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A POSTERIORI ERROR ESTIMATES FOR LOWEST-ORDER MIXED FINITE ELEMENT DISCRETIZATIONS OF CONVECTION-DIFFUSION-REACTION EQUATIONS*

MARTIN VOHRALÍK†

Abstract. We establish residual a posteriori error estimates for lowest-order Raviart–Thomas mixed finite element discretizations of convection-diffusion-reaction equations on simplicial meshes in two or three space dimensions. The upwind-mixed scheme is considered as well, and the emphasis is put on the presence of an inhomogeneous and anisotropic diffusion-dispersion tensor and on a possible convection dominance. Global upper bounds for the approximation error in the energy norm are derived, where in particular all constants are evaluated explicitly, so that the estimators are fully computable. Our estimators give local lower bounds for the error as well, and they hold from the cases where convection or reaction are not present to convection- or reaction-dominated problems; we prove that their local efficiency depends only on local variations in the coefficients and on the local Péclet number. Moreover, the developed general framework allows for asymptotic exactness and full robustness with respect to inhomogeneities and anisotropies. The main idea of the proof is a construction of a locally postprocessed approximate solution using the mean value and the flux in each element, known in the mixed finite element method, and a subsequent use of the abstract framework arising from the primal weak formulation of the continuous problem. Numerical experiments confirm the guaranteed upper bound and excellent efficiency and robustness of the derived estimators.

Key words. convection-diffusion-reaction equation, inhomogeneous and anisotropic diffusion, convection dominance, mixed finite element method, upwind weighting, a posteriori error estimates

AMS subject classifications. 65N15, 65N30, 76S05

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1. Introduction.

We consider the convection-diffusion-reaction problem

$$-\nabla \cdot (\mathbf{S}\nabla p) + \nabla \cdot (p\mathbf{w}) + rp = f \quad \text{in } \Omega, \tag{1.1a}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

where $\mathbf{S}$ is in general an inhomogeneous and anisotropic (nonconstant full-matrix) diffusion-dispersion tensor, $\mathbf{w}$ is a (dominating) velocity field, r a reaction function, f a source term, and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal (polyhedral) domain (open, bounded, and connected set). Our purpose is to derive a posteriori error estimates for the lowest-order Raviart–Thomas mixed finite element discretization of the problem (1.1a)–(1.1b) on simplicial meshes (consisting of triangles if $d = 2$ and of tetrahedra if $d = 3$), as well as for its upwind variant; cf. Douglas and Roberts [17] and Dawson [16].

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†Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie (Paris 6), B.C. 187, 4 place Jussieu, 75252 Paris, France (vohralik@ann.jussieu.fr).

A posteriori error estimates, pioneered by Babuška and Rheinboldt [7], are nowadays well established for primal discretizations of second-order elliptic problems involving only a diffusion term; cf., for example, the survey by Verfürth [32] for the conforming finite element method. An approach encompassing all conforming, nonconforming, and discontinuous finite element methods was recently proposed by Ainsworth [3], using a Helmholtz-like decomposition of the error in the numerical solution into its conforming and nonconforming parts in order to give a computable error bound. In most cases the analysis is given only for $\mathbf{S}$ being an identity matrix; an in-depth analysis for the general inhomogeneous and anisotropic diffusion tensor in the framework of the finite element method was presented by Bernardi and Verfürth [9]. Similar results have been obtained by Petzoldt [28], for nonconforming finite elements by Ainsworth [4], and some developments for the finite volume box scheme (in the given case actually equivalent to the lowest-order Raviart–Thomas mixed finite element method) are presented by El Alaoui and Ern [19]. In all these references, a hypothesis of the type "monotonicity around vertices" on the distribution of the inhomogeneities is necessary. In recent years a posteriori error estimates have been extended to convection-diffusion problems as well. We cite in particular Verfürth [33], who derived estimates in the energy norm for the conforming Galerkin method and its stabilized SUPG (streamline upwind Petrov–Galerkin) version. His estimates are both reliable (yielding a global upper bound on the error between the exact and approximate solutions) and locally efficient (giving a local lower bound). Moreover, they are semirobust in the sense that the lower and upper bounds differ by constants whose dependence on the local mesh discretization parameter vanishes as this approaches the ratio of the smallest eigenvalue of $\mathbf{S}$ to the local size of the velocity field (i.e., when the local Péclet number gets sufficiently small). Recently, Verfürth [34] improved his results while giving estimates which are fully robust with respect to convection dominance in a norm incorporating a dual norm of the convective derivative. The new norm is not, however, easily computable, there is no local lower bound, and the estimators do not change with respect to [33], and hence the adaptive strategies will remain the same. Finally, a different approach, yielding an estimate in the L^1-norm, independent of the size of the diffusion tensor, is given by Ohlberger [26] in the framework of the vertex-centered finite volume method.

In comparison with primal methods, the literature on a posteriori error estimates in the mixed finite element method is much less extensive. Most of the results have been obtained for the Poisson equation (i.e., $\mathbf{w} = r = 0$ in (1.1a)–(1.1b)) in two space dimensions: Alonso [5] derived estimates for the error in the flux $\mathbf{u} := -\mathbf{S}\nabla p$ of the scalar variable p and either Raviart–Thomas [29] or Brezzi–Douglas–Marini [11] mixed finite elements. Braess and Verfürth [10] proved estimates for both $\mathbf{u}$ and p for Raviart–Thomas elements, based on mesh-dependent norms and a saturation assumption. Carstensen [13] derived rigorous estimates for various mixed finite element schemes and for both $\mathbf{u}$ and p. Achchab et al. [1] can imbed Raviart–Thomas elements in their hierarchical a posteriori error estimates, whereas Carstensen and Bartels [14] give an upper bound using averaging techniques. Kirby [24] proposed simple residual-based estimates for Raviart–Thomas elements, where, however, the flux estimator is not proved to yield a lower bound and is, moreover, obtained under a saturation assumption. Wheeler and Yotov [39] were able to obtain a posteriori error estimates for the mortar version of all families of mixed finite elements, also including the three-dimensional case; a saturation assumption was, however, necessary for the velocity estimate. Recently, Lovadina and Stenberg [25] employed an idea of postprocessing similar to that used in this paper (with, however, the postprocessed scalar unknown of one degree lower than the one used here) in order to prove reliable and efficient a posteriori error estimates for both the scalar and flux variables in a mesh-dependent norm. Finally, Hoppe and Wohlmuth [22] treat a diffusion-reaction problem in two space dimensions and use the relation of lowest-order Raviart–Thomas mixed finite elements to nonconforming finite elements derived by Arnold and Brezzi in [6] in order to control, under a saturation assumption, the L^2-norm error in the primal variable p.

To the author's knowledge, no a posteriori estimates for mixed finite element discretizations of convection-diffusion(-reaction) problems have been presented in the literature so far. We do this in section 4 of this paper, after stating the assumptions on the data and formulating the continuous problem in section 2 and after defining the schemes in section 3. The estimates are derived in the energy norm for a new locally (on each element) postprocessed scalar variable $\tilde{p}_h$ such that its flux $-\mathbf{S}\nabla\tilde{p}_h$ is equal to $\mathbf{u}_h$ and such that its mean on each element is equal to p_h. By this construction, we actually have the $L^2(\Omega)$ control over both $\mathbf{u}_h-\mathbf{u}$ and $\tilde{p}_h-p$. Our estimates, in contrast to the usual practice, do not include any undetermined multiplicative constants, so that they are fully (and locally and easily) computable. They represent local lower bounds for the error as well, with efficiency constants of the form $c_1 + c_2 \min\{\mathrm{Pe}, \varrho\}$, where Pe (the local Péclet number) and ϱ are given below by (4.8) and where c_1, c_2 depend only on local variations in $\mathbf{S}$ (i.e., on local inhomogeneities and anisotropies), on local variations in $\mathbf{w}$ and r, on the space dimension, on the polynomial degree of f, and on the shape-regularity parameter of the mesh. They hold from the cases where convection or reaction are not present to convection- or reaction-dominated problems and are in particular semirobust as in [33] with respect to convection dominance. Next, in the pure diffusion case, we can write the general framework for our estimators in a form of an infimum over all $H_0^1(\Omega)$ functions plus a higher-order residual term, which yields asymptotic exactness and full robustness with respect to inhomogeneities and anisotropies, and this without any "monotonicity" hypothesis. Although in numerical experiments we use only local discrete evaluations of the estimators, they remain almost asymptotically exact (the ratio of the estimated and actual error is close to one, and this even in the convection-diffusion-reaction case) and quite robust. Finally, as an interesting consequence of our analysis, we find that in the pure diffusion case with piecewise constant coefficients, the lowest-order mixed finite elements represent an exact three-point scheme in one space dimension, and in two or three space dimensions, the postprocessed approximation is exact with respect to some generalized continuous solution. All these issues are discussed in detail in section 5.

Next, section 6 presents some discrete properties of the schemes and of the postprocessed scalar variable $\tilde{p}_h$. Namely, we show that $\tilde{p}_h$ is nonconforming in the sense that it is not included in $H_0^1(\Omega)$, but we prove that the means of its traces are continuous across interior sides (edges if $d = 2$, faces if $d = 3$) and equal to zero on exterior sides of the mesh; they are, in fact, shown to equal the Lagrange multipliers from the hybridized forms of the schemes. The actual proofs of our a posteriori error estimates and of their local efficiency are then given in section 7. The key element is Lemma 7.1 which states a primal weak formulation-based abstract framework allowing for the above-discussed asymptotic exactness and asymptotic robustness. The nonconformity of $\tilde{p}_h$ is then treated by the techniques developed in [2, 23, 19]. Neither any additional regularity of the weak solution nor any saturation assumption is needed. Finally, we illustrate the accuracy of the derived estimates in section 8 in several numerical experiments.

In this paper we focus only on lowest-order methods since in practice they are by far the most commonly used and hence we believe they deserve a special treatment; on the other hand, we do cover the three-dimensional case. Moreover, we have shown in [36] that there exists a local flux-expression formula in lowest-order mixed finite elements and that they can namely be implemented with only one unknown per element, which enables us to significantly decrease their traditional increased computational cost. The extension to higher-order schemes is an ongoing work. Finally, we have also generalized the presented type of a posteriori error estimates to the finite volume method in the forthcoming paper [38]. We treat there among other questions a larger variety of meshes and general inhomogeneous Dirichlet or Neumann boundary conditions. This paper is a detailed description of the results previously announced in [37].

2. Notation, assumptions, and the continuous problem.

We introduce here the notation, define admissible triangulations to which the space $W_0(\mathcal{T}_h)$ and the data will be related, and finally give details on the continuous problem (1.1a)–(1.1b).

2.1. Notation. For a domain $S \subset \mathbb{R}^d$, we denote by $L^2(S)$ and $\mathbf{L}^2(S) = [L^2(S)]^d$ the Lebesgue spaces, by $(\cdot,\cdot)_S$ the $L^2(S)$ or $\mathbf{L}^2(S)$ inner product, and by $\|\cdot\|_S$ the associated norm; $|S|$ stands for the Lebesgue measure of S. Next, $H^1(S)$ and $H^1_0(S)$ are the Sobolev spaces of functions with square-integrable weak derivatives, $\mathbf{H}(\mathrm{div}, S) = \{\mathbf{v} \in \mathbf{L}^2(S); \nabla\cdot\mathbf{v} \in L^2(S)\}$ is the space of functions with square-integrable weak divergences, and $\langle\cdot,\cdot\rangle_{\partial S}$ stands for $(d-1)$-dimensional inner product on ∂S or for the duality pairing between $H^{-\frac{1}{2}}(\partial S)$ and $H^{\frac{1}{2}}(\partial S)$. We will also use the "broken Sobolev space" $H^1(\mathcal{T}_h) := \{\varphi \in L^2(\Omega); \varphi|_K \in H^1(K)\ \forall K \in \mathcal{T}_h\}$. In what follows we conceptually denote by C_A, c_A constants dependent only on a quantity A.

2.2. Triangulation, Poincaré and Friedrichs inequalities, and the space $W_0(\mathcal{T}_h)$. We suppose that $\mathcal{T}_h$ for all $h > 0$ consists of closed simplices such that $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$ and such that if $K, L \in \mathcal{T}_h$, $K \neq L$, then $K \cap L$ is either an empty set or a common face, edge, or vertex of K and L. Let h_K denote the diameter of K and let $h := \max_{K\in\mathcal{T}_h} h_K$. We make the following shape-regularity assumption on the family of triangulations $\{\mathcal{T}_h\}_h$, denoting $\kappa_K := |K|/h_K^d$.

Assumption A (shape-regularity of the meshes). There exists a constant $\kappa_\mathcal{T} > 0$ such that $\min_{K\in\mathcal{T}_h} \kappa_K \geq \kappa_\mathcal{T}$ for all $h > 0$.

Let ρ_K denote the diameter of the largest ball inscribed in K. Then Assumption A is equivalent to the usual requirement of the existence of a constant $\theta_\mathcal{T} > 0$ such that $\max_{K\in\mathcal{T}_h} h_K/\rho_K \leq \theta_\mathcal{T}$ for all $h > 0$. We next denote by $\mathcal{E}_h$ the set of all sides of $\mathcal{T}_h$, by $\mathcal{E}_h^{\mathrm{int}}$ the set of interior, by $\mathcal{E}_h^{\mathrm{ext}}$ the set of exterior, and by $\mathcal{E}_K$ the set of all the sides of an element $K \in \mathcal{T}_h$. Finally, h_σ stands for the diameter of $\sigma \in \mathcal{E}_h$.

Let $K \in \mathcal{T}_h$ and $\varphi \in H^1(K)$. Two inequalities play an essential role in our analysis. First, the Poincaré inequality states that

$$\|\varphi - \varphi_K\|_K^2 \leq C_{\mathrm{P},d} h_K^2 \|\nabla\varphi\|_K^2\,, \tag{2.1}$$

where φ_K is the mean of φ over K, $\varphi_K := (\varphi, 1)_K/|K|$, and where the constant $C_{\mathrm{P},d}$ can for a simplex (using its convexity) be evaluated as d/π; cf. [27, 8]. Next, the following generalized Friedrichs inequalities have been proved in [35, Lemma 4.1]:

$$(\varphi_K - \varphi_\sigma)^2 \leq C_{\mathrm{F},d} \frac{h_K^2}{|K|} \|\nabla\varphi\|_K^2\,, \qquad \|\varphi - \varphi_\sigma\|_K^2 \leq C_{\mathrm{F},d} h_K^2 \|\nabla\varphi\|_K^2\,. \tag{2.2}$$

Here φ_σ is the mean of φ over $\sigma \in \mathcal{E}_K$, $\varphi_\sigma := \langle\varphi, 1\rangle_\sigma/|\sigma|$, and $C_{\mathrm{F},d} = 3d$.

We finally define the space $W_0(\mathcal{T}_h)$ of functions with mean values of the traces continuous across interior sides and zero on exterior sides,

$$W_0(\mathcal{T}_h) := \{\varphi \in L^2(\Omega)\,;\varphi|_K \in H^1(K)\ \forall K \in \mathcal{T}_h,\\ \langle \varphi|_K - \varphi|_L, 1\rangle_{\sigma_{K,L}} = 0\ \forall \sigma_{K,L} \in \mathcal{E}_h^{\rm int},\\ \langle \varphi, 1\rangle_\sigma = 0\ \forall \sigma \in \mathcal{E}_h^{\rm ext}\}, \tag{2.3}$$

and recall the discrete Friedrichs inequality

$$\|\varphi\|_\Omega^2 \le C_{\rm DF} \sum_{K\in\mathcal{T}_h} \|\nabla\varphi\|_K^2 \qquad \forall \varphi \in W_0(\mathcal{T}_h),\ \forall h > 0, \tag{2.4}$$

where $C_{\rm DF}$ depends only on $\kappa_{\mathcal{T}}$ and $\inf_{\mathbf{b}\in\mathbb{R}^d}\{\text{thick}_{\mathbf{b}}(\Omega)\}$; cf. [35, Theorem 5.4].

