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DISCRETE POINCARÉ INEQUALITIES: A REVIEW ON PROOFS, EQUIVALENT FORMULATIONS, AND BEHAVIOR OF CONSTANTS

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ABSTRACT. We investigate discrete Poincaré inequalities on piecewise polynomial subspaces of the Sobolev spaces $\boldsymbol{H}(\mathbf{curl},\omega)$ and $\boldsymbol{H}(\mathrm{div},\omega)$ in three space dimensions. We characterize the dependence of the constants on the continuous-level constants, the shape regularity and cardinality of the underlying tetrahedral mesh, and the polynomial degree. One important focus is on meshes being local patches (stars) of tetrahedra from a larger tetrahedral mesh. We also review various equivalent results to the discrete Poincaré inequalities, namely stability of discrete constrained minimization problems, discrete inf-sup conditions, bounds on operator norms of piecewise polynomial vector potential operators (Poincaré maps), and existence of graph-stable commuting projections.

1. INTRODUCTION

Let ω be a three-dimensional, open, bounded, connected, Lipschitz polyhedral domain with diameter h_ω. The L^2-inner product in ω is denoted as $\langle \cdot,\cdot\rangle_\omega$ and the corresponding norm as $\|\cdot\|_{L^2(\omega)}$ or $\|\cdot\|_{\boldsymbol{L}^2(\omega)}$ (notation is set in details in Section 2 below). Let $\mathcal{T}_\omega$ be a tetrahedral mesh of ω. Our main motivation is the case where $\mathcal{T}_\omega$ is some local collection (patch, star) of tetrahedra from a mesh of some larger fixed three-dimensional domain, say Ω.

1.1. Poincaré and discrete Poincaré inequalities on $H(\mathbf{grad},\omega) = H^1(\omega)$.

The Poincaré inequality

(1.1) $$\|u\|_{L^2(\omega)} \le C_{\mathrm{P}}^0 h_\omega \|\mathbf{grad}\, u\|_{\boldsymbol{L}^2(\omega)}, \qquad \forall u \in H(\mathbf{grad},\omega) \text{ such that } \langle u, 1\rangle_\omega = 0$$

is well known and omnipresent in the analysis of partial differential equations. Crucially, C_{P}^0 is a generic constant that only depends on the shape of ω. For convex ω, in particular, $C_{\mathrm{P}}^0 \le 1/\pi$ following Payne and Weinberger [44] and Bebendorf [5]. The discrete version of (1.1) for $H(\mathbf{grad},\omega)$-conforming piecewise polynomials of degree $(p+1)$, $p \ge 0$, on the tetrahedral mesh $\mathcal{T}_\omega$ of ω writes

(1.2) $$\|u_{\mathcal{T}}\|_{L^2(\omega)} \le C_{\mathrm{P}}^{\mathrm{d},0} h_\omega \|\mathbf{grad}\, u_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)}, \qquad \forall u_{\mathcal{T}} \in \mathcal{P}_{p+1}(\mathcal{T}_\omega) \cap H(\mathbf{grad},\omega) \text{ such that } \langle u_{\mathcal{T}}, 1\rangle_\omega = 0.$$

As the functions considered in (1.2) form a subset of the functions considered in (1.1), (1.2) trivially holds with $C_{\mathrm{P}}^{\mathrm{d},0} \le C_{\mathrm{P}}^0$. Moreover, similar results hold with (homogeneous) boundary condition on the boundary $\partial\omega$.

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1.2. **Poincaré and discrete Poincaré inequalities on $\boldsymbol{H}(\mathbf{curl},\omega)$ and $\boldsymbol{H}(\mathrm{div},\omega)$.** The Poincaré inequalities

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)} \le C_{\mathrm{P}}^1 h_\omega \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)}, \qquad \forall \boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},\omega) \text{ such that } \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\omega) \text{ with } \mathbf{curl}\,\boldsymbol{v} = \boldsymbol{0}, \tag{1.3a}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)} \le C_{\mathrm{P}}^2 h_\omega \|\mathrm{div}\,\boldsymbol{u}\|_{L^2(\omega)}, \qquad \forall \boldsymbol{u} \in \boldsymbol{H}(\mathrm{div},\omega) \text{ such that } \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\omega) \text{ with } \mathrm{div}\,\boldsymbol{v} = 0 \tag{1.3b}$$

arise when the differential operators employ the curl or the divergence in place of the gradient. They are also well known, though a bit more complicated to establish. When ω is simply connected, the orthogonality in (1.3a) means that $\boldsymbol{u}$ is orthogonal to gradients of $H(\mathbf{grad},\omega)$ functions and thus belongs to $\boldsymbol{H}(\mathrm{div},\omega)$, is divergence-free, and has zero normal component on the boundary $\partial\omega$. Thus, (1.3a) is the so-called Poincaré–Friedrichs–Weber inequality, see Fernandes and Gilardi [33, Proposition 7.4] or Chaumont-Frelet et al. [10, Theorem A.1]. Similarly, when $\partial\omega$ is connected, the orthogonality in (1.3b) means that $\boldsymbol{u}$ is orthogonal to curls of $\boldsymbol{H}(\mathbf{curl},\omega)$ functions and thus belongs to $\boldsymbol{H}(\mathbf{curl},\omega)$, is curl-free, and has zero tangential component on the boundary $\partial\omega$.

In this paper, we are interested in the following *discrete Poincaré inequalities* for $\boldsymbol{H}(\mathbf{curl},\omega)$- and $\boldsymbol{H}(\mathrm{div},\omega)$-conforming piecewise polynomials in the Nédélec and Raviart–Thomas finite element spaces of order p, $p \ge 0$, on the tetrahedral mesh $\mathcal{T}_\omega$ of ω:

$$\|\boldsymbol{u}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)} \le C_{\mathrm{P}}^{\mathrm{d},1} h_\omega \|\mathbf{curl}\,\boldsymbol{u}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}, \qquad \forall \boldsymbol{u}_\mathcal{T} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega) \text{ such that } \langle \boldsymbol{u}_\mathcal{T}, \boldsymbol{v}_\mathcal{T}\rangle_\omega = 0 \quad \forall \boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega) \text{ with } \mathbf{curl}\,\boldsymbol{v}_\mathcal{T} = \boldsymbol{0}, \tag{1.4a}$$

$$\|\boldsymbol{u}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)} \le C_{\mathrm{P}}^{\mathrm{d},2} h_\omega \|\mathrm{div}\,\boldsymbol{u}_\mathcal{T}\|_{L^2(\omega)}, \qquad \forall \boldsymbol{u}_\mathcal{T} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega) \text{ such that } \langle \boldsymbol{u}_\mathcal{T}, \boldsymbol{v}_\mathcal{T}\rangle_\omega = 0 \quad \forall \boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega) \text{ with } \mathrm{div}\,\boldsymbol{v}_\mathcal{T} = 0. \tag{1.4b}$$

We will also consider the counterparts with homogeneous boundary condition on the boundary $\partial\omega$. Here, unfortunately, the inequalities (1.4) do not follow from (1.3) and $C_{\mathrm{P}}^{\mathrm{d},1}$, $C_{\mathrm{P}}^{\mathrm{d},2}$ cannot be trivially bounded by C_{P}^1, C_{P}^2. Indeed, the kernel of the **curl** operator restricted to $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega)$ is different from its kernel on $\boldsymbol{H}(\mathbf{curl},\omega)$, and similarly, the kernel of the div operator restricted to $\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega)$ is different from its kernel on $\boldsymbol{H}(\mathrm{div},\omega)$ (the former being nontrivial and the latter being infinite-dimensional in both cases). In contrast, the kernel of the gradient operator is trivial (composed of constant functions) and is the same on $\mathcal{P}_{p+1}(\mathcal{T}_\omega) \cap H(\mathbf{grad},\omega)$ and $H(\mathbf{grad},\omega)$, which leads to the trivial passage from (1.1) to (1.2).

Remark 1.1 (Orthogonality constraint)**.** When ω is simply connected, a vector-valued piecewise polynomial $\boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega)$ with $\mathbf{curl}\,\boldsymbol{v}_\mathcal{T} = \boldsymbol{0}$ is a gradient of a scalar-valued piecewise polynomial $v_\mathcal{T} \in \mathcal{P}_{p+1}(\mathcal{T}_\omega) \cap H(\mathbf{grad},\omega)$; the orthogonality constraint in (1.4a) is often stated in the literature using gradients. We prefer the writing (1.4), where the orthogonality constraints in (1.4) are valid for a general topology.

1.3. **Focus of the paper.** In the literature, one often finds assertions that the discrete Poincaré inequalities (1.4) are "known". The purpose of this paper is to recall several equivalent reformulations of (1.4), discuss the available references, formulate some possible proofs of (1.4) in an abstract way with generic assumptions, and to establish some new results on (1.4). Our main focus is on the *characterization* of the *behavior* of the constants $C_{\mathrm{P}}^{\mathrm{d},1}$, $C_{\mathrm{P}}^{\mathrm{d},2}$ with respect to the *constants* C_{P}^1, C_{P}^2, the *shape-regularity parameter* of the mesh $\mathcal{T}_\omega$, the *number of elements* in $\mathcal{T}_\omega$, and the *polynomial degree* p. The motivation for writing explicitly the scaling with h_ω in (1.2) and (1.4) is twofold: (i) it is important when ω corresponds to a local collection of tetrahedra from a larger mesh; (ii) it makes the constants C_{P}^{0} and $C_{\mathrm{P}}^{\mathrm{d},0}$ dimensionless.

1.4. **Available results.** As discussed below in more detail, the discrete Poincaré inequalities (1.4) are equivalent to: (i) stability of discrete constrained minimization problems; (ii) discrete inf-sup conditions; (iii) bounds on operator norms of piecewise polynomial vector potential operators (that is, piecewise polynomial right-inverses for the curl and divergence operators, also called Poincaré maps); and (iv) existence of graph-stable commuting projections. There are also links to lower bounds on eigenvalues of vector Laplacians. Numerous results are available in the literature in one of these settings.

The discrete Poincaré inequality (1.4a), with a generic constant C in place of $C_{\mathrm{P}}^{\mathrm{d},1}h_\omega$, is presented in Girault and Raviart [36, Chapter 3, Proposition 5.1] and Monk and Demkowicz [41, Corollary 4.2] in three space dimensions and in Arnold *et al.* [3, Theorem 5.11] and Arnold *et al.* [4, Theorem 3.6] more abstractly in the finite element exterior calculus setting, covering both bounds in (1.4). The discrete Poincaré inequality in the precise form (1.4a) is established in Ern and Guermond [26, Theorem 44.6], with $C_{\mathrm{P}}^{\mathrm{d},1}$ at worst depending on the continuous-level constant C_{P}^{1} from (1.3a), the shape-regularity parameter of $\mathcal{T}_\omega$, and the polynomial degree p, see also [26, Remark 44.7] for further bibliographical ressources.

Discrete inf-sup conditions are extensively discussed in the mixed finite element literature. For instance, (1.4b) as a discrete inf-sup condition is established, with a generic constant C in place of $C_{\mathrm{P}}^{\mathrm{d},2}h_\omega$, in Raviart and Thomas [45, Theorem 4], see also Fortin [34], Boffi *et al.* [7, Theorem 4.2.3 and Propositions 5.4.3 and 7.1.1], or Gatica [35, Lemmas 2.6 and 4.4.]. The form leading precisely to (1.4b) can be found in [26, Remark 51.12], with $C_{\mathrm{P}}^{\mathrm{d},2}$ at worst depending on the continuous-level constant C_{P}^{2} from (1.3b), the shape-regularity parameter of $\mathcal{T}_\omega$, and the polynomial degree p.

Considering the operator norm of a piecewise polynomial vector potential operator, Demkowicz and Babuška [18, Theorem 1], Gopalakrishnan and Demkowicz [38, Theorems 4.1, 5.1, and 6.1], and Demkowicz and Buffa [19, Lemmas 6 and 8] establish (1.4) with a generic constant C independent of the polynomial degree p (*p-robustness*) on a single triangle or tetrahedron. Similar results hold on a cube and more generally on starlike domains with respect to a ball, see Costabel *et al.* [16] and Costabel and McIntosh [17]. Unfortunately, none of these results addresses piecewise polynomials with respect to a mesh $\mathcal{T}_\omega$. This issue is discussed in Boffi *et al.* [8, Lemma 2.5] for the p-version finite element method on a fixed mesh.

Piecewise polynomials on patches of tetrahedra sharing a given subsimplex (vertex, edge, or face) seem to have been addressed only more recently. Corresponding proofs employ the above-discussed results for polynomials on one element together with polynomial extension operators from the boundary of a tetrahedron (Demkowicz *et al.* [20, 21] for, respectively, the tangential or normal trace lifting in the $\boldsymbol{H}(\mathbf{curl},\omega)$ or $\boldsymbol{H}(\mathrm{div},\omega)$ context; cf. also the recent work of Falk and Winther [32] for a d-simplex). Following some early contributions like Gopalakrishnan *et al.* [39, Lemma 3.1 and Appendix], Braess *et al.* [9] address vertex stars in 2D and Ern and Vohralík [29, Corollaries 3.3 and 3.8] consider vertex stars in 3D in the $\boldsymbol{H}(\mathrm{div},\omega)$ context, whereas the $\boldsymbol{H}(\mathbf{curl},\omega)$ context is developed in Chaumont-Frelet et al. [10, Theorem 3.1] and Chaumont-Frelet and Vohralík [12, Theorem 3.3 and Corollary 4.3] (respectively edge and vertex stars in 3D). As we shall see, these results imply (1.4) with $C_{\mathrm{P}}^{\mathrm{d},1}$, $C_{\mathrm{P}}^{\mathrm{d},2}$ being p-robust, but possibly depending on the number of elements in the mesh $\mathcal{T}_\omega$. Finally, simultaneous independence of the number of elements in the mesh $\mathcal{T}_\omega$ and of the polynomial degree p follows from the recent result of Demkowicz and Vohralík [22] under a shellability assumption (this notion is further discussed in Section 5.1).

1.5. **Main results and organization of the paper.** We introduce some basic necessary notation in Section 2 in order to recall in Section 3 that discrete Poincaré inequalities are equivalent with stability of discrete constrained minimization problems, discrete inf-sup conditions, and bounds on operator norms of piecewise polynomial vector potential operators. Section 4 then wraps up known results on the continuous Poincaré inequalities (1.1) and (1.3) and their variants with boundary conditions on $\partial\omega$. Turning next to the discrete Poincaré inqualities in Section 5, our main result is Theorem 5.1, establishing (1.4) and its variants with boundary conditions on $\partial\omega$. In particular, we thoroughly discuss the dependencies of $C_{\mathrm{P}}^{\mathrm{d},1}$, $C_{\mathrm{P}}^{\mathrm{d},2}$ on the constants C_{P}^1, C_{P}^2, the shape-regularity parameter of $\mathcal{T}_\omega$, the number of elements in $\mathcal{T}_\omega$, and the polynomial degree p. Three different proofs, leading to various dependencies, are presented in Section 6, relying either on available results from the literature (invoking equivalences between discrete and continuous minimizers or stable commuting projections) or on a self-standing proof invoking piecewise Piola transformations. In Section 6, we also recall the equivalence of discrete Poincaré inequalities with the existence of graph-stable commuting projections.

