=0}} \|\nabla(\underline{\tau_{hp}^K} - v_p)\|_{\widetilde{\omega}_K}; \tag{5.21}$$

this is a lowered uniform polynomial degree version of (5.20) which employs $\underline{\tau_{hp}^K}$ from (5.19) in place of τ_{hp} from (3.4).

Since $\underline{\iota_p^K}$ belongs to the minimization set in (5.20), the triangle inequality, Corollary D.2 (applied with the uniform lowered polynomial degree $\underline{p}_K$), and the trivial estimate $\|\nabla(\tau_{hp} - v)\|_L \le \|\nabla(\underline{\tau_{hp}^K} - v)\|_L$ for all $L \in \widetilde{\mathcal{T}}_K$ give, in remplacement of (5.15),

$$\begin{aligned}
&\min_{v_p \in \mathcal{P}_{\underline{p}}(\widetilde{\mathcal{T}}_K) \cap H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)} \|\nabla(\tau_{hp} - v_p)\|_{\widetilde{\omega}_K} \\
&\quad = \|\nabla(\tau_{hp} - \iota_p^K)\|_{\widetilde{\omega}_K} \le \|\nabla(\tau_{hp} - \underline{\iota_p^K})\|_{\widetilde{\omega}_K} \\
&\quad \le \|\nabla(\tau_{hp} - \underline{\tau_{hp}^K})\|_{\widetilde{\omega}_K} + \|\nabla(\underline{\tau_{hp}^K} - \underline{\iota_p^K})\|_{\widetilde{\omega}_K} \\
&\quad \le \|\nabla(\tau_{hp} - v)\|_{\widetilde{\omega}_K} + \|\nabla(\underline{\tau_{hp}^K} - v)\|_{\widetilde{\omega}_K} + \|\nabla(\underline{\tau_{hp}^K} - \underline{\iota_p^K})\|_{\widetilde{\omega}_K} \\
&\quad \lesssim \|\nabla(\tau_{hp} - v)\|_{\widetilde{\omega}_K} + \|\nabla(\underline{\tau_{hp}^K} - v)\|_{\widetilde{\omega}_K} + \min_{w \in H^1_{0,\mathrm{D}}(\widetilde{\omega}_K)} \|\nabla(\underline{\tau_{hp}^K} - w)\|_{\widetilde{\omega}_K} \\
&\quad \le \|\nabla(\tau_{hp} - v)\|_{\widetilde{\omega}_K} + \|\nabla(\underline{\tau_{hp}^K} - v)\|_{\widetilde{\omega}_K} + \|\nabla(\underline{\tau_{hp}^K} - v)\|_{\widetilde{\omega}_K} \\
&\le 3\|\nabla(\underline{\tau_{hp}^K} - v)\|_{\widetilde{\omega}_K}.
\end{aligned}$$

We have also played in the target function $v \in H^1_{0,\mathrm{D}}(\Omega)$ from the announcement of Theorem 3.7. Thus, (3.20) follows from the definition (5.19).

(iv) Case $p_K \ge d+1$ and $\zeta_p^{\boldsymbol{b}} = \zeta_p^{\boldsymbol{b},\boldsymbol{b}} = 0$ for some $\boldsymbol{b} \in \mathcal{V}_h$ such that $K \in \widetilde{\mathcal{T}}_{\boldsymbol{b}}$, resulting from the prescriptions (3.12a), (3.13a) since $p_L < d+1$ for some $L \in \widetilde{\mathcal{T}}_{\boldsymbol{b}}$, whereas $\zeta_p^{\boldsymbol{a}}, \zeta_p^{\boldsymbol{a},\boldsymbol{a}}$ non (necessarily) zero for some other $\boldsymbol{a} \in \mathcal{V}_h$ such that $K \in \widetilde{\mathcal{T}}_{\boldsymbol{a}}$, resulting from the prescriptions (3.12b), (3.13b). Here, by (3.16), $(P_{hp}(v))|_K = s_{hp}|_K + \delta_{hp}|_K + \sum_{\boldsymbol{a} \in \widehat{\mathcal{V}}_K} \zeta_p^{\boldsymbol{a},\boldsymbol{a}}$, where s_{hp} is given by (3.6), δ_{hp} is given by (3.9b), $\zeta_p^{\boldsymbol{a},\boldsymbol{a}}$ is given by (3.12b), (3.13b), and $\widehat{\mathcal{V}}_K$ collects the vertices of the element K with $\zeta_p^{\boldsymbol{a}}, \zeta_p^{\boldsymbol{a},\boldsymbol{a}}$ prescribed by (3.12b), (3.13b). We cannot proceed exactly as in the above step (iii): the problem (5.2) is well-defined, but the elementwise mean values (5.5) cannot be fixed for those $L \in \widetilde{\mathcal{T}}_{\boldsymbol{b}} \cap \widetilde{\mathcal{T}}_K$ where the polynomial degree is too small, $p_L < d+1$.

The fix is, fortunately, easy. Define $\widehat{\mathcal{T}}_K := \cup_{\boldsymbol{a} \in \widehat{\mathcal{V}}_K} \{\widetilde{\mathcal{T}}_{\boldsymbol{a}}\}$, which is a subset of the extended element patch $\widetilde{\mathcal{T}}_K$; the subdomain corresponding to $\widehat{\mathcal{T}}_K$, denoted by $\widehat{\omega}_K$, is then a subset of $\widetilde{\omega}_K$. We then introduce ζ_p^K as in (5.2) but restricting the minimization to $\widehat{\mathcal{T}}_K$ and $\widehat{\omega}_K$. Then the mean value fixes (5.5) are well defined for all $L \in \widehat{\mathcal{T}}_K$ and one can verify that the proof proceeds similarly, merely employing the reduced set of vertices $\widehat{\mathcal{V}}_K$ in place of $\mathcal{V}_K$. □

A Stable H^1 decomposition on patches

We now state a stable decomposition result which immediately follows from Schöberl *et al.* [35] for a uniform polynomial degree and, in two space dimensions, Karkulik *et al.* [32] for a variable polynomial degree. We consider subdomains $\omega \subset \Omega$ and the corresponding meshes $\mathcal{T}_\omega$ as in Section 2.1 which will later be taken "small", typically the extended vertex patch $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ with the corresponding subdomain $\widetilde{\omega}_{\boldsymbol{a}}$ or the extended element patch $\widetilde{\mathcal{T}}_K$ with $\widetilde{\omega}_K$. Recall the notation from Sections 2.6 and 2.7. There holds:

Corollary A.1 (Stable H^1 decomposition on patches)**.** *Let a simplicial mesh $\mathcal{T}_h$ of Ω, a (variable) polynomial degree $p_K \ge 1$ for all $K \in \mathcal{T}_h$, and $\omega \subset \mathbb{R}^d$, an open, bounded, and connected Lipschitz polygonal or polyhedral subdomain of Ω corresponding to a face-connected submesh (patch) of $\mathcal{T}_h$, denoted* *by $\mathcal{T}_\omega$, with vertex set $\mathcal{V}_\omega$, be given. For $d = 3$ and variable polynomial degree, let Assumption 3.2 hold. Let*

$$s_p \in \underline{\mathcal{P}_p(\mathcal{T}_\omega)} \cap H^1_{0,\mathrm{D}}(\omega), \tag{A.1a}$$

$$(s_p, 1)_\omega = 0 \text{ if } |\partial\omega \cap \Gamma_{\mathrm{D}}| = 0 \tag{A.1b}$$

be a p-degree continuous piecewise polynomial on ω, respecting the zero trace condition on Γ_{D} if $\partial\omega$ contains faces from $\overline{\Gamma_{\mathrm{D}}}$, or mean-value free. Then there exists a decomposition of s_p as

$$s_p = \sum_{\boldsymbol{b} \in \mathcal{V}_\omega} s_p^{\boldsymbol{b}} \tag{A.2}$$

where the contributions

$$\begin{aligned} s_p^{\boldsymbol{b}} &\text{ are supported on the (truncated) vertex patch subdomains } \omega_{\boldsymbol{b}} \cap \omega, \text{ linearly} \\ &\text{depend on } s_p \text{ on the (truncated) extended vertex patch subdomains } \widetilde{\omega}_{\boldsymbol{b}} \cap \omega, \end{aligned} \tag{A.3}$$

and satisfy

$$s_p^{\boldsymbol{b}} \in \underline{\mathcal{P}_p(\mathcal{T}_{\boldsymbol{b}} \cap \mathcal{T}_\omega)} \cap H^1_{0,\mathrm{D},\psi^{\boldsymbol{b}}}(\omega_{\boldsymbol{b}} \cap \omega), \tag{A.4}$$

i.e., recalling (2.9) and (2.5), are such that $s_p^{\boldsymbol{b}}$ are p-degree continuous piecewise polynomial on $\omega_{\boldsymbol{b}} \cap \omega$ and $s_p^{\boldsymbol{b}} = 0$ on that faces in $\partial(\omega_{\boldsymbol{b}} \cap \omega)$ where the hat function $\psi^{\boldsymbol{b}}$ vanishes or which lie in the Dirichlet boundary $\overline{\Gamma_{\mathrm{D}}}$. Moreover, the decomposition is stable in that

$$\|\nabla s_p^{\boldsymbol{b}}\|_{\omega_{\boldsymbol{b}} \cap \omega} \lesssim \|\nabla s_p\|_{\widetilde{\omega}_{\boldsymbol{b}} \cap \omega} \qquad \forall \boldsymbol{b} \in \mathcal{V}_\omega, \tag{A.5}$$

where the constant hidden in $\lesssim$ only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_\omega}$ of the mesh $\mathcal{T}_\omega$ and the ratio $h_\omega / \min_{K \in \mathcal{T}_\omega} h_K$.

Remark A.2 (No global low order component and no local orthogonality constraints)**.** *In stable decompositions, there usually appears a lowest-order component supported over the whole domain ω (as in Schöberl* et al. *[35, Section 3.1]), or some local orthogonality constraints are imposed (as in [15, Appendix B]. This is avoided here under the zero trace or mean value condition in (A.1) and for the price of the ratio $h_\omega / \min_{K \in \mathcal{T}_\omega} h_K$. Note that, for $\mathcal{T}_\omega$ the local patches $\widetilde{\mathcal{T}}_{\boldsymbol{a}}$ or $\widetilde{\mathcal{T}}_K$, this ratio is bounded solely in function of the mesh shape regularity parameter κ_h from (2.1).*

Proof, uniform polynomial degree $p_K = p \geq 1$ for all $K \in \mathcal{T}_h$. Let s_p satisfy (A.1) and consider the decomposition of Schöberl *et al.* [35] on the domain ω and mesh $\mathcal{T}_\omega$. Importantly, we choose the "coarse grid contribution" (u_0 in equation (2) or $\Pi_h u$ in Section 4.1 of [35]) as zero. This is eligible in terms of [35, Lemma 3.1], since

$$\|\nabla 0\|_\omega \leq \|\nabla s_p\|_\omega, \tag{A.6a}$$

$$\|\nabla s_p\|_\omega = \|\nabla s_p\|_\omega, \tag{A.6b}$$

$$\|h^{-1} s_p\|_\omega \leq \frac{1}{\min_{K \in \mathcal{T}_\omega} h_K} \|s_p\|_\omega \overset{\text{Poincaré}}{\lesssim} \frac{h_\omega}{\min_{K \in \mathcal{T}_\omega} h_K} \|\nabla s_p\|_\omega. \tag{A.6c}$$

Consequently, there is no global low order component, cf. Remark A.2. The construction of [35] then gives the decomposition, see equations (11) and (24) (after associating the face, edge, and element contributions with the vertex contributions),

$$s_p = \sum_{\boldsymbol{b} \in \mathcal{V}_\omega} s_p^{\boldsymbol{b}},$$

stable as

$$\sum_{\boldsymbol{b} \in \mathcal{V}_\omega} \|\nabla s_p^{\boldsymbol{b}}\|^2_{\omega_{\boldsymbol{b}} \cap \omega} \lesssim \|\nabla s_p\|^2_\omega.$$

The inspection of the developments of [35] shows that $s_p^{\boldsymbol{b}}$ are supported on $\omega_{\boldsymbol{b}} \cap \omega$ and solely constructed from and linearly dependent on the values of s_p on the (truncated) extended patches $\widetilde{\omega}_{\boldsymbol{b}} \cap \omega$ as expressed in (A.3) and satisfy more precisely the local stability as expressed in (A.5). Crucially, the constant hidden in $\lesssim$ above only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_\omega}$ of the mesh $\mathcal{T}_\omega$ and, through (A.6c), on the ratio $h_\omega / \min_{K \in \mathcal{T}_\omega} h_K$. □

Proof, variable polynomial degrees $p_K \geq 1$ for $K \in \mathcal{T}_h$. The proof is as above, relying, in two space dimensions, on Karkulik *et al.* [32, Theorem 2.5], and, in three space dimensions, on Assumption 3.2. □

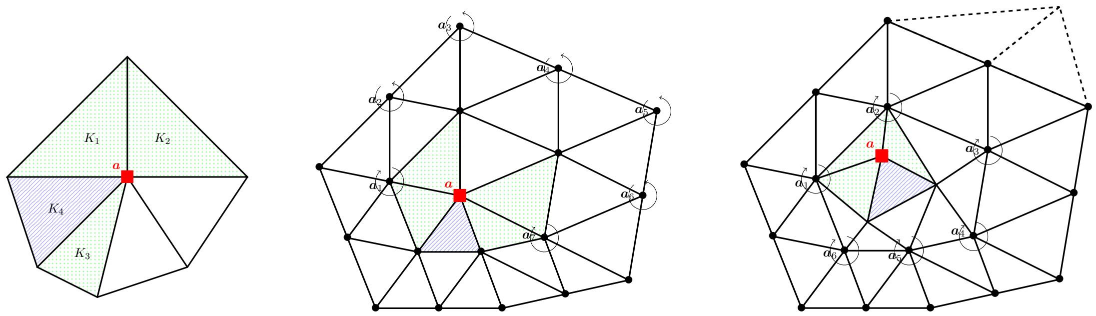

Figure 4: Examples where Definition B.1, property (i), point 1, is not satisfied (left and middle) and where Definition B.1, property (ii) is not satisfied (right). For the marked vertex $\boldsymbol{a}$ and the hatched triangle, where the already enumerated triangles sharing $\boldsymbol{a}$ are dotted. Enumeration shown explicitly or indicated by the order of vertices and direction of rotation.

