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Localization of the $W^{-1,q}$ norm for local a posteriori efficiency*

Jan Blechta[§†] Josef Málek[†] Martin Vohralík[‡¶]

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Abstract

This paper gives a direct proof of localization of dual norms of bounded linear functionals on the Sobolev space $W_0^{1,p}(\Omega)$, $1 \le p \le \infty$. The basic condition is that the functional in question vanishes over locally supported test functions from $W_0^{1,p}(\Omega)$ which form a partition of unity in Ω, apart from close to the boundary $\partial\Omega$. We also study how to weaken this condition. The results allow in particular to establish local efficiency and robustness with respect to the exponent p of a posteriori estimates for nonlinear partial differential equations in divergence form, including the case of inexact solvers. Numerical illustrations support the theory.

Key words: Sobolev space $W_0^{1,p}(\Omega)$, functional, dual norm, local structure, nonlinear partial differential equation, residual, finite element method, a posteriori error estimate

1 Introduction

The weak solution of the Dirichlet problem associated with the Laplace equation is a function u characterized by

$$u - u^{\mathrm{D}} \in W_0^{1,2}(\Omega), \tag{1.1a}$$

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in W_0^{1,2}(\Omega). \tag{1.1b}$$

Here $\Omega \subset \mathbb{R}^d$, $d \ge 1$, $f \in L^2(\Omega)$, and $u^{\mathrm{D}} \in W^{1,2}(\Omega)$. A typical numerical approximation of u gives u_h such that $u_h - u^{\mathrm{D}} \in V_h^0 \subset W_0^{1,2}(\Omega)$; we assume for simplicity that u^{D} lies in the same discrete space $V_h \subset W_0^{1,2}(\Omega)$ as u_h, so that there is no Dirichlet datum interpolation error.

The *intrinsic distance* of u_h to u is the $W_0^{1,2}(\Omega)$-*norm error* given by $\|\nabla(u - u_h)\|$. This distance is *localizable* in the sense that it is equal to a Hilbertian sum of the $W^{1,2}(\Omega)$-seminorm errors $\|\nabla(u - u_h)\|_K$ over elements K of a partition $\mathcal{T}_h$ of Ω, i.e.,

$$\|\nabla(u - u_h)\| = \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(u - u_h)\|_K^2 \right\}^{\frac{1}{2}}. \tag{1.2}$$

It is this problem-given intrinsic distance that is the most suitable for a posteriori error control. Under appropriate conditions, namely when the orthogonality $(f, \psi_{\boldsymbol{a}}) - (\nabla u_h, \nabla \psi_{\boldsymbol{a}}) = 0$ is fulfilled for the "hat"

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functions $\psi_{\boldsymbol{a}}$ associated with the interior vertices $\boldsymbol{a}$ of the partition $\mathcal{T}_h$, there exist a posteriori estimators $\eta_K(u_h)$, fully and locally computable from u_h, such that

$$\|\nabla(u-u_h)\| \leq \left\{ \sum_{K\in\mathcal{T}_h} \eta_K(u_h)^2 \right\}^{\frac{1}{2}} \tag{1.3}$$

and such that

$$\eta_K(u_h) \leq C \left\{ \sum_{K'\in\mathcal{T}_K} \|\nabla(u-u_h)\|_{K'}^2 \right\}^{\frac{1}{2}}, \tag{1.4}$$

where C is a generic constant and $\mathcal{T}_K$ is some local neighborhood of the element K, see Carstensen and Funken [22], Braess *et al.* [14], Veeser and Verfürth [49], Ern and Vohralík [33], or Verfürth [52] and the references therein. Property (1.4) is called *local efficiency* and is clearly only possible thanks to (1.2), the local structure of the $W_0^{1,2}(\Omega)$-norm distance. A different equivalence result where locality plays a central role is that of Veeser [48] who recently proved that the local- and global-best approximation errors in the $W_0^{1,2}(\Omega)$-norm are equivalent.

Many problems are nonlinear; the basic model that represents one example of a general class of nonlinear models considered here is the Dirichlet problem associated with the p-Laplace equation, where, in place of (1.1), one looks for function u such that

$$\begin{aligned} u-u^{\mathrm{D}} &\in W_0^{1,p}(\Omega), \\ (\boldsymbol{\sigma}(\nabla u), \nabla v) &= (f,v) \qquad \forall v \in W_0^{1,p}(\Omega), \\ \boldsymbol{\sigma}(\boldsymbol{g}) &= |\boldsymbol{g}|^{p-2}\boldsymbol{g} \qquad \boldsymbol{g}\in\mathbb{R}^d \end{aligned}$$

for some $p\in(1,\infty)$, $u^{\mathrm{D}}\in W^{1,p}(\Omega)$, and $f\in L^q(\Omega)$ with $\frac{1}{p}+\frac{1}{q}=1$. Let $u_h\in V_h\subset W^{1,p}(\Omega)$ fulfilling $u_h-u^{\mathrm{D}}\in V_h^0\subset W_0^{1,p}(\Omega)$ be a numerical approximation of the exact solution u and let $\mathcal{R}(u_h)$ be the *residual* of u_h given by

$$\langle \mathcal{R}(u_h), v\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)} := (f,v) - (\boldsymbol{\sigma}(\nabla u_h), \nabla v) \qquad v\in W_0^{1,p}(\Omega); \tag{1.6}$$

$\mathcal{R}(u_h)$ belongs to $W^{-1,q}(\Omega) := \left(W_0^{1,p}(\Omega)\right)'$, the set of bounded linear functionals on $W_0^{1,p}(\Omega)$, see Example 3.2 below for more details. In the present paper, we take for the intrinsic distance of u_h to u the *dual norm* of the residual $\mathcal{R}(u_h)$

$$\|\mathcal{R}(u_h)\|_{W^{-1,q}(\Omega)} := \sup_{v\in W_0^{1,p}(\Omega);\, \|\nabla v\|_p=1} \langle \mathcal{R}(u_h), v\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)}; \tag{1.7}$$

of course $\|\mathcal{R}(u_h)\|_{W^{-1,2}(\Omega)} = \|\nabla(u-u_h)\|$ when $p=2$ and $\boldsymbol{\sigma}(\boldsymbol{g})=\boldsymbol{g}$. Note, however, that other distances might be called intrinsic. Considering for simplicity $u^{\mathrm{D}}=0$ and defining the energy by

$$\mathcal{E}(v) := \frac{1}{p}\|\nabla v\|_p^p - (f,v) = \int_\Omega \left(\frac{1}{p}|\nabla v|^p - fv\right) \mathrm{d}\boldsymbol{x} \qquad v\in W_0^{1,p}(\Omega), \tag{1.8}$$

the energy difference $\mathcal{E}(u_h)-\mathcal{E}(u)$ is often considered as the intrinsic distance, see, e.g., Repin [45, Section 8.4.1], and is actually proportional to the squared quasi-norm error (that again can be used as an intrinsic distance) introduced by Barrett and Liu in [5, 6], see Diening and Kreuzer [28, Lemma 16] or Belenki *et al.* [9, Lemma 3.2], cf. also Remark 3.5 below.