2. Basic notation

Let ω be a three-dimensional, open, bounded, connected, Lipschitz polyhedral domain with boundary $\partial\omega$ and unit outward normal $\boldsymbol{n}_\omega$. Let h_ω denote the diameter of ω. We use boldface font for vector-valued quantities, vector-valued fields, and functional spaces composed of such fields. For simplicity, the inner product in $L^2(\omega)$ and $\boldsymbol{L}^2(\omega)$ is abbreviated as $\langle\cdot,\cdot\rangle_\omega$, whereas the norms are written as $\|\cdot\|_{L^2(\omega)}$, $\|\cdot\|_{\boldsymbol{L}^2(\omega)}$. Let $H(\mathbf{grad},\omega) := H^1(\omega)$ be the standard Sobolev space of scalar-valued functions from $L^2(\omega)$ with weak gradient in $\boldsymbol{L}^2(\omega)$, $\boldsymbol{H}(\mathbf{curl},\omega)$ the Sobolev space of vector-valued functions from $\boldsymbol{L}^2(\omega)$ with weak curl in $\boldsymbol{L}^2(\omega)$, and $\boldsymbol{H}(\mathrm{div},\omega)$ the Sobolev space of vector-valued functions from $\boldsymbol{L}^2(\omega)$ with weak divergence in $L^2(\omega)$, cf., e.g., [36, 25]. These spaces are Hilbert spaces when equipped with the graph norms

$$\|u\|^2_{H(\mathbf{grad},\omega)} := \|u\|^2_{L^2(\omega)} + h_\omega^2\|\mathbf{grad}\,u\|^2_{\boldsymbol{L}^2(\omega)}, \tag{2.1a}$$

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathbf{curl},\omega)} := \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\omega)} + h_\omega^2\|\mathbf{curl}\,\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\omega)}, \tag{2.1b}$$

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathrm{div},\omega)} := \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\omega)} + h_\omega^2\|\mathrm{div}\,\boldsymbol{u}\|^2_{L^2(\omega)}. \tag{2.1c}$$

The length scale h_ω is used for dimensional consistency (in terms of physical units) and corresponds to the scaling in the Poincaré inequalities (1.3) and (1.4). We denote by $\mathring{H}(\mathbf{grad},\omega) := H_0^1(\omega)$, $\mathring{\boldsymbol{H}}(\mathbf{curl},\omega)$, and $\mathring{\boldsymbol{H}}(\mathrm{div},\omega)$ the subspaces with homogeneous boundary conditions imposed along $\partial\omega$ with the usual trace maps associated with the trace, the trace of the tangential component, and the trace of the normal component on $\partial\omega$. Specifically, for a smooth function or field, the trace maps are $\gamma^0_{\partial\omega}(u) = u|_{\partial\omega}$, $\gamma^1_{\partial\omega}(\boldsymbol{u}) = \boldsymbol{u}|_{\partial\omega}\times\boldsymbol{n}_\omega$, and $\gamma^2_{\partial\omega}(\boldsymbol{u}) = \boldsymbol{u}|_{\partial\omega}\cdot\boldsymbol{n}_\omega$.

Let $\mathcal{T}_\omega$ be a triangulation of ω consisting of a finite number of tetrahedra. The shape-regularity parameter of $\mathcal{T}_\omega$ is defined as

$$\rho_{\mathcal{T}_\omega} := \max_{\tau\in\mathcal{T}_\omega} h_\tau/\iota_\tau, \tag{2.2}$$

where h_τ is the diameter of τ and ι_τ the diameter of the largest ball inscribed in τ. We also denote by $|\mathcal{T}_\omega|$ the cardinal number of $\mathcal{T}_\omega$, i.e., the number of elements in $\mathcal{T}_\omega$. Let $p \geq 0$ be a fixed polynomial degree. For a tetrahedron $\tau \in \mathcal{T}_\omega$, let $\mathcal{P}_p(\tau)$ denote the space of polynomials of total degree at most p on τ, $\mathcal{P}_p(\tau;\mathbb{R}^3)$ its vector-valued counterpart,

$$\boldsymbol{\mathcal{N}}_p(\tau) := \{\boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{x}\times\boldsymbol{v}(\boldsymbol{x}) : \boldsymbol{u},\boldsymbol{v}\in\mathcal{P}_p(\tau;\mathbb{R}^3)\} \tag{2.3}$$

the p-th order Nédélec space [42], and

$$\boldsymbol{\mathcal{RT}}_p(\tau) := \{\boldsymbol{u}(\boldsymbol{x}) + v(\boldsymbol{x})\boldsymbol{x} : \boldsymbol{u} \in \mathcal{P}_p(\tau;\mathbb{R}^3), v \in \mathcal{P}_p(\tau)\} \tag{2.4}$$

the p-th order Raviart–Thomas space [45]. We denote the broken spaces (that is, piecewise polynomial, i.e., without any continuity requirement across the mesh interfaces) as

$$\mathcal{P}_{p+1}(\mathcal{T}_\omega) := \{u_\mathcal{T} \in L^2(\omega) : u_\mathcal{T}|_\tau \in \mathcal{P}_{p+1}(\tau), \forall \tau \in \mathcal{T}_\omega\}, \tag{2.5a}$$

$$\boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) := \{\boldsymbol{u}_\mathcal{T} \in \boldsymbol{L}^2(\omega) : \boldsymbol{u}_\mathcal{T}|_\tau \in \boldsymbol{\mathcal{N}}_p(\tau), \forall \tau \in \mathcal{T}_\omega\}, \tag{2.5b}$$

$$\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) := \{\boldsymbol{u}_\mathcal{T} \in \boldsymbol{L}^2(\omega) : \boldsymbol{u}_\mathcal{T}|_\tau \in \boldsymbol{\mathcal{RT}}_p(\tau), \forall \tau \in \mathcal{T}_\omega\}. \tag{2.5c}$$

The usual subspaces with continuous trace, tangential trace, and normal trace are $\mathcal{P}_{p+1}(\mathcal{T}_\omega) \cap H(\mathbf{grad},\omega)$, $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega)$, and $\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega)$. We proceed similarly for the homogeneous-trace subspaces. Here and in what follows, the subscript $\mathcal{T}$ generically refers to functions and fields that sit in the above finite-dimensional spaces.

3. Equivalent statements for discrete Poincaré inequalities

In this section, we recall that the discrete Poincaré inequalities (1.4) are equivalent to: (i) stability of discrete constrained minimization problems; (ii) discrete inf-sup conditions; and (iii) bounds on operator norms of piecewise polynomial vector potential operators. All these equivalences are known from the literature, but possibly not that well known, and definitely seldom presented together. We find it instructive to briefly recall them, including proofs. Similar equivalences hold when homogeneous boundary conditions are imposed on the boundary $\partial\omega$ and are not detailed for brevity. These equivalences only consider finite-dimensional spaces and are rather easy to expose. A further equivalence with the existence of graph-stable commuting projections includes the infinite-dimensional spaces $\boldsymbol{H}(\mathbf{curl},\omega)$ and $\boldsymbol{H}(\mathrm{div},\omega)$ and requires a bit more setup; we postpone it to Lemma 6.6 below.

To proceed, it is convenient to define the subspaces

$$\boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega) := \mathbf{curl}(\boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega)) \subset \{\boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega) \text{ with } \mathrm{div}\, \boldsymbol{v}_\mathcal{T} = 0\}, \tag{3.1a}$$

$$\mathcal{D}_p(\mathcal{T}_\omega) := \mathrm{div}(\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega)) = \mathcal{P}_p(\mathcal{T}_\omega). \tag{3.1b}$$

When the boundary of ω is connected, we have $\boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega) = \{\boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega) \text{ with } \mathrm{div}\, \boldsymbol{v}_\mathcal{T} = 0\}$.

3.1. Equivalence with stability of discrete constrained minimization problems.

Let $\boldsymbol{r}_\mathcal{T} \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)$ and consider the *constrained quadratic minimization problem*

$$\boldsymbol{u}^*_\mathcal{T} = \underset{\substack{\boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega) \\ \mathbf{curl}\, \boldsymbol{v}_\mathcal{T} = \boldsymbol{r}_\mathcal{T}}}{\arg\min} \|\boldsymbol{v}_\mathcal{T}\|^2_{\boldsymbol{L}^2(\omega)}. \tag{3.2}$$

Since the minimization set is closed, convex, and nonempty by definition of $\boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)$ and the minimized functional is continuous and strongly convex, the above problem has a unique solution.

Lemma 3.1 (Equivalence of (1.4a) with stability of discrete constrained minimization in $\boldsymbol{H}(\mathbf{curl},\omega)$)**.** *The discrete Poincaré inequality* (1.4a) *is equivalent to the stability of* (3.2) *in the sense that*

$$\|\boldsymbol{u}^*_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)} \leq C_{\mathrm{P}}^{\mathrm{d},1} h_\omega \|\boldsymbol{r}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}. \tag{3.3}$$

Proof. The Euler optimality conditions for (3.2) allow for the following equivalent rewriting:

$$\begin{cases}\text{Find } \boldsymbol{u}_{\mathcal{T}}^* \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega) \text{ with } \mathbf{curl}\, \boldsymbol{u}_{\mathcal{T}}^* = \boldsymbol{r}_{\mathcal{T}} \text{ such that} \\ \left\langle \boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{v}_{\mathcal{T}} \right\rangle_\omega = 0 \quad \forall \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega) \text{ with } \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} = \mathbf{0}.\end{cases} \tag{3.4}$$

Thus, (1.4a) readily implies (3.3). Conversely, if (3.3) holds, given any $\boldsymbol{u}_{\mathcal{T}}$ satisfying the orthogonality constraints in (1.4a), one considers the constrained minimization problem (3.2) with data $\boldsymbol{r}_{\mathcal{T}} := \mathbf{curl}\, \boldsymbol{u}_{\mathcal{T}}$. Since $\boldsymbol{u}_{\mathcal{T}}^* = \boldsymbol{u}_{\mathcal{T}}$, this proves (1.4a). □

Similarly, let $r_{\mathcal{T}} \in \mathcal{D}_p(\mathcal{T}_\omega)$ and consider the constrained quadratic minimization problem

$$\boldsymbol{u}_{\mathcal{T}}^* = \underset{\substack{\boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega) \\ \mathrm{div}\, \boldsymbol{v}_{\mathcal{T}} = r_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\omega)}. \tag{3.5}$$

Lemma 3.2 (Equivalence of (1.4b) with stability of discrete constrained minimization in $\boldsymbol{H}(\mathrm{div}, \omega)$)**.** *The discrete Poincaré inequality* (1.4b) *is equivalent to the stability of* (3.5) *in the sense that*

$$\|\boldsymbol{u}_{\mathcal{T}}^*\|_{\boldsymbol{L}^2(\omega)} \le C_{\mathrm{P}}^{\mathrm{d},2} h_\omega \|r_{\mathcal{T}}\|_{L^2(\omega)}. \tag{3.6}$$

Proof. The equivalent rewriting of (3.5) using the Euler optimality conditions is

$$\begin{cases}\text{Find } \boldsymbol{u}_{\mathcal{T}}^* \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega) \text{ with } \mathrm{div}\, \boldsymbol{u}_{\mathcal{T}} = r_{\mathcal{T}} \text{ such that} \\ \left\langle \boldsymbol{u}_{\mathcal{T}}, \boldsymbol{v}_{\mathcal{T}} \right\rangle_\omega = 0 \quad \forall \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega) \text{ with } \mathrm{div}\, \boldsymbol{v}_{\mathcal{T}} = 0.\end{cases} \tag{3.7}$$

The rest of the proof proceeds as above. □

3.2. **Equivalence with discrete inf-sup conditions.** Let $\boldsymbol{r}_{\mathcal{T}} \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)$ and consider the following problem:

$$\begin{cases}\text{Find } \boldsymbol{u}_{\mathcal{T}}^* \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega) \text{ and } \boldsymbol{s}_{\mathcal{T}}^* \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega) \text{ such that} \\ \left\langle \boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{v}_{\mathcal{T}} \right\rangle_\omega - \left\langle \boldsymbol{s}_{\mathcal{T}}^*, \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} \right\rangle_\omega = 0 \qquad \forall \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega), \\ \left\langle \mathbf{curl}\, \boldsymbol{u}_{\mathcal{T}}^*, \boldsymbol{t}_{\mathcal{T}} \right\rangle_\omega = \left\langle \boldsymbol{r}_{\mathcal{T}}, \boldsymbol{t}_{\mathcal{T}} \right\rangle_\omega \qquad \forall \boldsymbol{t}_{\mathcal{T}} \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega),\end{cases} \tag{3.8}$$

which is called a mixed formulation of (3.4). As the curl operator is surjective from $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega)$ onto $\boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)$ by definition (see (3.1a)), the Euler conditions (3.4) are equivalent to the mixed formulation (3.8). We now recall that the stability property (3.3) is *equivalent* to the *discrete inf-sup condition*

$$\inf_{\boldsymbol{t}_{\mathcal{T}} \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)} \sup_{\boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega)} \frac{\left\langle \boldsymbol{t}_{\mathcal{T}}, \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} \right\rangle_\omega}{\|\boldsymbol{t}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)} \|\boldsymbol{v}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)}} \ge \frac{1}{C_{\mathrm{P}}^{\mathrm{d},1} h_\omega}, \tag{3.9}$$

leading via Lemma 3.1 to the following equivalence.

Lemma 3.3 (Equivalence of (1.4a) with the discrete inf-sup condition (3.9) in $\boldsymbol{H}(\mathbf{curl}, \omega)$)**.** *The discrete Poincaré inequality* (1.4a) *is equivalent to the discrete inf-sup condition* (3.9).

Proof. Since (1.4a) is equivalent to the stability property (3.3) as per Lemma 3.1, we prove the equivalence between (3.3) and (3.9).