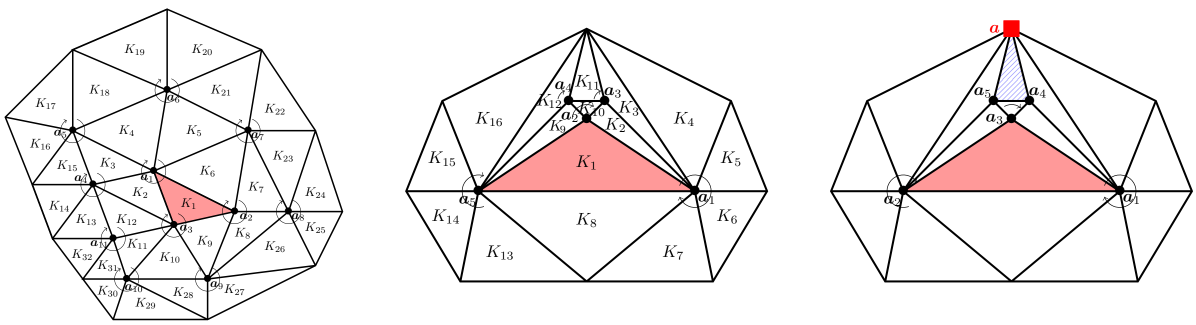

Figure 5: Example of enumeration of Algorithm B.3 (left and middle). Illustration of the necessity of the prioritization in the queue on step 3 of Algorithm B.3 (right; the running submesh $\mathcal{T}_i$ after the enumeration of the elements sharing the vertices $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ would create a non contractible (not simply connected) subdomain $\overline{\omega_i}$ corresponding to $\mathcal{T}_i$ (a hole including the hatched triangle), later causing violation of Definition B.1, property (i) for the marked vertex and the hatched triangle).

B Patch enumeration in two space dimensions

The following definition will be central:

Definition B.1 (Suitable patch enumeration in two space dimensions)**.** *Let $\mathcal{T}_\omega$ be an edge-connected triangular mesh with the corresponding open and bounded polygon $\omega \subset \mathbb{R}^2$ such that $\overline{\omega}$ is contractible. An enumeration $\{K_1, \ldots, K_{|\mathcal{T}_\omega|}\}$ of the triangles in $\mathcal{T}_\omega$ is suitable if:*

(i) For all $1 < i \leq |\mathcal{T}_\omega|$, if there are 2 edges of K_i shared with previously enumerated triangles, intersecting in a vertex $\boldsymbol{a}$, then 1) all the triangles sharing the vertex $\boldsymbol{a}$ come sooner in the enumeration; 2) the vertex $\boldsymbol{a}$ lies in the interior of ω.

(ii) For all $1 < i \leq |\mathcal{T}_\omega|$, there are between 1 and 2 edge neighbors of K_i which have been already enumerated and correspondingly, there is at least 1 edge neighbor which has not been enumerated yet, or K_i has an edge on the boundary $\partial\omega$. In particular, there is no enumerated edge neighbor only for K_1 and all edge neighbors are already enumerated for $K_{|\mathcal{T}_\omega|}$, which moreover has an edge on the boundary $\partial\omega$.

Property (i), point 1, is illustrated in Figure 4 (left and middle) and property (ii) in Figure 4 (right). Importantly, we have:

Lemma B.2 (Definition B.1 on extended patches $\widehat{\mathcal{T}}_K$ in 2D)**.** *Let $K \in \mathcal{T}_h$. Consider a patch given by a collection of extended vertex patches $\widehat{\mathcal{T}}_K := \cup_{\boldsymbol{a} \in \widehat{\mathcal{V}}_K} \{\widetilde{\mathcal{T}}_{\boldsymbol{a}}\}$ as per Section 2.2, where $\widehat{\mathcal{V}}_K$ is a (sub)set of vertices $\mathcal{V}_K$. Then $\widehat{\mathcal{T}}_K$ can be enumerated as per Definition B.1.*

In order to prove Lemma B.2, we will apply an algorithm of breadth-first search type. We enumerate the triangles one by one, starting from K, with an enumeration index i. To do so, we consider, also one by one and also yielding an enumeration, with an enumeration index j, all vertices $\boldsymbol{a}$ lying in the interior of $\mathcal{T}_\omega$[1]. Every vertex will have an associated clockwise direction of rotation. We also create a vertex queue, with a first-in first-out organization but with a possible application of a prioritization rule in order to avoid violations of the properties of Definition B.1 such as in Figure 5 (right). The algorithm is illustrated in Figure 5 (left and middle) and reads as follows:

Algorithm B.3 (Enumeration of an extended patch $\widehat{\mathcal{T}}_K$ in 2D)**.**

1. *Enumerate the element K as K_1.*
2. *Form the initial vertex queue by the vertices of the element K_1 lying in the interior of the domain $\widehat{\omega}_K$. The first vertex is arbitrary and the other vertices are added in the clockwise direction. Set the vertex counter j to* 0.
3. *Increase the vertex counter, $j := j + 1$. Take the next vertex from the queue (first-in first-out organization), and enumerate it as $\boldsymbol{a}_j$. If, however, the next step 4 shall create a non contractible subdomain $\overline{\omega_i}$ corresponding to the running submesh $\mathcal{T}_i$ given by all enumerated triangles $K_1 \ldots K_i$ (a hole in $\mathcal{T}_i$), then discard this and take instead (repeatedly) the subsequent vertex from the queue.*
4. *Number those triangles which share the vertex $\boldsymbol{a}_j$ and have not been enumerated yet, in the clockwise direction, starting from the neighbor of an already enumerated triangle.*
5. *Add those interior vertices of all triangles sharing the vertex $\boldsymbol{a}_j$ which are not there yet, in the clockwise direction, into the vertex queue.*
6. *If all triangles and interior vertices have been enumerated, then stop. Otherwise go to step 3.*

Proof of Lemma B.2. We employ Algorithm B.3 and verify (i)–(ii) of Definition B.1 by proceeding by induction on the vertex counter j.

1) Consider $j = 1$. Then $\boldsymbol{a}_1$ is the first vertex of the element K_1, since the prioritization rule does not apply. Then, any triangle K_i enumerated on step 4 of Algorithm B.3, $i \geq 2$, is an edge neighbor (in the clockwise direction) of K_{i-1}, both sharing the vertex $\boldsymbol{a}_1$. Hence, Definition B.1 is clearly satisfied after enumerating all triangles sharing $\boldsymbol{a}_1$. In particular, the situation of Figure 4 (left) cannot happen with respect to the vertex $\boldsymbol{a}_1$, since we proceed continuously in one (clockwise) direction.

2) Consider the enumeration at the end of step 4 of Algorithm B.3, while enumerating all triangles sharing the vertex $\boldsymbol{a}_j$. Then property (i), point 1, is satisfied for $\boldsymbol{a}_j$, since by the induction hypothesis, (i), point 1, is satisfied up to $\boldsymbol{a}_{j-1}$ and since we start from an edge neighbor of an already enumerated triangle and enumerate in a fixed direction of rotation; situation of Figure 4 (left) cannot happen with respect to the vertex $\boldsymbol{a}_j$. Property (i), point 2, is automatically satisfied for $\boldsymbol{a}_j$, since $\boldsymbol{a}_j$ is an interior vertex in Algorithm B.3.