Sticking to (1.7), the analog of (1.3) can then be obtained: there are a posteriori estimators $\eta_K(u_h)$, fully and locally computable from u_h, such that

$$\|\mathcal{R}(u_h)\|_{W^{-1,q}(\Omega)} \leq \left\{ \sum_{K\in\mathcal{T}_h} \eta_K(u_h)^q \right\}^{\frac{1}{q}}, \tag{1.9}$$

see, e.g., Verfürth [51, 52], Veeser and Verfürth [49], El Alaoui *et al.* [30], Ern and Vohralík [32], or Kreuzer and Süli [39]. This can typically be proved under the *orthogonality condition*

$$\langle \mathcal{R}(u_h), \psi_{\boldsymbol{a}}\rangle_{W^{-1,q}(\Omega),W_0^{1,p}(\Omega)} = 0 \qquad \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}, \tag{1.10}$$

where $\mathcal{V}_h^{\text{int}}$ stands for interior vertices of the mesh $\mathcal{T}_h$ and $\psi_{\boldsymbol{a}}$ are test functions forming a partition of unity over all vertices $\boldsymbol{a} \in \mathcal{V}_h$. However, the analog of the local efficiency (1.4) for $p \neq 2$ does not seem to be obvious. The foremost reason is that the intrinsic dual error measure (1.7) does not seem to be localizable in the sense that

$$\|\mathcal{R}(u_h)\|_{W^{-1,q}(\Omega)} \neq \left\{ \sum_{K \in \mathcal{T}_h} \|\mathcal{R}(u_h)\|^q_{W^{-1,q}(K)} \right\}^{\frac{1}{q}},$$

in contrast to (1.2).

For certain estimators from (1.9) with piecewise polynomial u_h, global efficiency in the form

$$\left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^q \right\}^{\frac{1}{q}} \leq C \|\mathcal{R}(u_h)\|_{W^{-1,q}(\Omega)}$$

has been shown previously, cf. [52, 30, 32, 39] and the references therein. Carrying on the results and the proofs in [51, 52], it is, in fact, possible to show that

$$\eta_K(u_h) \leq C \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_K} \|\mathcal{R}(u_h)\|^q_{W^{-1,q}(\omega_{\boldsymbol{a}})} \right\}^{\frac{1}{q}}, \tag{1.11}$$

where $\mathcal{V}_K$ stands for the vertices of the element $K \in \mathcal{T}_h$ and $\omega_{\boldsymbol{a}}$ is a patch of mesh elements around the vertex $\boldsymbol{a}$, see for example [32, Theorem 5.3], [30, proof of Lemma 4.3], [39, proof of Theorem 21] for general $p \in (1, \infty)$, or [27, equation (3.10)] for the Hilbertian setting $p = 2$. Note, however, that all these results are connected with a certain class of PDE problems considered in these studies as well as with a certain appropriately constructed error estimator. To conclude, we observe the following points:

1. Inequality (1.9) together with (1.11) imply

$$\|\mathcal{R}(u_h)\|_{W^{-1,q}(\Omega)} \leq C_1 \left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}(u_h)\|^q_{W^{-1,q}(\omega_{\boldsymbol{a}})} \right\}^{\frac{1}{q}}. \tag{1.12a}$$

   For $p = q = 2$, this has probably been first shown in Babuška and Miller [4, Theorem 2.1.1].

2. It can also be shown that

$$\left\{ \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}(u_h)\|^q_{W^{-1,q}(\omega_{\boldsymbol{a}})} \right\}^{\frac{1}{q}} \leq C_2 \|\mathcal{R}(u_h)\|_{W^{-1,q}(\Omega)}. \tag{1.12b}$$

   See in particular [4, Theorem 2.1.1], Cohen *et al.* [27, equation (3.23)], Ciarlet and Vohralík [26, Theorem 3.2], and the revised version of Ern and Guermond [31] for $p = q = 2$.

3. Thus, for the error measure $\|\mathcal{R}(u_h)\|_{W^{-1,q}(\Omega)}$, the a posteriori estimators $\eta_K(u_h)$ lead to an a posteriori analysis framework where one has localization of the error measure (1.12), global reliability (1.9), and local efficiency (1.11). This is thus a fully consistent and analogous situation to (1.2), (1.3), and (1.4) of the $W_0^{1,2}(\Omega)$ setting.

The main purpose of the present paper is to give a minimalist and direct proof of the two inequalities (1.12) for general exponent p, including also the limiting cases $p = 1$ and $p = \infty$, and without considering any particular partial differential equation or a posteriori error estimators. In particular, Theorem 3.7 shows that, under the orthogonality condition (1.10), *dual norms of all functionals in* $W^{-1,q}(\Omega)$ are *localizable* in the sense that (1.12) holds, with C_1 and C_2 *only depending* on the *regularity* of the partition $\mathcal{T}_h$; in particular the constants are robust with respect to exponent $p \in [1, \infty]$. The orthogonality condition (1.10) is only necessary for robustness of C_1 with respect to the mesh size h; the constant C_2 depends merely on maximal overlap of the partition $\cup_{\boldsymbol{a} \in \mathcal{V}_h} \omega_{\boldsymbol{a}}$. The result of Theorem 3.7 applies to, but is not limited to, dual norms of residuals of (nonlinear) partial differential equations of the form (1.6). Besides implying

local a posteriori error efficiency, the localization of a seemingly only global distance of the form (1.12) may have important consequences in the adaptive approximation theory or for equivalence of local-best and global-best approximations as in [48]. We discuss localization of the $W_0^{1,p}(\Omega)$-norm error in Remark 3.4 and the localization of the global lifting of $\mathcal{R}(u_h)$ into $W_0^{1,p}(\Omega)$ in Remark 3.6. Remark 3.10 further shows that (1.12b) can be strengthened to hold patch by patch $\omega_{\boldsymbol{a}}$, with a global lifting of $\mathcal{R}(u_h)$ on the right-hand side. All these results are presented in Section 3, after we set up the notation and gather the preliminaries in Section 2.

Localization concepts that take form similar to (1.12) also appear in the theory of function spaces, cf. Triebel [47], where they are of independent interest. Consider the Whitney covering of the domain Ω, which we here denote as $\{\omega_{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathbb{N}}$, and a subordinate partition of unity $\sum_{\boldsymbol{a}\in\mathbb{N}}\psi_{\boldsymbol{a}}=1$ with $0\le\psi_{\boldsymbol{a}}\le 1$, $\psi_{\boldsymbol{a}}$ smooth, compactly supported in $\omega_{\boldsymbol{a}}$, and all derivatives controlled by distance to boundary: $\|\nabla^M\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\le C_M\operatorname{dist}(\omega_{\boldsymbol{a}},\partial\Omega)^{-M}$ for all $M\in\mathbb{N}$. For E-thick domain Ω (which includes bounded Lipschitz domains), there holds the so-called *refined localization property*

$$\|v\|_{W^{-1,q}(\Omega)}\approx\left\{\sum_{\boldsymbol{a}\in\mathbb{N}}\|\psi_{\boldsymbol{a}}v\|^q_{W^{-1,q}(\omega_{\boldsymbol{a}})}\right\}^{\frac{1}{q}}\qquad\forall v\in W^{-1,q}(\Omega)\,,\tag{1.13}$$

i.e., the term on the right-hand side is an equivalent quasi-norm. For precise definitions and statements, see [47, Theorem 3.28]. This result holds for spaces of F-scale comprising Lizorkin–Triebel and classical Sobolev spaces, including negative differentiability and specially the case $W^{-1,q}(\Omega)$, which is incidentally of interest here and which we only indicate in (1.13). Note that there is no sequence of partitions here (the partition $\{\omega_{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathbb{N}}$ is fixed, arbitrarily fine close to the boundary $\partial\Omega$). In contrast, the aim of this study is robustness of the constants C_1 and C_2 in (1.12) with respect to all possible partitions $\mathcal{T}_h$ (subject only to regularity), including arbitrary refinement in the interior of the domain Ω.