(1) Assume the stability property (3.3). Let $\boldsymbol{t}_{\mathcal{T}} \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)$. Consider, as in (3.4), the following well-posed problem:

$$\begin{cases}\text{Find } \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega) \text{ with } \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} = \boldsymbol{t}_{\mathcal{T}} \text{ such that} \\ \left\langle \boldsymbol{v}_{\mathcal{T}}, \boldsymbol{w}_{\mathcal{T}} \right\rangle_\omega = 0 \quad \forall \boldsymbol{w}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega) \text{ with } \mathbf{curl}\, \boldsymbol{w}_{\mathcal{T}} = \mathbf{0}.\end{cases}$$

The stability property (3.3) gives $\|\boldsymbol{v}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)} \leq C_{\mathrm{P}}^{\mathrm{d},1} h_\omega \|\boldsymbol{t}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}$. Now, since $\mathbf{curl}\, \boldsymbol{v}_\mathcal{T} = \boldsymbol{t}_\mathcal{T}$, we infer from this bound that

$$\left\langle \boldsymbol{t}_\mathcal{T}, \mathbf{curl}\, \boldsymbol{v}_\mathcal{T} \right\rangle_\omega = \|\boldsymbol{t}_\mathcal{T}\|^2_{\boldsymbol{L}^2(\omega)} \geq \frac{\|\boldsymbol{v}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)} \|\boldsymbol{t}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}}{C_{\mathrm{P}}^{\mathrm{d},1} h_\omega},$$

which gives the discrete inf-sup condition (3.9).

(2) Conversely, we now suppose (3.9) and show that this implies (3.3). Let $\boldsymbol{r}_\mathcal{T} \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)$ and let $\boldsymbol{u}^*_\mathcal{T}$ solve (3.2). Since (3.2) is equivalent to (3.4) which is in turn equivalent to (3.8), we can consider $\boldsymbol{s}^*_\mathcal{T} \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)$ so that the pair $(\boldsymbol{u}^*_\mathcal{T}, \boldsymbol{s}^*_\mathcal{T})$ solves (3.8). Using in (3.8) the test functions $\boldsymbol{v}_\mathcal{T} = \boldsymbol{u}^*_\mathcal{T}$ and $\boldsymbol{t}_\mathcal{T} = \boldsymbol{s}^*_\mathcal{T}$ and summing the two equations, we infer that

$$\|\boldsymbol{u}^*_\mathcal{T}\|^2_{\boldsymbol{L}^2(\omega)} = \left\langle \boldsymbol{r}_\mathcal{T}, \boldsymbol{s}^*_\mathcal{T} \right\rangle_\omega \leq \|\boldsymbol{r}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)} \|\boldsymbol{s}^*_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)},$$

where we used the Cauchy–Schwarz inequality. Now, the discrete inf-sup condition (3.9) gives the existence of $\boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega)$ such that

$$\|\boldsymbol{s}^*_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)} \leq C_{\mathrm{P}}^{\mathrm{d},1} h_\omega \frac{\left\langle \boldsymbol{s}^*_\mathcal{T}, \mathbf{curl}\, \boldsymbol{v}_\mathcal{T} \right\rangle_\omega}{\|\boldsymbol{v}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}}.$$

From the first equation in (3.8) and the Cauchy–Schwarz inequality, we obtain

$$\frac{\left\langle \boldsymbol{s}^*_\mathcal{T}, \mathbf{curl}\, \boldsymbol{v}_\mathcal{T} \right\rangle_\omega}{\|\boldsymbol{v}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}} = \frac{\left\langle \boldsymbol{u}^*_\mathcal{T}, \boldsymbol{v}_\mathcal{T} \right\rangle_\omega}{\|\boldsymbol{v}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}} \leq \|\boldsymbol{u}^*_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}.$$

Combining the three above inequalities, (3.3) follows. □

Similarly, let $r_\mathcal{T} \in \mathcal{D}_p(\mathcal{T}_\omega) = \mathcal{P}_p(\mathcal{T}_\omega)$ and consider the following mixed formulation of (3.7):

$$\begin{cases} \text{Find } \boldsymbol{u}^*_\mathcal{T} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega) \text{ and } s^*_\mathcal{T} \in \mathcal{P}_p(\mathcal{T}_\omega) \text{ such that} \\ \left\langle \boldsymbol{u}^*_\mathcal{T}, \boldsymbol{v}_\mathcal{T} \right\rangle_\omega - \left\langle s^*_\mathcal{T}, \mathrm{div}\, \boldsymbol{v}_\mathcal{T} \right\rangle_\omega = 0 \qquad \forall \boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega), \\ \left\langle \mathrm{div}\, \boldsymbol{u}^*_\mathcal{T}, t_\mathcal{T} \right\rangle_\omega = \left\langle r_\mathcal{T}, t_\mathcal{T} \right\rangle_\omega \qquad \forall t_\mathcal{T} \in \mathcal{P}_p(\mathcal{T}_\omega). \end{cases} \tag{3.10}$$

The associated discrete inf-sup condition reads as follows:

$$\inf_{t_\mathcal{T} \in \mathcal{P}_p(\mathcal{T}_\omega)} \sup_{\boldsymbol{v}_\mathcal{T} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega)} \frac{\left\langle t_\mathcal{T}, \mathrm{div}\, \boldsymbol{v}_\mathcal{T} \right\rangle_\omega}{\|t_\mathcal{T}\|_{L^2(\omega)} \|\boldsymbol{v}_\mathcal{T}\|_{\boldsymbol{L}^2(\omega)}} \geq \frac{1}{C_{\mathrm{P}}^{\mathrm{d},2} h_\omega}. \tag{3.11}$$

Lemma 3.4 (Equivalence of (1.4b) with the discrete inf-sup condition (3.11) in $\boldsymbol{H}(\mathrm{div}, \omega)$)**.** *The discrete Poincaré inequality* (1.4b) *is equivalent to the discrete inf-sup condition* (3.11)*.*

Proof. Proceed as in the proof of Lemma 3.3. □

Remark 3.5 (Norms)**.** We stress that we do not use here the norms for which the spaces are Hilbert spaces, but merely $L^2(\omega)$- or $\boldsymbol{L}^2(\omega)$-norms, in contrast to the usual practice, see, e.g., [7, Theorem 4.2.3] or [26, Theorem 49.13], but similarly to, e.g., [46, Theorem 5.9] or [26, Remark 51.12].

3.3. **Equivalence with bounds on operator norms of piecewise polynomial vector potential operators.** Let $\boldsymbol{r}_{\mathcal{T}} \in \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega)$. Then a piecewise polynomial vector potential is any field $\Phi_{\mathcal{T}}^{\mathrm{curl}}(\boldsymbol{r}_{\mathcal{T}}) \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega)$ such that $\mathbf{curl}\, \Phi_{\mathcal{T}}^{\mathrm{curl}}(\boldsymbol{r}_{\mathcal{T}}) = \boldsymbol{r}_{\mathcal{T}}$, and we say that

$$\Phi_{\mathcal{T}}^{\mathrm{curl}} : \boldsymbol{\mathcal{C}}_p(\mathcal{T}_\omega) \to \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega) \tag{3.12}$$

is a *piecewise polynomial vector potential operator* (piecewise polynomial right-inverse of the curl operator). We are particularly interested in the $\boldsymbol{L}^2(\omega)$-norm minimizing operator

$$\Phi_{\mathcal{T}}^{\mathrm{curl},*}(\boldsymbol{r}_{\mathcal{T}}) := \underset{\substack{\boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega) \\ \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} = \boldsymbol{r}_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\omega)}, \tag{3.13}$$

with operator norm

$$\left\|\left|\Phi_{\mathcal{T}}^{\mathrm{curl},*}\right|\right\| := \max_{\boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl}, \omega)} \frac{\|\Phi_{\mathcal{T}}^{\mathrm{curl},*}(\mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}})\|_{\boldsymbol{L}^2(\omega)}}{\|\mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)}}. \tag{3.14}$$

Lemma 3.6 (Equivalence of the best constant in (1.4a) with the operator norm of the minimal discrete vector potential operator in $\boldsymbol{H}(\mathbf{curl}, \omega)$)**.** *The operator norm* $\left\|\left|\Phi_{\mathcal{T}}^{\mathrm{curl},*}\right|\right\|$ *from* (3.14) equals *the best discrete Poincaré inequality constant* $C_{\mathrm{P}}^{\mathrm{d},1} h_\omega$ *from* (1.4a).

Proof. Observe that (3.13) matches exactly the form of the constrained minimization (3.2). □

Again, the situation in the $\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega)$ setting is identical. Let $r_{\mathcal{T}} \in \mathcal{D}_p(\mathcal{T}_\omega) = \mathcal{P}_p(\mathcal{T}_\omega)$. Then a piecewise polynomial vector potential is any field $\Phi_{\mathcal{T}}^{\mathrm{div}}(r_{\mathcal{T}}) \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega)$ such that $\mathrm{div}\, \Phi_{\mathcal{T}}^{\mathrm{div}}(r_{\mathcal{T}}) = r_{\mathcal{T}}$, and we say that

$$\Phi_{\mathcal{T}}^{\mathrm{div}} : \mathcal{P}_p(\mathcal{T}_\omega) \to \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega) \tag{3.15}$$

is a piecewise polynomial vector potential operator (piecewise polynomial right inverse of the divergence operator). We are particularly interested in the $\boldsymbol{L}^2(\omega)$-norm minimizing operator

$$\Phi_{\mathcal{T}}^{\mathrm{div},*}(r_{\mathcal{T}}) := \underset{\substack{\boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega) \\ \mathrm{div}\, \boldsymbol{v}_{\mathcal{T}} = r_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\omega)}, \tag{3.16}$$

with operator norm

$$\left\|\left|\Phi_{\mathcal{T}}^{\mathrm{div},*}\right|\right\| := \max_{\boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div}, \omega)} \frac{\|\Phi_{\mathcal{T}}^{\mathrm{div},*}(\mathrm{div}\, \boldsymbol{v}_{\mathcal{T}})\|_{\boldsymbol{L}^2(\omega)}}{\|\mathrm{div}\, \boldsymbol{v}_{\mathcal{T}}\|_{L^2(\omega)}}. \tag{3.17}$$

Lemma 3.7 (Equivalence of the best constant in (1.4b) with the operator norm of the minimal discrete vector potential operator in $\boldsymbol{H}(\mathrm{div}, \omega)$)**.** *The operator norm* $\left\|\left|\Phi_{\mathcal{T}}^{\mathrm{div},*}\right|\right\|$ *from* (3.17) equals *the best discrete Poincaré inequality constant* $C_{\mathrm{P}}^{\mathrm{d},2} h_\omega$ *from* (1.4b).

Proof. Identical to the above proof. □

4. Continuous Poincaré inequalities

In this section, we state the continuous Poincaré inequalities and give some pointers to the literature for bounds on the continuous Poincaré constants. This will pave the way to our main topic, the discrete Poincaré inequalities.

Proposition 4.1 (Continuous Poincaré inequalities)**.** (i) *Continuous Poincaré inequalities without boundary conditions: There exist constants* $C_{\rm P}^l$, $l \in \{0{:}2\}$, *only depending on the shape of* ω, *such that*

$$\|u\|_{L^2(\omega)} \le C_{\rm P}^0 h_\omega \|\mathbf{grad}\, u\|_{\boldsymbol{L}^2(\omega)}, \qquad \forall u \in H(\mathbf{grad},\omega) \text{ such that } \langle u, 1\rangle_\omega = 0, \tag{4.1a}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)} \le C_{\rm P}^1 h_\omega \|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)}, \qquad \forall \boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},\omega) \text{ such that } \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\omega) \text{ with } \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{0}, \tag{4.1b}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)} \le C_{\rm P}^2 h_\omega \|\operatorname{div} \boldsymbol{u}\|_{L^2(\omega)}, \qquad \forall \boldsymbol{u} \in \boldsymbol{H}(\operatorname{div},\omega) \text{ such that } \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div},\omega) \text{ with } \operatorname{div} \boldsymbol{v} = 0. \tag{4.1c}$$

(ii) *Continuous Poincaré inequalities with boundary conditions: There exist constants* $\mathring{C}_{\rm P}^l$, $l \in \{0{:}2\}$, *only depending on the shape of* ω, *such that*

$$\|u\|_{L^2(\omega)} \le \mathring{C}_{\rm P}^0 h_\omega \|\mathbf{grad}\, u\|_{\boldsymbol{L}^2(\omega)}, \qquad \forall u \in \mathring{H}(\mathbf{grad},\omega), \tag{4.2a}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)} \le \mathring{C}_{\rm P}^1 h_\omega \|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)}, \qquad \forall \boldsymbol{u} \in \mathring{\boldsymbol{H}}(\mathbf{curl},\omega) \text{ such that } \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0 \quad \forall \boldsymbol{v} \in \mathring{\boldsymbol{H}}(\mathbf{curl},\omega) \text{ with } \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{0}, \tag{4.2b}$$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\omega)} \le \mathring{C}_{\rm P}^2 h_\omega \|\operatorname{div} \boldsymbol{u}\|_{L^2(\omega)}, \qquad \forall \boldsymbol{u} \in \mathring{\boldsymbol{H}}(\operatorname{div},\omega) \text{ such that } \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0 \quad \forall \boldsymbol{v} \in \mathring{\boldsymbol{H}}(\operatorname{div},\omega) \text{ with } \operatorname{div} \boldsymbol{v} = 0. \tag{4.2c}$$