2.3. Data. We suppose that there exists a basic triangulation $\widetilde{\mathcal{T}}_h$ of Ω such that the data of the problem (1.1a)–(1.1b) are related to $\widetilde{\mathcal{T}}_h$ in the following way.

Assumption B (data).

(B1) $\mathbf{S}_K := \mathbf{S}|_K$ *is a constant, symmetric, bounded, and uniformly positive definite tensor such that* $c_{\mathbf{S},K}\,\mathbf{v}\cdot\mathbf{v} \le \mathbf{S}_K\mathbf{v}\cdot\mathbf{v} \le C_{\mathbf{S},K}\,\mathbf{v}\cdot\mathbf{v}$, $c_{\mathbf{S},K} > 0$, $C_{\mathbf{S},K} > 0$, *for all* $\mathbf{v} \in \mathbb{R}^d$ *and all* $K \in \widetilde{\mathcal{T}}_h$*;*

(B2) $\mathbf{w} \in \mathbf{RTN}^0(\widetilde{\mathcal{T}}_h)$ *satisfies* $|\mathbf{w}|_K| \le C_{\mathbf{w},K}$, $C_{\mathbf{w},K} \ge 0$, *for all* $K \in \widetilde{\mathcal{T}}_h$*;*

(B3) $r_K := r|_K$ *is a constant for all* $K \in \widetilde{\mathcal{T}}_h$*;*

(B4) $\frac{1}{2}\nabla\cdot\mathbf{w}|_K + r|_K = c_{\mathbf{w},r,K}$ *and* $|\nabla\cdot\mathbf{w}|_K + r_K| = C_{\mathbf{w},r,K}$, $c_{\mathbf{w},r,K} \ge 0$, $C_{\mathbf{w},r,K} \ge 0$, *for all* $K \in \widetilde{\mathcal{T}}_h$*;*

(B5) $f|_K$ *is a polynomial of degree at most* k *for each* $K \in \widetilde{\mathcal{T}}_h$*;*

(B6) *if* $c_{\mathbf{w},r,K} = 0$, *then* $C_{\mathbf{w},r,K} = 0$.

The assumptions that $\mathbf{S}$ and r are piecewise constant on $\widetilde{\mathcal{T}}_h$, that $\mathbf{w} \in \mathbf{RTN}^0(\widetilde{\mathcal{T}}_h)$ (cf. section 3.1 below for the definition of this space), and that f is a piecewise polynomial are made for the sake of simplicity and are usually satisfied in practice. If the functions at hand do not fulfill these requirements, interpolation can be used. Finally, note that Assumption (B6) allows $c_{\mathbf{w},r,K} = 0$ but $\mathbf{w}|_K \neq 0$.

2.4. Continuous problem. Let $\mathcal{T}_h$ be, as throughout the whole paper, a refinement of $\widetilde{\mathcal{T}}_h$. We define a bilinear form $\mathcal{B}$ by

$$\mathcal{B}(p,\varphi) := \sum_{K\in\mathcal{T}_h} \{(\mathbf{S}\nabla p, \nabla\varphi)_K + (\nabla\cdot(p\mathbf{w}), \varphi)_K + (rp, \varphi)_K\}, \qquad p, \varphi \in H^1(\mathcal{T}_h), \tag{2.5}$$

and the corresponding energy (semi)norm by

$$|||\varphi|||_\Omega^2 := \sum_{K\in\mathcal{T}_h} |||\varphi|||_K^2, \quad |||\varphi|||_K^2 := (\mathbf{S}\nabla\varphi, \nabla\varphi)_K + c_{\mathbf{w},r,K}\|\varphi\|_K^2, \qquad \varphi \in H^1(\mathcal{T}_h). \tag{2.6}$$

In this way $\mathcal{B}(\cdot,\cdot)$ and $|||\cdot|||_\Omega$ are well defined for $p, \varphi \in H^1(\Omega)$ as well as for p, φ that are only piecewise regular. Note also that $|||\cdot|||_\Omega$ is a norm on $W_0(\mathcal{T}_h)$ even if there exists $K \in \mathcal{T}_h$ such that $c_{\mathbf{w},r,K} = 0$ because of the discrete Friedrichs inequality (2.4) and Assumption (B1). The weak formulation of the problem (1.1a)–(1.1b) is then to find $p \in H_0^1(\Omega)$ such that

$$\mathcal{B}(p,\varphi) = (f,\varphi)_\Omega \qquad \forall \varphi \in H_0^1(\Omega). \tag{2.7}$$

Assumption B, the Green theorem, and the Cauchy–Schwarz inequality imply that

$$\mathcal{B}(\varphi,\varphi) = |||\varphi|||_\Omega^2 \qquad \forall \varphi \in H_0^1(\Omega), \tag{2.8}$$

$$\mathcal{B}(\varphi,\varphi) = |||\varphi|||_\Omega^2 + \frac{1}{2}\sum_{K\in\mathcal{T}_h} \langle \varphi^2, \mathbf{w}\cdot\mathbf{n}\rangle_{\partial K} \qquad \forall \varphi \in H^1(\mathcal{T}_h), \tag{2.9}$$

$$\begin{aligned}\mathcal{B}(p,\varphi) \le{}& \max\left\{1, \max_{K\in\mathcal{T}_h}\left\{\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right\}\right\} |||p|||_\Omega |||\varphi|||_\Omega \\ &+ \max_{K\in\mathcal{T}_h}\left\{\frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}}\right\} |||p|||_\Omega \|\varphi\|_\Omega \qquad \forall p,\varphi \in H^1(\mathcal{T}_h),\end{aligned} \tag{2.10}$$

and problem (2.7) under Assumption B, in particular, admits a unique solution.

Remark 2.1 (notation). In estimate (2.10), if $c_{\mathbf{w},r,K} = 0$, the term $C_{\mathbf{w},r,K}/c_{\mathbf{w},r,K}$ should be evaluated as zero, since Assumption (B6) in this case gives $C_{\mathbf{w},r,K} = 0$. To simplify notation, we systematically use the convention $0/0 = 0$ throughout the text.

3. Mixed finite element schemes.

We define in this section the centered and upwind-weighted mixed finite element schemes.

3.1. Function spaces.

Let $\mathbf{RTN}_{-1}^0(\mathcal{T}_h)$ be the space of elementwise linear vector functions $\mathbf{u}_h$ such that, on each $K \in \mathcal{T}_h$, $\mathbf{u}_h|_K = (a_K + d_K x, b_K + d_K y)$ if $d = 2$ and $\mathbf{u}_h|_K = (a_K + d_K x, b_K + d_K y, c_K + d_K z)$ if $d = 3$. The Raviart–Thomas–Nédélec space $\mathbf{RTN}^0(\mathcal{T}_h)$ imposes the continuity of the normal trace across all $\sigma \in \mathcal{E}_h^{\mathrm{int}}$ and is given by $\mathbf{RTN}^0(\mathcal{T}_h) := \mathbf{RTN}_{-1}^0(\mathcal{T}_h) \cap \mathbf{H}(\mathrm{div},\Omega)$. There is one basis function $\mathbf{v}_\sigma$ associated with each $\sigma \in \mathcal{E}_h$. For $\sigma_{K,L} \in \mathcal{E}_h^{\mathrm{int}}$, $\mathbf{v}_{\sigma_{K,L}}(\mathbf{x}) = \frac{1}{d|K|}(\mathbf{x} - V_K)$, $\mathbf{x} \in K$; $\mathbf{v}_{\sigma_{K,L}}(\mathbf{x}) = \frac{1}{d|L|}(V_L - \mathbf{x})$, $\mathbf{x} \in L$; $\mathbf{v}_{\sigma_{K,L}}(\mathbf{x}) = 0$ otherwise, where V_K is the vertex of K opposite to σ and V_L the vertex of L opposite to σ. We suppose that the orientation of $\mathbf{v}_{\sigma_{K,L}}$, i.e., the order of K and L, is fixed. For a boundary side σ, the support of $\mathbf{v}_\sigma$ consists only of $K \in \mathcal{T}_h$ such that $\sigma \in \mathcal{E}_K$. Next, the space $\Phi(\mathcal{T}_h)$ consists of elementwise constant scalar functions; we denote $p_h|_K = p_K$ for $p_h \in \Phi(\mathcal{T}_h)$. Recall also that $\nabla\cdot\mathbf{u}_h \in \Phi(\mathcal{T}_h)$ for each $\mathbf{u}_h \in \mathbf{RTN}_{-1}^0(\mathcal{T}_h)$.

3.2. Centered scheme.

The centered mixed finite element scheme (cf. [17]) reads: find $\mathbf{u}_h \in \mathbf{RTN}^0(\mathcal{T}_h)$ and $p_h \in \Phi(\mathcal{T}_h)$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla\cdot\mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RTN}^0(\mathcal{T}_h), \tag{3.1a}$$

$$(\nabla\cdot\mathbf{u}_h, \phi_h)_\Omega - (\mathbf{S}^{-1}\mathbf{u}_h\cdot\mathbf{w}, \phi_h)_\Omega + ((r + \nabla\cdot\mathbf{w})p_h, \phi_h)_\Omega = (f, \phi_h)_\Omega \qquad \forall \phi_h \in \Phi(\mathcal{T}_h). \tag{3.1b}$$

3.3. Upwind-weighted scheme.

The upwind-weighted mixed finite element scheme reads: find $\mathbf{u}_h \in \mathbf{RTN}^0(\mathcal{T}_h)$ and $p_h \in \Phi(\mathcal{T}_h)$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla\cdot\mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RTN}^0(\mathcal{T}_h), \tag{3.2a}$$

$$(\nabla\cdot\mathbf{u}_h, \phi_h)_\Omega + \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_K} \hat{p}_\sigma w_{K,\sigma}\phi_K + (rp_h, \phi_h)_\Omega = (f, \phi_h)_\Omega \qquad \forall \phi_h \in \Phi_h(\mathcal{T}_h), \tag{3.2b}$$

where $w_{K,\sigma} := \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma$, $\sigma \in \mathcal{E}_K$, with $\mathbf{n}$ being the unit normal vector of the side σ, outward to K, and where $\hat{p}_\sigma$ is the weighted upwind value defined by

$$\hat{p}_\sigma := \begin{cases} (1-\nu_\sigma)p_K + \nu_\sigma p_L & \text{if } w_{K,\sigma} \geq 0, \\ (1-\nu_\sigma)p_L + \nu_\sigma p_K & \text{if } w_{K,\sigma} < 0, \end{cases} \tag{3.3}$$

if σ is an interior side between elements K and L, and

$$\hat{p}_\sigma := \begin{cases} (1-\nu_\sigma)p_K & \text{if } w_{K,\sigma} \geq 0, \\ \nu_\sigma p_K & \text{if } w_{K,\sigma} < 0, \end{cases} \tag{3.4}$$

if σ is a boundary side. Here, $\nu_\sigma \in [0, 1/2]$ is the coefficient of the amount of upstream weighting which may be, in order to reduce the excessive numerical diffusion added by the full upstream weighting used in [16], chosen as

$$\nu_\sigma := \begin{cases} \min\left\{c_{\mathbf{S},\sigma}\dfrac{|\sigma|}{h_\sigma|w_{K,\sigma}|}, \dfrac{1}{2}\right\} & \text{if } w_{K,\sigma} \neq 0 \text{ and } \sigma \in \mathcal{E}_h^{\text{int}}, \\ & \text{or if } \sigma \in \mathcal{E}_h^{\text{ext}} \text{ and } w_{K,\sigma} > 0, \\ 0 & \text{if } w_{K,\sigma} = 0 \text{ or if } \sigma \in \mathcal{E}_h^{\text{ext}} \text{ and } w_{K,\sigma} < 0, \end{cases} \tag{3.5}$$

where $c_{\mathbf{S},\sigma}$ is the harmonic average of $c_{\mathbf{S},K}$ and $c_{\mathbf{S},L}$ if $\sigma = \partial K \cap \partial L$ and $c_{\mathbf{S},K}$ otherwise.

4. A posteriori error estimates.

We summarize in this section our a posteriori estimates on the error between the weak solution p and a postprocessed variable $\tilde{p}_h$, which we shall define first, along with its modified Oswald interpolate.

4.1. A postprocessed scalar variable $\tilde{p}_h$.

In standard mixed finite element theory (see, e.g., Brezzi and Fortin [12] or Roberts and Thomas [31]) the two variables p_h and $\mathbf{u}_h$ are considered as independent. In contrast, the basis for our a posteriori error estimates is a construction of a postprocessed scalar variable $\tilde{p}_h$ which links p_h and $\mathbf{u}_h$ on each simplex in the following way:

$$-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K \quad \forall K \in \mathcal{T}_h, \tag{4.1a}$$

$$\frac{(\tilde{p}_h, 1)_K}{|K|} = p_K \quad \forall K \in \mathcal{T}_h. \tag{4.1b}$$

Note that, in particular, if $\mathbf{S} = Id$, $\tilde{p}_h|_K = -d_K/2\,(x^2+y^2) - a_K x - b_K y - e_K$ if $d = 2$ and $\tilde{p}_h|_K = -d_K/2\,(x^2+y^2+z^2) - a_K x - b_K y - c_K z - e_K$ if $d = 3$. Here a_K–d_K are the coefficients from section 3.1, and e_K is given so that (4.1b) was satisfied. If $\mathbf{S} \neq Id$, then $\tilde{p}_h$ verifying (4.1a)–(4.1b) still exists due to the symmetry of $\mathbf{S}$ and is this time a full second-order polynomial on each $K \in \mathcal{T}_h$. The new variable $\tilde{p}_h$ is nonconforming, $\tilde{p}_h \notin H_0^1(\Omega)$, but, by Lemma 6.1 below, $\tilde{p}_h \in W_0(\mathcal{T}_h)$; i.e., its means on interior sides are continuous and its means on exterior sides are equal to zero. In fact, by Lemma 6.4 below, these means coincide with the Lagrange multipliers of hybridized schemes. Moreover, the centered scheme can equivalently be rewritten with the help of $\tilde{p}_h$ (see Lemma 6.2 below), which corresponds to the employment of the Lagrange multipliers in the convection term. Note that the proposed postprocessing is local on each element and its cost is negligible.

4.2. A modified Oswald interpolation operator. Let $\mathbb{P}_l(\mathcal{T}_h)$ denote the space of polynomials of degree at most l on each simplex, not necessary continuous. The Oswald interpolation operator $\mathcal{I}_{\rm Os} : \mathbb{P}_l(\mathcal{T}_h) \to \mathbb{P}_l(\mathcal{T}_h) \cap H_0^1(\Omega)$ has been considered, e.g., in [2, 23, 19]. Given a function $\varphi_h \in \mathbb{P}_l(\mathcal{T}_h)$, $\mathcal{I}_{\rm Os}(\varphi_h)$ is prescribed at the Lagrangian nodes (degrees of freedom; cf. [15, section 2.2]) of $\mathbb{P}_l(\mathcal{T}_h) \cap H_0^1(\Omega)$ by the average of the values of φ_h at this node. We will now construct its modification which preserves the means of $\tilde{p}_h$ over the sides, since this will appear crucial when convection is present.