Finally, we are also interested in the situations where the orthogonality condition (1.10) is not satisfied. In practical applications, this is typically connected with inexact algebraic/nonlinear solvers. Our Theorems 4.1 and 4.3 give two-sided bounds on $\|\mathcal{R}(u_h)\|_{W^{-1,q}(\Omega)}$ in this setting and Corollary 4.8 proves therefrom that the h- and p-*robust* localization result of Theorem 3.7 can be recovered provided that the loss of orthogonality is small with respect to the leading term. In Remark 4.2, we comment that (1.12) holds even without orthogonality condition (1.10), but with C_1 deteriorating with mesh refinement (for decreasing h). This is intuitively consistent with the result (1.13), where the fixed partition is coarse in the interior of Ω and arbitrarily fine only close to the boundary $\partial\Omega$. We collect these results in Section 4, including Theorem 4.10 that presents an extension to vectorial setting. Its typical practical applications stem from fluid mechanics or elasticity; the results established here indeed represent one of the key tools used in [13] for deriving a complete theory of a posteriori error estimation for implicit constitutive relations in the generalized Stokes setting, capturing the most common nonlinear fluid models in a unified way. To conclude, Section 5 illustrates our theoretical findings on several numerical experiments.

2 Setting

We describe the setting and notation in this section, detailing the partition of unity that will be central in our developments. We then state cut-off estimates based on Poincaré–Friedrichs inequalities necessary later.

2.1 Notation, assumptions, and a partition of unity

We suppose that $\Omega\subset\mathbb{R}^d$, $d\ge 1$, is a domain (open, bounded, and connected set) with a Lipschitz-continuous boundary and diameter h_Ω. Let $1\le p\le\infty$ with $\frac{1}{p}+\frac{1}{q}=1$. We will work with standard Sobolev spaces $W^{1,p}(\Omega)$ of functions with $L^p(\Omega)$-integrable weak derivatives, see, e.g., Evans [34], Brenner and Scott [15], and the references therein. The space $W_0^{1,p}(\Omega)$ then stands for functions that are zero in the sense of traces on $\partial\Omega$. Similar notation is used on subdomains of Ω.

For measurable subset $\omega\subset\Omega$ and functions $u\in L^q(\omega)$, $v\in L^p(\omega)$, $(u,v)_\omega$ stands for $\int_\omega u(\boldsymbol{x})\,v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ and similarly $(\boldsymbol{u},\boldsymbol{v})_\omega:=\int_\omega\boldsymbol{u}(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ for $\boldsymbol{u}\in[L^q(\omega)]^d$ and $\boldsymbol{v}\in[L^p(\omega)]^d$; we simply write (u,v) instead of $(u,v)_\Omega$ when $\omega=\Omega$ and similarly in the vectorial case. We follow the convention $\|v\|_{p,\omega}:=$

$\left(\int_\omega |v(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{p}}$ for $1 \le p < \infty$, $\|v\|_{\infty,\omega} := \operatorname{ess\,sup}_{\boldsymbol{x}\in\omega} |v(\boldsymbol{x})|$, $\|\boldsymbol{v}\|_{p,\omega} := \left(\int_\omega |\boldsymbol{v}(\boldsymbol{x})|^p \,\mathrm{d}\boldsymbol{x}\right)^{\frac{1}{p}}$ for $1 \le p < \infty$, and $\|\boldsymbol{v}\|_{\infty,\omega} := \operatorname{ess\,sup}_{\boldsymbol{x}\in\omega} |\boldsymbol{v}(\boldsymbol{x})|$, where $|\boldsymbol{v}| = \left(\sum_{i=1}^d |\boldsymbol{v}_i|^2\right)^{\frac{1}{2}}$ is the Euclidean norm in $\mathbb{R}^d$. Note that, when $p \neq 2$, $\|\nabla v\|_{p,\omega}$ is different from (but equivalent to) $|v|_{1,p,\omega} = \left(\sum_{i=1}^d \|\partial_{\boldsymbol{x}_i} v\|_{p,\omega}^p\right)^{\frac{1}{p}}$ if $1 \le p < \infty$, $|v|_{1,\infty,\omega} = \max_{i=1,\dots,d} \|\partial_{\boldsymbol{x}_i} v\|_{\infty,\omega}$ for $v \in W^{1,p}(\omega)$; we will often use below the equivalence of $l^p(\mathbb{R}^m)$ norms $|\mathrm{v}|_p := \left(\sum_{i=1}^m |\mathrm{v}_i|^p\right)^{\frac{1}{p}}$ if $1 \le p < \infty$, $|\mathrm{v}|_\infty := \max_{i=1,\dots,d} |\mathrm{v}_i|$

$$|\mathrm{v}|_p \le |\mathrm{v}|_q \le m^{\frac{1}{q}-\frac{1}{p}} |\mathrm{v}|_p \qquad \forall \mathrm{v} \in \mathbb{R}^m,\ 1 \le q \le p \le \infty. \tag{2.1}$$

We also denote by $|\cdot|_2$ the spectral matrix norm, given by $|\mathbb{A}|_2 := \max_{\mathrm{v}\in\mathbb{R}^m;\ |\mathrm{v}|_2=1} |\mathbb{A}\mathrm{v}|_2$ for a matrix $\mathbb{A} \in \mathbb{R}^{m\times m}$.

We suppose that there exists a partition of unity

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi_{\boldsymbol{a}} = 1 \qquad \text{a.e. in } \Omega \tag{2.2}$$

by functions $\psi_{\boldsymbol{a}} \in W^{1,\infty}(\Omega)$ with a local support denoted by $\overline{\omega_{\boldsymbol{a}}}$. More precisely, $\omega_{\boldsymbol{a}}$ are open subdomains of the domain Ω of nonzero d-dimensional measure, diameter $h_{\omega_{\boldsymbol{a}}}$, with a Lipschitz-continuous boundary, and satisfying $\cup_{\boldsymbol{a}\in\mathcal{V}_h} \omega_{\boldsymbol{a}} = \Omega$; $\omega_{\boldsymbol{a}}$ is called a *patch*. The index $\boldsymbol{a}$ denotes a point in $\overline{\omega_{\boldsymbol{a}}}$ called a *vertex*, termed interior if $\boldsymbol{a} \in \Omega$ and termed boundary if $\boldsymbol{a} \in \partial\Omega$; the corresponding index sets are $\mathcal{V}_h = \mathcal{V}_h^{\mathrm{int}} \cup \mathcal{V}_h^{\mathrm{ext}}$, $\mathcal{V}_h^{\mathrm{int}} \cap \mathcal{V}_h^{\mathrm{ext}} = \emptyset$. For $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{ext}}$, $\partial\omega_{\boldsymbol{a}} \cap \partial\Omega$ is supposed to have a nonzero $(d-1)$-dimensional measure. We identify $\psi_{\boldsymbol{a}}$ with $\psi_{\boldsymbol{a}}|_{\omega_{\boldsymbol{a}}}$ and suppose that $\psi_{\boldsymbol{a}}$ takes values between 0 and 1 on $\omega_{\boldsymbol{a}}$, $\|\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} = 1$; $\psi_{\boldsymbol{a}}$ is zero in the sense of traces on the whole boundary $\partial\omega_{\boldsymbol{a}}$ for $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$ and on $\partial\omega_{\boldsymbol{a}} \setminus \partial\Omega$ for $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{ext}}$.