Remark 4.2 (Proofs without explicit bounds on constants)**.** (i) One known route from the literature to establish the inequalities (4.1)–(4.2) is to invoke a *compactness* argument, which can be formalized in the following Peetre–Tartar lemma [25, Lemma A.20]: Let X, Y, Z be three Banach spaces, let $A \in \mathcal{L}(X;Y)$ be an injective operator, and let $T \in \mathcal{L}(X;Z)$ be a compact operator. Assume that there is $\gamma > 0$ such that $\gamma\|u\|_X \le \|A(u)\|_Y + \|T(u)\|_Z$ for all $u \in X$. Then there is $\alpha > 0$ such that

$$\alpha\|u\|_X \le \|A(u)\|_Y, \qquad \forall u \in X. \tag{4.3}$$

To briefly illustrate the application of the Peetre–Tartar lemma to prove the continuous Poincaré inequalities, let us prove (4.1a). We set $X := \{u \in H(\mathbf{grad},\omega) \mid \langle u, 1\rangle_\omega = 0\}$, $\boldsymbol{Y} := \boldsymbol{L}^2(\omega)$, $Z := L^2(\omega)$, $\boldsymbol{A}(u) := h_\omega \mathbf{grad}\, u$, and $T(u) := u$. The operator $\boldsymbol{A}$ is injective since any $u \in X$ such that $\boldsymbol{A}(u) = \boldsymbol{0}$ is L^2-orthogonal to itself and thus vanishes identically. Moreover, T is compact since the embedding $H^1(\omega) \hookrightarrow L^2(\omega)$ is compact. Finally, we have $\|u\|_X^2 = \|u\|^2_{H(\mathbf{grad},\omega)} = \|u\|^2_{L^2(\omega)} + h_\omega^2 \|\mathbf{grad}\, u\|^2_{\boldsymbol{L}^2(\omega)} = \|T(u)\|_Z^2 + \|\boldsymbol{A}(u)\|_{\boldsymbol{Y}}^2$. Thus, by the Peetre–Tartar Lemma, (4.1a) holds true. The proof for the other Poincaré inequalities is similar. In particular, for the curl and divergence operators, one invokes the compactness of the embeddings $\boldsymbol{H}(\mathbf{curl},\omega) \cap \mathring{\boldsymbol{H}}(\operatorname{div},\omega) \hookrightarrow \boldsymbol{L}^2(\omega)$ and $\mathring{\boldsymbol{H}}(\mathbf{curl},\omega) \cap \boldsymbol{H}(\operatorname{div},\omega) \hookrightarrow \boldsymbol{L}^2(\omega)$, see [15, Theorem 2], [6, Theorem 3.1], [1, Proposition 3.7] and [48]. (ii) Another, somewhat related, route to prove the Poincaré inequalities hinges on Helmholtz decompositions which show that the following operators are isomorphisms (see, e.g., [27, Lemma 2.8 & Remark 2.11] and the references therein):

$$\mathbf{grad} : \{u \in H(\mathbf{grad},\omega) \mid \langle u, 1\rangle_\omega = 0\} \longrightarrow \{\boldsymbol{w} \in \boldsymbol{L}^2(\omega) \mid \langle \boldsymbol{w}, \boldsymbol{v}\rangle_\omega = 0, \forall \boldsymbol{v} \in \mathring{\boldsymbol{H}}(\operatorname{div},\omega) \text{ s.t. } \operatorname{div} \boldsymbol{v} = 0\}, \tag{4.4a}$$

$$\mathbf{curl} : \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},\omega) \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0, \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl},\omega) \text{ s.t. } \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{0}\} \longrightarrow \{\boldsymbol{w} \in \boldsymbol{L}^2(\omega) \mid \langle \boldsymbol{w}, \boldsymbol{v}\rangle_\omega = 0, \forall \boldsymbol{v} \in \mathring{\boldsymbol{H}}(\mathbf{curl},\omega) \text{ s.t. } \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{0}\}, \tag{4.4b}$$

$$\text{(4.4c)}\qquad \operatorname{div} : \{\boldsymbol{u} \in \boldsymbol{H}(\operatorname{div},\omega) \mid \langle \boldsymbol{u},\boldsymbol{v}\rangle_\omega = 0, \forall \boldsymbol{v} \in \boldsymbol{H}(\operatorname{div},\omega)\,\text{s.t.}\operatorname{div}\boldsymbol{v} = 0\} \longrightarrow \{w \in L^2(\omega)\},$$

with similar isomorphisms in the case of prescribed boundary conditions. Then, the range of all these operators is closed, and Banach's Closed Range theorem (see, e.g., [26, Lemma C.39]) implies the Poincaré inequalities (4.1)–(4.2). (iii) If ω is star-shaped with respect to a ball, upper bounds on the continuous Poincaré constants $C_{\rm P}^l$, $\mathring{C}_{\rm P}^l$, $l \in \{0{:}2\}$ can be derived from estimates on suitable right inverses (Bogovskii/Poincaré integral operators) of the adjoint differential operator (see, e.g.,[23]). A generalization of the results in [23] to other differential operators can be found in [40].

Remark 4.3 (Proofs with explicit bounds on constants)**.** (i) Inequalities (4.1a) and (4.2a) are the well-known Poincaré inequalities. They can be shown constructively, as, e.g., in [44, 5] or [25, Exercise 22.3], from where it follows that $C_{\rm P}^0 = 1/\pi$ if ω is convex and $\mathring{C}_{\rm P}^0 \leq 1$. For general nonconvex domains with a finite convex cover, upper bounds on $C_{\rm P}^0$ can be found in, e.g. [30, Lemma 3.7]. (ii) Computable upper bounds on the continuous Poincaré constants $C_{\rm P}^l$, $\mathring{C}_{\rm P}^l$, $l \in \{1{:}2\}$ can be derived by considering a shape-regular mesh $\mathcal{T}_\omega$ of ω and determining these bounds in terms of the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$ and the number of elements $|\mathcal{T}_\omega|$. This approach is detailed in [11], see also the references therein.

Remark 4.4 (Comparison of the continuous Poincaré constants)**.** One has $C_{\rm P}^2 = \mathring{C}_{\rm P}^0$, $\mathring{C}_{\rm P}^2 = C_{\rm P}^0$, and $C_{\rm P}^1 = \mathring{C}_{\rm P}^1$. We refer the reader to [43] and the references therein for further insight into the relations between, and values of, the constants in (4.1) and (4.2), including the case where boundary conditions are enforced only on part of the boundary of ω.

5. Discrete Poincaré inequalities

In this section, we present our main result on the discrete Poincaré inequalities. We focus on the dependency of the discrete Poincaré constants on the continuous-level constants $C_{\rm P}^l$, $\mathring{C}_{\rm P}^l$, $l \in \{0{:}2\}$, and the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$, the number of elements $|\mathcal{T}_\omega|$, and the polynomial degree p. We shall consider in Section 6 three routes to prove these inequalities, each one leading to different dependencies of the constants.

5.1. Shellable triangulations and finite element stars.

Some specific tetrahedral meshes $\mathcal{T}_\omega$ will be of particular interest. We either look at $\mathcal{T}_\omega$ as a triangulation of some computational domain ω, or we consider $\mathcal{T}_\omega$ as some local (vertex, edge, face) star of a shape-regular simplicial mesh $\mathcal{T}_h$ of some larger three-dimensional computational domain Ω (open, bounded, connected, Lipschitz polyhedral set).

One relevant case is when the pure 3-dimensional simplicial complex represented by $\mathcal{T}_\omega$ is shellable in the sense of [49, Section 8.1]. This means that there exists an enumeration $K_1, \ldots, K_{|\mathcal{T}_\omega|}$ of the tetrahedra from $\mathcal{T}_\omega$ such that, for all $1 < i \leq |\mathcal{T}_\omega|$, the intersection of the tetrahedron K_i with the union of the previously enumerated tetrahedra $\cup_{j=1}^{i-1} K_j$ is a non-empty collection of faces of K_i (in particular does not contain an isolated vertex or edge of K_i). A specific example of a shellable triangulation is the one where $\mathcal{T}_\omega$ is a so-called *vertex (edge, face) star*, where all the tetrahedra in $\mathcal{T}_\omega$ share a given vertex (edge, face), see [29, Lemma B.1]. This example is of particular interest in what follows. We refer the reader to [50, 37, 47, 11] for further insight into shellability and enumeration procedures for general simplicial complexes.

Let K be a tetrahedron from $\mathcal{T}_\omega$. We will call a "twice-extended element star" a collection of such tetrahedra K' from $\mathcal{T}_\omega$ which either share a vertex with K, $K \cap K' \neq \emptyset$, or such that there exists a tetrahedron K'' from $\mathcal{T}_\omega$ such that K' shares a vertex with K'' and K'' shares a vertex with K. We will also consider triangulations $\mathcal{T}_\omega$ where the twice-extended element stars are shellable.

5.2. **Main result.** Our main result is as follows.

Theorem 5.1 (Discrete Poincaré inequalities)**.** (i) *Discrete Poincaré inequalities without boundary conditions: There exist constants $C_{\mathrm{P}}^{\mathrm{d},l}$, $l \in \{0{:}2\}$, such that*

$$\|u_{\mathcal{T}}\|_{L^2(\omega)} \le C_{\mathrm{P}}^{\mathrm{d},0} h_\omega \|\mathbf{grad}\, u_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)}, \quad \forall u_{\mathcal{T}} \in \mathcal{P}_{p+1}(\mathcal{T}_\omega) \cap H(\mathbf{grad},\omega) \text{ such that } \langle u_{\mathcal{T}}, 1\rangle_\omega = 0, \tag{5.1a}$$

$$\|\boldsymbol{u}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)} \le C_{\mathrm{P}}^{\mathrm{d},1} h_\omega \|\mathbf{curl}\, \boldsymbol{u}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)}, \quad \forall \boldsymbol{u}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega) \text{ such that } \langle \boldsymbol{u}_{\mathcal{T}}, \boldsymbol{v}_{\mathcal{T}}\rangle_\omega = 0 \quad \forall \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega) \text{ with } \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} = \mathbf{0}, \tag{5.1b}$$

$$\|\boldsymbol{u}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)} \le C_{\mathrm{P}}^{\mathrm{d},2} h_\omega \|\mathrm{div}\, \boldsymbol{u}_{\mathcal{T}}\|_{L^2(\omega)}, \quad \forall \boldsymbol{u}_{\mathcal{T}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega) \text{ such that } \langle \boldsymbol{u}_{\mathcal{T}}, \boldsymbol{v}_{\mathcal{T}}\rangle_\omega = 0 \quad \forall \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega) \text{ with } \mathrm{div}\, \boldsymbol{v}_{\mathcal{T}} = 0. \tag{5.1c}$$

(ii) *Discrete Poincaré inequalities with boundary conditions: There exist constants $\mathring{C}_{\mathrm{P}}^{\mathrm{d},l}$, $l \in \{0{:}2\}$, such that*

$$\|u_{\mathcal{T}}\|_{L^2(\omega)} \le \mathring{C}_{\mathrm{P}}^{\mathrm{d},0} h_\omega \|\mathbf{grad}\, u_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)}, \quad \forall u_{\mathcal{T}} \in \mathcal{P}_{p+1}(\mathcal{T}_\omega) \cap \mathring{H}(\mathbf{grad},\omega), \tag{5.2a}$$

$$\|\boldsymbol{u}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)} \le \mathring{C}_{\mathrm{P}}^{\mathrm{d},1} h_\omega \|\mathbf{curl}\, \boldsymbol{u}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)}, \quad \forall \boldsymbol{u}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \mathring{\boldsymbol{H}}(\mathbf{curl},\omega) \text{ such that } \langle \boldsymbol{u}_{\mathcal{T}}, \boldsymbol{v}_{\mathcal{T}}\rangle_\omega = 0 \quad \forall \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \mathring{\boldsymbol{H}}(\mathbf{curl},\omega) \text{ with } \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} = \mathbf{0}, \tag{5.2b}$$

$$\|\boldsymbol{u}_{\mathcal{T}}\|_{\boldsymbol{L}^2(\omega)} \le \mathring{C}_{\mathrm{P}}^{\mathrm{d},2} h_\omega \|\mathrm{div}\, \boldsymbol{u}_{\mathcal{T}}\|_{L^2(\omega)}, \quad \forall \boldsymbol{u}_{\mathcal{T}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \mathring{\boldsymbol{H}}(\mathrm{div},\omega) \text{ such that } \langle \boldsymbol{u}_{\mathcal{T}}, \boldsymbol{v}_{\mathcal{T}}\rangle_\omega = 0 \quad \forall \boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \mathring{\boldsymbol{H}}(\mathrm{div},\omega) \text{ with } \mathrm{div}\, \boldsymbol{v}_{\mathcal{T}} = 0. \tag{5.2c}$$

Here, the constants $C_{\mathrm{P}}^{\mathrm{d},l}$, $\mathring{C}_{\mathrm{P}}^{\mathrm{d},l}$ have the following properties:

(1) *$C_{\mathrm{P}}^{\mathrm{d},0} \le C_{\mathrm{P}}^{0}$ and $\mathring{C}_{\mathrm{P}}^{\mathrm{d},0} \le \mathring{C}_{\mathrm{P}}^{0}$. Thus, $C_{\mathrm{P}}^{\mathrm{d},0} \le 1/\pi$ if ω is convex, and $\mathring{C}_{\mathrm{P}}^{\mathrm{d},0} \le 1$ for any ω, see the discussion in Remark 4.3.*

(2) *If the triangulation $\mathcal{T}_\omega$ is shellable, then there exist constants $C_{\min}^{l}$, $l \in \{1{:}2\}$, only depending on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$ and the number of elements $|\mathcal{T}_\omega|$, such that $C_{\mathrm{P}}^{\mathrm{d},l} \le C_{\min}^{l} C_{\mathrm{P}}^{l}$. If $\mathcal{T}_\omega$ is a vertex or edge star, then there exist constants $C_{\min}^{l}$, $l \in \{1{:}2\}$, only depending on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$, such that $C_{\mathrm{P}}^{\mathrm{d},l} \le C_{\min}^{l} C_{\mathrm{P}}^{l}$ and $\mathring{C}_{\mathrm{P}}^{\mathrm{d},l} \le C_{\min}^{l} \mathring{C}_{\mathrm{P}}^{l}$.*

(3) *There exist constants C_{st}^{l}, $l \in \{1{:}2\}$, only depending on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$ and the polynomial degree p, such that $C_{\mathrm{P}}^{\mathrm{d},l} \le C_{\mathrm{st}}^{l} C_{\mathrm{P}}^{l}$ and $\mathring{C}_{\mathrm{P}}^{\mathrm{d},l} \le C_{\mathrm{st}}^{l} \mathring{C}_{\mathrm{P}}^{l}$. Moreover, if all twice-extended element stars in $\mathcal{T}_\omega$ are shellable, then C_{st}^{2} only depends on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$.*

(4) *The constants $C_{\mathrm{P}}^{\mathrm{d},l}$, $\mathring{C}_{\mathrm{P}}^{\mathrm{d},l}$, $l \in \{1{:}2\}$, admit upper bounds that only depend on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$, the number of elements $|\mathcal{T}_\omega|$, and the polynomial degree p, but that do not need to invoke the constants C_{P}^{l}, $\mathring{C}_{\mathrm{P}}^{l}$.*

5.3. **Discussion.** Let us discuss items (2) to (4) of Theorem 5.1:

- **Discussion of** (2)**.** This result is established in Section 6.2 below and relies on piecewise polynomial extension operators. The constants here are systematically p-robust, but can unfavorably depend on the number of elements in $\mathcal{T}_\omega$. In stars or extended stars or any local patches, $|\mathcal{T}_\omega|$ is bounded as a function of the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$, leading to discrete Poincaré constants only depending on $\rho_{\mathcal{T}_\omega}$ and the continuous Poincaré constants

C_{P}^{l} or $\mathring{C}_{\mathrm{P}}^{l}$, $l \in \{1{:}2\}$ (for which upper bounds only depending on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$ can be derived as discussed in Remark 4.3). Since vertex and edge stars are shellable (see [29, Lemma B.1]), the assumption that $\mathcal{T}_\omega$ is shellable is automatically granted in this case.