The modified Oswald interpolation operator $\mathcal{I}_{\rm MO} : \mathbb{P}_2(\mathcal{T}_h) \cap W_0(\mathcal{T}_h) \to \mathbb{P}_d(\mathcal{T}_h) \cap H_0^1(\Omega)$ is defined as follows: at all Lagrangian nodes of $\mathbb{P}_d(\mathcal{T}_h) \cap H_0^1(\Omega)$, except for those lying at the barycenters of the sides, the value of $\mathcal{I}_{\rm MO}(\varphi_h)$ is given by the average of the values of φ_h at this node (as in the standard Oswald interpolation operator). The values at the barycenters of the sides are then established so that the means of $\mathcal{I}_{\rm MO}(\varphi_h)$ over the sides were given by the means of φ_h. (The space $\mathbb{P}_2(\mathcal{T}_h) \cap H_0^1(\Omega)$ in three space dimensions does not have Lagrangian nodes at side barycenters; this is the reason to use $\mathbb{P}_3(\mathcal{T}_h) \cap H_0^1(\Omega)$ in this case.) It is easily verified that, as in the case of the Oswald interpolation operator, $\mathcal{I}_{\rm MO}(\varphi_h)$ is a uniquely defined piecewise polynomial continuous function. Let $[\varphi_h]$ be the jump of a function φ_h across a side σ: if $\sigma = \partial K \cap \partial L$, then $[\varphi_h]$ is the difference of the value of φ_h in K and L (the order of K and L has no influence on what follows), and if $\sigma \in \mathcal{E}_h^{\rm ext}$, then $[\varphi_h] = \varphi_h$. Then the following lemma is an easy modification of [23, Theorem 2.2] ($\sigma \cap K \neq \emptyset$ when σ contains a vertex of K).

LEMMA 4.1 (modified Oswald interpolation operator). *Let $\varphi_h \in \mathbb{P}_2(\mathcal{T}_h) \cap W_0(\mathcal{T}_h)$, and let $\mathcal{I}_{\rm MO}(\varphi_h) \in \mathbb{P}_d(\mathcal{T}_h) \cap H_0^1(\Omega)$ be constructed as described above. Then*

$$\|\nabla(\varphi_h - \mathcal{I}_{\rm MO}(\varphi_h))\|_K^2 \le C_1 \sum_{\sigma;\sigma\cap K\neq\emptyset} h_\sigma^{-1} \|[\varphi_h]\|_\sigma^2 ,$$

where the constant C_1 depends only on d and $\kappa_{\mathcal{T}}$.

4.3. A posteriori error estimates. We now finally state the a posteriori error estimates. Let $K \in \mathcal{T}_h$. Let us first set

$$m_K^2 := \min\left\{ C_{{\rm P},d} \frac{h_K^2}{c_{\mathbf{S},K}}, \frac{1}{c_{\mathbf{w},r,K}} \right\}.$$

We define the *residual estimator* $\eta_{{\rm R},K}$ by

$$\eta_{{\rm R},K} := m_K \| f + \nabla\cdot(\mathbf{S}\nabla\tilde{p}_h) - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h \|_K . \tag{4.2}$$

Next, denote $v := \tilde{p}_h - \mathcal{I}_{\rm MO}(\tilde{p}_h)$. The *nonconformity estimator* $\eta_{{\rm NC},K}$ is given by

$$\eta_{{\rm NC},K} := |||v|||_K \tag{4.3}$$

and the *convection estimator* $\eta_{{\rm C},K}$ by

(4.4)

$$\eta_{{\rm C},K} := \min\left\{ \frac{\|\nabla\cdot(v\mathbf{w}) - \frac{1}{2}v\nabla\cdot\mathbf{w}\|_K}{\sqrt{c_{\mathbf{w},r,K}}}, \left(\frac{C_{{\rm P},d} h_K^2 \|\nabla v\cdot\mathbf{w}\|_K^2}{c_{\mathbf{S},K}} + \frac{9\|v\nabla\cdot\mathbf{w}\|_K^2}{4c_{\mathbf{w},r,K}} \right)^{\frac{1}{2}} \right\} .$$

Finally, let

$$m_\sigma^2 := \min\left\{ \max_{K;\,\sigma\in\mathcal{E}_K} \left\{ C_{{\rm F},d} \frac{|\sigma| h_K^2}{|K| c_{\mathbf{S},K}} \right\}, \max_{K;\,\sigma\in\mathcal{E}_K} \left\{ \frac{|\sigma|}{|K| c_{\mathbf{w},r,K}} \right\} \right\} \tag{4.5}$$

for all $\sigma \in \mathcal{E}_h$. We set $\tilde{p}_\sigma := \langle \tilde{p}_h, 1\rangle_\sigma / |\sigma|$, the mean of the postprocessed scalar variable $\tilde{p}_h$ over a side $\sigma \in \mathcal{E}_h$; recall that $\hat{p}_\sigma$ is the upwind value given by (3.3) or (3.4); and define the *upwinding estimator* $\eta_{\mathrm{U},K}$ by

$$\eta_{\mathrm{U},K} := \sum_{\sigma\in\mathcal{E}_K} m_\sigma \|(\hat{p}_\sigma - \tilde{p}_\sigma)\mathbf{w}\cdot\mathbf{n}\|_\sigma \,. \tag{4.6}$$

We have the following a posteriori error estimates.

THEOREM 4.2 (a posteriori error estimate for the centered mixed finite element scheme). *Let p be the weak solution of the problem* (1.1a)–(1.1b) *given by* (2.7)*, and let $\tilde{p}_h$ be the postprocessed solution of the centered mixed finite element scheme* (3.1a)–(3.1b) *given by* (4.1a)–(4.1b). *Then*

$$|||p - \tilde{p}_h|||_\Omega \le \left\{ \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac12} + \left\{ \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{C},K})^2 \right\}^{\frac12} . \tag{4.7}$$

THEOREM 4.3 (a posteriori error estimate for the upwind-weighted mixed finite element scheme). *Let p be the weak solution of the problem* (1.1a)–(1.1b) *given by* (2.7)*, and let $\tilde{p}_h$ be the postprocessed solution of the upwind-weighted mixed finite element scheme* (3.2a)–(3.2b) *given by* (4.1a)–(4.1b). *Then*

$$|||p - \tilde{p}_h|||_\Omega \le \left\{ \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac12} + \left\{ \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{C},K} + \eta_{\mathrm{U},K})^2 \right\}^{\frac12} .$$

4.4. Local efficiency of the estimates. Let the local Péclet number Pe_K and ϱ_K be given by

$$\mathrm{Pe}_K := h_K \frac{C_{\mathbf{w},K}}{c_{\mathbf{S},K}}\,, \quad \varrho_K := \frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{w},r,K}}\sqrt{c_{\mathbf{S},K}}}\,. \tag{4.8}$$

Next, let, for $\varphi \in H^1(K)$,

$$\alpha_{*,K} := c_{\mathbf{S},K}\left(\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + 2\varrho_K^2\right), \quad \beta_{*,K} := c_{\mathbf{w},r,K} + \frac{\left|\nabla\cdot\mathbf{w}|_K\right|^2}{2c_{\mathbf{w},r,K}}\,,$$

$$\alpha_{\#,K} := c_{\mathbf{S},K}\left(\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}} + C_{\mathrm{P},d}\mathrm{Pe}_K^2\right), \quad \beta_{\#,K} := c_{\mathbf{w},r,K} + \frac{9\left|\nabla\cdot\mathbf{w}|_K\right|^2}{4c_{\mathbf{w},r,K}}\,,$$

$$|||\varphi|||_{*,K}^2 := \alpha_{*,K}\|\nabla\varphi\|_K^2 + \beta_{*,K}\|\varphi\|_K^2\,, \quad |||\varphi|||_{\#,K}^2 := \alpha_{\#,K}\|\nabla\varphi\|_K^2 + \beta_{\#,K}\|\varphi\|_K^2\,.$$

Finally, let

$$c_{\mathbf{S},\omega_K} := \min_{L;L\cap K\ne\emptyset} c_{\mathbf{S},L}\,, \quad c_{\mathbf{w},r,\omega_K} := \min_{L;L\cap K\ne\emptyset} c_{\mathbf{w},r,L}\,, \quad c_{\mathbf{S},\Omega} := \min_{K\in\mathcal{T}_h} c_{\mathbf{S},K}\,. \tag{4.9}$$

The theorem below discusses the local efficiency of our a posteriori error estimators.

THEOREM 4.4 (local efficiency of the a posteriori error estimators). *Let p be the weak solution of the problem* (1.1a)–(1.1b) *given by* (2.7)*, and let $\tilde{p}_h$ be the postprocessed solution of the centered mixed finite element scheme* (3.1a)–(3.1b) *or of the* *upwind-weighted mixed finite element scheme* (3.2a)–(3.2b) *given by* (4.1a)–(4.1b). *Then, for the residual estimator* $\eta_{\mathrm{R},K}$ *on each* $K \in \mathcal{T}_h$*, there holds*

(4.10)
$$\eta_{\mathrm{R},K} \le C_2 |||p - \tilde{p}_h|||_K \left\{ \sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}} \max\left\{1, \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right\} + \min\left\{\mathrm{Pe}_K, \sqrt{\frac{C_{\mathbf{S},K}}{c_{\mathbf{S},K}}}\varrho_K\right\}\right\},$$

where the constant C_2 *depends only on the space dimension* d*, on the shape-regularity parameter* $\kappa_{\mathcal{T}}$*, and on the polynomial degree* k *of* f *(see Lemma* 7.6 *below). Next, for the nonconformity and velocity estimators* $\eta_{\mathrm{NC},K}$ *and* $\eta_{\mathrm{C},K}$ *on each* $K \in \mathcal{T}_h$*, we have*

$$\begin{aligned}\eta_{\mathrm{NC},K}^2 + \eta_{\mathrm{C},K}^2 \le C_3 \min\Bigg\{ &\frac{\alpha_{*,K}}{c_{\mathbf{S},\omega_K}} + \min\left\{\frac{\beta_{*,K}}{c_{\mathbf{w},r,\omega_K}}, \frac{\beta_{*,K}h_K^2}{c_{\mathbf{S},\omega_K}}\right\}, \\ &\frac{\alpha_{\#,K}}{c_{\mathbf{S},\omega_K}} + \min\left\{\frac{\beta_{\#,K}}{c_{\mathbf{w},r,\omega_K}}, \frac{\beta_{\#,K}h_K^2}{c_{\mathbf{S},\omega_K}}\right\}\Bigg\} \sum_{L;L\cap K\neq\emptyset} |||p - \tilde{p}_h|||_L^2 \\ &+ C_3\beta_{\#,K} \inf_{s_h \in \mathbb{P}_2(\mathcal{T}_h)\cap H_0^1(\Omega)} \sum_{L;L\cap K\neq\emptyset} \|p - s_h\|_L^2,\end{aligned} \tag{4.11}$$

where the constant C_3 *depends only on* d *and* $\kappa_{\mathcal{T}}$ *(see Lemma* 7.7 *below). Finally, the upwinding estimator* $\eta_{\mathrm{U},K}$ *is not efficient and we have only*

$$\sum_{K\in\mathcal{T}_h} \eta_{\mathrm{U},K}^2 \le C_4 \max_{\sigma\in\mathcal{E}_h} \varrho_\sigma \max_{K\in\mathcal{T}_h} \tilde{\varrho}_K \min\left\{\frac{1}{2}\sum_{K\in\mathcal{T}_h} \frac{\|f\|_K^2}{c_{\mathbf{w},r,K}}, \|f\|_\Omega^2 \frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}}\right\}, \tag{4.12}$$

where C_{DF} *is the constant from the discrete Friedrichs inequality* (2.4)*, the constant* C_4 *depends only on* d *and* $\kappa_{\mathcal{T}}$ *(see Lemma* 7.8 *below), and*

$$\varrho_\sigma := \left(\frac{\max_{K;\sigma\in\mathcal{E}_K} c_{\mathbf{S},K}}{\min_{K;\sigma\in\mathcal{E}_K} c_{\mathbf{S},K}}\right)^2, \quad \tilde{\varrho}_K := \min\left\{(\mathrm{Pe}_K)^2, (\varrho_K)^2 \frac{\max_{L;L\cap K\in\mathcal{E}_h} c_{\mathbf{w},r,L}}{\min_{L;L\cap K\in\mathcal{E}_h} c_{\mathbf{w},r,L}}\right\}.$$

5. Various remarks.

We give several remarks in this section.

5.1. Nature of the estimates.

The basis of the a posteriori error estimates derived in this paper is the construction of the postprocessed scalar variable $\tilde{p}_h$ and the consequent application of the abstract framework arising from the primal weak formulation (2.7) of the continuous problem; cf. Lemmas 7.1 and 7.2 below. Compared to Galerkin finite element approximations, the crucial advantage is that $\tilde{p}_h$, an elementwise quadratic polynomial, has the normal traces of its flux $-\mathbf{S}\nabla\tilde{p}_h$ (which is, by (4.1a), nothing else than the mixed finite element vector variable $\mathbf{u}_h$) continuous across interior sides. Hence the side error estimators penalizing the mass balance common in Galerkin finite element methods (cf. [33]) do not appear here at all. This advantage is, however, compensated by the fact that $\tilde{p}_h \notin H_0^1(\Omega)$, so that the estimators known from nonconforming and discontinuous Galerkin finite elements (cf. [19, 23]) appear. Next, whereas in the lowest-order Galerkin finite element method $\nabla\cdot(\mathbf{S}_K\nabla p_h)|_K$ is always equal to zero on all $K \in \mathcal{T}_h$, the element residuals (4.2) give a very good sense. We also notice that using (2.6), (4.1a), and (2.4),

(5.1)
$$
\begin{aligned}
|||p-\tilde{p}_h|||_\Omega^2 &= \sum_{K\in\mathcal{T}_h}\left\{\|\mathbf{S}^{-\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)\|_K^2 + c_{\mathbf{w},r,K}\|p-\tilde{p}_h\|_K^2\right\}\\
&\geq \sum_{K\in\mathcal{T}_h}\left\{\frac{1}{2}\|\mathbf{S}^{-\frac{1}{2}}(\mathbf{u}-\mathbf{u}_h)\|_K^2 + c_{\mathbf{w},r,K}\|p-\tilde{p}_h\|_K^2\right\} + \frac{c_{\mathbf{S},\Omega}}{2C_{\mathrm{DF}}}\|p-\tilde{p}_h\|_\Omega^2,
\end{aligned}
$$

so that we have the usual mixed finite element $L^2(\Omega)$ control over the error in both the scalar and vector unknowns even if $c_{\mathbf{w},r,K}=0$ for some $K\in\mathcal{T}_h$.

5.2. The estimates and their local efficiency with respect to S and w. We discuss here our a posteriori error estimates and their local efficiency that we have been able to prove in Theorem 4.4. For further remarks, see the next section.

The minimum in the definition of the residual estimator $\eta_{\mathrm{R},K}$ (4.2) prevents it from growing to extreme values on coarse elements with a small value $c_{\mathbf{S},K}$ when $c_{\mathbf{w},r,K}>0$. Its local efficiency depends only on anisotropy in its element expressed by the ratio $\sqrt{C_{\mathbf{S},K}/c_{\mathbf{S},K}}$ and there is no dependency on inhomogeneities. Next, under the given assumptions, $C_{\mathbf{w},r,K}/c_{\mathbf{w},r,K}\leq 2$ whenever r_K is nonnegative. Finally, the minimum of the local Péclet number Pe_K and ϱ_K ensures boundedness if $c_{\mathbf{w},r,K}\neq 0$ and if h_K is large and optimal efficiency as Pe_K becomes small.

The minimum in the definition of the convection estimator $\eta_{\mathrm{C},K}$ (4.4) prevents it from exploding when $c_{\mathbf{w},r,K}=0$ but $C_{\mathbf{w},K}\neq 0$. Together with the nonconformity estimator $\eta_{\mathrm{NC},K}$ (4.3), they give local efficiency, up to higher-order terms if $c_{\mathbf{w},r,K}\neq 0$ (the part $\inf_{s_h\in\mathbb{P}_2(\mathcal{T}_h)\cap H_0^1(\Omega)}$), which is shown to be a function of a local (meaning all elements sharing a vertex with the given one) maximal ratio of inhomogeneities (the term $\sqrt{\alpha_{*,K}/c_{\mathbf{S},\omega_K}}$) and of $\sqrt{C_{\mathbf{S},K}/c_{\mathbf{S},K}}$ in each element concerning anisotropy. For further remarks, see the next section. Finally, the efficiency gets into optimal values with respect to convection dominance as Pe_K gets sufficiently small. We note also that the estimate is robust (up to the higher-order term) in the reaction-dominated case as well, since the quantities $C_{\mathbf{w},r,K}/c_{\mathbf{w},r,K}$ and $\sqrt{\beta_{*,K}/c_{\mathbf{w},r,\omega_K}}$ remain well bounded in the limit.