An algorithm employing enumeration in the direction of rotation around each vertex $\boldsymbol{a}_j$ without additional rules may, however, potentially violate property (i), point 1, with respect to another vertex with not all elements sharing it already enumerated, say $\boldsymbol{a}$, as illustrated in Figure 4 (middle) for $j = 7$. The vertex queue helps to prevent this. Yet another problem can, however, arrive even in presence of the vertex queue. This is depicted in Figure 5 (right). There, the enumeration would violate property (i), point 1 if $\boldsymbol{a}$ is on the interior of $\widehat{\omega}_K$ and property (i), point 2 if $\boldsymbol{a}$ is on the boundary of $\widehat{\omega}_K$. Indeed, on step 2 of Algorithm B.3, the initial queue would be formed by the vertices $\boldsymbol{a}_2, \boldsymbol{a}_3$ and after the first passage through steps 3–5, this would become $\boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4$. However, enumerating around $\boldsymbol{a}_2$ would create a non contractible (not simply connected) subdomain $\overline{\omega_i}$ corresponding to the running submesh $\mathcal{T}_i$ given by all enumerated triangles around $\boldsymbol{a}_1, \boldsymbol{a}_2$, so that rather enumeration around $\boldsymbol{a}_3$ is performed first. Now the queue becomes $\boldsymbol{a}_2, \boldsymbol{a}_4, \boldsymbol{a}_5$. Taking $\boldsymbol{a}_2$ would still create a hole in $\mathcal{T}_i$, so that rather $\boldsymbol{a}_4$ is employed first. Finally, the queue becomes $\boldsymbol{a}_2, \boldsymbol{a}_5$, still $\boldsymbol{a}_2$ is inadmissible, $\boldsymbol{a}_5$ passes first, and only then, last, $\boldsymbol{a}_2$ is treated, leading to the enumeration of Figure 5 (middle). To avoid this, property (i) for the vertices different from $\boldsymbol{a}_j$ is satisfied by the construction of Algorithm B.3 (there is always a suitable vertex in the queue: this may not be the case for a domain $\widehat{\omega}_K$ such that $\overline{\widehat{\omega}_K}$ is not contractible, but here we consider extended patches of elements from Section 2.2).

3) We finally treat property (ii). Triangle K_1 clearly has no enumerated edge neighbor and $K_{|\mathcal{T}_\omega|}$ has all edge neighbors already enumerated. Further, by construction of Algorithm B.3, for K_i, $1 < i \leq |\mathcal{T}_\omega|$, there is at least 1 edge neighbor which has been already enumerated (we always attribute a subsequent number to an edge neighbor). We are left to show the two remaining properties. First, we need to show that any K_i, $1 < i < |\mathcal{T}_\omega|$ (excluding the first and last elements), has at most 2 edge neighbors which have been already enumerated. (Then, clearly, there is at least 1 edge neighbor which has not been enumerated yet, or K_i has an edge on the boundary $\partial\widehat{\omega}_K$.) Second, we need to show that the last triangle

[1] To simplify the presentation, we do not describe here the treatment of the irregular cases caused by trimming by the boundary $\partial\Omega$, like when there are no interior vertices in $\mathcal{T}_\omega$, when $\mathcal{T}_\omega = \mathcal{T}_h$ is only composed of a single element, or when $\mathcal{T}_\omega$ contains a stripe of triangles.

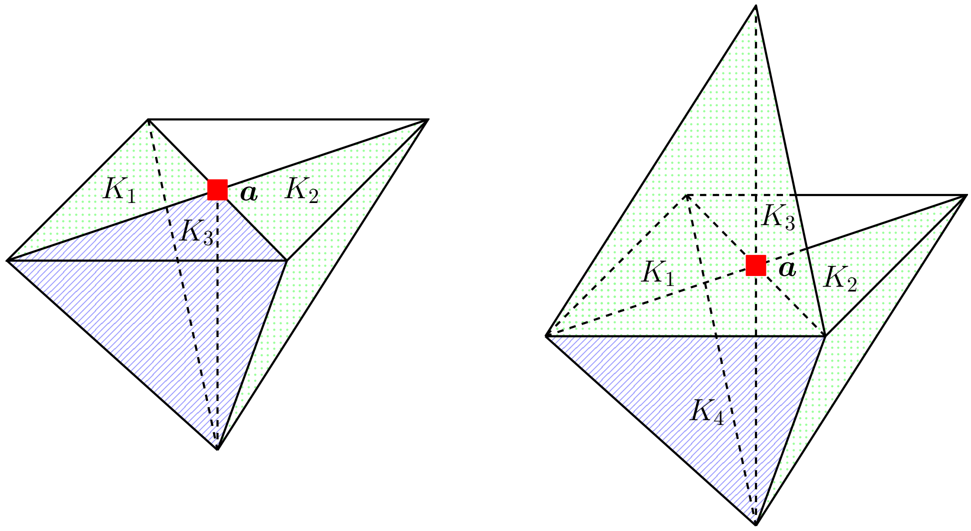

Figure 6: Example where Definition C.1, property (i), point 1, (left) and Definition C.1, property (ii), point 1, (right) is not satisfied . For the marked vertex $\boldsymbol{a}$ and the hatched tetrahedron, where the already enumerated tetrahedra sharing $\boldsymbol{a}$ are dotted.

$K_{|\mathcal{T}_\omega|}$ has an edge on the boundary $\partial\widehat{\omega}_K$. Both these situations are illustrated in Figure 4 (right): either with the hatched triangle enumerated before the dashed triangles (it has 3 edge neighbors which have been already enumerated but only one of the dashed triangles is the last)(first case) or with the hatched triangle enumerated last (it has no edge on the boundary $\partial\widehat{\omega}_K$) (omitting the dashed triangles) (second case). This is again ensured in Algorithm B.3 by construction by the use of the priority vertex queue. $\square$

C Patch enumeration in three space dimensions

The equivalent of Definition B.1 in three space dimensions is:

Definition C.1 (Suitable patch enumeration in three space dimensions)**.** *Let $\mathcal{T}_\omega$ be a face-connected tetrahedral mesh with the corresponding open and bounded polyhedron $\omega \subset \mathbb{R}^3$ such that $\overline{\omega}$ is contractible. An enumeration $\{K_1, \ldots, K_{|\mathcal{T}_\omega|}\}$ of the tetrahedra in $\mathcal{T}_\omega$ is suitable if:*

(i) For all $1 < i \leq |\mathcal{T}_\omega|$, if there are at least 2 faces of K_i shared with previously enumerated tetrahedra, intersecting in an edge e, then 1) all the tetrahedra sharing the edge e come sooner in the enumeration; 2) the edge e lies in the interior of ω.

(ii) For all $1 < i \leq |\mathcal{T}_\omega|$, if there are 3 faces of K_i shared with previously enumerated tetrahedra, intersecting in a vertex $\boldsymbol{a}$, then 1) all the tetrahedra sharing the vertex $\boldsymbol{a}$ come sooner in the enumeration; 2) the vertex $\boldsymbol{a}$ lies in the interior of ω.