- **Discussion of** (3)**.** This result is proven in Section 6.3 below upon relying on stable commuting projections. This is probably the most common way of proving the discrete Poincaré inequalities. In this case, the constants $C_{\mathrm{P}}^{\mathrm{d},l}$, $\mathring{C}_{\mathrm{P}}^{\mathrm{d},l}$, $l \in \{1{:}2\}$, are independent of the number of elements in $\mathcal{T}_\omega$ (i.e., this number can be arbitrarily high), but may (unfavorably) depend on the polynomial degree p. In the $\boldsymbol{H}(\mathrm{div},\omega)$ setting ($l=2$), the p-robust projector from [22, Definition 3.5] gives a constant $C_{\mathrm{P}}^{\mathrm{d},2}$ independent of both the number of elements in $\mathcal{T}_\omega$ and the polynomial degree p if all twice-extended element stars in $\mathcal{T}_\omega$ are shellable, which is, to our knowledge, the best result available so far. Once again, upper bounds on the continuous Poincaré constants only depending on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$ can be derived as discussed in Remark 4.3.
- **Discussion of** (4)**.** This result is established in Section 6.4 below. The technique of proof does not rely on the continuous Poincaré inequalities. Therefore, the upper bounds on the discrete Poincaré constants do not involve the constants C_{P}^{l}, $\mathring{C}_{\mathrm{P}}^{l}$, $l \in \{1{:}2\}$. There are no requirements on the triangulation $\mathcal{T}_\omega$ either (shellable or not, local star or not). The direct proof argument leads to discrete Poincaré constants depending on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$, the number of elements $|\mathcal{T}_\omega|$, and the polynomial degree p.

6. Proofs of discrete Poincaré inequalities

In this section, we describe the three routes mentioned above to prove the discrete Poincaré inequalities (5.1b)–(5.1c) and (5.2b)–(5.2c), leading to Theorem 5.1. Recall that these three routes respectively consist in:

- Invoking equivalence between discrete and continuous minimizers (piecewise polynomial extension operators);
- Invoking stable commuting projections with stability in L^2 (we also comment on stability in graph spaces and fractional-order Sobolev spaces);
- Invoking piecewise Piola transformations.

We observe that the first two routes hinge on the continuous Poincaré inequalities (4.1b)–(4.1c) and (4.2b)–(4.2c), whereas the third route employs only a finite-dimensional argument. The different routes give the different dependencies of the discrete Poincaré constant on the parameters $\rho_{\mathcal{T}_\omega}$, $|\mathcal{T}_\omega|$, and p, as summarized in items 2–4 of Theorem 5.1. For routes 1 and 2, we give pointers to the literature providing tools to realize the proofs, whereas we present a stand-alone proof for route 3.

6.1. **Unified presentation.** To avoid the proliferation of cases, we henceforth write the continuous and discrete Poincaré inequalities in a unified setting.

We consider the well-known de Rham sequences

$$\mathbb{R} \xrightarrow{\subset} V^0(\omega) \xrightarrow{\mathbf{grad}} \boldsymbol{V}^1(\omega) \xrightarrow{\mathbf{curl}} \boldsymbol{V}^2(\omega) \xrightarrow{\mathrm{div}} V^3(\omega) \longrightarrow 0, \tag{6.1a}$$

$$0 \xrightarrow{\subset} \mathring{V}^0(\omega) \xrightarrow{\mathbf{grad}} \mathring{\boldsymbol{V}}^1(\omega) \xrightarrow{\mathbf{curl}} \mathring{\boldsymbol{V}}^2(\omega) \xrightarrow{\mathrm{div}} \mathring{V}^3(\omega) \xrightarrow{\int_\omega} 0, \tag{6.1b}$$

where the relevant graph spaces are

(6.2a) $$V^0(\omega) := H(\mathbf{grad},\omega), \qquad \mathring{V}^0(\omega) := \mathring{H}(\mathbf{grad},\omega),$$

(6.2b) $$\boldsymbol{V}^1(\omega) := \boldsymbol{H}(\mathbf{curl},\omega), \qquad \mathring{\boldsymbol{V}}^1(\omega) := \mathring{\boldsymbol{H}}(\mathbf{curl},\omega),$$

(6.2c) $$\boldsymbol{V}^2(\omega) := \boldsymbol{H}(\mathrm{div},\omega), \qquad \mathring{\boldsymbol{V}}^2(\omega) := \mathring{\boldsymbol{H}}(\mathrm{div},\omega),$$

(6.2d) $$V^3(\omega) := L^2(\omega), \qquad \mathring{V}^3(\omega) := \mathring{L}^2(\omega) := \{u \in L^2(\omega) \ : \ \langle u, 1\rangle_\omega = 0\}.$$

We define the kernels of the differential operators

(6.3a) $$\mathfrak{Z}V^0(\omega) := \{u \in V^0(\omega) \ : \ \mathbf{grad}\, u = \mathbf{0}\},$$

(6.3b) $$\mathfrak{Z}\boldsymbol{V}^1(\omega) := \{u \in \boldsymbol{V}^1(\omega) \ : \ \mathbf{curl}\, \boldsymbol{u} = \mathbf{0}\},$$

(6.3c) $$\mathfrak{Z}\boldsymbol{V}^2(\omega) := \{u \in \boldsymbol{V}^2(\omega) \ : \ \mathrm{div}\, \boldsymbol{u} = 0\},$$

(notice that $\mathfrak{Z}V^0(\omega) = \{u \in V^0(\omega) \ : \ u = \text{constant}\}$), as well as their L^2-orthogonal complements

(6.4a) $$\mathfrak{Z}^\perp V^0(\omega) := \{u \in V^0(\omega) \ : \ \langle u, v\rangle_\omega = 0 \quad \forall v \in \mathfrak{Z}V^0(\omega)\},$$

(6.4b) $$\mathfrak{Z}^\perp \boldsymbol{V}^1(\omega) := \{\boldsymbol{u} \in \boldsymbol{V}^1(\omega) \ : \ \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0 \quad \forall \boldsymbol{v} \in \mathfrak{Z}\boldsymbol{V}^1(\omega)\},$$

(6.4c) $$\mathfrak{Z}^\perp \boldsymbol{V}^2(\omega) := \{\boldsymbol{u} \in \boldsymbol{V}^2(\omega) \ : \ \langle \boldsymbol{u}, \boldsymbol{v}\rangle_\omega = 0 \quad \forall \boldsymbol{v} \in \mathfrak{Z}\boldsymbol{V}^2(\omega)\},$$

(notice that $\mathfrak{Z}^\perp V^0(\omega) = \{u \in V^0(\omega) \ : \ \langle u, 1\rangle_\omega = 0\}$). We define the spaces $\mathfrak{Z}\mathring{V}^0(\omega)$, $\mathfrak{Z}\mathring{\boldsymbol{V}}^1(\omega)$, and $\mathfrak{Z}\mathring{\boldsymbol{V}}^2(\omega)$ as in (6.3) (notice that $\mathfrak{Z}\mathring{V}^0(\omega) = \{0\}$), and their L^2-orthogonal complements as in (6.4) (notice that $\mathfrak{Z}^\perp\mathring{V}^0(\omega) = \mathring{V}^0(\omega)$).

At the discrete level, we henceforth use the following notation:

(6.5a) $$V_p^0(\mathcal{T}_\omega) := \mathcal{P}_{p+1}(\mathcal{T}_\omega) \cap H(\mathbf{grad},\omega), \qquad \mathring{V}_p^0(\mathcal{T}_\omega) := \mathcal{P}_{p+1}(\mathcal{T}_\omega) \cap \mathring{H}(\mathbf{grad},\omega),$$

(6.5b) $$\boldsymbol{V}_p^1(\mathcal{T}_\omega) := \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathbf{curl},\omega), \qquad \mathring{\boldsymbol{V}}_p^1(\mathcal{T}_\omega) := \boldsymbol{\mathcal{N}}_p(\mathcal{T}_\omega) \cap \mathring{\boldsymbol{H}}(\mathbf{curl},\omega),$$

(6.5c) $$\boldsymbol{V}_p^2(\mathcal{T}_\omega) := \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \boldsymbol{H}(\mathrm{div},\omega), \qquad \mathring{\boldsymbol{V}}_p^2(\mathcal{T}_\omega) := \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_\omega) \cap \mathring{\boldsymbol{H}}(\mathrm{div},\omega),$$

(6.5d) $$V_p^3(\mathcal{T}_\omega) := \mathcal{P}_p(\mathcal{T}_\omega), \qquad \mathring{V}_p^3(\mathcal{T}_\omega) := \mathcal{P}_p(\mathcal{T}_\omega) \cap \mathring{L}^2(\omega).$$

As in (6.1), the discrete spaces are related by the following two discrete de Rham sequences:

(6.6a) $$\mathbb{R} \xrightarrow{\subset} V_p^0(\mathcal{T}_\omega) \xrightarrow{\mathbf{grad}} \boldsymbol{V}_p^1(\mathcal{T}_\omega) \xrightarrow{\mathbf{curl}} \boldsymbol{V}_p^2(\mathcal{T}_\omega) \xrightarrow{\mathrm{div}} V_p^3(\mathcal{T}_\omega) \longrightarrow 0,$$

(6.6b) $$0 \xrightarrow{\subset} \mathring{V}_p^0(\mathcal{T}_\omega) \xrightarrow{\mathbf{grad}} \mathring{\boldsymbol{V}}_p^1(\mathcal{T}_\omega) \xrightarrow{\mathbf{curl}} \mathring{\boldsymbol{V}}_p^2(\mathcal{T}_\omega) \xrightarrow{\mathrm{div}} \mathring{V}_p^3(\mathcal{T}_\omega) \xrightarrow{\int_\omega} 0.$$

We define the kernels of the differential operators in the discrete spaces (6.5) as

(6.7a) $$\mathfrak{Z}V_p^0(\mathcal{T}_\omega) := \{u_\mathcal{T} \in V_p^0(\mathcal{T}_\omega) : \mathbf{grad}\, u_\mathcal{T} = \mathbf{0}\},$$

(6.7b) $$\mathfrak{Z}\boldsymbol{V}_p^1(\mathcal{T}_\omega) := \{\boldsymbol{u}_\mathcal{T} \in \boldsymbol{V}_p^1(\mathcal{T}_\omega) : \mathbf{curl}\, \boldsymbol{u}_\mathcal{T} = \mathbf{0}\},$$

(6.7c) $$\mathfrak{Z}\boldsymbol{V}_p^2(\mathcal{T}_\omega) := \{\boldsymbol{u}_\mathcal{T} \in \boldsymbol{V}_p^2(\mathcal{T}_\omega) : \mathrm{div}\, \boldsymbol{u}_\mathcal{T} = 0\},$$

whereas the L^2-orthogonal complements are defined as follows:

(6.8a) $$\mathfrak{Z}^\perp V_p^0(\mathcal{T}_\omega) := \{u_\mathcal{T} \in V_p^0(\mathcal{T}_\omega) : \langle u_\mathcal{T}, v_\mathcal{T}\rangle_\omega = 0, \forall v_\mathcal{T} \in \mathfrak{Z}V_p^0(\mathcal{T}_\omega)\},$$

(6.8b) $$\mathfrak{Z}^\perp \boldsymbol{V}_p^1(\mathcal{T}_\omega) := \{\boldsymbol{u}_\mathcal{T} \in \boldsymbol{V}_p^1(\mathcal{T}_\omega) : \langle \boldsymbol{u}_\mathcal{T}, \boldsymbol{v}_\mathcal{T}\rangle_\omega = 0, \forall \boldsymbol{v}_\mathcal{T} \in \mathfrak{Z}\boldsymbol{V}_p^1(\mathcal{T}_\omega)\},$$

(6.8c) $$\mathfrak{Z}^\perp \boldsymbol{V}_p^2(\mathcal{T}_\omega) := \{\boldsymbol{u}_\mathcal{T} \in \boldsymbol{V}_p^2(\mathcal{T}_\omega) : \langle \boldsymbol{u}_\mathcal{T}, \boldsymbol{v}_\mathcal{T}\rangle_\omega = 0, \forall \boldsymbol{v}_\mathcal{T} \in \mathfrak{Z}\boldsymbol{V}_p^2(\mathcal{T}_\omega)\}.$$

We define the subspaces $\mathfrak{Z}\mathring{V}_p^0(\mathcal{T}_\omega)$, $\mathfrak{Z}\mathring{\boldsymbol{V}}_p^1(\mathcal{T}_\omega)$, and $\mathfrak{Z}\mathring{\boldsymbol{V}}_p^2(\mathcal{T}_\omega)$ as well as $\mathfrak{Z}^\perp\mathring{V}_p^0(\mathcal{T}_\omega)$, $\mathfrak{Z}^\perp\mathring{\boldsymbol{V}}_p^1(\mathcal{T}_\omega)$, and $\mathfrak{Z}^\perp\mathring{\boldsymbol{V}}_p^2(\mathcal{T}_\omega)$ similarly.