The partition of the domain Ω by the patches $\omega_{\boldsymbol{a}}$ needs to be overlapping, i.e., the intersection of several different patches has a nonzero d-dimensional measure. We collect the closures of the minimal intersections into a nonoverlapping partition $\mathcal{T}_h$ of Ω with closed *elements* denoted by K, with diameter h_K. We suppose that each point in Ω lies in at most N_{ov} patches. Equivalently, each $K \in \mathcal{T}_h$ corresponds to the closure of intersection of at most N_{ov} patches, and we collect their vertices $\boldsymbol{a}$ in the set $\mathcal{V}_K$. Vice-versa, the elements $K \in \mathcal{T}_{\boldsymbol{a}}$ cover $\overline{\omega_{\boldsymbol{a}}}$. There in particular holds

$$\left\{\frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|v\|_{p,\omega_{\boldsymbol{a}}}^p\right\}^{\frac{1}{p}} \le \|v\|_p \qquad \forall v \in L^p(\Omega),\ 1 \le p < \infty, \tag{2.3a}$$

$$\max_{\boldsymbol{a}\in\mathcal{V}_h} \|v\|_{\infty,\omega_{\boldsymbol{a}}} = \|v\|_\infty \qquad \forall v \in L^\infty(\Omega). \tag{2.3b}$$

We shall frequently use the patchwise Sobolev spaces given by

$$W_*^{1,p}(\omega_{\boldsymbol{a}}) := \begin{cases} \{v \in W^{1,p}(\omega_{\boldsymbol{a}});\ (v,1)_{\omega_{\boldsymbol{a}}} = 0\} & \text{if } \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}, \\ \{v \in W^{1,p}(\omega_{\boldsymbol{a}});\ v = 0 \text{ on } \partial\omega_{\boldsymbol{a}} \cap \partial\Omega\} & \text{if } \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{ext}}, \end{cases} \tag{2.4}$$

having zero mean value over $\omega_{\boldsymbol{a}}$ in the first case and vanishing trace on the boundary of Ω in the second case. The Poincaré–Friedrichs inequality then states that

$$\|v\|_{p,\omega_{\boldsymbol{a}}} \le C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla v\|_{p,\omega_{\boldsymbol{a}}} \qquad \forall v \in W_*^{1,p}(\omega_{\boldsymbol{a}}), \tag{2.5}$$

where, recall, $h_{\omega_{\boldsymbol{a}}}$ stands for the diameter of the patch $\omega_{\boldsymbol{a}}$. In particular, for $1 < p < \infty$, $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, and $\omega_{\boldsymbol{a}}$ convex, $C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} = 2\left(\frac{p}{2}\right)^{\frac{1}{p}}$, see Chua and Wheeden [25]; for $p = 1$, $C_{\mathrm{PF},1,\omega_{\boldsymbol{a}}} = \frac{1}{2}$ in this setting, see Acosta and Durán [1] or [25], and for $p = \infty$, $C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} = 1$ is straightforward. This implies that $\frac{1}{2} \le C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} \le C_{\mathrm{PF},2\mathrm{e},\omega_{\boldsymbol{a}}} = 2\,\mathrm{e}^{\frac{1}{2\mathrm{e}}} \approx 2.404$ for all $1 \le p \le \infty$ and a convex interior patch. The values for $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$ and nonconvex patches $\omega_{\boldsymbol{a}}$ are identified in, e.g., Veeser and Verfürth [50, Theorems 3.1 and 3.2] for $1 \le p < \infty$; whenever $\omega_{\boldsymbol{a}}$ is star-shaped, $C_{\mathrm{PF},\infty,\omega_{\boldsymbol{a}}} = 2$. Finally, $C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} = 1$ for $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{ext}}$ when $\partial\omega_{\boldsymbol{a}} \cap \partial\Omega$ can be reached in a constant direction from any point inside $\omega_{\boldsymbol{a}}$; bounds in the general case can be obtained for instance as in [49, Lemma 5.1]. We describe the *regularity* of the partition by the number

$$C_{\mathrm{cont,PF}} := \max_{\boldsymbol{a}\in\mathcal{V}_h} \{1 + C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\}, \tag{2.6}$$

which we suppose to be uniformly bounded on families of the considered partitions indexed by the parameter $h := \max_{\boldsymbol{a}\in\mathcal{V}_h} h_{\omega_{\boldsymbol{a}}}$.

2.2 Examples of partitions of unity

We now give three examples of possible partitions of unity $\psi_{\boldsymbol{a}}$ and subdomains $\omega_{\boldsymbol{a}}$.

Example 2.1 (Simplicial or parallelepipedal meshes from the finite element context)**.** *A prototypal example we have in mind is the case where Ω is a polytope, $\cup_{K\in\mathcal{T}_h} K = \overline{\Omega}$, each element K is a closed d-dimensional simplex (triangle in $d = 2$, tetrahedron in $d = 3$) or a d-dimensional parallelepiped (quadrilateral in $d = 2$, hexahedron in $d = 3$), and the intersection of two different elements K is either empty or their d'-dimensional common face, $0 \le d' \le d-1$. Then $N_{\mathrm{ov}} = d+1$ for simplices and $N_{\mathrm{ov}} = 2^d$ for parallelepipeds, $\omega_{\boldsymbol{a}}$ is the patch of all elements sharing the given vertex $\boldsymbol{a} \in \mathcal{V}_h$, and (2.3a) takes form of equality. In particular, for the seminorm on $W^{1,p}(\Omega)$,*

$$\frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla v\|_{p,\omega_{\boldsymbol{a}}}^p = \sum_{K\in\mathcal{T}_h} \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_K} \|\nabla v\|_{p,K}^p = \sum_{K\in\mathcal{T}_h} \|\nabla v\|_{p,K}^p = \|\nabla v\|_p^p \qquad \forall v \in W^{1,p}(\Omega)\,,\ 1 \le p < \infty, \tag{2.7a}$$

$$\max_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla v\|_{\infty,\omega_{\boldsymbol{a}}} = \|\nabla v\|_\infty \qquad \forall v \in W^{1,\infty}(\Omega)\,. \tag{2.7b}$$

We then take $\psi_{\boldsymbol{a}}$ as the continuous, piecewise (d-)affine "hat" function of finite element analysis, taking value 1 at the vertex $\boldsymbol{a} \in \mathcal{V}_h$ and 0 in all other vertices from $\mathcal{V}_h$. Denoting by $\kappa_{\mathcal{T}_h}$ the mesh shape-regularity parameter given by the maximal ratio of the diameter of K to the diameter of the largest ball inscribed into K over all $K \in \mathcal{T}_h$, it follows from Veeser and Verfürth [50, Theorems 3.1 and 3.2], Carstensen and Funken [22], or Braess et al. *[14] that both $C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}}$ of (2.5) and $C_{\mathrm{cont,PF}}$ of (2.6) only depend on $\kappa_{\mathcal{T}_h}$. Note further that in the context of approximation of the solution of a partial differential equation by the finite element method, with the residual $\mathcal{R}$ described in Remark 3.2 below, the crucial orthogonality condition (3.20) amounts to requesting the presence of the hat functions $\psi_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, in the finite element basis.*