The fact that the upwinding estimator $\eta_{\mathrm{U},K}$ (4.6) cannot in general give a lower bound for the error is quite obvious: it is not difficult to imagine a situation where $p=\tilde{p}_h$, whereas $(\hat{p}_\sigma-\tilde{p}_\sigma)$, the difference of the mean value of $\tilde{p}_h$ on a side σ and of the combination of the mean values of $\tilde{p}_h$ on the elements sharing σ, is generally nonzero. However, we at least show that there is an upper bound for the contributions of this estimator, which moreover decreases with the local Péclet numbers as $O(h)$. It should be noted that this estimator does not change the limit optimality of the schemes and estimates—see section 5.5 below for a remark on this point.

5.3. Asymptotic exactness and asymptotic robustness with respect to inhomogeneities and anisotropies. We show in this remark that the (global asymptotic) efficiency of our estimates is indeed even better than that proved in Theorem 4.4 and discussed in the previous section.

5.3.1. Pure diffusion problems. Let us first consider a pure diffusion problem, i.e., $r=\mathbf{w}=0$ in (1.1a)–(1.1b). Using that in this case $-\nabla\cdot(\mathbf{S}_K\nabla\tilde{p}_h|_K)=\nabla\cdot\mathbf{u}_h|_K=f_K$ for all $K\in\mathcal{T}_h$, where f_K is the mean value of f over K, the analysis for the general case simplifies to the a posteriori error estimate (4.7) with $\eta_{\mathrm{C},K} = 0$ and

$$\eta^2_{\mathrm{R},K} := C_{\mathrm{P},d} \frac{h_K^2}{c_{\mathbf{S},K}} \|f - f_K\|_K^2 \,, \tag{5.2}$$

$$\eta^2_{\mathrm{NC},K} := \|\mathbf{S}^{\frac{1}{2}} \nabla(\tilde{p}_h - s)\|_K^2 \,, \tag{5.3}$$

where in particular $s \in H_0^1(\Omega)$ can be chosen arbitrarily (cf. Lemma 7.2 below). Examples are the Oswald or the modified Oswald interpolates of $\tilde{p}_h$—in the pure diffusion case, all the presented results hold similarly for these two operators. Also note that since $\nabla \cdot (\mathbf{u} - \mathbf{u}_h)|_K = f - f_K$ is fully computable for all $K \in \mathcal{T}_h$, the control over $\|\mathbf{u} - \mathbf{u}_h\|_\Omega + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_\Omega$ immediately follows using (5.1).

Our main point is, however, that the above developments in fact imply

$$|||p - \tilde{p}_h|||_\Omega \leq \inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega + \left\{ \sum_{K \in \mathcal{T}_h} C_{\mathrm{P},d} \frac{h_K^2}{c_{\mathbf{S},K}} \|f - f_K\|_K^2 \right\}^{\frac{1}{2}} , \tag{5.4}$$

which, in the case where f is piecewise constant, by virtue of

$$\inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega \leq |||\tilde{p}_h - p|||_\Omega,$$

gives asymptotic global efficiency of such an estimator with a constant 1, i.e., asymptotic exactness and asymptotic full robustness with respect to inhomogeneities and anisotropies (asymptotic with respect to the approximation of $\tilde{p}_h$ by some, e.g., polynomial, $s \in H_0^1(\Omega)$ on a fixed grid $\mathcal{T}_h$). In the general case, if, e.g., $f \in H^1(\mathcal{T}_h)$, then $\|f - f_K\|_K^2 \leq C_{\mathrm{P},d} h_K^2 \|\nabla f\|_K^2$, and asymptotic exactness and asymptotic robustness still hold true (this time asymptotic also with respect to $h \to 0$). Although we use only the Oswald or the modified Oswald interpolates of $\tilde{p}_h$ instead of evaluating or approximating the infimum in (5.4), the numerical experiments of section 8.1 below show that estimators of section 4.3 remain almost asymptotically exact and robust with respect to inhomogeneities and anisotropies.

5.3.2. Convection-diffusion-reaction problems. The above considerations roughly extend to the convection-diffusion-reaction case in the following sense: for the centered mixed finite element scheme (3.1a)–(3.1b), one has (7.4) and consequently a superconvergence of the residual estimators $\eta_{\mathrm{R},K}$ (4.2) to zero. Next, for divergence-free velocity fields $\mathbf{w}$, the second arguments of the convection estimators $\eta_{\mathrm{C},K}$ in (4.4) again superconverge to zero since $\tilde{p}_h \in W_0(\mathcal{T}_h)$ (both as $h \to 0$). Hence the estimate will be asymptotically given only by the nonconformity estimators $\eta_{\mathrm{NC},K}$ of (4.3) and thus by the best approximation of $\tilde{p}_h$ by $s \in H_0^1(\Omega)$ such that its means are given by the means of $\tilde{p}_h$. (This property is needed when convection is present; see Lemma 7.4 below.) This asymptotic almost optimal efficiency is again observed below in numerical experiments in section 8.2.

5.4. Pure diffusion problems: Mixed finite elements and a generalized weak solution. Let us in this remark consider $r = \mathbf{w} = 0$ in (1.1a)–(1.1b) and generalize the classical weak solution to a function $\tilde{p} \in W_0(\mathcal{T}_h)$ such that

$$\mathcal{B}(\tilde{p}, \varphi) = (f, \varphi)_\Omega \qquad \forall \varphi \in W_0(\mathcal{T}_h) \,. \tag{5.5}$$

(In)equalities (2.9) and (2.10) together with the discrete Friedrichs inequality (2.4) ensure the existence of a unique solution of (5.5).

We thus have

$$|||\tilde{p} - \tilde{p}_h|||_\Omega = \frac{\mathcal{B}(\tilde{p} - \tilde{p}_h, \tilde{p} - \tilde{p}_h)}{|||\tilde{p} - \tilde{p}_h|||_\Omega} \leq \sup_{\varphi \in W_0(\mathcal{T}_h),\, |||\varphi|||_\Omega = 1} \mathcal{B}(\tilde{p} - \tilde{p}_h, \varphi)$$

and develop, similarly as in the proof of Lemma 7.2 below,

$$\begin{aligned}
\mathcal{B}(\tilde{p} - \tilde{p}_h, \varphi) &= (f, \varphi)_\Omega + \sum_{K \in \mathcal{T}_h} \left\{ (\nabla \cdot (\mathbf{S} \nabla \tilde{p}_h), \varphi)_K - \langle \mathbf{S} \nabla \tilde{p}_h \cdot \mathbf{n}, \varphi \rangle_{\partial K} \right\} \\
&= \sum_{K \in \mathcal{T}_h} (f - \nabla \cdot \mathbf{u}_h, \varphi)_K + \sum_{\sigma \in \mathcal{E}_h} \langle \mathbf{u}_h \cdot \mathbf{n}, [\varphi] \rangle_\sigma \\
&= \sum_{K \in \mathcal{T}_h} (f - \nabla \cdot \mathbf{u}_h, \varphi)_K = \sum_{K \in \mathcal{T}_h} (f - f_K, \varphi - \varphi_K)_K,
\end{aligned}$$

using the bilinearity of $\mathcal{B}(\cdot, \cdot)$, the definition (5.5) of the generalized weak solution $\tilde{p}$, the Green theorem in each $K \in \mathcal{T}_h$, the relation (4.1a) between $\tilde{p}_h$ and $\mathbf{u}_h$, reordering the summation over the boundaries of elements to the summation over the sides, using the continuity of the normal trace of $\mathbf{u}_h$ expressed by $\mathbf{u}_h|_K \cdot \mathbf{n}_K = -\mathbf{u}_h|_L \cdot \mathbf{n}_L$ on $\sigma_{K,L} \in \mathcal{E}_h^{\mathrm{int}}$, the fact that $\mathbf{u}_h \cdot \mathbf{n}$ is constant on all sides $\sigma \in \mathcal{E}_h$ and the definition (2.3) of the space $W_0(\mathcal{T}_h)$, and finally the equation (3.1b) of the definition of the mixed finite element scheme (φ_K is the mean of φ over K). Next, estimate (7.5) given below holds true also in this case, so that finally the Cauchy–Schwarz inequality leads to

$$|||\tilde{p} - \tilde{p}_h|||_\Omega \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}}$$

with $\eta_{\mathrm{R},K}$ given by (5.2).

First, this is a completely data-dependent a posteriori error estimate, and second, this is in fact an a priori error estimate as well: it shows that the mixed finite element solutions $\tilde{p}_h$ and $\mathbf{u}_h$ (cf. (5.1), which still holds true) converge both as $O(h^2)$ in the $L^2(\Omega)$, $\mathbf{L}^2(\Omega)$, respectively, norms to the generalized weak solution $\tilde{p}$ given by (5.5) and its flux $\tilde{\mathbf{u}}$, $\tilde{\mathbf{u}}|_K := -\mathbf{S}\nabla \tilde{p}|_K$ (for $f \in H^1(\mathcal{T}_h)$). Moreover, as soon as f is piecewise constant, $\tilde{p}_h$ is directly equal to the generalized solution! We emphasize that these results hold true for $\mathbf{S}$ piecewise constant but arbitrarily inhomogeneous and anisotropic; they apparently confirm the observations of a very good behavior of mixed methods in these circumstances. There are also very interesting consequences in one space dimension; cf. section 5.6 below.

5.5. A combination of the centered and upwind-weighted schemes. The scheme (3.2a)–(3.2b) guarantees stability in the convection-dominated case, but the additional upwinding estimator $\eta_{\mathrm{U},K}$ given by (4.6) is unfortunately not efficient. On the other hand, the scheme (3.1a)–(3.1b), however precise if h is sufficiently small, may give completely wrong results on coarse meshes. Hence a good idea may be a smooth transition from the upwind-weighted to the centered scheme under the form

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla \cdot \mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RTN}^0(\mathcal{T}_h),$$

$$(\nabla \cdot \mathbf{u}_h, \phi_K)_K + \sum_{\sigma \in \mathcal{E}_K} \left\{ (\mu_\sigma \hat{p}_\sigma + (1 - \mu_\sigma) \tilde{p}_\sigma) w_{K,\sigma} \phi_K \right\} + (r p_h, \phi_K)_K = (f, \phi_K)_K \qquad \forall K \in \mathcal{T}_h,$$

where $\hat{p}_\sigma$ is the upstream value and μ_σ is set to $1-2\nu_\sigma$ with ν_σ given by (3.5). Notice that such a scheme is fully rewritable in terms of the original unknowns p_h, $\mathbf{u}_h$, using that $\sum_{\sigma\in\mathcal{E}_K} \tilde{p}_\sigma w_{K,\sigma}\phi_K = \langle \tilde{p}_h \mathbf{w}\cdot\mathbf{n}, \phi_K\rangle_{\partial K}$ and Lemma 6.2 below.

5.6. The estimates in one space dimension. As the last remark, it appears that the above results have interesting particular consequences in one space dimension, where the two schemes (3.1a)–(3.1b) and (3.2a)–(3.2b) can likewise be defined.

5.6.1. One dimension: No nonconformity. First of all, Lemma 6.1 below reduces in one space dimension to the assertion that the postprocessed variable $\tilde{p}_h$ given by (4.1a)–(4.1b) is continuous, i.e., that in this case $\tilde{p}_h \in H_0^1(\Omega)$. An immediate consequence is that the parts of the a posteriori error estimates of Theorems 4.2–4.3 related to nonconformity disappear.

5.6.2. Lowest-order mixed finite elements: An exact three-point scheme for one-dimensional diffusion problems with piecewise constant coefficients. Another quite interesting consequence is related to the remark of section 5.4 and results of [36]. As there is no nonconformity, the superconvergence $O(h^2)$ of both $\tilde{p}_h$ and $\mathbf{u}_h$ (this time towards the weak solution and its flux, coinciding with the generalized one) always holds true, and, moreover, it appears that in one space dimension, one can always rewrite the schemes with only p_K, $K\in\mathcal{T}_h$, as unknowns. Hence the lowest-order mixed finite elements represent a scheme with a three-point stencil which is exact for one-dimensional pure diffusion problems, where the diffusion tensor $\mathbf{S}$ (this time a scalar function) and the right-hand side f are piecewise constant (and hence possibly arbitrarily discontinuous). This should be compared to the known results for the finite volume/finite difference method. In particular, the (best known?) scheme proposed by Ewing, Iliev, and Lazarov in [21] is exact only when the right-hand side is constant (the diffusion tensor may be piecewise constant); cf. Remark 2.4 in [21].

6. Discrete properties of the schemes. In this section we prove different properties of the schemes (3.1a)–(3.1b) and (3.2a)–(3.2b) and of the postprocessed scalar variable $\tilde{p}_h$ needed in the paper.

LEMMA 6.1 (continuity of the means of traces of $\tilde{p}_h$). *It holds that $\tilde{p}_h \in W_0(\mathcal{T}_h)$; i.e.,*

$$\langle \tilde{p}_h|_K - \tilde{p}_h|_L, 1\rangle_{\sigma_{K,L}} = 0 \qquad \forall \sigma_{K,L}\in\mathcal{E}_h^{\mathrm{int}},$$

$$\langle \tilde{p}_h, 1\rangle_\sigma = 0 \qquad \forall \sigma\in\mathcal{E}_h^{\mathrm{ext}}.$$

Proof. Let us consider a side $\sigma_{K,L}\in\mathcal{E}_h^{\mathrm{int}}$. Then taking $\mathbf{v}_h$ equal to the basis function $\mathbf{v}_{\sigma_{K,L}}$ (cf. section 3.1) in (3.1a) or (3.2a) yields

$$\begin{aligned}
0 &= -(\nabla\tilde{p}_h, \mathbf{v}_{\sigma_{K,L}})_{K\cup L} - (\tilde{p}_h, \nabla\cdot\mathbf{v}_{\sigma_{K,L}})_{K\cup L}\\
&= -\langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}, \tilde{p}_h\rangle_{\partial K} - \langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}, \tilde{p}_h\rangle_{\partial L} = \langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}_K, \tilde{p}_h|_L - \tilde{p}_h|_K\rangle_{\sigma_{K,L}},
\end{aligned}$$

using the definition (4.1a)–(4.1b) of $\tilde{p}_h$, the fact that $\nabla\cdot\mathbf{v}_h$ for $\mathbf{v}_h\in\mathbf{RTN}^0(\mathcal{T}_h)$ is constant in each simplex (which allows us to replace p_h by $\tilde{p}_h$), the Green theorem, and the fact that $\mathbf{v}_{\sigma_{K,L}}$ has a nonzero normal flux only through $\sigma_{K,L}$. The first assertion of the lemma follows by the fact that $\mathbf{v}_h\cdot\mathbf{n}$ for $\mathbf{v}_h\in\mathbf{RTN}^0(\mathcal{T}_h)$ is constant on each side $\sigma\in\mathcal{E}_h$. The proof for boundary sides is completely similar. ∎

LEMMA 6.2 (equivalent form of the centered scheme). *The scheme* (3.1a)–(3.1b) *can be equivalently written: find* $\mathbf{u}_h \in \mathbf{RTN}^0(\mathcal{T}_h)$ *and* $p_h \in \Phi(\mathcal{T}_h)$ *such that*

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (\tilde{p}_h, \nabla\cdot\mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RTN}^0(\mathcal{T}_h), \tag{6.1a}$$

$$(\nabla\cdot\mathbf{u}_h, \phi_K)_K + \langle \tilde{p}_h \mathbf{w}\cdot\mathbf{n}, \phi_K\rangle_{\partial K} + (r\tilde{p}_h, \phi_K)_K = (f, \phi_K)_K \qquad \forall K \in \mathcal{T}_h, \tag{6.1b}$$

where $\tilde{p}_h$ *is defined by* (4.1a)–(4.1b).