We use the generic notation $V^l(\omega)$, $\mathring{V}^l(\omega)$ for the continuous spaces defined in (6.2) and $V_p^l(\mathcal{T}_\omega)$, $\mathring{V}_p^l(\mathcal{T}_\omega)$ for their discrete subspaces defined in (6.5). The kernels and L^2-orthogonal complements are denoted as $\mathfrak{Z}V^l(\omega)$, $\mathfrak{Z}\mathring{V}^l(\omega)$ and $\mathfrak{Z}^\perp V^l(\omega)$, $\mathfrak{Z}^\perp\mathring{V}^l(\omega)$ at the continuous level, and as $\mathfrak{Z}V_p^l(\mathcal{T}_\omega)$, $\mathfrak{Z}\mathring{V}_p^l(\mathcal{T}_\omega)$ and $\mathfrak{Z}^\perp V_p^l(\mathcal{T}_\omega)$, $\mathfrak{Z}^\perp\mathring{V}_p^l(\mathcal{T}_\omega)$ at the discrete level. Moreover, we write $d^0 := \mathbf{grad}$, $d^1 := \mathbf{curl}$, $d^2 := \mathrm{div}$. Finally, to unify the notation regarding boundary conditions, we set, for all $l \in \{1{:}2\}$,

$$\widetilde{V}^l(\omega) := V^l(\omega) \quad \text{or } \mathring{V}^l(\omega), \qquad \widetilde{V}_p^l(\mathcal{T}_\omega) := V_p^l(\mathcal{T}_\omega) \quad \text{or } \mathring{V}_p^l(\mathcal{T}_\omega), \tag{6.9a}$$

$$\mathfrak{Z}^\perp\widetilde{V}^l(\omega) := \mathfrak{Z}^\perp V^l(\omega) \text{ or } \mathfrak{Z}^\perp\mathring{V}^l(\omega), \qquad \mathfrak{Z}^\perp\widetilde{V}^l_p(\mathcal{T}_\omega) := \mathfrak{Z}^\perp V_p^l(\mathcal{T}_\omega) \text{ or } \mathfrak{Z}^\perp\mathring{V}_p^l(\mathcal{T}_\omega). \tag{6.9b}$$

Then, the continuous Poincaré inequalities (4.1)–(4.2) are rewritten as follows:

$$\|u\|_{L^2(\omega)} \le \mathcal{C}_{\mathrm{P}}^l h_\omega \|d^l u\|_{L^2(\omega)}, \qquad \forall u \in \mathfrak{Z}^\perp\widetilde{V}^l(\omega), \qquad \forall l \in \{0{:}2\}, \tag{6.10}$$

with $\mathcal{C}_{\mathrm{P}}^l := C_{\mathrm{P}}^l$ or $\mathring{C}_{\mathrm{P}}^l$ depending on the context, and the discrete Poincaré inequalities (5.1)–(5.2) are rewritten as follows:

$$\|u_\mathcal{T}\|_{L^2(\omega)} \le \mathcal{C}_{\mathrm{P}}^{\mathrm{d},l} h_\omega \|d^l u_\mathcal{T}\|_{L^2(\omega)}, \qquad \forall u_\mathcal{T} \in \mathfrak{Z}^\perp\widetilde{V}^l_p(\mathcal{T}_\omega), \qquad \forall l \in \{0{:}2\}, \tag{6.11}$$

with $\mathcal{C}_{\mathrm{P}}^{\mathrm{d},l} := C_{\mathrm{P}}^{\mathrm{d},l}$ or $\mathring{C}_{\mathrm{P}}^{\mathrm{d},l}$ depending on the context. Here, $\|\cdot\|_{L^2(\omega)}$ generically refers to the $L^2(\omega)$-norm of functions or fields depending on the context.

6.2. **Route 1: Invoking equivalence between discrete and continuous minimizers.** Let $l \in \{1{:}2\}$. For all $r_\mathcal{T} \in d^l(\widetilde{V}_p^l(\mathcal{T}_\omega)) \subset \widetilde{V}_p^{l+1}(\mathcal{T}_\omega)$, as in (3.2) and (3.5), we consider the following two constrained quadratic minimization problems:

$$u_\mathcal{T}^* := \underset{\substack{v_\mathcal{T} \in \widetilde{V}_p^l(\mathcal{T}_\omega) \\ d^l v_\mathcal{T} = r_\mathcal{T}}}{\arg\min} \|v_\mathcal{T}\|^2_{L^2(\omega)}, \tag{6.12a}$$

$$u^* := \underset{\substack{v \in \widetilde{V}^l(\omega) \\ d^l v = r_\mathcal{T}}}{\arg\min} \|v\|^2_{L^2(\omega)}. \tag{6.12b}$$

We notice that the finite-dimensional minimization set $\{v_\mathcal{T} \in \widetilde{V}_p^l(\mathcal{T}_\omega) \; : \; d^l v_\mathcal{T} = r_\mathcal{T}\}$ is nonempty, closed, and convex, and so is also the larger, infinite-dimensional set $\{v \in \widetilde{V}^l(\omega) \; : \; d^l v = r_\mathcal{T}\}$. Thus, as for (3.2), both problems admit a unique minimizer. Moreover, we trivially have

$$\|u^*\|_{L^2(\omega)} \le \|u_\mathcal{T}^*\|_{L^2(\omega)}.$$

The Euler optimality conditions respectively read, cf. (3.4) and (3.7):

$$\begin{cases} \text{Find } u_\mathcal{T}^* \in \widetilde{V}_p^l(\mathcal{T}_\omega) \text{ with } d^l u_\mathcal{T}^* = r_\mathcal{T} \text{ such that} \\ \langle u_\mathcal{T}^*, v_\mathcal{T}\rangle_\omega = 0 \quad \forall v_\mathcal{T} \in \widetilde{V}_p^l(\mathcal{T}_\omega) \text{ with } d^l v_\mathcal{T} = 0, \end{cases} \tag{6.13}$$

and

$$\begin{cases} \text{Find } u^* \in \widetilde{V}^l(\omega) \text{ with } d^l u^* = r_\mathcal{T} \text{ such that} \\ \langle u^*, v\rangle_\omega = 0 \quad \forall v \in \widetilde{V}^l(\omega) \text{ with } d^l v = 0. \end{cases} \tag{6.14}$$

Lemma 6.1 (Discrete Poincaré inequalities invoking equivalence between discrete and continuous minimizers)**.** *Let $l \in \{1{:}2\}$. Assume that there is $\mathcal{C}^l_{\min}$ such that, for all $r_{\mathcal{T}} \in d^l(\widetilde{V}^l_p(\mathcal{T}_\omega))$, the solutions to (6.12) satisfy*

$$\|u^*_{\mathcal{T}}\|_{L^2(\omega)} \leq \mathcal{C}^l_{\min}\|u^*\|_{L^2(\omega)}. \tag{6.15}$$

Then (6.11) holds true with constant $\mathcal{C}^{\mathrm{d},l}_{\mathrm{P}} \leq \mathcal{C}^l_{\min}\mathcal{C}^l_{\mathrm{P}}$.

Proof. Let $u_{\mathcal{T}} \in \mathfrak{Z}^\perp\widetilde{V}^l_p(\mathcal{T}_\omega)$. Set $r_{\mathcal{T}} := d^l u_{\mathcal{T}}$. Since $\mathfrak{Z}^\perp\widetilde{V}^l_p(\mathcal{T}_\omega) \subset \widetilde{V}^l_p(\mathcal{T}_\omega)$, we have $r_{\mathcal{T}} \in d^l(\widetilde{V}^l_p(\mathcal{T}_\omega))$. Moreover, by considering the Euler conditions (6.13) and (6.14), we infer that $u^*_{\mathcal{T}} \in \mathfrak{Z}^\perp\widetilde{V}^l_p(\mathcal{T}_\omega)$ and $u^* \in \mathfrak{Z}^\perp\widetilde{V}^l(\omega)$. In addition, since the minimization problems admit a unique solution, and since $u_{\mathcal{T}}$ satisfies the Euler conditions for the discrete problem, we have $u_{\mathcal{T}} = u^*_{\mathcal{T}}$. Invoking (6.15) followed by the continuous Poincaré inequality (6.10) gives

$$\|u_{\mathcal{T}}\|_{L^2(\omega)} = \|u^*_{\mathcal{T}}\|_{L^2(\omega)} \leq \mathcal{C}^l_{\min}\|u^*\|_{L^2(\omega)} \leq \mathcal{C}^l_{\min}\mathcal{C}^l_{\mathrm{P}} h_\omega \|d^l u^*\|_{L^2(\omega)}.$$

Since, from (6.12), $\|d^l u^*\|_{L^2(\omega)} = \|r_{\mathcal{T}}\|_{L^2(\omega)} = \|d^l u_{\mathcal{T}}\|_{L^2(\omega)}$, we conclude that (6.11) holds true with constant $\mathcal{C}^{\mathrm{d},l}_{\mathrm{P}} \leq \mathcal{C}^l_{\min}\mathcal{C}^l_{\mathrm{P}}$. □

In the case of the divergence operator and homogeneous boundary conditions, [29, Corollaries 3.3 and 3.8] established (6.15) whenever $\mathcal{T}_\omega$ is a vertex star, see also [12, Proposition 3.1 and Corollary 4.1], by considering the minimization problems

$$\boldsymbol{u}^*_{\mathcal{T}} := \underset{\substack{\boldsymbol{v}_{\mathcal{T}} \in \mathring{\boldsymbol{V}}^2_p(\mathcal{T}_\omega) \\ \operatorname{div} \boldsymbol{v}_{\mathcal{T}} = r_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\omega)}, \qquad \boldsymbol{u}^* := \underset{\substack{\boldsymbol{v} \in \mathring{\boldsymbol{V}}^2(\omega) \\ \operatorname{div} \boldsymbol{v} = r_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\omega)} \tag{6.16}$$

with $r_{\mathcal{T}} \in \mathring{V}^3_p(\mathcal{T}_\omega)$. When the mesh $\mathcal{T}_\omega$ is more complex than a vertex star, a crucial notion to establish the equivalence of finite- and infinite-dimensional minimization problems is the notion of shellability recalled in Section 5.1. In this setting, [22, Theorem B.2 and Corollary B.3] established (6.15) by considering the minimization problems

$$\boldsymbol{u}^*_{\mathcal{T}} := \underset{\substack{\boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{V}^2_p(\mathcal{T}_\omega) \\ \operatorname{div} \boldsymbol{v}_{\mathcal{T}} = r_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\omega)}, \qquad \boldsymbol{u}^* := \underset{\substack{\boldsymbol{v} \in \boldsymbol{V}^2(\omega) \\ \operatorname{div} \boldsymbol{v} = r_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\omega)} \tag{6.17}$$

with $r_{\mathcal{T}} \in V^3_p(\mathcal{T}_\omega)$.

In the case of the curl operator with homogeneous boundary conditions, (6.15) was established in [10, Proposition 6.6] for edge stars and in [12, Theorem 3.3 and Corollary 4.3] for vertex stars by considering the minimization problems

$$\boldsymbol{u}^*_{\mathcal{T}} := \underset{\substack{\boldsymbol{v}_{\mathcal{T}} \in \mathring{\boldsymbol{V}}^1_p(\mathcal{T}_\omega) \\ \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} = \boldsymbol{r}_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\omega)}, \qquad \boldsymbol{u}^* := \underset{\substack{\boldsymbol{v} \in \mathring{\boldsymbol{V}}^1(\omega) \\ \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{r}_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\omega)} \tag{6.18}$$

with $\boldsymbol{r}_{\mathcal{T}} \in \mathbf{curl}\, \mathring{\boldsymbol{V}}^1_p(\mathcal{T}_\omega)$. Moreover, for a shellable triangulation, (6.15) can be established following the ideas in [22, Theorem B.2 and Corollary B.3] by considering the minimization problems

$$\boldsymbol{u}^*_{\mathcal{T}} := \underset{\substack{\boldsymbol{v}_{\mathcal{T}} \in \boldsymbol{V}^1_p(\mathcal{T}_\omega) \\ \mathbf{curl}\, \boldsymbol{v}_{\mathcal{T}} = \boldsymbol{r}_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}_{\mathcal{T}}\|^2_{\boldsymbol{L}^2(\omega)}, \qquad \boldsymbol{u}^* := \underset{\substack{\boldsymbol{v} \in \boldsymbol{V}^1(\omega) \\ \mathbf{curl}\, \boldsymbol{v} = \boldsymbol{r}_{\mathcal{T}}}}{\arg\min} \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\omega)} \tag{6.19}$$

with $\boldsymbol{r}_{\mathcal{T}} \in \mathbf{curl}\, \boldsymbol{V}^1_p(\mathcal{T}_\omega)$.

One interesting outcome of the proofs based on route 1 is that the constant $\mathcal{C}^l_{\min}$, and consequently $\mathcal{C}^{\mathrm{d},l}_{\mathrm{P}}$, is independent of the polynomial degree p. Still, $\mathcal{C}^l_{\min}$ and $\mathcal{C}^{\mathrm{d},l}_{\mathrm{P}}$ depend on $|\mathcal{T}_\omega|$ and the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$, leading to item (2) of Theorem 5.1 (recall that for stars or any local patch, $|\mathcal{T}_\omega|$ only depends on $\rho_{\mathcal{T}_\omega}$).

6.3. **Route 2: Invoking stable commuting projections.** Here, we proceed as in [34], [3, Theorem 5.11], [4, Theorem 3.6], [14], [7, Proposition 5.4.2], [31], and [26, Theorem 44.6 & Remark 51.12].

Lemma 6.2 (Discrete Poincaré inequalities invoking stable commuting projections)**.** *Assume that there are projections* $\Pi^m_p : \widetilde{V}^m(\omega) \to \widetilde{V}^m_p(\mathcal{T}_\omega)$*,* $m \in \{1{:}3\}$*, satisfying, for all* $l \in \{1{:}2\}$*, the commuting property*

$$d^l(\Pi^l_p(u)) = \Pi^{l+1}_p(d^l u), \qquad \forall u \in \widetilde{V}^l(\omega), \tag{6.20}$$

and the L^2*-stability property*

$$\|\Pi^l_p(u)\|_{L^2(\omega)} \le C_{\mathrm{st}}\|u\|_{L^2(\omega)}, \qquad \forall u \in \widetilde{V}^l(\omega) \text{ such that } d^l u \in \widetilde{V}^{l+1}_p(\mathcal{T}_\omega). \tag{6.21}$$

Then (6.11) *holds true with constant* $\mathcal{C}^{\mathrm{d},l}_{\mathrm{P}} \le C_{\mathrm{st}}\mathcal{C}^l_{\mathrm{P}}$*.*

Proof. Let $u_{\mathcal{T}} \in \mathfrak{Z}^{\perp}\widetilde{V}^l_p(\mathcal{T}_\omega)$. Set $r_{\mathcal{T}} := d^l u_{\mathcal{T}}$. We consider again the minimization problems in (6.12). Recall that both problems are well-posed and that $u_{\mathcal{T}} = u^*_{\mathcal{T}}$. We observe that $\Pi^l_p(u^*) \in \widetilde{V}^l_p(\mathcal{T}_\omega)$ by definition, that $d^l u^* = r_{\mathcal{T}} \in \widetilde{V}^{l+1}_p(\mathcal{T}_\omega)$, and that

$$d^l(\Pi^l_p(u^*)) = \Pi^{l+1}_p(d^l u^*) = \Pi^{l+1}_p(r_{\mathcal{T}}) = r_{\mathcal{T}},$$

where we used the commuting property (6.20), the fact that $r_{\mathcal{T}} \in \widetilde{V}^{l+1}_p(\mathcal{T}_\omega)$, and that Π^{l+1}_p is a projection. This shows that $\Pi^l_p(u^*)$ is in the discrete minimization set. Using the L^2-stability property (6.21) and the continuous Poincaré inequality (6.10), we infer that

$$\begin{aligned}\|u_{\mathcal{T}}\|_{L^2(\omega)} = \|u^*_{\mathcal{T}}\|_{L^2(\omega)} &\le \|\Pi^l_p(u^*)\|_{L^2(\omega)}\\ &\le C_{\mathrm{st}}\|u^*\|_{L^2(\omega)}\\ &\le C_{\mathrm{st}}\mathcal{C}^l_{\mathrm{P}} h_\omega \|d^l u^*\|_{L^2(\omega)}.\end{aligned} \tag{6.22}$$

Since $\|d^l u^*\|_{L^2(\omega)} = \|r_{\mathcal{T}}\|_{L^2(\omega)} = \|d^l u_{\mathcal{T}}\|_{L^2(\omega)}$, we conclude that (6.11) holds true with constant $\mathcal{C}^{\mathrm{d},l}_{\mathrm{P}} \le C_{\mathrm{st}}\mathcal{C}^l_{\mathrm{P}}$. □

Operators satisfying (6.20)–(6.21) have been constructed in [24, Definition 3.1] (for $l = 2$) and in [13, Definition 2] for $l = 1$. In all these cases, C_{st} depends on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$ and on the polynomial degree p, but is independent of the number of elements in $\mathcal{T}_\omega$. In the $\boldsymbol{H}(\mathrm{div}, \omega)$ setting ($l = 2$), the p-robust projector of [22, Definition 3.5] gives a constant C_{st} only depending on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$ if all twice-extended element stars in $\mathcal{T}_\omega$ are shellable. All these cases are summarized in item (3) of Theorem 5.1.