Example 2.2 (B-splines supports from the isogeometric analysis context)**.** *Let the space dimension $d = 1$, let Ω be an interval, and let $\mathcal{T}_h$ be a mesh of Ω consisting of intervals K of size h_K, $\cup_{K\in\mathcal{T}_h} K = \overline{\Omega}$. B-splines are non-negative piecewise (with respect to $\mathcal{T}_h$) polynomials of degree k and class C^l, $k \ge 1$, $0 \le l \le k-1$, with smallest possible support and given scaling; typically $l = k-1$, i.e., one requests continuity of the derivatives up to order $k-1$. Denoting them $\psi_{\boldsymbol{a}}$, the subdomains $\omega_{\boldsymbol{a}}$ can simply be taken as the supports of the B-splines $\psi_{\boldsymbol{a}}$. Then the vertices $\boldsymbol{a}$ that form the set $\mathcal{V}_h$ lie inside $\omega_{\boldsymbol{a}}$ if the value of $\psi_{\boldsymbol{a}}$ on the boundary of the domain Ω is zero, and are the corresponding endpoints of Ω otherwise. Crucially, the partition of unity (2.2) holds for B-splines. For $k = 1$ and $l = 0$ (piecewise affine functions with C^0 continuity), this setting coincides with the finite element context of Remark 2.1. In general, however, the subdomains $\omega_{\boldsymbol{a}}$ are larger here, leading to increased overlap between $\omega_{\boldsymbol{a}}$ and higher value of the overlap parameter N_{ov}, in dependence on the continuity parameter l. In the context of the partial differential equation residual $\mathcal{R}$ of Remark 3.2 below, the orthogonality condition (3.20) amounts to the use of the B-splines/isogeometric analysis approximation, see Bazilevs* et al. *[7] or Buffa and Giannelli [17] and the references therein. Extension to higher space dimensions $d > 1$ is straightforward by tensor products for Ω being a rectangular parallelepiped; general domains can be treated via non-uniform rational basis splines (NURBS) and transformation from the parametric space into the physical space.*

Example 2.3 (Meshfree methods)**.** *In general, the approach developed here can be applied to any setting that is based on the idea of basis functions having local (small, compact) support and forming the partition of unity (2.2). The partition of unity method, see Babuška and Melenk [42, 3], and in general meshfree methods, see [36] and the references therein, can serve as examples.*

2.3 Poincaré–Friedrichs cut-off estimates

The forthcoming result, following the lines of Carstensen and Funken [22, Theorem 3.1] or Braess *et al.* [14, Section 3], with $W^{1,p}(\omega_{\boldsymbol{a}})$-Poincaré–Friedrichs inequalities of Chua and Wheeden [25] and Veeser and Verfürth [50], will form the basic building block for our considerations:

Lemma 2.4 (Cut-off estimate)**.** *For the constant $C_{\mathrm{cont,PF}}$ from (2.6), there holds, for all $\boldsymbol{a} \in \mathcal{V}_h$,*

$$\|\nabla(\psi_{\boldsymbol{a}} v)\|_{p,\omega_{\boldsymbol{a}}} \le C_{\mathrm{cont,PF}} \|\nabla v\|_{p,\omega_{\boldsymbol{a}}} \qquad \forall v \in W_*^{1,p}(\omega_{\boldsymbol{a}}),\ 1 \le p \le \infty.$$

Proof. Let $\boldsymbol{a} \in \mathcal{V}_h$. We have, employing the triangle inequality, $\|\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} = 1$, and (2.5),

$$\begin{aligned}
\|\nabla(\psi_{\boldsymbol{a}} v)\|_{p,\omega_{\boldsymbol{a}}} &= \|\nabla\psi_{\boldsymbol{a}} v + \psi_{\boldsymbol{a}} \nabla v\|_{p,\omega_{\boldsymbol{a}}} \\
&\le \|\nabla\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \|v\|_{p,\omega_{\boldsymbol{a}}} + \|\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \|\nabla v\|_{p,\omega_{\boldsymbol{a}}} \\
&\le \left(1 + C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\right) \|\nabla v\|_{p,\omega_{\boldsymbol{a}}},
\end{aligned}$$

and the assertion follows from the definition (2.6). □

2.4 An overlapping-patches estimate

We finally provide an auxiliary coloring-type estimate for a sum of functions from $W_0^{1,p}(\omega_{\boldsymbol{a}})$ that will be used later.

Lemma 2.5 (An overlapping-patches estimate)**.** *Let $1 \le p \le \infty$. Assume there is a collection of functions $\{v^{\boldsymbol{a}}\}_{\boldsymbol{a}\in\mathcal{V}_h}$ with $v^{\boldsymbol{a}} \in W_0^{1,p}(\omega_{\boldsymbol{a}})$, extended by zero to $W_0^{1,p}(\Omega)$. Then $\sum_{\boldsymbol{a}\in\mathcal{V}_h} v^{\boldsymbol{a}} \in W_0^{1,p}(\Omega)$ and*

$$\left\|\nabla \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} v^{\boldsymbol{a}}\right\|_p \le \left\{\frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla v^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}}^p\right\}^{\frac{1}{p}} \qquad \text{if } 1 \le p < \infty, \tag{2.8a}$$

$$\left\|\nabla \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} v^{\boldsymbol{a}}\right\|_\infty \le \max_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla v^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \qquad \text{if } p = \infty. \tag{2.8b}$$

Proof. Let $1 \le p < \infty$. Fix $K \in \mathcal{T}_h$ and denote the number of vertices in K by $|\mathcal{V}_K|$. The triangle and Hölder inequalities give, a.e. in K,

$$\left|\sum_{\boldsymbol{a}\in\mathcal{V}_K} \nabla v^{\boldsymbol{a}}\right| \le \sum_{\boldsymbol{a}\in\mathcal{V}_K} |\nabla v^{\boldsymbol{a}}| \le \left\{\sum_{\boldsymbol{a}\in\mathcal{V}_K} |\nabla v^{\boldsymbol{a}}|^p\right\}^{\frac{1}{p}} |\mathcal{V}_K|^{\frac{1}{q}}.$$

By assumption it holds $|\mathcal{V}_K| \le N_{\mathrm{ov}}$ for every $K \in \mathcal{T}_h$, so that, since $\frac{p}{q} = p - 1$,

$$\left|\nabla \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_K} v^{\boldsymbol{a}}\right|^p \le \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_K} |\nabla v^{\boldsymbol{a}}|^p. \tag{2.9}$$

Integrating both sides of (2.9) over K, summing over all $K \in \mathcal{T}_h$, and taking $\frac{1}{p}$-th power of the result gives (2.8a). Estimate (2.8b) is trivial. □

3 Localization of dual functional norms

In this section we state and prove our main localization result; we also present some of its consequences. We first fix some notations and introduce the overall context in more detail.

3.1 Context

For given $p \in [1, \infty]$, denote

$$V := W_0^{1,p}(\Omega) \tag{3.1}$$

and consider a bounded linear functional $\mathcal{R} \in V'$. We denote the action of $\mathcal{R}$ on $v \in V$ by $\langle \mathcal{R}, v\rangle_{V',V}$ and define

$$\|\mathcal{R}\|_{V'} := \sup_{v\in V;\, \|\nabla v\|_p = 1} \langle \mathcal{R}, v\rangle_{V',V}. \tag{3.2}$$

Similarly, for vertex $\boldsymbol{a} \in \mathcal{V}_h$ and the corresponding patch subdomain $\omega_{\boldsymbol{a}}$, set

$$V^{\boldsymbol{a}} := W_0^{1,p}(\omega_{\boldsymbol{a}})$$

and define the restriction of the functional $\mathcal{R}$ to $V^{\boldsymbol{a}}$, still denoted by $\mathcal{R}$, via

$$\langle \mathcal{R}, v \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}} := \langle \mathcal{R}, v \rangle_{V', V} \qquad v \in V^{\boldsymbol{a}}, \tag{3.3}$$

where $v \in V^{\boldsymbol{a}}$ is extended by zero outside of the patch $\omega_{\boldsymbol{a}}$ to $v \in V$. Let

$$\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} := \sup_{v \in V^{\boldsymbol{a}};\, \|\nabla v\|_{p,\omega_{\boldsymbol{a}}} = 1} \langle \mathcal{R}, v \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}}. \tag{3.4}$$

3.2 Examples of functionals $\mathcal{R}$

To fix ideas, we give two examples fitting in the context of Section 3.1.