(iii) For all $1 < i \leq |\mathcal{T}_\omega|$, there are between 1 and 3 face neighbors of K_i which have been already enumerated and correspondingly, there is at least 1 face neighbor which has not been enumerated yet, or K_i has a face on the boundary $\partial\omega$. In particular, there is no enumerated face neighbor only for K_1 and all face neighbors are already enumerated for $K_{|\mathcal{T}_\omega|}$, which moreover has a face on the boundary $\partial\omega$.

We currently do not see how to prove the existence of enumeration satisfying Definition C.1 for an arbitrary patch given by a collection of extended vertex patches $\widehat{\mathcal{T}}_K := \cup_{\boldsymbol{a}\in\widehat{\mathcal{V}}_K}\{\widetilde{\mathcal{T}}_{\boldsymbol{a}}\}$ for a tetrahedron $K \in \mathcal{T}_h$. This lead us to Assumption 3.1. We, however, as in Appendix B, can construct an algorithm that produces, in most cases, the requested enumeration. This again employs a queue of vertices lying in the interior of $\widehat{\omega}_K$ with priority and a breadth-first search procedure. The idea is to employ therein a 2D enumeration of surface triangular meshes in the spirit of Algorithm B.3, where, however, three adjustments are needed: 1) one needs to extend it to arbitrary patch subdomains; 2) one needs to ensure that "or K_i has an edge on the boundary $\partial\omega$" from Definition B.1, point (ii) is removed (so that for all K_i, $1 < i \leq |\mathcal{T}_\omega|$, there is at least 1 edge neighbor which has not been enumerated yet); 3) one needs to start the enumeration in a way that K_1 touches the boundary $\partial\omega$. Various variants of Algorithm B.3 can be designed but for the moment we do not see any that works in all possible geometrical situations.

To describe the 3D enumeration algorithm, we will need a notion of "interior-like vertex patch" $\mathcal{T}_{\boldsymbol{a}}$ which is formed by tetrahedra sharing the vertex $\boldsymbol{a}$ such that the corresponding patch subdomain

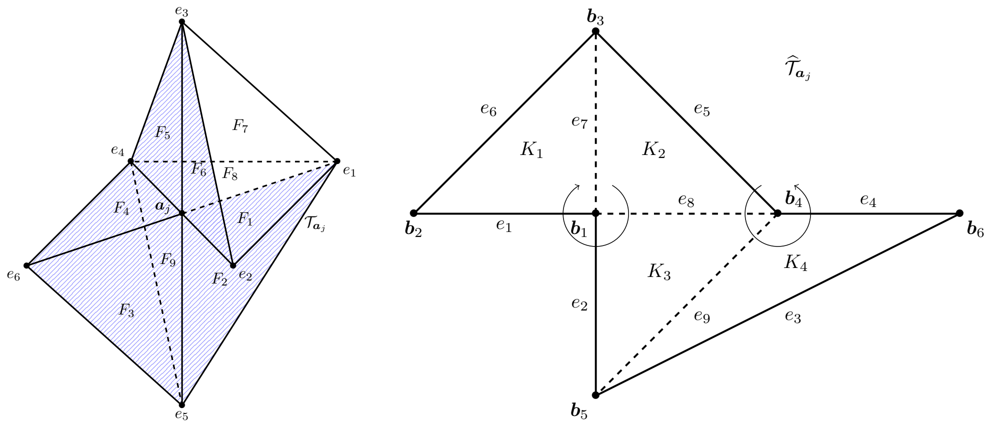

Figure 7: Example of enumeration of Algorithm C.2 at the stage of a boundary-like vertex patch $\mathcal{T}_{\boldsymbol{a}_j}$ with 4 tetrahedra, 6 faces F_1–F_6 on $\partial\omega_{\boldsymbol{a}_j}$ sharing the vertex $\boldsymbol{a}_j$ (hatched), 3 faces F_7–F_9 inside $\omega_{\boldsymbol{a}_j}$ sharing the vertex $\boldsymbol{a}_j$, and 6 edges e_1–e_6 (on $\partial\omega_{\boldsymbol{a}_j}$) sharing the vertex $\boldsymbol{a}_j$ (left); the corresponding planar triangular mesh $\widehat{\mathcal{T}}_{\boldsymbol{a}_j}$ with 4 triangles corresponding to the 4 tetrahedra, 6 boundary edges e_1–e_6 on $\partial\widehat{\omega}_{\boldsymbol{a}_j}$ corresponding to the faces F_1–F_6, 3 interior edges e_7–e_9 inside $\widehat{\omega}_{\boldsymbol{a}_j}$ corresponding to the faces F_7–F_9, and 6 (boundary) vertices $\boldsymbol{b}_1$–$\boldsymbol{b}_6$ on $\partial\widehat{\omega}_{\boldsymbol{a}_j}$ corresponding to the edges e_1–e_6 (right)

$\omega_{\boldsymbol{a}}$ contains an open ball around $\boldsymbol{a}$, cf. [26, Section 2.1]. This corresponds to a vertex patch $\mathcal{T}_{\boldsymbol{a}}$ from Section 2.2 for the vertex $\boldsymbol{a}$ inside Ω. We will also need a notion of a "boundary-like vertex patch" $\mathcal{T}_{\boldsymbol{a}}$ where the corresponding patch subdomain $\omega_{\boldsymbol{a}}$ only contains an open ball around $\boldsymbol{a}$ minus a sector with a solid angle $\theta_{\boldsymbol{a}} \in (0, 4\pi)$, cf. [26, Section 2.4]. This corresponds to a vertex patch $\mathcal{T}_{\boldsymbol{a}}$ from Section 2.2 for the vertex $\boldsymbol{a}$ inside Ω but where only some tetrahedra sharing $\boldsymbol{a}$ are considered. We will only employ face-connected boundary-like vertex patches with one solid angle opening, where the closure of the corresponding subdomain is contractible.

Let $\mathcal{T}_{\boldsymbol{a}}$ be a "boundary-like vertex patch". Consider the surface triangular mesh, say $\widehat{\mathcal{T}}_{\boldsymbol{a}}$, made by the faces of $\mathcal{T}_{\boldsymbol{a}}$ lying on the boundary $\partial\omega_{\boldsymbol{a}}$ but not sharing the vertex $\boldsymbol{a}$. There is a one-to-one correspondence between the tetrahedra of $\mathcal{T}_{\boldsymbol{a}}$ and the triangles of $\widehat{\mathcal{T}}_{\boldsymbol{a}}$, between the faces of $\mathcal{T}_{\boldsymbol{a}}$ sharing the vertex $\boldsymbol{a}$ and the edges of $\widehat{\mathcal{T}}_{\boldsymbol{a}}$, and between the edges of $\mathcal{T}_{\boldsymbol{a}}$ sharing the vertex $\boldsymbol{a}$ and the vertices of $\widehat{\mathcal{T}}_{\boldsymbol{a}}$. In particular, enumerating $\mathcal{T}_{\boldsymbol{a}}$ is equivalent to enumerating $\widehat{\mathcal{T}}_{\boldsymbol{a}}$. Moreover, by a homeomorphism, $\widehat{\mathcal{T}}_{\boldsymbol{a}}$ can be identified with a two-dimensional planar triangular mesh, where we take the "outside" viewpoint with respect to the vertex $\boldsymbol{a}$. This mesh is edge connected and the closure of the corresponding domain is contractible.