Remark 6.3 (L^2-stability of Π^l_p)**.** The assumptions in Lemma 6.2 on the projection Π^l_p do not ask for full stability in $L^2(\omega)$. Indeed, it suffices that Π^l_p be defined on the graph space $\widetilde{V}^l(\omega)$ and that the L^2-stability property (6.21) holds true for functions so that $d^l u \in \widetilde{V}^{l+1}_p(\mathcal{T}_\omega)$ (and $d^l u$ is, in particular, a polynomial).

Remark 6.4 (Graph-norm stability of Π_p^l)**.** Actually, the proof still works if one considers commuting projections that are stable in the graph norm

$$\|v\|_{\widetilde{V}^l(\omega)} := \big(\|v\|^2_{L^2(\omega)} + h_\omega^2 \|d^l v\|^2_{L^2(\omega)}\big)^{\frac{1}{2}}, \tag{6.23}$$

leading to the bound $\mathcal{C}_{\mathrm{P}}^{\mathrm{d},l} \le C_{\mathrm{st}}\big(1 + (\mathcal{C}_{\mathrm{P}}^l)^2\big)^{\frac{1}{2}}$. Indeed, the final step of the above proofs now writes

$$\begin{aligned}\|u_{\mathcal{T}}\|_{L^2(\omega)} = \|u^*_{\mathcal{T}}\|_{L^2(\omega)} &\le \|\Pi_p^l(u^*)\|_{L^2(\omega)} \\ &\le C_{\mathrm{st}}\|u^*\|_{\widetilde{V}^l(\omega)} \\ &\le C_{\mathrm{st}}\big(1 + (\mathcal{C}_{\mathrm{P}}^l)^2\big)^{\frac{1}{2}} h_\omega \|d^l u_{\mathcal{T}}\|_{L^2(\omega)}.\end{aligned}$$

Remark 6.5 (Fractional-order Sobolev stability of Π_p^l)**.** It is also possible to invoke regularity results stating that $\mathfrak{Z}^\perp \widetilde{V}^l(\omega) \hookrightarrow H^s(\omega)$, $s > \frac{1}{2}$, with embedding constant C_{emb} so that

$$\|v\|_{H^s(\omega)} \le C_{\mathrm{emb}} h_\omega \|d^l v\|_{L^2(\omega)}, \qquad \forall v \in \mathfrak{Z}^\perp \widetilde{V}^l(\omega),$$

where

$$\|v\|^2_{H^s(\omega)} = \|v\|^2_{L^2(\omega)} + h_\omega^s |v|^2_{H^s(\omega)}, \qquad |v|^2_{H^s(\omega)} = \int_\omega \int_\omega \frac{|v(x) - v(y)|^2}{|x-y|^{3+2s}} dxdy.$$

(Again, the scaling by h_ω is introduced for dimensional consistency.) This allows one to consider commuting projections that are stable only in $H^s(\omega)$, $s > \frac{1}{2}$, i.e.,

$$\|\Pi_p^l(z)\|_{L^2(\omega)} \le C_{\mathrm{st}} \|z\|_{H^s(\omega)}, \qquad \forall z \in H^s(\omega).$$

The proof of the discrete Poincaré inequality then runs as follows. For all $u_{\mathcal{T}} \in \mathfrak{Z}^\perp \widetilde{V}_p^l(\mathcal{T}_\omega)$, there exists $z \in \mathfrak{Z}^\perp \widetilde{V}^l(\omega)$ such that $d^l z = d^l u_{\mathcal{T}}$ (indeed, take $z := u_{\mathcal{T}} - m$, where m is the L^2-orthogonal projection of $u_{\mathcal{T}}$ onto $\mathfrak{Z}\widetilde{V}^l(\omega)$). We have

$$\|u_{\mathcal{T}}\|^2_{L^2(\omega)} = \big\langle u_{\mathcal{T}}, \Pi_p^l(z)\big\rangle_\omega + \big\langle u_{\mathcal{T}}, u_{\mathcal{T}} - \Pi_p^l(z)\big\rangle_\omega = \big\langle u_{\mathcal{T}}, \Pi_p^l(z)\big\rangle_\omega,$$

since $u_{\mathcal{T}} - \Pi_p^l(z) \in \mathfrak{Z}\widetilde{V}_p^l(\mathcal{T}_\omega)$ (indeed, $d^l(u_{\mathcal{T}} - \Pi_p^l(z)) = d^l u_{\mathcal{T}} - \Pi_p^{l+1}(d^l z) = d^l u_{\mathcal{T}} - \Pi_p^{l+1}(d^l u_{\mathcal{T}}) = 0$ since Π_p^{l+1} leaves $\widetilde{V}_p^{l+1}(\mathcal{T}_\omega)$ pointwise invariant). The above identity together with the Cauchy–Schwarz inequality gives

$$\|u_{\mathcal{T}}\|_{L^2(\omega)} \le \|\Pi_p^l(z)\|_{L^2(\omega)}.$$

Observing that $z \in H^s(\omega)$, we infer that

$$\|u_{\mathcal{T}}\|_{L^2(\omega)} \le \|\Pi_p^l(z)\|_{L^2(\omega)} \le C_{\mathrm{st}}\|z\|_{H^s(\omega)} \le C_{\mathrm{st}} C_{\mathrm{emb}} h_\omega \|d^l z\|_{L^2(\omega)} = C_{\mathrm{st}} C_{\mathrm{emb}} h_\omega \|d^l u_{\mathcal{T}}\|_{L^2(\omega)},$$

which proves (6.11) with constant $\mathcal{C}_{\mathrm{P}}^{\mathrm{d},l} \le C_{\mathrm{st}} C_{\mathrm{emb}}$. The above approach was considered in early works where L^2-stable or graph-stable commuting projections were not yet available. The idea is to trade some stability of Π_l^p by invoking subtle regularity results on the curl and divergence operators. On the downside, estimating $\mathcal{C}_{\mathrm{P}}^{\mathrm{d},l}$ now requires upper bounds on C_{st} and C_{emb}.

With the above developments, we can now add one more equivalent statement for discrete Poincaré inequalities, in the spirit of [4, Theorems 3.6 and 3.7]. This completes the results on equivalent statements given in Section 3.

Lemma 6.6 (Equivalence of discrete Poincaré inequalities with the existence of graph-stable commuting projections)**.** *The discrete Poincaré inequalities* (6.11) *for* $l \in \{1{:}2\}$ *are equivalent to the existence of projections* $\Pi_p^m : \widetilde{V}^m(\omega) \to \widetilde{V}_p^m(\mathcal{T}_\omega)$*,* $m \in \{1{:}3\}$*, satisfying, for all* $l \in \{1{:}2\}$*, the commuting property*

$$d^l(\Pi_p^l(u)) = \Pi_p^{l+1}(d^l u), \qquad \forall u \in \widetilde{V}^l(\omega), \tag{6.24}$$

and the graph-stability property

$$\|\Pi_p^l(u)\|_{\widetilde{V}^l(\omega)} \leq C_{\mathrm{st}} \|u\|_{\widetilde{V}^l(\omega)}, \qquad \forall u \in \widetilde{V}^l(\omega). \tag{6.25}$$

Proof. We show the two implications.

(i) That the existence of graph-stable projections satisfying (6.24)–(6.25) implies the discrete Poincaré inequalities (6.11) for $l \in \{1{:}2\}$ follows from Remark 6.4.

(ii) Suppose the validity of the discrete Poincaré inequalities (6.11). We show that this implies the existence of projections satisfying (6.24)–(6.25). A generic way is to take Π_p^3 as the L^2-orthogonal projection onto $\widetilde{V}_p^3(\mathcal{T}_\omega)$ and to define $\Pi_p^l : \widetilde{V}^l(\omega) \to \widetilde{V}_p^l(\mathcal{T}_\omega)$ for all $l \in \{1{:}2\}$ by the following constrained quadratic minimization problems, similar to (3.2) and (3.5): For all $u \in \widetilde{V}^l(\omega)$,

$$\Pi_p^l(u) := \mathop{\arg\min}_{\substack{v_\mathcal{T} \in \widetilde{V}_p^l(\mathcal{T}_\omega) \\ d^l v_\mathcal{T} = \Pi_p^{l+1}(d^l u)}} \|u - v_\mathcal{T}\|^2_{L^2(\omega)}, \tag{6.26}$$

first for $l = 2$ and then for $l = 1$. Notice that the commuting property (6.24) is built in the definition of Π_l^p, so that only the stability in the graph norm (6.25) needs to be verified. To this purpose, we notice that the Euler optimality conditions for (6.26), as in (3.4), read as follows: Find $\Pi_p^l(u) \in \widetilde{V}_p^l(\mathcal{T}_\omega)$ with $d^l \Pi_p^l(u) = \Pi_p^{l+1}(d^l u)$ such that

$$\left\langle \Pi_p^l(u) - u, v_\mathcal{T} \right\rangle_\omega = 0, \qquad \forall v_\mathcal{T} \in \widetilde{V}_p^l(\mathcal{T}_\omega) \text{ with } d^l v_\mathcal{T} = 0.$$

The mixed formulation using a Lagrange multiplier, as in (3.8), reads as follows: Find $\Pi_p^l(u) \in \widetilde{V}_p^l(\mathcal{T}_\omega)$ and $s_\mathcal{T} \in d^l(\widetilde{V}_p^l(\mathcal{T}_\omega))$ such that

$$\begin{aligned} \left\langle \Pi_p^l(u), v_\mathcal{T} \right\rangle_\omega - \left\langle s_\mathcal{T}, d^l v_\mathcal{T} \right\rangle_\omega &= \left\langle u, v_\mathcal{T} \right\rangle_\omega \qquad && \forall v_\mathcal{T} \in \widetilde{V}_p^l(\mathcal{T}_\omega), \\ \left\langle d^l \Pi_p^l(u), t_\mathcal{T} \right\rangle_\omega &= \left\langle d^l u, t_\mathcal{T} \right\rangle_\omega && \forall t_\mathcal{T} \in d^l(\widetilde{V}_p^l(\mathcal{T}_\omega)). \end{aligned}$$

(Notice that $\left\langle \Pi_p^{l+1}(d^l u), t_\mathcal{T} \right\rangle_\omega = \left\langle d^l u, t_\mathcal{T} \right\rangle_\omega$ owing to the Euler optimality conditions for Π_p^{l+1} and the fact that $d^{l+1} t_\mathcal{T} = 0$). As highlighted in Section 3.2, the discrete Poincaré inequality (6.11) is equivalent to to the discrete inf-sup condition formulated using L^2-norms, see Lemmas 3.3 and 3.4. The inf-sup conditions in the form of (3.9) or (3.11) readily imply the discrete inf-sup condition in the graph norm

$$\inf_{t_\mathcal{T} \in d^l(\widetilde{V}_p^l(\mathcal{T}_\omega))} \sup_{v_\mathcal{T} \in \widetilde{V}_p^l(\mathcal{T}_\omega)} \frac{\left\langle t_\mathcal{T}, d^l v_\mathcal{T} \right\rangle_\omega}{\|t_\mathcal{T}\|_{L^2(\omega)} \|v_\mathcal{T}\|_{\widetilde{V}^l(\omega)}} \geq \frac{1}{\left(1 + (C_{\mathrm{P}}^{\mathrm{d},l})^2\right)^{\frac{1}{2}} h_\omega}.$$

Then, invoking [7, Theorem 4.2.3] or [26, Theorem 49.13], we obtain

$$\begin{aligned} \|\Pi_p^l(u)\|_{\widetilde{V}^l(\omega)} &\leq \|u\|_{\widetilde{V}^l(\omega)} + 2\left(1 + (C_{\mathrm{P}}^{\mathrm{d},l})^2\right)^{\frac{1}{2}} h_\omega \|d^l u\|_{L^2(\omega)} \\ &\leq \left(10 + 8(C_{\mathrm{P}}^{\mathrm{d},l})^2\right)^{\frac{1}{2}} \|u\|_{\widetilde{V}^l(\omega)}. \end{aligned}$$

This proves that the commuting projection Π_p^l defined above is indeed stable in the graph norm. ☐

Remark 6.7 (Locality)**.** The above graph-stable commuting projections are not necessarily locally defined and locally stable. Stable *local* commuting projections are designed in [31, 2, 24, 13, 22], see also the references therein.

6.4. **Route 3: Invoking piecewise Piola transformations.** In this section, we prove the discrete Poincaré inequality by a direct argument, thereby circumventing the need to invoke the continuous Poincaré inequalities. The discrete Poincaré constants resulting from the present proofs depend on the shape-regularity parameter $\rho_{\mathcal{T}_\omega}$, the number of elements $|\mathcal{T}_\omega|$, and the polynomial degree p, as summarized in item (4) of Theorem 5.1. The proof shares ideas with the one given in [28], but eventually employs a different argument to conclude.

The starting point, shared with [28], is to introduce reference meshes and piecewise Piola transformations on those meshes. We enumerate the set of vertices (resp., edges, faces, and cells) in $\mathcal{T}_\omega$ as $\mathcal{V}_\omega := \{v_1, \dots, v_{N^{\mathrm{v}}}\}$ (resp., $\mathcal{E}_\omega := \{e_1, \dots, e_{N^{\mathrm{e}}}\}$, $\mathcal{F}_\omega := \{f_1, \dots, f_{N^{\mathrm{f}}}\}$, and $\mathcal{T}_\omega := \{\tau_1, \dots, \tau_{N^{\mathrm{c}}}\}$ with $N^{\mathrm{c}} = |\mathcal{T}_\omega|$. All these geometric objects are oriented by increasing vertex enumeration (see, e.g., [25, Chapter 10]). The topology of the mesh $\mathcal{T}_\omega$ is completely described by the connectivity arrays

$$\texttt{j\_ev} : \{1{:}N^{\mathrm{e}}\} \times \{0{:}1\} \to \{1{:}N^{\mathrm{v}}\}, \tag{6.27a}$$

$$\texttt{j\_fv} : \{1{:}N^{\mathrm{f}}\} \times \{0{:}2\} \to \{1{:}N^{\mathrm{v}}\}, \tag{6.27b}$$

$$\texttt{j\_cv} : \{1{:}N^{\mathrm{c}}\} \times \{0{:}3\} \to \{1{:}N^{\mathrm{v}}\}, \tag{6.27c}$$

such that $\texttt{j\_ev}(m, n)$ is the global vertex number of the vertex n of the edge e_m, and so on (the local enumeration of vertices is by increasing enumeration order). Notice that the connectivity arrays only take integer values and are independent of the actual coordinates of the vertices in the physical space $\mathbb{R}^3$.