Example 3.1 ($\mathcal{R}$ being divergence of an integrable function)**.** *Let $\boldsymbol{\xi} \in [L^q(\Omega)]^d$. A simple example of $\mathcal{R} \in V'$ is*

$$\langle \mathcal{R}, v \rangle_{V', V} := (\boldsymbol{\xi}, \nabla v) \qquad v \in V. \tag{3.5}$$

In this case, immediately, for any $\boldsymbol{a} \in \mathcal{V}_h$,

$$\langle \mathcal{R}, v \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}} = (\boldsymbol{\xi}, \nabla v)_{\omega_{\boldsymbol{a}}} \qquad v \in V^{\boldsymbol{a}}.$$

Moreover, using definitions (3.2) and (3.4), we easily obtain via the Hölder inequality the bounds

$$\|\mathcal{R}\|_{V'} \le \|\boldsymbol{\xi}\|_q, \tag{3.6a}$$

$$\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \le \|\boldsymbol{\xi}\|_{q,\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{3.6b}$$

Example 3.2 ($\mathcal{R}$ given by a residual of a partial differential equation)**.** *Let $1 \le p \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $u^{\mathrm{D}} \in W^{1,p}(\Omega)$, $f \in L^q(\Omega)$, and let $(u, \boldsymbol{\sigma})$ be a weak solution*[1] *to the problem*

$$-\operatorname{div} \boldsymbol{\sigma} = f \qquad \text{in } \Omega, \tag{3.9a}$$

$$u = u^{\mathrm{D}} \qquad \text{on } \partial\Omega, \tag{3.9b}$$

$$\boldsymbol{h}(\boldsymbol{\sigma}, \nabla u) = \boldsymbol{0} \qquad \text{in } \Omega. \tag{3.9c}$$

[1] Assuming $1 < p < \infty$, weak solution to problem (3.9) can be defined as: to find $u : \Omega \to \mathbb{R}$ and $\boldsymbol{\sigma} : \Omega \to \mathbb{R}^d$ such that

$$u - u^{\mathrm{D}} \in V, \tag{3.7a}$$

$$\boldsymbol{\sigma} \in [L^q(\Omega)]^d, \tag{3.7b}$$

$$(\boldsymbol{\sigma}, \nabla v) = (f, v) \qquad \forall v \in V, \tag{3.7c}$$

$$\boldsymbol{h}(\boldsymbol{\sigma}, \nabla u) = \boldsymbol{0} \qquad \text{almost everywhere in } \Omega. \tag{3.7d}$$

This problem has at least one solution if, for example, the function $\boldsymbol{h}$ fulfills, with some $\alpha, \beta > 0$, the following conditions:

1. $\boldsymbol{h}(\boldsymbol{0}, \boldsymbol{0}) = \boldsymbol{0}$;
2. if $\boldsymbol{s}^1, \boldsymbol{s}^2, \boldsymbol{d}^1, \boldsymbol{d}^2 \in \mathbb{R}^d$ and $\boldsymbol{h}(\boldsymbol{s}^1, \boldsymbol{d}^1) = \boldsymbol{h}(\boldsymbol{s}^2, \boldsymbol{d}^2) = \boldsymbol{0}$ then
$$(\boldsymbol{s}^1 - \boldsymbol{s}^2) \cdot (\boldsymbol{d}^1 - \boldsymbol{d}^2) \ge 0; \tag{3.8}$$
3. if the couple $(\boldsymbol{s}, \boldsymbol{d}) \in \mathbb{R}^d \times \mathbb{R}^d$ fulfills
$$(\boldsymbol{s} - \tilde{\boldsymbol{s}}) \cdot (\boldsymbol{d} - \tilde{\boldsymbol{d}}) \ge 0 \qquad \text{for all } \tilde{\boldsymbol{s}}, \tilde{\boldsymbol{d}} \in \mathbb{R}^d \text{ with } \boldsymbol{h}\left(\tilde{\boldsymbol{s}}, \tilde{\boldsymbol{d}}\right) = \boldsymbol{0},$$
then $(\boldsymbol{s}, \boldsymbol{d})$ also fulfills $\boldsymbol{h}(\boldsymbol{s}, \boldsymbol{d}) = \boldsymbol{0}$;
4. if $\boldsymbol{h}(\boldsymbol{s}, \boldsymbol{d}) = \boldsymbol{0}$ then (3.10) holds;

see [35] and also [18, 20, 19] for fluid mechanics context. If, in addition, inequality (3.8) is strict whenever $\boldsymbol{s}^1 \ne \boldsymbol{s}^2$ and $\boldsymbol{d}^1 \ne \boldsymbol{d}^2$, then such a solution is unique.

For a novel theory of weak solutions in the non-reflexive case $p = \infty$ and within the context of solid mechanics, we refer the interested reader to [8].

Here $\boldsymbol{\sigma} \in [L^q(\Omega)]^d$, $u \in W^{1,p}(\Omega)$ *such that* $u - u^{\mathrm{D}} \in W_0^{1,p}(\Omega)$, *and a nonlinear function* $\boldsymbol{h} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *is such that it holds, with some* $\alpha, \beta > 0$,

$$\boldsymbol{s}\cdot\boldsymbol{d} \geq \alpha\big(|\boldsymbol{s}|^q + |\boldsymbol{d}|^p\big) - \beta \qquad \text{whenever } \boldsymbol{s}, \boldsymbol{d} \in \mathbb{R}^d \text{ and } \boldsymbol{h}(\boldsymbol{s}, \boldsymbol{d}) = \boldsymbol{0}. \tag{3.10}$$

Typical examples of function $\boldsymbol{h}$ *are*

$$\left.\begin{aligned} \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{s} - \big(1 + |\boldsymbol{d}|^2\big)^{\frac{p-2}{2}}\boldsymbol{d}, \\ \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{s} - \big(1 + |\boldsymbol{d}|\big)^{p-2}\boldsymbol{d}, \end{aligned}\right\} \text{regularized } p\text{-Laplace}$$

$$\left.\begin{aligned} \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{d} - \big(1 + |\boldsymbol{s}|^2\big)^{\frac{q-2}{2}}\boldsymbol{s}, \\ \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{d} - \big(1 + |\boldsymbol{s}|\big)^{q-2}\boldsymbol{s}, \end{aligned}\right\} \text{generalized } p\text{-Laplace}$$

$$\left.\begin{aligned} \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{d} - \frac{\big(|\boldsymbol{s}| - \sigma_*\big)^+}{|\boldsymbol{s}|}\big(1 + |\boldsymbol{s}|^2\big)^{\frac{q-2}{2}}\boldsymbol{s}, \\ \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{d} - \frac{\big(|\boldsymbol{s}| - \sigma_*\big)^+}{|\boldsymbol{s}|}\big(1 + |\boldsymbol{d}|^2\big)^{-\frac{p-2}{2}}\boldsymbol{s}, \\ \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{s} - \frac{\big(|\boldsymbol{d}| - \delta_*\big)^+}{|\boldsymbol{d}|}\big(1 + |\boldsymbol{d}|^2\big)^{\frac{p-2}{2}}\boldsymbol{d}, \end{aligned}\right\} \text{activated } p\text{-Laplace}$$

$$\left.\begin{aligned} \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{s} - |\boldsymbol{d}|^{p-2}\boldsymbol{d}, \\ \boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) &= \boldsymbol{d} - |\boldsymbol{s}|^{q-2}\boldsymbol{s}, \end{aligned}\right\} \text{classical } p\text{-Laplace}$$

where $(t)^+ = \max(t, 0)$ *and* $\sigma_*, \delta_* \geq 0$ *are given real parameters. Note that the last two examples give identical response since*

$$\boldsymbol{s} = |\boldsymbol{d}|^{p-2}\boldsymbol{d} \qquad \Longleftrightarrow \qquad \boldsymbol{d} = |\boldsymbol{s}|^{q-2}\boldsymbol{s}.$$