Figure 7 shows an example of the enumeration of a boundary-like vertex patch $\mathcal{T}_{\boldsymbol{a}_j}$ by applying the above-discussed adjustment of Algorithm B.3 to the triangular mesh $\widehat{\mathcal{T}}_{\boldsymbol{a}_j}$. We employ it to enumerate the tetrahedra sharing the interior vertex $\boldsymbol{a}_j$ form the queue in the algorithm that reads as follows:

Algorithm C.2 (Enumeration of an extended patch $\widehat{\mathcal{T}}_K$ in 3D)**.**

1. *Enumerate the element K as K_1.*
2. *Form the initial vertex queue by the vertices of the element K_1 lying in the interior of the domain $\widehat{\omega}_K$. The order of the vertices is arbitrary. Set the vertex counter j to 0.*
3. *Increase the vertex counter, $j := j + 1$. Take the next vertex from the queue (first-in first-out organization), and enumerate it as $\boldsymbol{a}_j$. If, however, the next step 4 shall create a non contractible subdomain $\overline{\omega_i}$ corresponding to the running submesh $\mathcal{T}_i$ given by all enumerated tetrahedra $K_1 \ldots K_i$ (a hole in $\mathcal{T}_i$), then discard this and take instead (repeatedly) the subsequent vertex from the queue.*
4. *Consider all tetrahedra from $\widehat{\mathcal{T}}_K$ sharing the vertex $\boldsymbol{a}_j$ which have not been enumerated yet. This forms a boundary-like vertex patch $\mathcal{T}_{\boldsymbol{a}_j}$ with the associated patch subdomain $\omega_{\boldsymbol{a}_j}$. Consider the surface triangular mesh $\widehat{\mathcal{T}}_{\boldsymbol{a}_j}$ corresponding to $\mathcal{T}_{\boldsymbol{a}_j}$. Enumerate $\widehat{\mathcal{T}}_{\boldsymbol{a}_j}$ using Algorithm B.3 with the adjustments discussed above; this also creates an enumeration of all vertices of $\widehat{\mathcal{T}}_{\boldsymbol{a}_j}$. Herein, start the enumeration from a neighbor of an already enumerated tetrahedron. The one-to-one correspondence between the triangles of $\widehat{\mathcal{T}}_{\boldsymbol{a}_j}$ and the tetrahedra of $\mathcal{T}_{\boldsymbol{a}_j}$ creates the continuation of the enumeration of the tetrahedra of $\widehat{\mathcal{T}}_K$. Moreover, the one-to-one correspondence between the vertices of $\widehat{\mathcal{T}}_{\boldsymbol{a}_j}$ and the edges of $\mathcal{T}_{\boldsymbol{a}_j}$ sharing the vertex $\boldsymbol{a}_j$ enumerates all edges of $\mathcal{T}_{\boldsymbol{a}_j}$ sharing the vertex $\boldsymbol{a}_j$.*
5. *Add those interior vertices of all tetrahedra sharing the vertex $\boldsymbol{a}_j$ which are not there yet into the*

vertex queue. The adding is done following the enumeration of all edges of $\mathcal{T}_{\boldsymbol{a}_j}$ sharing the vertex $\boldsymbol{a}_j$.

6. *If all tetrahedra and interior vertices have been enumerated, then stop. Otherwise go to step 3.*

D Stable broken polynomial extension on patch subdomains

We summarize here our results on stable broken polynomial extensions on patch subdomains. We only consider here a uniform polynomial degree p.

D.1 Available results

Stable H^1 polynomial extensions on a single triangle or tetrahedron have been achieved in Muñoz-Sola [34], Ainsworth and Demkowicz [2], and Demkowicz *et al.* [19], see also the references therein. Let K be a triangle or a tetrahedron and let $p \geq 1$. Let $\tau_K \in \mathcal{P}_p(K)$ be a volume datum and $r_F \in \mathcal{P}_p(F)$ a target trace; the latter is prescribed on $\mathcal{F}_K^{\mathrm{D}}$, a subset of all $(d-1)$-dimensional faces of K, possibly empty or containing some or all faces of K. Importantly, r_F has to be compatible in that it is a trace on $\mathcal{F}_K^{\mathrm{D}}$ of some p-degree polynomial in K. The combination of the above-cited trace liftings allows to prove, see [26, Lemma A.1], that

$$\min_{\substack{v_p \in \mathcal{P}_p(K) \\ v_p = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla(\tau_K - v_p)\|_K \lesssim \min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla(\tau_K - v)\|_K, \tag{D.1}$$

where the hidden constant only depends on the shape-regularity parameter of the element K and the space dimension d (the form (D.1) follows from [26, Lemma A.1] by a shift by τ_K).

Stable broken polynomial extensions achieve a similar result to (D.1) but on patches of elements, where, crucially, the datum τ_K is a broken piecewise polynomial. For patches of elements sharing a vertex and prescribed trace boundary conditions in three space dimensions, they have been established in [26, Corollaries 3.1 and 3.7], see also [14, Corollaries 3.4 and 4.4].

D.2 General meshes and no trace boundary conditions

We now extend the above results in two directions. First, we consider possibly larger patches than merely all elements sharing a given vertex. Second, we consider the case with no trace boundary conditions prescribed. Recall that by "face", we mean "$(d-1)$-dimensional face".

Our main result is:

Theorem D.1 (Stable broken polynomial extension on extended patches and without boundary conditions)**.** *Let $\mathcal{T}_\omega$ be a face-connected simplicial mesh with the corresponding open, bounded, and Lipschitz polygon or polyhedron $\omega \subset \mathbb{R}^d$, $d = 2, 3$, with $\overline{\omega}$ contractible, where $\mathcal{T}_\omega$ can be enumerated as per Definition B.1 or Definition C.1. Let $\tau_{hp} \in \mathcal{P}_p(\mathcal{T}_\omega)$ be a volume datum, a broken piecewise polynomial of (a uniform) degree $p \geq 1$ for all $K \in \mathcal{T}_\omega$. Then*

$$\min_{v_p \in \mathcal{P}_p(\mathcal{T}_\omega) \cap H^1(\omega)} \|\nabla(\tau_{hp} - v_p)\|_\omega \lesssim \min_{v \in H^1(\omega)} \|\nabla(\tau_{hp} - v)\|_\omega, \tag{D.2}$$

where the constant hidden in $\lesssim$ only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_\omega}$ of the mesh $\mathcal{T}_\omega$, the ratio $h_\omega / \min_{K \in \mathcal{T}_\omega} h_K$, and the space dimension d.