Proof. Since $\nabla\cdot\mathbf{v}_h$ for $\mathbf{v}_h \in \mathbf{RTN}^0(\mathcal{T}_h)$ is constant in each simplex and since r was in Assumption (B3) supposed piecewise constant as well, one can replace p_h by $\tilde{p}_h$ in the terms $(p_h, \nabla\cdot\mathbf{v}_h)_\Omega$ and $(rp_h, \phi_K)_K$ using (4.1b). Similarly, using in addition the Green theorem,

$$\begin{aligned} -(\mathbf{S}_K^{-1}\mathbf{u}_h\cdot\mathbf{w}, \phi_K)_K + (p_K\nabla\cdot\mathbf{w}, \phi_K)_K &= (\nabla\tilde{p}_h\cdot\mathbf{w}, \phi_K)_K + (\tilde{p}_h\nabla\cdot\mathbf{w}, \phi_K)_K \\ &= (\nabla\cdot(\tilde{p}_h\mathbf{w}), \phi_K)_K = \langle \tilde{p}_h\mathbf{w}\cdot\mathbf{n}, \phi_K\rangle_{\partial K}. \qquad \square \end{aligned}$$

Remark 6.3 (hybridization of the schemes). Mixed finite element schemes can equivalently be reformulated while relaxing the continuity of the normal trace of $\mathbf{u}_h$ required in the definition of the space $\mathbf{RTN}^0(\mathcal{T}_h)$ and imposing it instead with the help of Lagrange multipliers λ_σ, $\sigma \in \mathcal{E}_h^{\text{int}}$; cf. [12, section V.1.2]. The centered scheme (3.1a)–(3.1b), taking into account its equivalent form given by Lemma 6.2, then changes to: find $\mathbf{u}_h \in \mathbf{RTN}_{-1}^0(\mathcal{T}_h)$, $p_h \in \Phi(\mathcal{T}_h)$, and λ_σ, $\sigma \in \mathcal{E}_h^{\text{int}}$, with $\tilde{p}_h$ defined by (4.1a)–(4.1b), such that

$$\sum_{K\in\mathcal{T}_h} \left\{ (\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K - (\tilde{p}_h, \nabla\cdot\mathbf{v}_h)_K + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\text{int}}} \langle \mathbf{v}_h\cdot\mathbf{n}, \lambda_\sigma\rangle_\sigma \right\} = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RTN}_{-1}^0(\mathcal{T}_h), \tag{6.2a}$$

$$(\nabla\cdot\mathbf{u}_h, \phi_K)_K + \langle \tilde{p}_h\mathbf{w}\cdot\mathbf{n}, \phi_K\rangle_{\partial K} + (r\tilde{p}_h, \phi_K)_K = (f, \phi_K)_K \qquad \forall K \in \mathcal{T}_h, \tag{6.2b}$$

$$\langle (\mathbf{u}_h\cdot\mathbf{n})|_K + (\mathbf{u}_h\cdot\mathbf{n})|_L, 1\rangle_{\sigma_{K,L}} = 0 \qquad \forall \sigma_{K,L} \in \mathcal{E}_h^{\text{int}}, \tag{6.2c}$$

whereas the upwind-weighted scheme (3.2a)–(3.2b) becomes: find $\mathbf{u}_h \in \mathbf{RTN}_{-1}^0(\mathcal{T}_h)$, $p_h \in \Phi(\mathcal{T}_h)$, and λ_σ, $\sigma \in \mathcal{E}_h^{\text{int}}$ such that

$$\sum_{K\in\mathcal{T}_h} \left\{ (\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_K - (p_h, \nabla\cdot\mathbf{v}_h)_K + \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\text{int}}} \langle \mathbf{v}_h\cdot\mathbf{n}, \lambda_\sigma\rangle_\sigma \right\} = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RTN}_{-1}^0(\mathcal{T}_h), \tag{6.3a}$$

$$(\nabla\cdot\mathbf{u}_h, \phi_K)_K + \sum_{\sigma\in\mathcal{E}_K} \hat{p}_\sigma w_{K,\sigma}\phi_K + (rp_h, \phi_K)_K = (f, \phi_K)_K \qquad \forall K \in \mathcal{T}_h, \tag{6.3b}$$

$$\langle (\mathbf{u}_h\cdot\mathbf{n})|_K + (\mathbf{u}_h\cdot\mathbf{n})|_L, 1\rangle_{\sigma_{K,L}} = 0 \qquad \forall \sigma_{K,L} \in \mathcal{E}_h^{\text{int}}. \tag{6.3c}$$

LEMMA 6.4 (relation of $\tilde{p}_h$ to the Lagrange multipliers λ_σ). *It holds that*

$$\lambda_\sigma = \tilde{p}_\sigma = \frac{\langle \tilde{p}_h, 1\rangle_\sigma}{|\sigma|} \qquad \forall \sigma \in \mathcal{E}_h^{\text{int}}.$$

Proof. The proof is similar to that of Lemma 6.1. Let $K \in \mathcal{T}_h$ and $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\rm int}$. Then taking $\mathbf{v}_h = \mathbf{v}_\sigma$ in (6.2a) or (6.3a), we have

$$0 = -(\nabla \tilde{p}_h, \mathbf{v}_\sigma)_K - (\tilde{p}_h, \nabla \cdot \mathbf{v}_\sigma)_K + \langle \mathbf{v}_\sigma \cdot \mathbf{n}, \lambda_\sigma \rangle_\sigma = \langle \mathbf{v}_\sigma \cdot \mathbf{n}, \lambda_\sigma - \tilde{p}_h \rangle_\sigma \,,$$

using the definition (4.1a)–(4.1b) of $\tilde{p}_h$, the fact that $\nabla \cdot \mathbf{v}_\sigma$ is constant in each simplex, the fact that $\mathbf{v}_\sigma$ has a nonzero normal flux only through σ, and the Green theorem. The assertion of the lemma follows by the fact that $\mathbf{v}_\sigma \cdot \mathbf{n}$ is constant on σ. ∎

LEMMA 6.5 (a priori estimate for the upwind-weighted scheme). *Let $\mathbf{u}_h$, p_h be the solutions of the upwind-weighted scheme* (3.2a)–(3.2b)*, and let $\tilde{p}_h$ be the postprocessed scalar variable given by* (4.1a)–(4.1b)*. Then*

$$\sum_{K \in \mathcal{T}_h} \left\{ c_{\mathbf{S},K} \|\nabla \tilde{p}_h\|_K^2 + \frac{1}{2} c_{\mathbf{w},r,K} \, \|p_h\|_K^2 \right\} \le \frac{1}{2} \sum_{K \in \mathcal{T}_h} \frac{\|f\|_K^2}{c_{\mathbf{w},r,K}}$$

if $c_{\mathbf{w},r,K} > 0$ for all $K \in \mathcal{T}_h$ and

$$\sum_{K \in \mathcal{T}_h} \left\{ \frac{1}{2} c_{\mathbf{S},K} \|\nabla \tilde{p}_h\|_K^2 + c_{\mathbf{w},r,K} \, \|p_h\|_K^2 \right\} \le \frac{\|f\|_\Omega^2}{2} \frac{C_{\rm DF}}{c_{\mathbf{S},\Omega}},$$

where $c_{\mathbf{S},\Omega}$ is given by (4.9) *and $C_{\rm DF}$ is the constant from the discrete Friedrichs inequality* (2.4).

Proof. Let us set $\phi_h = p_h$ in (3.2b). We then can rewrite the first term of the left-hand side of (3.2b) as

$$\sum_{K \in \mathcal{T}_h} (\nabla \cdot \mathbf{u}_h, p_K)_K = \sum_{K \in \mathcal{T}_h} \left\{ -(\mathbf{u}_h, \nabla \tilde{p}_h)_K + \langle \mathbf{u}_h \cdot \mathbf{n}, \tilde{p}_h \rangle_{\partial K} \right\} = \sum_{K \in \mathcal{T}_h} (\mathbf{S}_K \nabla \tilde{p}_h, \nabla \tilde{p}_h)_K$$
$$+ \sum_{\sigma_{K,L} \in \mathcal{E}_h^{\rm int}} \langle \mathbf{u}_h \cdot \mathbf{n}_K, \tilde{p}_h|_K - \tilde{p}_h|_L \rangle_{\sigma_{K,L}} + \sum_{\sigma \in \mathcal{E}_h^{\rm ext}} \langle \mathbf{u}_h \cdot \mathbf{n}, \tilde{p}_h \rangle_\sigma \ge \sum_{K \in \mathcal{T}_h} c_{\mathbf{S},K} \|\nabla \tilde{p}_h\|_K^2 \,,$$

using the fact that $\nabla \cdot \mathbf{u}_h$ is constant on each $K \in \mathcal{T}_h$ and we thus can replace p_h by $\tilde{p}_h$ employing (4.1b), the Green theorem, (4.1a), the fact that $\mathbf{u}_h \cdot \mathbf{n}$ is constant on each $\sigma \in \mathcal{E}_h$, the continuity of the means of the traces of $\tilde{p}_h$ given by Lemma 6.1, and finally Assumption (B1). Next,

$$\sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \hat{p}_\sigma w_{K,\sigma} p_K = \sum_{\sigma_{K,L} \in \mathcal{E}_h^{\rm int}} \{ \hat{p}_\sigma w_{K,\sigma} p_K + \hat{p}_\sigma w_{L,\sigma} p_L \} + \sum_{\sigma_K \in \mathcal{E}_h^{\rm ext}} \hat{p}_\sigma w_{K,\sigma} p_K$$
$$= \sum_{\sigma_{K,L} \in \mathcal{E}_h^{\rm int},\, w_{K,\sigma} \ge 0} w_{K,\sigma} \left(p_K (p_K - p_L) - \nu_\sigma (p_L - p_K)^2 \right) + \sum_{\sigma_K \in \mathcal{E}_h^{\rm ext}} \hat{p}_\sigma w_{K,\sigma} p_K$$
$$= \frac{1}{2} \sum_{\sigma_{K,L} \in \mathcal{E}_h^{\rm int},\, w_{K,\sigma} \ge 0} w_{K,\sigma} (p_K^2 - p_L^2) + \sum_{\sigma_{K,L} \in \mathcal{E}_h^{\rm int}} |w_{K,\sigma}| (p_L - p_K)^2 \left(\frac{1}{2} - \nu_\sigma \right)$$
$$+ \sum_{\sigma_K \in \mathcal{E}_h^{\rm ext}} \left\{ \frac{1}{2} p_K^2 w_{K,\sigma} + |w_{K,\sigma}| p_K^2 \left(\frac{1}{2} - \nu_\sigma \right) \right\} \ge \frac{1}{2} \sum_{K \in \mathcal{T}_h} p_K^2 (\nabla \cdot \mathbf{w}, 1)_K,$$

where we have rewritten the summation over the sides and fixed denotation of $K, L \in \mathcal{T}_h$ sharing a side $\sigma_{K,L} \in \mathcal{E}_h^{\rm int}$ such that $w_{K,\sigma} \ge 0$; used that $w_{K,\sigma} = -w_{L,\sigma}$, the definition (3.3)–(3.4) of $\hat{p}_\sigma$, and the relation $2a(a-b) = (a-b)^2 + a^2 - b^2$; estimated using $0 \le \nu_\sigma \le 1/2$, which follows from (3.5); rewritten the summation back over the elements and their sides; and finally employed the Green theorem, giving $\sum_{\sigma\in\mathcal{E}_K} w_{K,\sigma} = (\nabla\cdot\mathbf{w}, 1)_K$. Finally, $(rp_h, p_h)_\Omega = \sum_{K\in\mathcal{T}_h} p_K^2 (r,1)_K$.

The right-hand side of (3.2b) with $\phi_h = p_h$ can be estimated either by

$$(f, p_h)_\Omega \le \sum_{K\in\mathcal{T}_h} \|f\|_K \frac{\sqrt{c_{\mathbf{w},r,K}}}{\sqrt{c_{\mathbf{w},r,K}}} \|p_h\|_K \le \frac{1}{2}\sum_{K\in\mathcal{T}_h} \frac{\|f\|_K^2}{c_{\mathbf{w},r,K}} + \frac{1}{2}\sum_{K\in\mathcal{T}_h} c_{\mathbf{w},r,K}\|p_h\|_K^2$$

or by

$$(f,p_h)_\Omega \le \|f\|_\Omega \|p_h\|_\Omega \le \frac{\|f\|_\Omega^2}{2}\frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}} + \frac{c_{\mathbf{S},\Omega}}{C_{\mathrm{DF}}}\frac{\|\tilde{p}_h\|_\Omega^2}{2} \le \frac{\|f\|_\Omega^2}{2}\frac{C_{\mathrm{DF}}}{c_{\mathbf{S},\Omega}} + \frac{c_{\mathbf{S},\Omega}}{2}\sum_{K\in\mathcal{T}_h} \|\nabla\tilde{p}_h\|_K^2,$$

using the Cauchy–Schwarz, $ab \le \varepsilon a^2/2 + b^2/(2\varepsilon)$, $\varepsilon > 0$, $\|p_h\|_K \le \|\tilde{p}_h\|_K$, and the discrete Friedrichs (2.4) inequalities. The assertion follows by combining the above estimates. □

Remark 6.6 (existence and uniqueness for the upwind-weighted scheme). From Lemma 6.5, existence and uniqueness for the upwind-weighted scheme (3.2a)–(3.2b) easily follows. Indeed, let $f = 0$. Then $p_h = 0$ and $\mathbf{u}_h = -\mathbf{S}\nabla\tilde{p}_h = 0$ for all $K \in \mathcal{T}_h$.

Remark 6.7 (existence and uniqueness for the centered scheme). In contrast with the upwind-weighted scheme, existence and uniqueness for the centered scheme (3.1a)–(3.1b) is in [17] guaranteed only for "h sufficiently small." Alternatively, there exists a unique solution if $C_{\mathbf{w},K} \le 2(1-\mu)\sqrt{c_{\mathbf{S},K}}\sqrt{\tilde{c}_{\mathbf{w},r,K}}$ for some $\mu \in (0,1)$ and all $K \in \mathcal{T}_h$, where $(\nabla\cdot\mathbf{w} + r)|_K = \tilde{c}_{\mathbf{w},r,K} > 0$, which corresponds to the case that is not convection-dominated.

7. Proofs of the a posteriori error estimates and of their local efficiency. We shall prove in this section the a posteriori error estimates stated by Theorems 4.2–4.3, as well as their local efficiency discussed in Theorem 4.4.

7.1. Proofs of the a posteriori error estimates. To begin with, we state the following result, the purpose of which is to give an optimal abstract bound on the error between $p \in H^1(\Omega)$ and $\tilde{p} \in H^1(\mathcal{T}_h)$ in the energy (semi)norm $|||\cdot|||_\Omega$. ($H^1_{\mathrm{D}}(\Omega)$ is the subspace of $H^1(\Omega)$ of functions with traces vanishing on $\Gamma_{\mathrm{D}} \subset \partial\Omega$.)