Consequently

$$\boldsymbol{s}\cdot\boldsymbol{d} = \left(\frac{1}{p} + \frac{1}{q}\right)\boldsymbol{s}\cdot\boldsymbol{d} = \frac{|\boldsymbol{s}|^q}{q} + \frac{|\boldsymbol{d}|^p}{p}$$

which not only verifies, but also motivates the assumption (3.10) *above. To verify that the other models fulfill* (3.10), *we refer to [19, Lemma 1.1]. Finally, the responses given by*

$$\boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) = \boldsymbol{s} - \frac{\boldsymbol{d}}{\big(1 + |\boldsymbol{d}|^a\big)^{\frac{1}{a}}}, \Big\} \text{ flux limiting } p\text{-Laplace}$$

$$\boldsymbol{h}(\boldsymbol{s},\boldsymbol{d}) = \boldsymbol{d} - \frac{\boldsymbol{s}}{\big(1 + |\boldsymbol{s}|^b\big)^{\frac{1}{b}}}, \Big\} \text{ gradient limiting } p\text{-Laplace}$$

with some $a, b \in (0, \infty)$ *give automatically* $\boldsymbol{\sigma} \in [L^\infty(\Omega)]^d$, $\nabla u \in [L^1(\Omega)]^d$, *respectively* $\boldsymbol{\sigma} \in [L^1(\Omega)]^d$, $\nabla u \in [L^\infty(\Omega)]^d$ *and concern the limit cases* $p = 1$, $p = \infty$. *We refer to [12], where such models are summarized in the context of fluid mechanics, and [21, 40] for examples from solid mechanics. This general setting with implicit function* $\boldsymbol{h}$ *is, for example, interesting to employ mixed finite element methods. In fluid mechanics context, this has been studied in [37, 13].*

The above rather complex example still fits perfectly into our setting. Indeed, let $\boldsymbol{\sigma}_h \in [L^q(\Omega)]^d$ *be an arbitrary approximation to* $\boldsymbol{\sigma}$. *Then we can define a linear functional* $\mathcal{R}$ *on the space* V *as*

$$\langle \mathcal{R}, v\rangle_{V',V} := (f, v) - (\boldsymbol{\sigma}_h, \nabla v) \qquad v \in V.$$

Note that the Hölder inequality and the Poincaré–Friedrichs inequality (2.5), *used in the entire domain* Ω *on the space* V, *imply that*

$$|\langle \mathcal{R}, v\rangle| \leq \big(\|f\|_q C_{\mathrm{PF},p,\Omega} h_\Omega + \|\boldsymbol{\sigma}_h\|_q\big)\,\|\nabla v\|_p.$$

Consequently, $\mathcal{R}$ *is indeed bounded,* $\mathcal{R} \in V'$. *To complement, let also* $u_h \in W^{1,p}(\Omega)$, $u_h - u^{\mathrm{D}} \in V$, *be an arbitrary approximation to* u. *Then one in general also wishes to measure a deviation from equality* (3.9c)

when $\boldsymbol{\sigma}_h$ together with u_h are plugged therein in place of $\boldsymbol{\sigma}$ and u. There are various ways to evaluate this error; compare, e.g., [39, 13].

For the rest of this example, we limit ourselves to the following specific but important subcase: $1 < p < \infty$ and the implicit function $\boldsymbol{h}$ admits an explicit continuous representation $\boldsymbol{s} = \boldsymbol{\sigma}(\boldsymbol{d})$;[2] more precisely we assume that all solutions $(\boldsymbol{s}, \boldsymbol{d}) \in \mathbb{R}^d \times \mathbb{R}^d$ of $\boldsymbol{h}(\boldsymbol{s}, \boldsymbol{d}) = \boldsymbol{0}$ are given by $\boldsymbol{s} = \boldsymbol{\sigma}(\boldsymbol{d})$ with continuous $\boldsymbol{\sigma} : \mathbb{R}^d \to \mathbb{R}^d$. Then the weak formulation of problem (3.9) simplifies to: find $u \in W^{1,p}(\Omega)$ such that

$$u - u^{\mathrm{D}} \in V, \tag{3.11a}$$

$$(\boldsymbol{\sigma}(\nabla u), \nabla v) = (f, v) \qquad \forall v \in V \tag{3.11b}$$

and admits at least one weak solution under classical assumptions.[3] This gives rise to the standard notion of the residual $\mathcal{R}$ of an arbitrary function $u_h \in W^{1,p}(\Omega)$ such that $u_h - u^{\mathrm{D}} \in V$, defined via

$$\langle \mathcal{R}, v \rangle_{V',V} := (f, v) - (\boldsymbol{\sigma}(\nabla u_h), \nabla v) \qquad v \in V. \tag{3.12}$$

The Hölder inequality and (2.5) again imply that $\mathcal{R} \in V'$, since

$$|\langle \mathcal{R}, v \rangle| \leq (\|f\|_q C_{\mathrm{PF},p,\Omega} h_\Omega + \|\boldsymbol{\sigma}(\nabla u_h)\|_q) \|\nabla v\|_p.$$

Here, actually, $\mathcal{R} = 0$ if and only if u_h solves (3.11b). Then $\|\mathcal{R}\|_{V'}$ is the intrinsic distance of u_h to u, *the* dual norm of the residual. *Remark that this problem can also be cast in the form of Example 3.1, taking $\boldsymbol{\xi} := \boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)$, with any $u \in W^{1,p}(\Omega)$ solving (3.11).*

3.3 Motivation

We now give four remarks motivating our main question whether $\|\mathcal{R}\|_{V'}$, a priori just a number defined for any $\mathcal{R} \in V'$, expressing its size over the entire computational domain Ω, can be bounded from above and from below by the sizes $\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}$ of $\mathcal{R}$ localized over the patches $\omega_{\boldsymbol{a}}$.

Remark 3.3 (Localization of the $\boldsymbol{L}^q(\Omega)$-norm error in the fluxes)**.** *Consider $\mathcal{R}$ given by (3.12) from Example 3.2 in the finite element context of Remark 2.1. We immediately obtain from (3.6a) and (3.6b)*

$$\|\mathcal{R}\|_{V'} \leq \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_q, \tag{3.13a}$$

$$\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \leq \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_{q,\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h, \tag{3.13b}$$

and observe that the flux error norm on the right-hand side of (3.13a) localizes, as in (2.7), into the right-hand sides of (3.13b) by the formula

$$\|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_q = \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_{q,\omega_{\boldsymbol{a}}}^q \right\}^{\frac{1}{q}}. \tag{3.14}$$

Note that, unfortunately, it is unclear when (3.14) is, up to a constant, bounded back by $\|\mathcal{R}\|_{V'}$, so that these considerations do not give an answer to the question of localization of $\|\mathcal{R}\|_{V'}$.