Proof. We present the proof for $d = 3$; the two-dimensional case is (much) easier. We follow [26, Section 6], see also [14, Section 6.4]. Let

$$v^\star := \arg \min_{\substack{v \in H^1(\omega) \\ (v,1)_\omega = 0}} \|\nabla(\tau_{hp} - v)\|_\omega \tag{D.3}$$

denote the infinite-dimensional $H^1(\omega)$ minimizer from the right-hand side of (D.2); the mean value constraint is employed merely for uniqueness. We present a constructive proof of (D.2) which proceeds along the enumeration of Definition C.1. On each element K_i, $1 \leq i \leq |\mathcal{T}_\omega|$, we in particular construct a suitable minimizer $\xi_i \in \mathcal{P}_p(K_i)$ and we gradually set

$$\xi_{hp}|_{K_i} := \xi_i. \tag{D.4}$$

We then verify that

$$\xi_{hp} \in \mathcal{P}_p(\mathcal{T}_\omega) \cap H^1(\omega) \tag{D.5}$$

and that

$$\|\nabla(\tau_{hp} - \xi_{hp})\|_\omega \lesssim \|\nabla(\tau_{hp} - v^\star)\|_\omega, \tag{D.6}$$

which establishes (D.2). More precisely, on each step $1 \le i \le |\mathcal{T}_\omega|$, we will verify that

$$\|\nabla(\tau_{hp} - \xi_i)\|_{K_i} \lesssim \|\nabla(\tau_{hp} - v^\star)\|_\omega, \tag{D.7}$$

which yields (D.6) up to a constant depending on the shape-regularity parameter $\kappa_{\mathcal{T}_\omega}$ of the mesh $\mathcal{T}_\omega$, the ratio $h_\omega / \min_{K\in\mathcal{T}_\omega} h_K$, and the space dimension d. Moreover, as $\xi_{hp}|_{K_i}$ will have its trace prescribed by $\xi_{hp}|_{K_j}$ of the previously enumerated K_j, it will have no jumps and (D.5) follows. We proceed along the enumeration $1 \le i \le |\mathcal{T}_\omega|$ of Definition C.1 and consider different cases.

(i) On the first element K_1, let

$$\xi_1 := \arg \min_{\substack{v_p \in \mathcal{P}_p(K_1) \\ (v_p,1)_{K_1}=0}} \|\nabla(\tau_{hp} - v_p)\|_{K_1}. \tag{D.8}$$

This is a well-posed problem. Actually, $\nabla\xi_1 = \nabla\tau_{hp}$ on K_1, so that (D.7) is trivial, but les us outline the path we will take subsequently. Since the datum $\tau_{hp}|_{K_1}$ in (D.8) is polynomial, we know from (D.1) that we can pass to the infinite-dimensional level,

$$\|\nabla(\tau_{hp} - \xi_1)\|_{K_1} \lesssim \min_{v\in H^1(K_1)} \|\nabla(\tau_{hp} - v)\|_{K_1}. \tag{D.9}$$

Then, since the infinite-dimensional minimizer $v^\star$ from (D.3) restricted to the element K_1, $v^\star|_{K_1}$, belongs to the minimization set on the right-hand side of (D.9), i.e., $v^\star|_{K_1} \in H^1(K_1)$ (please note that there are no trace conditions in (D.9)), we obtain

$$\|\nabla(\tau_{hp} - \xi_1)\|_{K_1} \lesssim \|\nabla(\tau_{hp} - v^\star)\|_{K_1}, \tag{D.10}$$

which gives (D.7) for $i = 1$.

(ii) On each element K_i with exactly one face shared with some previously enumerated simplex, say $F_{i,j}$ shared with K_j, $j < i$, we consider

$$\xi_i := \arg \min_{\substack{v_p \in \mathcal{P}_p(K_i) \\ v_p = \xi_{hp}|_{K_j} \text{ on } F_{i,j}}} \|\nabla(\tau_{hp} - v_p)\|_{K_i}. \tag{D.11}$$

Please note that since $j < i$ and by (D.4), $\xi_{hp}|_{K_j}$ is known. Then (D.11) is well-posed; there is in particular no compatibility condition to verify, since the trace is only imposed on one face. We now again employ (D.1). This yields

$$\|\nabla(\tau_{hp} - \xi_i)\|_{K_i} \lesssim \min_{\substack{v \in H^1(K_i) \\ v = \xi_{hp}|_{K_j} \text{ on } F_{i,j}}} \|\nabla(\tau_{hp} - v)\|_{K_i}. \tag{D.12}$$

Unfortunately, now $v^\star|_{K_i}$ does not belong to the minimization set on the right-hand side of (D.12) since there is a trace condition on the face $F_{i,j}$ imposed. The fix is, for the moment, easy. Consider the face neighbor K_j and let $\boldsymbol{T} : K_j \to K_i$ be the unique affine geometric mapping that maps K_j to K_i, leaving the face $F_{i,j}$ invariant. Now we "bring" the function $v^\star - \xi_{hp}$ from K_j to K_i, forming

$$v := v^\star|_{K_i} - (v^\star - \xi_{hp})|_{K_j} \circ \boldsymbol{T}^{-1}, \tag{D.13}$$

see [26, Proof of (5.11b), (1)] for the details. On $F_{i,j}$, this removes the trace of $v^\star$ and brings instead the requested $\xi_{hp}|_{K_j}$ (in appropriate weak sense), so that v from (D.13) now crucially belongs to the minimization set on the right-hand side of (D.12). Consequently, we obtain

$$\|\nabla(\tau_{hp} - \xi_i)\|_{K_i} \lesssim \|\nabla(\tau_{hp} - v^\star|_{K_i} + (v^\star - \xi_{hp})|_{K_j} \circ \boldsymbol{T}^{-1})\|_{K_i}. \tag{D.14}$$

Finally, by the triangle inequality and the properties of the geometric map $\boldsymbol{T}$ (recall that we suppose shape regularity of $\mathcal{T}_\omega$)

$$\begin{aligned}
\|\nabla(\tau_{hp} - \xi_i)\|_{K_i} &\le \|\nabla(\tau_{hp} - v^\star)\|_{K_i} + \|\nabla((v^\star - \xi_{hp})|_{K_j} \circ \boldsymbol{T}^{-1})\|_{K_i} \\
&\lesssim \|\nabla(\tau_{hp} - v^\star)\|_{K_i} + \|\nabla(v^\star - \xi_{hp})\|_{K_j} \\
&\le \|\nabla(\tau_{hp} - v^\star)\|_{K_i} + \|\nabla(\tau_{hp} - v^\star)\|_{K_j} + \|\nabla(\tau_{hp} - \xi_{hp})\|_{K_j} \\
&\lesssim \|\nabla(\tau_{hp} - v^\star)\|_\omega,
\end{aligned} \tag{D.15}$$

where, in the last estimate, we have employed (D.7) in K_j, which has been established previously since $j < i$. Thus (D.7) is established.

(iii) On each element K_i with exactly two faces shared with some previously enumerated simplices, say $F_{i,j}$ shared with K_j, $j < i$, and $F_{i,k}$ shared with K_k, $k < i$, we consider

$$\xi_i := \arg \min_{\substack{v_p \in \mathcal{P}_p(K_i) \\ v_p = \xi_{hp}|_{K_j} \text{ on } F_{i,j} \\ v_p = \xi_{hp}|_{K_k} \text{ on } F_{i,k}}} \|\nabla(\tau_{hp} - v_p)\|_{K_i}. \tag{D.16}$$