LEMMA 7.1 (abstract framework). *Let $\Gamma_{\mathrm{D}} \subset \partial\Omega$, $|\Gamma_{\mathrm{D}}| \ne 0$, let $\Gamma_{\mathrm{in}} := \{\mathbf{x} \in \partial\Omega; \mathbf{w}\cdot\mathbf{n} < 0\} \subset \Gamma_{\mathrm{D}}$, let $p, s \in H^1(\Omega)$ be such that $p - s \in H^1_{\mathrm{D}}(\Omega)$, and let $\tilde{p} \in H^1(\mathcal{T}_h)$ be arbitrary. Then*

$$|||p-\tilde{p}|||_\Omega \le |||\tilde{p}-s|||_\Omega + \Bigg|\mathcal{B}\left(p-\tilde{p}, \frac{p-s}{|||p-s|||_\Omega}\right) + \sum_{K\in\mathcal{T}_h}\left(\nabla\cdot((\tilde{p}-s)\mathbf{w}) - \frac{1}{2}(\tilde{p}-s)\nabla\cdot\mathbf{w}, \frac{p-s}{|||p-s|||_\Omega}\right)_K\Bigg|.$$

Proof. Let us set, for $p, \varphi \in H^1(\mathcal{T}_h)$,

$$\mathcal{B}_{\mathrm{S}}(p,\varphi) := \sum_{K\in\mathcal{T}_h}\left\{(\mathbf{S}\nabla p, \nabla\varphi)_K + \left(\left(\frac{1}{2}\nabla\cdot\mathbf{w} + r\right)p, \varphi\right)_K\right\},$$

$$\mathcal{B}_{\mathrm{A}}(p,\varphi) := \sum_{K\in\mathcal{T}_h}\left(\nabla\cdot(p\mathbf{w}) - \frac{1}{2}p\nabla\cdot\mathbf{w}, \varphi\right)_K,$$

so that

$$\mathcal{B}(p,\varphi) = \mathcal{B}_{\mathrm{S}}(p,\varphi) + \mathcal{B}_{\mathrm{A}}(p,\varphi) \qquad \forall p,\varphi \in H^1(\mathcal{T}_h), \tag{7.1}$$

$$\mathcal{B}_{\mathrm{S}}(\varphi,\varphi) = |||\varphi|||_\Omega^2 \qquad \forall \varphi \in H^1(\mathcal{T}_h), \tag{7.2}$$

$$\mathcal{B}_{\mathrm{A}}(\varphi,\varphi) \geq 0 \qquad \forall \varphi \in H^1_{\mathrm{D}}(\Omega), \tag{7.3}$$

using (2.9) and $\sum_{K\in\mathcal{T}_h} \langle \varphi^2, \mathbf{w}\cdot\mathbf{n}\rangle_{\partial K} \geq 0$ for $\varphi \in H^1_{\mathrm{D}}(\Omega)$ in the estimate.

We then have, using that $p - s \in H^1_{\mathrm{D}}(\Omega)$,

$$\begin{aligned} |||p-s|||_\Omega^2 &\leq \mathcal{B}(p-s,p-s) = \mathcal{B}(p-\tilde{p},p-s) + \mathcal{B}(\tilde{p}-s,p-s) \\ &= \mathcal{B}_{\mathrm{S}}(\tilde{p}-s,p-s) + \mathcal{B}(p-\tilde{p},p-s) + \mathcal{B}_{\mathrm{A}}(\tilde{p}-s,p-s) \\ &\leq |||\tilde{p}-s|||_\Omega |||p-s|||_\Omega + |||p-s|||_\Omega \mathcal{B}\left(p-\tilde{p}, \frac{p-s}{|||p-s|||_\Omega}\right) \\ &\quad + |||p-s|||_\Omega \mathcal{B}_{\mathrm{A}}\left(\tilde{p}-s, \frac{p-s}{|||p-s|||_\Omega}\right), \end{aligned}$$

employing the Cauchy–Schwarz inequality in the first term. If $|||p-\tilde{p}|||_\Omega \leq |||p-s|||_\Omega$, this concludes the proof. In general, we could use the triangle inequality $|||p-\tilde{p}|||_\Omega \leq |||p-s|||_\Omega + |||s-\tilde{p}|||_\Omega$ and the above bound for $|||p-s|||_\Omega$, but this would lead to an estimate which is not optimal (the term $|||\tilde{p}-s|||_\Omega$ would be replaced by $2|||\tilde{p}-s|||_\Omega$). We thus show below that the same bound holds true also when $|||p-s|||_\Omega \leq |||p-\tilde{p}|||_\Omega$.

We have, using (7.3) and the Cauchy–Schwarz inequality,

$$\begin{aligned} |||p-\tilde{p}|||_\Omega^2 &= \mathcal{B}_{\mathrm{S}}(p-\tilde{p},p-\tilde{p}) = \mathcal{B}_{\mathrm{S}}(p-\tilde{p},p-s) + \mathcal{B}_{\mathrm{S}}(p-\tilde{p},s-\tilde{p}) \\ &= \mathcal{B}_{\mathrm{S}}(p-\tilde{p},s-\tilde{p}) + \mathcal{B}(p-\tilde{p},p-s) - \mathcal{B}_{\mathrm{A}}(p-\tilde{p},p-s) \\ &= \mathcal{B}_{\mathrm{S}}(p-\tilde{p},s-\tilde{p}) + \mathcal{B}(p-\tilde{p},p-s) - \mathcal{B}_{\mathrm{A}}(p-s,p-s) + \mathcal{B}_{\mathrm{A}}(\tilde{p}-s,p-s) \\ &\leq \mathcal{B}_{\mathrm{S}}(p-\tilde{p},s-\tilde{p}) + \mathcal{B}(p-\tilde{p},p-s) + \mathcal{B}_{\mathrm{A}}(\tilde{p}-s,p-s) \\ &\leq |||p-\tilde{p}|||_\Omega |||s-\tilde{p}|||_\Omega + |||p-s|||_\Omega \mathcal{B}\left(p-\tilde{p}, \frac{p-s}{|||p-s|||_\Omega}\right) \\ &\quad + |||p-s|||_\Omega \mathcal{B}_{\mathrm{A}}\left(\tilde{p}-s, \frac{p-s}{|||p-s|||_\Omega}\right), \end{aligned}$$

which, by virtue of $|||p-s|||_\Omega \leq |||p-\tilde{p}|||_\Omega$ supposed in this second case, concludes the proof. ∎

Consequently, the following bound for the error $|||p-\tilde{p}_h|||_\Omega$ holds.

LEMMA 7.2 (abstract error estimate). *Let p be the weak solution of the problem* (1.1a)–(1.1b) *given by* (2.7), *and let $s \in H^1_0(\Omega)$ be arbitrary. If $\tilde{p}_h$ is the postprocessed solution of the centered mixed finite element scheme* (3.1a)–(3.1b) *given by* (4.1a)–(4.1b), *then*

$$|||p-\tilde{p}_h|||_\Omega \leq |||\tilde{p}_h - s|||_\Omega + \sup_{\varphi\in H^1_0(\Omega),\, |||\varphi|||_\Omega = 1} \left\{T_{\mathrm{R}}(\varphi) + T_{\mathrm{C}}(\varphi)\right\},$$

and if $\tilde{p}_h$ *is the postprocessed solution of the upwind-weighted mixed finite element scheme* (3.2a)–(3.2b)*, given by* (4.1a)–(4.1b)*, then*

$$|||p - \tilde{p}_h|||_\Omega \le |||\tilde{p}_h - s|||_\Omega + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||_\Omega = 1} \big\{T_{\mathrm{R}}(\varphi) + T_{\mathrm{C}}(\varphi) + T_{\mathrm{U}}(\varphi)\big\},$$

where

$$T_{\mathrm{R}}(\varphi) := \sum_{K \in \mathcal{T}_h} \big(f + \nabla\cdot \mathbf{S}\nabla \tilde{p}_h - \nabla\cdot(\tilde{p}_h \mathbf{w}) - r\tilde{p}_h, \varphi - \varphi_K\big)_K,$$

$$T_{\mathrm{C}}(\varphi) := \sum_{K \in \mathcal{T}_h} \left(\nabla\cdot((\tilde{p}_h - s)\mathbf{w}) - \frac{1}{2}(\tilde{p}_h - s)\nabla\cdot\mathbf{w}, \varphi\right)_K,$$

$$T_{\mathrm{U}}(\varphi) := \sum_{K \in \mathcal{T}_h} \sum_{\sigma \in \mathcal{E}_K} \langle (\hat{p}_\sigma - \tilde{p}_h)\mathbf{w}\cdot\mathbf{n}, \varphi_K\rangle_\sigma,$$

and where φ_K *is the mean of* φ *over* $K \in \mathcal{T}_h$*,* $\varphi_K := (\varphi, 1)_K / |K|$.

Proof. Let us consider an arbitrary $\varphi \in H_0^1(\Omega)$. We have, using the bilinearity of $\mathcal{B}(\cdot,\cdot)$, the definition (2.7) of the weak solution p, and the Green theorem in each $K \in \mathcal{T}_h$,

$$\begin{aligned}\mathcal{B}(p - \tilde{p}_h, \varphi) &= (f, \varphi)_\Omega - \sum_{K \in \mathcal{T}_h} \big\{(\mathbf{S}\nabla\tilde{p}_h, \nabla\varphi)_K + \big(\nabla\cdot(\tilde{p}_h\mathbf{w}), \varphi\big)_K + (r\tilde{p}_h, \varphi)_K\big\} \\ &= \sum_{K \in \mathcal{T}_h} \big\{\big(f + \nabla\cdot(\mathbf{S}\nabla\tilde{p}_h) - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h, \varphi\big)_K - \langle \mathbf{S}\nabla\tilde{p}_h\cdot\mathbf{n}, \varphi\rangle_{\partial K}\big\} \\ &= \sum_{K \in \mathcal{T}_h} \big(f + \nabla\cdot(\mathbf{S}\nabla\tilde{p}_h) - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h, \varphi\big)_K.\end{aligned}$$

Note that we have, in particular, used the continuity of the normal trace of $\mathbf{S}\nabla\tilde{p}_h$ (i.e., by (4.1a), the mixed finite element continuity of the normal trace of $\mathbf{u}_h$), yielding

$$\langle (\mathbf{S}\nabla\tilde{p}_h\cdot\mathbf{n})|_K + (\mathbf{S}\nabla\tilde{p}_h\cdot\mathbf{n})|_L, \varphi\rangle_{\sigma_{K,L}} = \langle 0, \varphi\rangle_{\sigma_{K,L}} = 0 \qquad \forall \sigma_{K,L} \in \mathcal{E}_h^{\mathrm{int}}$$

(the fact that $\langle \mathbf{S}\nabla\tilde{p}_h\cdot\mathbf{n}, \varphi\rangle_\sigma = 0$ for $\sigma \in \mathcal{E}_h^{\mathrm{ext}}$ follows by $\varphi \in H_0^1(\Omega)$).

Now the equation (6.1b) of the equivalent form of the centered scheme by the definition of $\tilde{p}_h$ (4.1a)–(4.1b) and by the Green theorem implies that (recall that φ_K is the constant mean of φ over K)

$$\big(f + \nabla\cdot(\mathbf{S}\nabla\tilde{p}_h) - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h, \varphi_K\big)_K = 0 \qquad \forall K \in \mathcal{T}_h. \tag{7.4}$$

Hence in the case of the centered scheme,

$$\mathcal{B}(p - \tilde{p}_h, \varphi) = \sum_{K \in \mathcal{T}_h} \big(f + \nabla\cdot(\mathbf{S}\nabla\tilde{p}_h) - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h, \varphi - \varphi_K\big)_K = T_{\mathrm{R}}(\varphi).$$

For the upwind-weighted scheme, we have

$$\mathcal{B}(p - \tilde{p}_h, \varphi) = T_{\mathrm{R}}(\varphi) + T_{\mathrm{U}}(\varphi).$$

To conclude the proof, if now suffices to use Lemma 7.1. ∎

We now estimate the terms T_{R}, T_{C}, and T_{U} separately, setting $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$ in Lemma 7.2.

LEMMA 7.3 (residual estimate). *Let $\varphi \in H_0^1(\Omega)$ be arbitrary. Then*

$$T_{\mathrm{R}}(\varphi) \leq \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K} |||\varphi|||_K,$$

where $\eta_{\mathrm{R},K}$ is given by (4.2).

Proof. The Poincaré inequality (2.1) and the definition of $|||\cdot|||_K$ by (2.6) imply

$$\|\varphi - \varphi_K\|_K^2 \leq C_{\mathrm{P},d} h_K^2 \|\nabla \varphi\|_K^2 \leq C_{\mathrm{P},d} \frac{h_K^2}{c_{\mathbf{S},K}} |||\varphi|||_K^2 . \tag{7.5}$$

Next, the estimate

$$\|\varphi - \varphi_K\|_K^2 \leq \|\varphi\|_K^2 \leq \frac{1}{c_{\mathbf{w},r,K}} |||\varphi|||_K^2$$

is obvious using the definition of $|||\cdot|||_K$ by (2.6). Thus the Schwarz inequality implies

$$\begin{aligned} T_{\mathrm{R}}(\varphi) &\leq \sum_{K \in \mathcal{T}_h} \left\| f + \nabla \cdot (\mathbf{S} \nabla \tilde{p}_h) - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r \tilde{p}_h \right\|_K \|\varphi - \varphi_K\|_K \\ &\leq \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K} |||\varphi|||_K . \qquad \square \end{aligned}$$

LEMMA 7.4 (convection estimate). *Let $\varphi \in H_0^1(\Omega)$ be arbitrary. Then*

$$T_{\mathrm{C}}(\varphi) \leq \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{C},K} |||\varphi|||_K,$$

where $\eta_{\mathrm{C},K}$ is given by (4.4).

Proof. Denote $v := \tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$. Then, for each $K \in \mathcal{T}_h$,

$$\left(\nabla \cdot (v \mathbf{w}) - \frac{1}{2} v \nabla \cdot \mathbf{w}, \varphi \right)_K \leq \frac{\|\nabla \cdot (v\mathbf{w}) - \frac{1}{2} v \nabla \cdot \mathbf{w}\|_K}{\sqrt{c_{\mathbf{w},r,K}}} |||\varphi|||_K .$$

Note that this estimate is valid for an arbitrary $s \in H_0^1(\Omega)$ instead of $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$.

Next, the fact that the modified Oswald interpolation operator of section 4.2 preserves the means of $\tilde{p}_h$ over the sides and that $\mathbf{w} \cdot \mathbf{n}$ is constant on all sides implies

$$(\nabla \cdot (v \mathbf{w}), \varphi_K)_K = \langle v \mathbf{w} \cdot \mathbf{n}, \varphi_K \rangle_{\partial K} = 0, \tag{7.6}$$

where again $\varphi_K := (\varphi, 1)_K / |K|$. Thus we also have an alternative estimate

$$\begin{aligned} &\left(\nabla \cdot (v \mathbf{w}) - \frac{1}{2} v \nabla \cdot \mathbf{w}, \varphi \right)_K \\ &= (\nabla v \cdot \mathbf{w}, \varphi - \varphi_K)_K + \left(\frac{1}{2} v \nabla \cdot \mathbf{w}, \varphi \right)_K - (v \nabla \cdot \mathbf{w}, \varphi_K)_K \\ &\leq \frac{\sqrt{C_{\mathrm{P},d}} h_K \|\nabla v \cdot \mathbf{w}\|_K}{\sqrt{c_{\mathbf{S},K}}} \sqrt{c_{\mathbf{S},K}} \|\nabla \varphi\|_K + \frac{3 \|v \nabla \cdot \mathbf{w}\|_K}{2 \sqrt{c_{\mathbf{w},r,K}}} \sqrt{c_{\mathbf{w},r,K}} \|\varphi\|_K \\ &\leq \left(\frac{C_{\mathrm{P},d} h_K^2 \|\nabla v \cdot \mathbf{w}\|_K^2}{c_{\mathbf{S},K}} + \frac{9 \|v \nabla \cdot \mathbf{w}\|_K^2}{4 c_{\mathbf{w},r,K}} \right)^{\frac{1}{2}} |||\varphi|||_K, \end{aligned}$$

using the Cauchy–Schwarz inequality and the Poincaré inequality (2.1). ∎

Finally, the proof of the following lemma can be found in [38].