Let $\rho_\sharp > 0$ be a positive real number and let $T_\sharp$ be a (finite) integer number. The number of meshes with shape-regularity parameter bounded from above by $\rho_\sharp$ and cardinal number given by $T_\sharp$ with different possible realizations of the connectivity arrays is bounded from above by a constant $\hat{N}_\sharp := \hat{N}(\rho_\sharp, T_\sharp)$ only depending on $\rho_\sharp$ and $T_\sharp$. Thus, for each $\rho_\sharp$ and $T_\sharp$, there is a *finite* set of reference meshes, which we denote by $\widehat{\mathbb{T}} := \widehat{\mathbb{T}}(\rho_\sharp, T_\sharp)$, such that every mesh $\mathcal{T}$ with the shape-regularity parameter bounded from above by $\rho_\sharp$ and cardinal number given by $T_\sharp$ has the same connectivity arrays as those of one reference mesh in $\widehat{\mathbb{T}}$. We enumerate the reference meshes in $\widehat{\mathbb{T}}$ as $\{\widehat{\mathcal{T}}_1, \dots, \widehat{\mathcal{T}}_{\hat{N}_\sharp}\}$ and fix them once and for all. For each reference mesh, the element diameters are of order unity, and the shape-regularity parameter is chosen as small as possible (it is bounded from above by $\rho_\sharp$). For all $j \in \{1{:}\hat{N}_\sharp\}$, we let $\widehat{\omega}_j$ be the open, bounded, connected, Lipschitz polyhedral set covered by the reference mesh $\widehat{\mathcal{T}}_j$. For all $l \in \{1{:}2\}$, we define the piecewise polynomial spaces $V_p^l(\widehat{\mathcal{T}}_j)$ and $\mathring{V}_p^l(\widehat{\mathcal{T}}_j)$ as in (6.5), and set $\widetilde{V}_p^l(\widehat{\mathcal{T}}_j) := V_p^l(\widehat{\mathcal{T}}_j)$ or $\mathring{V}_p^l(\widehat{\mathcal{T}}_j)$ depending on whether boundary conditions are enforced or not. We also define $\mathfrak{Z}^\perp \widetilde{V}_p^l(\widehat{\mathcal{T}}_j)$ as the L^2-orthogonal complement of the kernel subspace $\{\widehat{u}_\mathcal{T} \in \widetilde{V}_p^l(\widehat{\mathcal{T}}_j) \; : \; d^l \widehat{u}_\mathcal{T} = 0\}$ in $\widetilde{V}_p^l(\widehat{\mathcal{T}}_j)$. Norm equivalence in finite dimension implies that, for all $j \in \{1{:}\hat{N}_\sharp\}$ and all $p \geq 0$, there exists a constant $\mathcal{C}_{\mathrm{P}}^l(\widehat{\mathcal{T}}_j, p)$ such that

$$\|\widehat{u}_\mathcal{T}\|_{L^2(\widehat{\omega}_j)} \leq \mathcal{C}_{\mathrm{P}}^l(\widehat{\mathcal{T}}_j, p) \|d^l \widehat{u}_\mathcal{T}\|_{L^2(\widehat{\omega}_j)}, \qquad \forall \widehat{u}_\mathcal{T} \in \mathfrak{Z}^\perp \widetilde{V}_p^l(\widehat{\mathcal{T}}_j). \tag{6.28}$$

Consider an arbitrary mesh $\mathcal{T}_\omega$ with shape-regularity parameter bounded from above by $\rho_\sharp$ and cardinal number given by $T_\sharp$. Then there is an index $j(\mathcal{T}_\omega) \in \{1{:}\hat{N}_\sharp\}$ so that $\mathcal{T}_\omega$ and $\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)}$ share the same connectivity arrays. Therefore, $\mathcal{T}_\omega$ can be generated from $\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)}$ by a piecewise-affine geometric mapping $\boldsymbol{F}_{\mathcal{T}_\omega} := \{\boldsymbol{F}_\tau : \widehat{\tau} \to \tau\}_{\tau\in\mathcal{T}_\omega}$, where all the geometric mappings $\boldsymbol{F}_\tau$ are affine, invertible, with positive Jacobian, and $\bigcup_{\tau\in\mathcal{T}_\omega} \boldsymbol{F}_\tau^{-1}(\tau) = \widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)}$. For all $\tau \in \mathcal{T}_\omega$, let $\boldsymbol{J}_\tau$ be the Jacobian matrix of $\boldsymbol{F}_\tau$. We consider the Piola transformations $\psi^l_{\mathcal{T}_\omega} : L^2(\omega) \to L^2(\widehat{\omega}_{j(\mathcal{T}_\omega)})$, for all $l \in \{1{:}3\}$, such that $\psi^l_\tau := \psi^l_{\mathcal{T}_\omega}|_\tau$ is defined as follows: For all $v \in L^2(\tau)$,

$$\psi^1_\tau(v) := \boldsymbol{J}_\tau^{\mathrm{T}}(v \circ \boldsymbol{F}_\tau), \tag{6.29a}$$

$$\psi^2_\tau(v) := \det(\boldsymbol{J}_\tau)\boldsymbol{J}_\tau^{-1}(v \circ \boldsymbol{F}_\tau), \tag{6.29b}$$

$$\psi^3_\tau(v) := \det(\boldsymbol{J}_\tau)(v \circ \boldsymbol{F}_\tau). \tag{6.29c}$$

The restricted Piola transformations (we keep the same notation for simplicity) $\psi^l_{\mathcal{T}_\omega} : \widetilde{V}^l_p(\mathcal{T}_\omega) \to \widetilde{V}^l_p(\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)})$ are isomorphisms. This follows from the fact that $\mathcal{T}_\omega$ and $\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)}$ have the same connectivity arrays, that $\boldsymbol{F}_{\mathcal{T}_\omega}$ maps any edge (face, tetrahedron) in $\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)}$ to an edge (face, tetrahedron) of $\mathcal{T}_\omega$, and that, for each tetrahedron $\tau \in \mathcal{T}_\omega$, ψ^l_τ is an isomorphism that preserves appropriate moments [25, Lemma 9.13 & Exercise 9.4]. Moreover, the Piola transformations satisfy the following bounds:

$$\|\psi^l_{\mathcal{T}_\omega}\|_{\mathcal{L}} := \|\psi^l_{\mathcal{T}_\omega}\|_{\mathcal{L}(L^2(\omega);L^2(\widehat{\omega}_{j(\mathcal{T}_\omega)}))} \le C(\rho_\sharp)(\overline{h}_{\mathcal{T}_\omega})^l, \tag{6.30}$$

and they satisfy the following commuting properties:

$$d^l(\psi^l_{\mathcal{T}_\omega}(v)) = \psi^{l+1}_{\mathcal{T}_\omega}(d^l v), \qquad \forall v \in \widetilde{V}^l(\omega). \tag{6.31}$$

We use the shorthand notation $\psi^{-l}_{\mathcal{T}_\omega}$ for the inverse of the Piola transformations. We have

$$\|\psi^{-l}_{\mathcal{T}_\omega}\|_{\mathcal{L}} := \|\psi^{-l}_{\mathcal{T}_\omega}\|_{\mathcal{L}(L^2(\widehat{\omega}_{j(\mathcal{T}_\omega)});L^2(\omega))} \le C(\rho_\sharp)(\underline{h}_{\mathcal{T}_\omega})^{-l}, \tag{6.32}$$

where $\underline{h}_{\mathcal{T}_\omega}$ denotes the smallest diameter of a cell in $\mathcal{T}_\omega$. The commuting property (6.31) readily gives

$$\psi^{-(l+1)}_{\mathcal{T}_\omega}(d^l\widehat{v}) = d^l(\psi^{-l}_{\mathcal{T}_\omega}(\widehat{v})), \qquad \forall \widehat{v} \in \widetilde{V}^l(\widehat{\omega}_{j(\mathcal{T}_\omega)}). \tag{6.33}$$

Lemma 6.8 (Discrete Poincaré inequalities invoking piecewise Piola transformations)**.** *The discrete Poincaré inequalities* (6.11) *hold true for all* $l \in \{1{:}2\}$ *with a constant* $\mathcal{C}^{\mathrm{d},l}_{\mathrm{P}}$ *only depending on the shape-regularity parameter* $\rho_{\mathcal{T}_\omega}$*, the number of elements* $|\mathcal{T}_\omega|$*, and the polynomial degree* p*.*

Proof. Let $u_{\mathcal{T}} \in \mathfrak{Z}^\perp\widetilde{V}^l_p(\mathcal{T}_\omega)$ and set $r_{\mathcal{T}} := d^l u_{\mathcal{T}}$. As in (3.2) and (3.5), we consider the following two (well-posed) constrained quadratic minimization problems:

$$u^*_{\mathcal{T}} := \mathop{\arg\min}_{\substack{v_{\mathcal{T}}\in\widetilde{V}^l_p(\mathcal{T}_\omega)\\ d^l v_{\mathcal{T}}=r_{\mathcal{T}}}} \|v_{\mathcal{T}}\|^2_{L^2(\omega)}, \qquad \widehat{u}^*_{\mathcal{T}} := \mathop{\arg\min}_{\substack{\widehat{v}_{\mathcal{T}}\in\widetilde{V}^l_p(\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)})\\ d^l \widehat{v}_{\mathcal{T}}=\widehat{r}_{\mathcal{T}}}} \|\widehat{v}_{\mathcal{T}}\|^2_{L^2(\widehat{\omega}_{j(\mathcal{T}_\omega)})}, \tag{6.34}$$

with

$$\widehat{r}_{\mathcal{T}} := \psi^{l+1}_{\mathcal{T}_\omega}(r_{\mathcal{T}}). \tag{6.35}$$

The Euler conditions for the second minimization problem imply that $\widehat{u}^*_{\mathcal{T}} \in \mathfrak{Z}^\perp\widetilde{V}^l_p(\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)})$. Owing to the discrete Poincaré inequality (6.28), we infer that

$$\|\widehat{u}^*_{\mathcal{T}}\|_{L^2(\widehat{\omega}_{j(\mathcal{T}_\omega)})} \le \mathcal{C}^l_{\mathrm{P}}(\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)}, p)\|\widehat{r}_{\mathcal{T}}\|_{L^2(\widehat{\omega}_{j(\mathcal{T}_\omega)})}.$$

Moreover, we observe that $\psi_{\mathcal{T}_\omega}^{-l}(\widehat{u}_{\mathcal{T}}^*) \in \widetilde{V}_p^l(\mathcal{T}_\omega)$ and, owing to (6.33), the constraint in the second problem in (6.34), and (6.35), we have

$$d^l(\psi_{\mathcal{T}_\omega}^{-l}(\widehat{u}_{\mathcal{T}}^*)) = \psi_{\mathcal{T}_\omega}^{-(l+1)}(d^l\widehat{u}_{\mathcal{T}}^*) = \psi_{\mathcal{T}_\omega}^{-(l+1)}(\widehat{r}_{\mathcal{T}}) = r_{\mathcal{T}}.$$

Hence, $\psi_{\mathcal{T}_\omega}^{-l}(\widehat{u}_{\mathcal{T}}^*)$ is in the minimization set of the first problem in (6.34). This implies that

$$\begin{aligned}
\|u_{\mathcal{T}}\|_{L^2(\omega)} = \|u_{\mathcal{T}}^*\|_{L^2(\omega)} &\le \|\psi_{\mathcal{T}_\omega}^{-l}(\widehat{u}_{\mathcal{T}}^*)\|_{L^2(\omega)} \\
&\le \|\psi_{\mathcal{T}_\omega}^{-l}\|_{\mathcal{L}}\|\widehat{u}_{\mathcal{T}}^*\|_{L^2(\widehat{\omega}_{j(\mathcal{T}_\omega)})} \\
&\le \|\psi_{\mathcal{T}_\omega}^{-l}\|_{\mathcal{L}}\mathcal{C}_{\mathrm{P}}^l(\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)},p)\|\widehat{r}_{\mathcal{T}}\|_{L^2(\widehat{\omega}_{j(\mathcal{T}_\omega)})} \\
&\le \|\psi_{\mathcal{T}_\omega}^{-l}\|_{\mathcal{L}}\|\psi_{\mathcal{T}_\omega}^{l+1}\|_{\mathcal{L}}\mathcal{C}_{\mathrm{P}}^l(\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)},p)\|r_{\mathcal{T}}\|_{L^2(\omega)} \\
&= \|\psi_{\mathcal{T}_\omega}^{-l}\|_{\mathcal{L}}\|\psi_{\mathcal{T}_\omega}^{l+1}\|_{\mathcal{L}}\mathcal{C}_{\mathrm{P}}^l(\widehat{\mathcal{T}}_{j(\mathcal{T}_\omega)},p)\|d^l u_{\mathcal{T}}\|_{L^2(\omega)}.
\end{aligned}$$

The bounds (6.30)–(6.32) on the operator norm of the Piola maps and their inverse together give $\|\psi_{\mathcal{T}_\omega}^{-l}\|_{\mathcal{L}}\|\psi_{\mathcal{T}_\omega}^{l+1}\|_{\mathcal{L}} \le C(\rho_{\mathcal{T}_\omega},|\mathcal{T}_\omega|)h_\omega$, where we used that $\overline{h}_{\mathcal{T}_\omega} \le h_\omega$ and $\overline{h}_{\mathcal{T}_\omega}/\underline{h}_{\mathcal{T}_\omega} \le C(\rho_{\mathcal{T}_\omega},|\mathcal{T}_\omega|)$. This implies that

$$\|u_{\mathcal{T}}\|_{L^2(\omega)} \le \left(C(\rho_{\mathcal{T}_\omega},|\mathcal{T}_\omega|)\max_{j\in\{1:\hat{N}_\sharp\}}\mathcal{C}_{\mathrm{P}}^l(\widehat{\mathcal{T}}_j,p)\right)h_\omega\|d^l u_{\mathcal{T}}\|_{L^2(\omega)}.$$

This completes the proof. □

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