Again, since $j < i$ and $k < i$ and by (D.4), $\xi_{hp}|_{K_j}$ and $\xi_{hp}|_{K_k}$ are known. Since the trace is imposed on two faces, problem (D.16) is well-posed if the two data $\xi_{hp}|_{K_j}$ and $\xi_{hp}|_{K_k}$ are compatible, i.e., match along the common edge of $F_{i,j}$ and $F_{i,k}$, say e. This is crucially implied by Definition C.1, property (i). Indeed, by property (i), point 1, of Definition C.1 on the enumeration, all the simplices sharing the edge e come sooner in the enumeration and by property (i), point 2, the edge e does not lie on the boundary of ω. Thus $\xi_{hp}|_{K_j}$ and $\xi_{hp}|_{K_k}$ match along e as ξ_{hp} is trace-continuous on all faces sharing the edge e different from $F_{i,j}$ and $F_{i,k}$. We then again employ (D.1), which now yields

$$\|\nabla(\tau_{hp} - \xi_i)\|_{K_i} \lesssim \min_{\substack{v \in H^1(K_i) \\ v = \xi_{hp}|_{K_j} \text{ on } F_{i,j} \\ v = \xi_{hp}|_{K_k} \text{ on } F_{i,k}}} \|\nabla(\tau_{hp} - v)\|_{K_i}. \tag{D.17}$$

As above in step (ii), the continuous-level minimizer $v^\star$ from (D.3) restricted to K_i does not belong to the minimization set on the right-hand side of (D.17) since there are two trace conditions on the two faces $F_{i,j}$ and $F_{i,k}$ imposed. Here again, Definition C.1, property (i) is crucial: it enables to construct a suitable v in this sprit of (D.13) but which now involves the geometric mappings from all the simplices sharing the edge e except for K_i. This is done in a “2-folding” way which replaces $v^\star$ on $F_{i,j}$ and $F_{i,k}$ (in a proper weak sense) by respectively $\xi_{hp}|_{K_j}$ and $\xi_{hp}|_{K_k}$; the precise formula is [26, equation (5.12)]. Existence of a two-color refinement around edges of [26, Lemma B.2] is crucial at this step. Then (D.7) is established similarly to (D.15).

(iv) Finally, on each element K_i with exactly three faces shared with some previously enumerated simplices, say $F_{i,j}$ shared with K_j, $j < i$, $F_{i,k}$ shared with K_k, $k < i$, and $F_{i,l}$ shared with K_l, $l < i$, we consider

$$\xi_i := \arg \min_{\substack{v_p \in \mathcal{P}_p(K_i) \\ v_p = \xi_{hp}|_{K_j} \text{ on } F_{i,j} \\ v_p = \xi_{hp}|_{K_k} \text{ on } F_{i,k} \\ v_p = \xi_{hp}|_{K_l} \text{ on } F_{i,l}}} \|\nabla(\tau_{hp} - v_p)\|_{K_i}. \tag{D.18}$$

Again, all $\xi_{hp}|_{K_j}$, $\xi_{hp}|_{K_k}$, and $\xi_{hp}|_{K_l}$ are known at this stage. Then (D.18) is well-posed; as above, by consequence of Definition C.1, property (i), for any edge e common to two of the three above faces $F_{i,j}$, $F_{i,k}$, $F_{i,l}$, the face data given by ξ_{hp} are compatible, i.e., match along e. Employing once more (D.1), we have

$$\|\nabla(\tau_{hp} - \xi_i)\|_{K_i} \lesssim \min_{\substack{v \in H^1(K_i) \\ v = \xi_{hp}|_{K_j} \text{ on } F_{i,j} \\ v = \xi_{hp}|_{K_k} \text{ on } F_{i,k} \\ v = \xi_{hp}|_{K_l} \text{ on } F_{i,l}}} \|\nabla(\tau_{hp} - v)\|_{K_i}. \tag{D.19}$$

As above in steps (ii) and (iii), the infinite-dimensional minimizer $v^\star|_{K_i}$ does not belong to the minimization set on the right-hand side of (D.19) since there are three trace conditions on the three faces $F_{i,j}$, $F_{i,k}$, and $F_{i,l}$ imposed. Crucially, by property (ii), point 1, of Definition C.1 on the enumeration, all the simplices sharing the vertex $\boldsymbol{a}$ common to the three faces $F_{i,j}$, $F_{i,k}$, and $F_{i,l}$ come sooner in the enumeration and by property (ii), point 2, the vertex $\boldsymbol{a}$ does not lie on the boundary of ω. This enables to construct a suitable v in this sprit of (D.13) but which now involves the geometric mappings from all the simplices sharing the vertex $\boldsymbol{a}$ except for K_i. This is done in a “3-folding” way; the precise formula is [26, equation (5.14)]. Existence of a three-color refinement around vertices of [26, Lemma B.3] is crucial at this step. Then (D.7) is established similarly to (D.15). □

D.3 Application to extended patches of elements and imposition of trace boundary conditions

We now finally formulate the result precisely in the form needed in the proof of Lemmas 5.3 and 5.4 in Section 5.3. We again only consider a uniform polynomial degree p.

Corollary D.2 (Stable broken polynomial extension on extended patches $\widehat{\mathcal{T}}_K$)**.** *Let $K \in \mathcal{T}_h$. Consider a patch given by a collection of extended vertex patches $\widehat{\mathcal{T}}_K := \cup_{\boldsymbol{a}\in\widehat{\mathcal{V}}_K}\{\widetilde{\mathcal{T}}_{\boldsymbol{a}}\}$ as per Section 2.2, where $\widehat{\mathcal{V}}_K$ is a (sub)set of vertices $\mathcal{V}_K$. Let $\widehat{\omega}_K$ be the associated open subdomain. Let Assumption 3.1 hold. Let $\tau_{hp} \in \mathcal{P}_p(\widehat{\mathcal{T}}_K)$ be a volume datum, a broken piecewise polynomial of a uniform polynomial degree $p \geq 1$. Then*

$$\min_{v_p\in\mathcal{P}_p(\widehat{\mathcal{T}}_K)\cap H^1_{0,\mathrm{D}}(\widehat{\omega}_K)} \|\nabla(\tau_{hp} - v_p)\|_{\widehat{\omega}_K} \lesssim \min_{v\in H^1_{0,\mathrm{D}}(\widehat{\omega}_K)} \|\nabla(\tau_{hp} - v)\|_{\widehat{\omega}_K}, \tag{D.20}$$

where the constant hidden in $\lesssim$ only depends on the shape-regularity parameter $\kappa_{\widehat{\mathcal{T}}_K}$ of the mesh $\widehat{\mathcal{T}}_K$ and the space dimension d.

Proof. Let $d = 2$ and let Γ_D be empty. Then (D.20) is a combination of Theorem D.1 together with Lemma B.2. Similarly, if $d = 3$ and if $\partial\widehat{\omega}_K$ does not contain any face from $\partial\Omega$, (D.20) is a combination of Theorem D.1 together with the first part of Assumption 3.1. Note that the ratio $h_{\widehat{\omega}_K}/\min_{L\in\widehat{\mathcal{T}}_K} h_L$ for an extended patch $\widehat{\mathcal{T}}_K$ only depends on the shape-regularity parameter $\kappa_{\widehat{\mathcal{T}}_K}$. If $d = 3$ and if $\partial\widehat{\omega}_K$ contains at least one face from $\partial\Omega$, or if $d = 2$ with Γ_D non-empty and if $\partial\widehat{\omega}_K$ contains at least one face $\overline{\Gamma_\mathrm{D}}$, using the second part of Assumption 3.1, $\widehat{\mathcal{T}}_K$ can be mapped by d symmetries as in [14] for boundary patches into a patch that can be enumerated as per Definition C.1 or Definition B.1, where we can branch with Theorem D.1. □

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