LEMMA 7.5 (upwinding estimate). *Let $\varphi \in H_0^1(\Omega)$ be arbitrary. Then*

$$T_{\mathrm{U}}(\varphi) \le \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{U},K} |||\varphi|||_K,$$

where $\eta_{\mathrm{U},K}$ is given by (4.6).

Lemmas 7.1–7.5 and the Cauchy–Schwarz inequality prove Theorems 4.2–4.3.

7.2. Proofs of the local efficiency of the estimates.

LEMMA 7.6 (local efficiency of the residual estimator). *Let $K \in \mathcal{T}_h$ and let $\eta_{\mathrm{R},K}$ be the residual estimator given by* (4.2). *Then* (4.10) *holds true.*

Proof. The proof follows that given in [33]. Let ψ_K be the bubble function on K, given as the product of the $d+1$ linear functions that take the value 1 at one vertex of K and vanish at the other vertices, and let us denote $v := (f+\nabla\cdot(\mathbf{S}\nabla\tilde{p}_h)-\nabla\cdot(\tilde{p}_h\mathbf{w})-r\tilde{p}_h)$ on a given $K \in \mathcal{T}_h$. Note that v is a polynomial in K by Assumption B. Then the equivalence of norms on finite-dimensional spaces, the inverse inequality (cf., e.g., [15, Theorem 3.2.6]), and the definition of $|||\cdot|||_K$ by (2.6) give

$$\begin{aligned} c\,\|v\|_K^2 &\le (v,\psi_K v)_K,\\ \|\psi_K v\|_K &\le \|v\|_K,\\ |||\psi_K v|||_K &\le C\min\left\{\frac{h_K}{\sqrt{C_{\mathbf{S},K}}},\frac{1}{\sqrt{c_{\mathbf{w},r,K}}}\right\}^{-1}\|v\|_K, \end{aligned}$$

with the constants c and C depending only on the polynomial degree k of f, d, and κ_K. Next, we immediately have (cf. the proof of Lemma 7.2)

$$\mathcal{B}(p-\tilde{p}_h,\psi_K v) = (v,\psi_K v)_K,$$

and, using (2.10),

$$\begin{aligned} \mathcal{B}(p-\tilde{p}_h,\psi_K v) \le{}& \max\left\{1,\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right\}|||p-\tilde{p}_h|||_K|||\psi_K v|||_K\\ &+\frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}}|||p-\tilde{p}_h|||_K\|\psi_K v\|_K. \end{aligned}$$

Combining the above estimates, one comes to

$$\begin{aligned} c\|v\|_K^2 \le{}& |||p-\tilde{p}_h|||_K\|v\|_K\\ &\cdot\left\{\max\left\{1,\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right\}C\min\left\{\frac{h_K}{\sqrt{C_{\mathbf{S},K}}},\frac{1}{\sqrt{c_{\mathbf{w},r,K}}}\right\}^{-1}+\frac{C_{\mathbf{w},K}}{\sqrt{c_{\mathbf{S},K}}}\right\}. \end{aligned}$$

Considering the definition of $\eta_{\mathrm{R},K}$ by (4.2) and that of Pe_K and ϱ_K by (4.8) concludes the proof. ∎

LEMMA 7.7 (local efficiency of the nonconformity and velocity estimators). *Let $K \in \mathcal{T}_h$ and let $\eta_{\mathrm{NC},K}$ and $\eta_{\mathrm{C},K}$ be the nonconformity and velocity estimators given, respectively, by* (4.3) *and* (4.4). *Then* (4.11) *holds true.*

Proof. One shows easily that (with $|||\cdot|||_{*,K}$ and $|||\cdot|||_{\#,K}$ defined in section 4.4)

$$\eta_{\mathrm{NC},K}^2 + \eta_{\mathrm{C},K}^2 \leq \min\left\{|||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_{*,K}^2, |||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_{\#,K}^2\right\}.$$

Throughout the rest of the proof, let C denote a constant depending only on d and on $\kappa_{\mathcal{T}}$, not necessarily the same at each occurrence. We first show that

(7.7)
$$|||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_{*,K}^2 \leq C\left(\alpha_{*,K} \sum_{\sigma;\sigma\cap K\neq\emptyset} h_\sigma^{-1}\|[\tilde{p}_h]\|_\sigma^2 + \beta_{*,K} \sum_{\sigma;\sigma\cap K\neq\emptyset} h_\sigma\|[\tilde{p}_h]\|_\sigma^2\right).$$

The first part of the estimate follows directly from Lemma 4.1 and the definition of $|||\cdot|||_{*,K}$. To estimate $\beta_{*,K}\|\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)\|_K^2$, we notice that the means of $\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$ over all sides of a simplex $K \in \mathcal{T}_h$ are by the construction of the modified Oswald interpolation operator equal to 0. Hence

$$\|\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)\|_K^2 \leq C_{\mathrm{F},d} h_K^2 \|\nabla(\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h))\|_K^2$$

by the generalized Friedrichs inequality (2.2). The fact that h_K/h_σ for $K \cap \sigma \neq \emptyset$ depends only on $\kappa_{\mathcal{T}}$, which will be used in what follows as well, and another use of Lemma 4.1 proves the second part of the estimate.

We will next use the inequality

$$h_\sigma^{-\frac{1}{2}}\|[\tilde{p}_h]\|_\sigma \leq C \sum_{L;\sigma\in\mathcal{E}_L} \|\nabla(\tilde{p}_h - \varphi)\|_L$$

established in [2, Theorem 10] for $\sigma \in \mathcal{E}_h^{\mathrm{int}}$ and an arbitrary $\varphi \in H^1(\Omega)$. It generalizes easily to the case $\sigma \in \mathcal{E}_h^{\mathrm{ext}}$ and $\varphi \in H_0^1(\Omega)$. This inequality implies that

$$h_\sigma^\gamma \|[\tilde{p}_h]\|_\sigma^2 \leq C \frac{h_\sigma^{\gamma+1}}{\min_{L;\sigma\in\mathcal{E}_L} c_{\mathbf{S},L}} \sum_{L;\sigma\in\mathcal{E}_L} c_{\mathbf{S},L}\|\nabla(\tilde{p}_h - p)\|_L^2\,, \tag{7.8}$$

where we set $\gamma = -1, 1$. Next, for an arbitrary $s_h \in \mathbb{P}_2(\mathcal{T}_h) \cap H_0^1(\Omega)$,

$$\begin{aligned} h_\sigma^{\frac{1}{2}}\|[\tilde{p}_h]\|_\sigma &\leq h_\sigma C \sum_{L;\sigma\in\mathcal{E}_L} \|\nabla(\tilde{p}_h - s_h)\|_L \leq C \sum_{L;\sigma\in\mathcal{E}_L} h_L\|\nabla(\tilde{p}_h - s_h)\|_L \\ &\leq C \sum_{L;\sigma\in\mathcal{E}_L} \|\tilde{p}_h - s_h\|_L \leq C \sum_{L;\sigma\in\mathcal{E}_L} \|\tilde{p}_h - p\|_L + C \sum_{L;\sigma\in\mathcal{E}_L} \|p - s_h\|_L, \end{aligned}$$

by the inverse inequality (cf. [15, Theorem 3.2.6]) and the triangle inequality. Hence

$$h_\sigma\|[\tilde{p}_h]\|_\sigma^2 \leq C \frac{1}{\min_{L;\sigma\in\mathcal{E}_L} c_{\mathbf{w},r,L}} \sum_{L;\sigma\in\mathcal{E}_L} c_{\mathbf{w},r,L}\|\tilde{p}_h - p\|_L^2 + C \sum_{L;\sigma\in\mathcal{E}_L} \|p - s_h\|_L^2 \tag{7.9}$$

holds as well, which gives a sense when all $c_{\mathbf{w},r,L}$ for L such that $\sigma \in \mathcal{E}_L$ are nonzero. Combining estimates (7.7)–(7.9) while estimating $\min_{L;\sigma\in\mathcal{E}_L} c_L$ for a side σ such that $\sigma \cap K \neq \emptyset$ from below by $\min_{L;L\cap K\neq\emptyset} c_L$ concludes the proof for $|||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_{*,K}$. The proof for $|||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_{\#,K}$ is completely similar. ∎

LEMMA 7.8 ((non)efficiency of the upwinding estimator). *Let $K \in \mathcal{T}_h$ and let $\eta_{\mathrm{U},K}$ be the upwinding estimator given by* (4.6). *Then* (4.12) *holds true.*

Proof. Let $K \in \mathcal{T}_h$, $\varphi \in H^1(K)$, and $\varphi_\sigma := \langle \varphi, 1\rangle_\sigma/|\sigma|$. Let us set $\tilde{\varphi} := \varphi - \varphi_\sigma$ and $\tilde{\varphi}_K := (\tilde{\varphi}, 1)_K/|K|$. We now note that $\tilde{\varphi}_\sigma := \langle \tilde{\varphi}, 1\rangle_\sigma/|\sigma| = 0$ and that $\nabla\tilde{\varphi} = \nabla\varphi$, which allows us to estimate

$$\|\varphi_K - \varphi_\sigma\|_\sigma^2 = \tilde{\varphi}_K^2|\sigma| \le \frac{|\sigma|}{|K|}\|\tilde{\varphi}\|_K^2 \le C_{\mathrm{F},d}\frac{|\sigma|h_K^2}{|K|}\|\nabla\varphi\|_K^2,$$

employing the generalized Friedrichs inequality (2.2). Now using the definition of $\hat{p}_\sigma$ for $\sigma \in \mathcal{E}_h^{\mathrm{int}}$ by (3.3), the fact that $0 \le \nu_\sigma \le 1/2$, (4.1b), and the above estimate,

$$\begin{aligned}\|\hat{p}_\sigma - \tilde{p}_\sigma\|_\sigma &= \|(1-\nu_\sigma)(p_K - \tilde{p}_\sigma) + \nu_\sigma(p_L - \tilde{p}_\sigma)\|_\sigma \\ &\le \max_{M;\sigma\in\mathcal{E}_M}\left\{\frac{C_{\mathrm{F},d}|\sigma|h_M^2}{|M|}\right\}^{\frac{1}{2}}(\|\nabla\tilde{p}_h\|_K + \|\nabla\tilde{p}_h\|_L)\end{aligned}$$

for suitable denotation K, L of the two elements sharing σ. For $\sigma \in \mathcal{E}_h^{\mathrm{ext}}$, a similar estimate holds. The assertion of the lemma follows by using the above estimate, (4.5), (4.6), the definition of κ_K, the estimate $|\sigma| \le h_K^{d-1}/(d-1)$, the Cauchy–Schwarz inequality, and estimating the term $\sum_{K\in\mathcal{T}_h} c_{\mathbf{S},K}\|\nabla\tilde{p}_h\|_K^2$ using Lemma 6.5. ∎

Lemmas 7.6–7.8 together prove Theorem 4.4.

8. Numerical experiments. We test our a posteriori error estimates on two model problems in this section. The first problem contains a strongly inhomogeneous diffusion-dispersion tensor, and the second one is convection-dominated; in both cases, the analytical solution is known. Estimators for inhomogeneous Dirichlet (and Neumann) boundary conditions are adapted from [38].

8.1. Model problem with strongly inhomogeneous diffusion-dispersion tensor. This model problem is taken from [30, 18] and is motivated by the fact that in real-life applications, the diffusion-dispersion tensor $\mathbf{S}$ may be discontinuous and strongly inhomogeneous. We consider in particular $\Omega = (-1,1)\times(-1,1)$ and (1.1a) with $\mathbf{w} = 0$, $r = 0$, and $f = 0$. We suppose that Ω is divided into four subdomains Ω_i corresponding to the axis quadrants (in the counterclockwise direction) and that $\mathbf{S}$ is constant and equal to $s_i\, Id$ in Ω_i. Under such conditions, an analytical solution writing

$$p(r,\theta) = r^\alpha(a_i\sin(\alpha\theta) + b_i\cos(\alpha\theta))$$

in each Ω_i can be found. Here (r,θ) are the polar coordinates in Ω, a_i and b_i are constants depending on Ω_i, and α is a parameter. This solution is continuous across the interfaces, but only the normal component of its flux $\mathbf{u} = -\mathbf{S}\nabla p$ is continuous; it finally exhibits a singularity at the origin. We assume Dirichlet boundary conditions given by this solution and consider two sets of the coefficients, with $s_1 = s_3 = 5$, $s_2 = s_4 = 1$ in the first case and $s_1 = s_3 = 100$, $s_2 = s_4 = 1$ in the second one:

$\alpha = 0.53544095$		$\alpha = 0.12690207$	
$a_1 = 0.44721360$	$b_1 = 1$	$a_1 = 0.1$	$b_1 = 1$
$a_2 = -0.74535599$	$b_2 = 2.33333333$	$a_2 = -9.60396040$	$b_2 = 2.96039604$
$a_3 = -0.94411759$	$b_3 = 0.55555556$	$a_3 = -0.48035487$	$b_3 = -0.88275659$
$a_4 = -2.40170264$	$b_4 = -0.48148148$	$a_4 = 7.70156488$	$b_4 = -6.45646175$

The original grid consisted of 24 right-angled triangles, and we have refined it either uniformly (up to five refinements) or adaptively on the basis of our estimator.

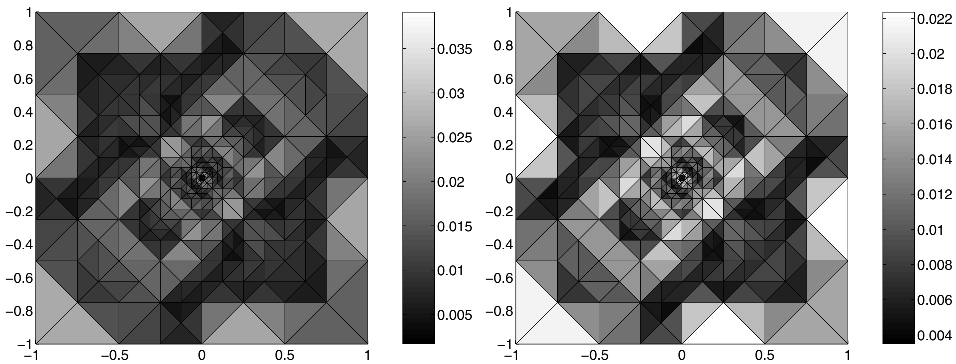

FIG. 8.1. *Estimated (left) and actual (right) error distribution, $\alpha = 0.53544095$ (the maximum is attained at the origin).*

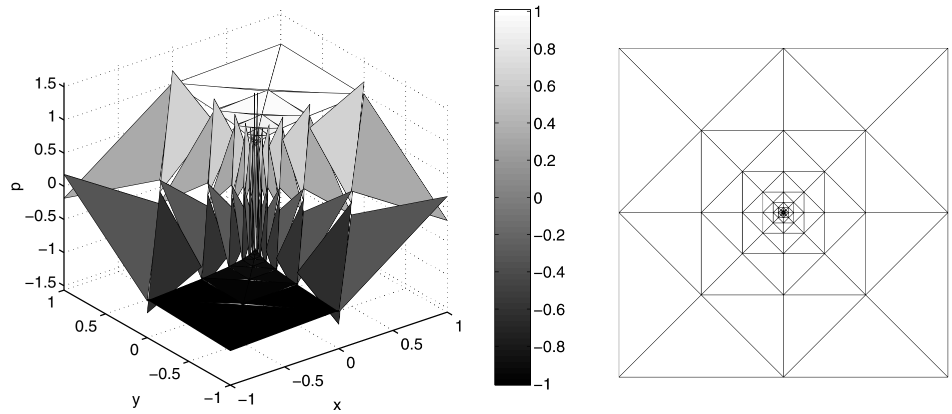

FIG. 8.2. *Approximate solution and the corresponding adaptively refined mesh, $\alpha = 0.12690207$.*

In the latter case, we refine each element where the estimated $|||\cdot|||_\Omega$-error is greater than the half of the maximum of the estimators regularly into four subelements and then use the "longest edge" refinement to recover an admissible mesh. In the given case, the residual estimators $\eta_{\mathrm{R},K}$ of (5.2) are zero for each $K \in \mathcal{T}_h$, and hence the a posteriori error estimate is entirely given by the nonconformity estimators $\eta_{\mathrm{NC},K}$ in (5.3). We have done numerical experiments with two choices, $s = \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$ and $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$, and present the results with the first one, which gives a slightly better efficiency.

We can see in Figure 8.1 that the predicted error distribution on an adaptively refined mesh for the first test case is excellent. In particular, even if the solution is smoother, the singularity is well recognized. Next, Figure 8.2 gives an example of the approximate solution on an adaptively refined mesh and this mesh in the second test case. Here, the singularity is much more important, and consequently the grid is highly refined around the origin (for 1800 triangles, the diameter of the smallest ones is 10^{-16}, and 73% of them are contained in the circle of radius 0.1). Figure 8.3 then reports the estimated and actual errors of the numerical solutions on uniformly/adaptively refined grids in the two test cases. The energy norm (2.6) was approximated with a 7-point quadrature formula in each triangle. It can be seen

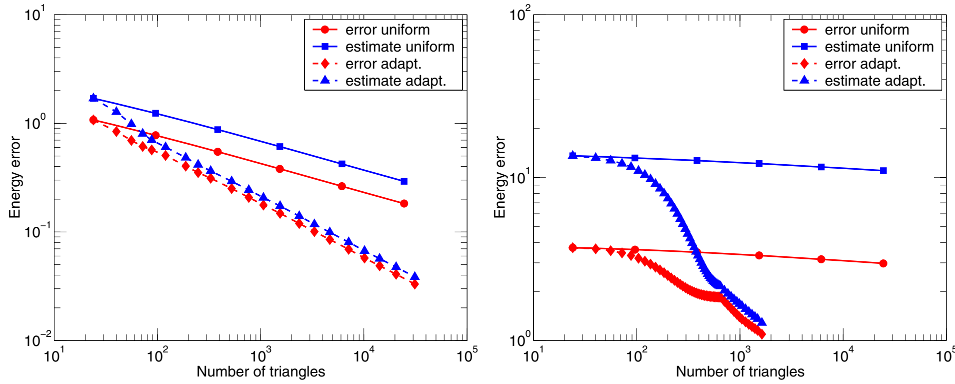

FIG. 8.3. *Estimated and actual error against the number of elements in uniformly/adaptively refined meshes for* $\alpha = 0.53544095$ *(left) and* $\alpha = 0.12690207$ *(right).*

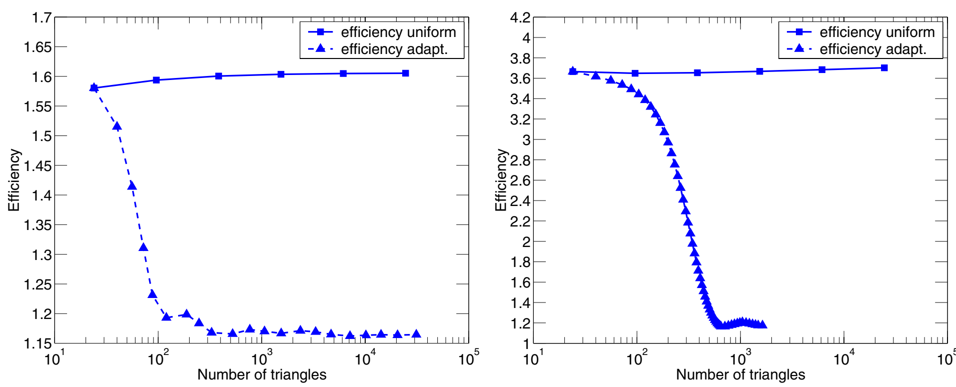

FIG. 8.4. *Overall efficiency of the a posteriori error estimates against the number of elements in uniformly/adaptively refined meshes for* $\alpha = 0.53544095$ *(left) and* $\alpha = 0.12690207$ *(right).*

from these plots that one can substantially reduce the number of unknowns necessary to attain the prescribed precision using the derived a posteriori error estimates and adaptively refined grids. Finally, Figure 8.4 gives the efficiency plots for the two cases, i.e., the ratio of the estimated $|||\cdot|||_\Omega$-error to the actual $|||\cdot|||_\Omega$-error. This quantity simply expresses how many times we have overestimated the error—recall that there are no undetermined multiplicative constants in our estimates. These plots confirm the theoretical results of section 5.3. Even while only using $\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$ instead of evaluating the infimum in (5.4), (approximate) asymptotic exactness and robustness with respect to inhomogeneities is confirmed.

8.2. Convection-dominated model problem. This problem is a modification of a problem considered in [20]. We set $\Omega = (0,1) \times (0,1)$, $\mathbf{w} = (0,1)$, and $r = 1$ in (1.1a) and consider three cases with $\mathbf{S} = \varepsilon\, Id$ and ε equal to, respectively, 1, 10^{-2}, and 10^{-4}. The right-hand-side term f, Neumann boundary conditions on the upper side, and Dirichlet boundary conditions elsewhere are chosen so that

$$p(x,y) = 0.5\left(1 - \tanh\left(\frac{0.5 - x}{a}\right)\right)$$

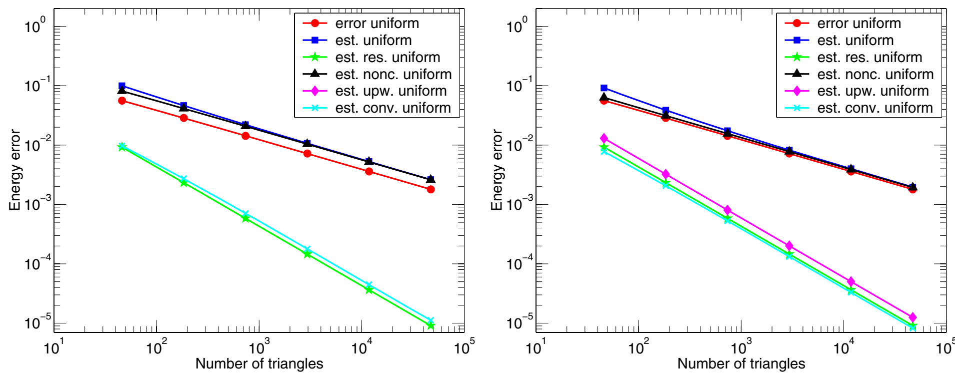

FIG. 8.5. *Estimated and actual error using* $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$ *(left) and* $s = \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$ *(right) against the number of elements,* $\varepsilon = 1$*,* $a = 0.5$*.*

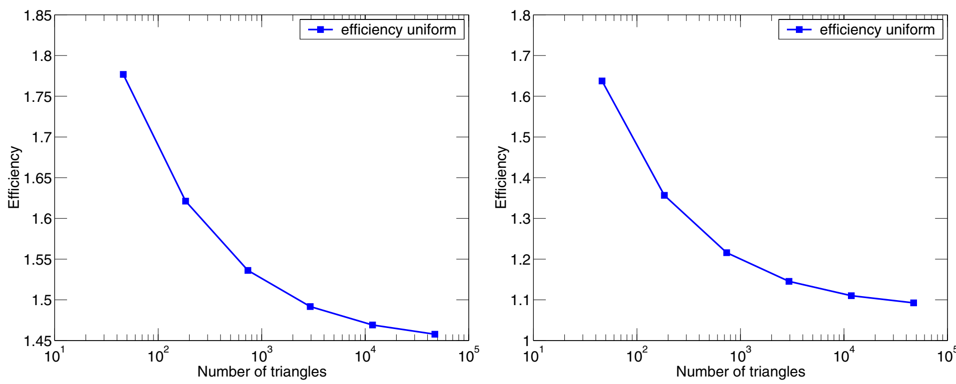

FIG. 8.6. *Overall efficiency using* $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$ *(left) and* $s = \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$ *(right) against the number of elements,* $\varepsilon = 1$*,* $a = 0.5$*.*

was the exact solution. It is, in fact, one-dimensional and possesses an internal layer of width a which we set, respectively, equal to 0.5, 0.05, and 0.02. We start the computations from an unstructured grid of Ω consisting of 46 triangles and refine it either uniformly (up to five refinements) or adaptively. We use the scheme described in section 5.5.

We first compare, for $\varepsilon = 1$ and $a = 0.5$, the estimates with $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$ as proposed in section 4.3 and a modification with $s = \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$, corresponding to the approach chosen in [38, 37], on uniformly refined grids. In the latter case, we no longer have the important property (7.6), and consequently there is an additional term which we associate with the upwinding estimator; it, however, turns out to be of higher order; see Figure 8.5. Note that the (approximate) asymptotic exactness observed in Figure 8.6 is in full correspondence with the theoretical considerations of section 5.3.2. In this case, $s = \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$ gives a slightly better efficiency. In the following examples, however, we use $s = \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$, since it turns out to be the better choice.

For $\varepsilon = 10^{-2}$ and $a = 0.05$ (convection-dominated regime on coarse meshes and diffusion-dominated regime with progressive refinement), still the distribution of the error is predicted very well; cf. Figure 8.7. Note in particular the correct localization of

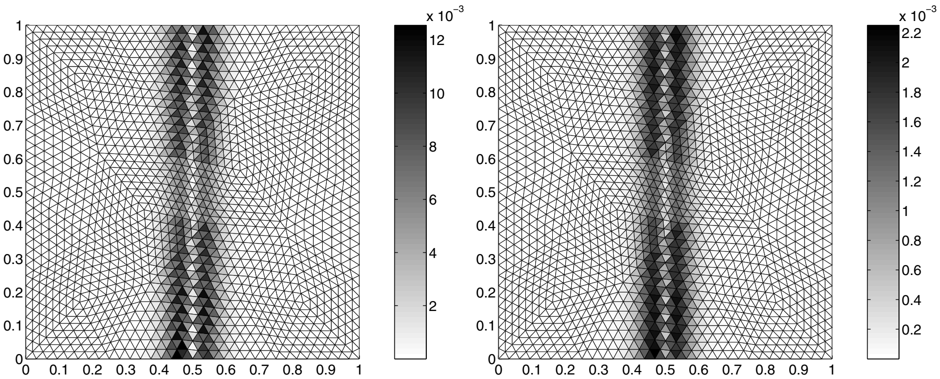

FIG. 8.7. *Estimated (left) and actual (right) error distribution,* $\varepsilon = 10^{-2}$, $a = 0.05$.

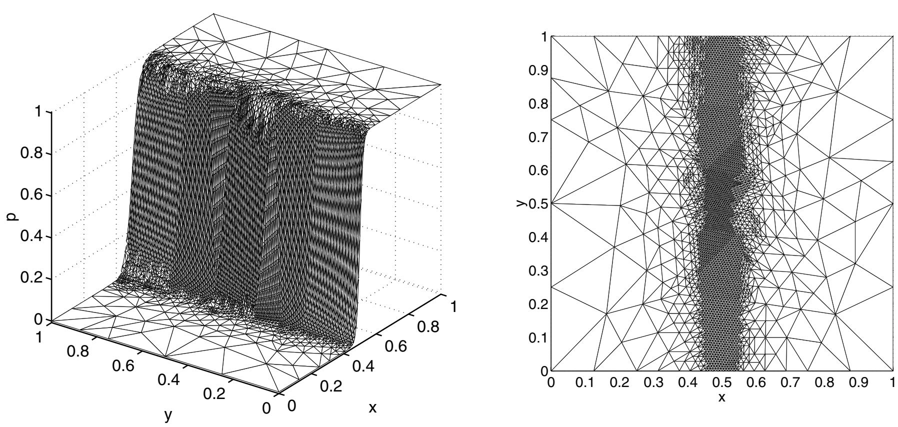

FIG. 8.8. *Approximate solution and the corresponding adaptively refined mesh,* $\varepsilon = 10^{-4}$, $a = 0.02$.

the error away from the center of the shock, as well as the sensitivity of our estimator to the shape of the elements. Next, an example of an adaptively refined mesh and of the corresponding solution for $\varepsilon = 10^{-4}$ and $a = 0.02$ is given in Figure 8.8. For these two test cases, we have used as a refinement criterion 0.2- and 0.05-times the maximum of the estimators, respectively. The estimated and actual errors are plotted against the number of elements in uniformly/adaptively refined meshes in Figure 8.9. Again, one can see that we can substantially reduce the number of unknowns necessary to attain the prescribed precision using the derived estimators and adaptively refined grids. Finally, the efficiency plots are given in Figure 8.10. In the first case, the efficiency is almost optimal for finest grids, whereas in the second one, only the elements in the refined shock region start to leave the convection-dominated regime, and thus the efficiency starts to decrease.

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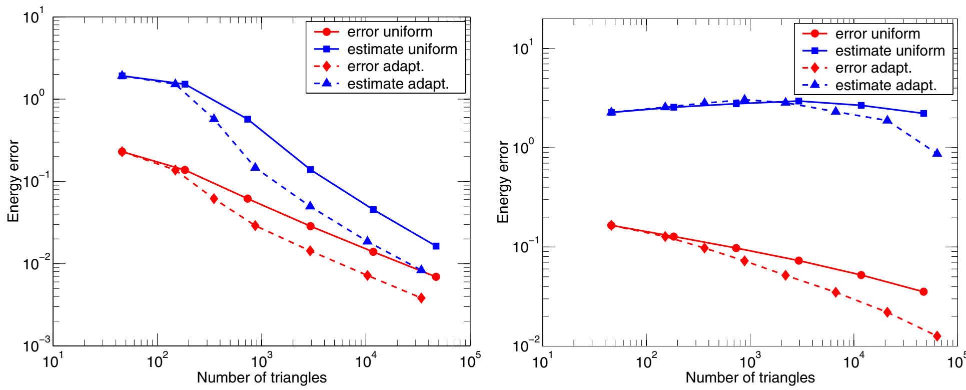

FIG. 8.9. *Estimated and actual error against the number of elements in uniformly/adaptively refined meshes for* $\varepsilon = 10^{-2}$, $a = 0.05$ *(left) and* $\varepsilon = 10^{-4}$, $a = 0.02$ *(right).*

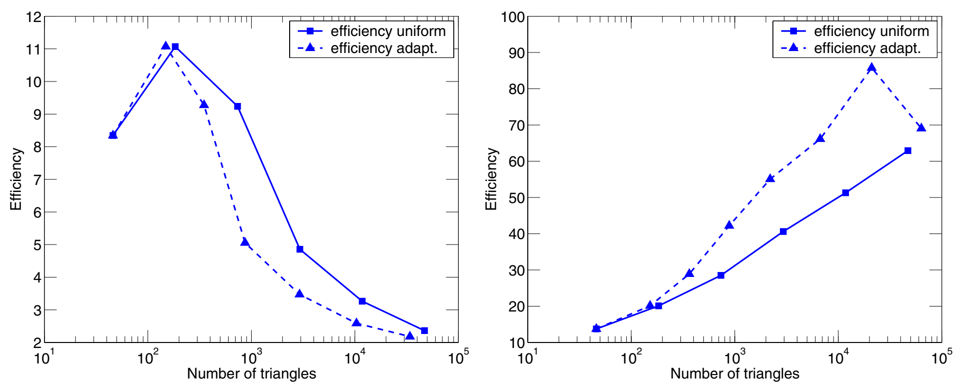

FIG. 8.10. *Overall efficiency of the a posteriori error estimates against the number of elements in uniformly/adaptively refined meshes for* $\varepsilon = 10^{-2}$, $a = 0.05$ *(left) and* $\varepsilon = 10^{-4}$, $a = 0.02$ *(right).*

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