[2] If $\boldsymbol{h}(\boldsymbol{s}, \boldsymbol{d}) = 0$ does not admit explicit solution $\boldsymbol{s} = \boldsymbol{\sigma}(\boldsymbol{d})$, which happens for some examples given above, one can approximate up to (in certain sense) arbitrary precision, by explicit relation $\boldsymbol{s} = \boldsymbol{\sigma}^\epsilon(\boldsymbol{d})$, and later pass in the limit $\epsilon \to 0+$. This is an approach of many studies, ranging from PDE analysis to a priori convergence of finite element schemes; see, e.g., [35, 18, 19, 29, 39].

[3] This holds, for example, if

1. $\boldsymbol{\sigma} : \mathbb{R}^d \to \mathbb{R}^d$ is continuous,
2. $\boldsymbol{\sigma}(\boldsymbol{0}) = \boldsymbol{0}$,
3. $(\boldsymbol{\sigma}(\boldsymbol{d}_1) - \boldsymbol{\sigma}(\boldsymbol{d}_2)) \cdot (\boldsymbol{d}_1 - \boldsymbol{d}_2) \geq 0$ for all $\boldsymbol{d}_1, \boldsymbol{d}_2 \in \mathbb{R}^d$,
4. $C_1 |\boldsymbol{d}|^p \leq \boldsymbol{\sigma}(\boldsymbol{d}) \cdot \boldsymbol{d}$, for all $\boldsymbol{d} \in \mathbb{R}^d$,
5. $|\boldsymbol{\sigma}(\boldsymbol{d})| \leq C_2 (1 + |\boldsymbol{d}|)^{p-1}$ for all $\boldsymbol{d} \in \mathbb{R}^d$.

See, e.g. [41].

Remark 3.4 ($W_0^{1,p}(\Omega)$-norm error localization)**.** *Remark that similarly to* (1.2)*, there always holds, for* $1 \le p < \infty$*,*

$$\|\nabla v\|_p = \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla v\|_{p,K}^p \right\}^{\frac{1}{p}}, \qquad v \in V.$$

In particular, in the context of Example 3.2, on meshes from Remark 2.1, for $1 < p < \infty$*,*

$$\|\nabla (u - u_h)\|_p = \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla (u - u_h)\|_{p,K}^p \right\}^{\frac{1}{p}}. \tag{3.15}$$

The $W_0^{1,p}(\Omega)$*-norm* $\|\nabla \cdot\|_p$ *is always localizable, but it seems difficult/suboptimal to derive a posteriori error estimates of the form* (1.9)*,* (1.11) *for* $\|\nabla (u - u_h)\|_p$ *in place of* $\|\mathcal{R}\|_{V'}$*, see, e.g., the discussions in Belenki* et al. *[9] and [32].*

Remark 3.5 (Energy difference/quasi-norm error localization)**.** *As mentioned in the introduction, still in the context* (3.11) *of Example 3.2, there are other possible substitutes used in both a priori and a posteriori error analysis. Besides* $W_0^{1,p}(\Omega)$*-norm error* $\|\nabla (u - u_h)\|_p$*, the energy difference* $\mathcal{E}(u_h) - \mathcal{E}(u)$*, where the energy is defined by* (1.8)*, is used mostly for the problem involving the p-Laplace or its nondegenerate/nonsingular modifications. Following Kreuzer [28, Lemma 16] or Belenki* et al. *[9, Lemma 3.2], there holds*

$$\mathcal{E}(u_h) - \mathcal{E}(u) \approx \|\nabla (u - u_h)\|_{(p)}^2 \approx \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\|^2, \tag{3.16}$$

where $\|\cdot\|_{(p)}$ *is the quasi-norm of Barrett and Liu [5, 6] and* $\boldsymbol{F}(\boldsymbol{v}) := |\boldsymbol{v}|^{\frac{p-2}{2}} \boldsymbol{v}$*. Here* $\|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\|^2 = \sum_{K \in \mathcal{T}_h} \|\boldsymbol{F}(\nabla u) - \boldsymbol{F}(\nabla u_h)\|_K^2$ *localizes immediately. However, unfortunately, the constants hidden in* (3.16) *depend on the Lebesgue exponent p.*

Remark 3.6 (Localization of the p-Laplacian lifting of $\mathcal{R}$)**.** *Let* $1 < p < \infty$*. Let* $\mathscr{z} \in V$ *be the analogue of the Riesz representation of the functional* $\mathcal{R}$ *by the p-Laplacian solve on* Ω*, i.e.,* $\mathscr{z} \in V$ *is such that*

$$\left(|\nabla \mathscr{z}|^{p-2} \nabla \mathscr{z}, \nabla v\right) = \langle \mathcal{R}, v \rangle_{V',V} \qquad \forall v \in V. \tag{3.17}$$

Then, we readily obtain

$$\|\nabla \mathscr{z}\|_p^p = \left(|\nabla \mathscr{z}|^{p-2} \nabla \mathscr{z}, \nabla \mathscr{z}\right) = \langle \mathcal{R}, \mathscr{z} \rangle_{V',V} = \|\mathcal{R}\|_{V'}^q. \tag{3.18}$$

Consequently, on meshes $\mathcal{T}_h$ *from Remark 2.1,*

$$\|\nabla \mathscr{z}\|_p = \left\{ \sum_{K \in \mathcal{T}_h} \|\nabla \mathscr{z}\|_{p,K}^p \right\}^{\frac{1}{p}} \tag{3.19}$$

suggests itself as a way to measure the error with localization and a posteriori estimate of the form (1.9)*. Also an equivalent of* (1.11)*,*

$$\eta_K(u_h) \le C \left\{ \sum_{K' \in \mathcal{T}_K} \|\nabla \mathscr{z}\|_{K'}^p \right\}^{\frac{1}{q}},$$

would hold. The trouble here is that (3.17) *is a* nonlocal *problem, obtained itself by a global solve. Remark also that the definition of the lifting* $\mathscr{z}$ *by* (3.17) *is dictated by the choice of the space* V *in* (3.1) *together with its norm* $\|\nabla \cdot\|_p$*.*[4]

[4] Consider an alternative choice of the space, $V := \{v \in W^{1,p}(\Omega); (v, 1) = 0\}$ with the norm $\|\nabla v\|_p$. We have $\|\mathcal{R}\|_{V'} := \sup_{v \in V; \|\nabla v\|_p = 1} \langle \mathcal{R}, v \rangle_{V',V}$, as in (3.2). For $\mathcal{R} \in V'$, one can define a lifting $\mathscr{z} \in V$ as a solution of the Neumann p-Laplace problem $-\operatorname{div}\left(|\nabla \mathscr{z}|^{p-2} \nabla \mathscr{z}\right) = \mathcal{R}$ in Ω; $|\nabla \mathscr{z}|^{p-2} \nabla \mathscr{z} \boldsymbol{n} = 0$ on $\partial\Omega$; $(\mathscr{z}, 1) = 0$. The weak formulation (3.17), the $W^{1,p}(\Omega)$-norm equality (3.18), and the localization (3.19) hold with the appropriate replacement of V.

3.4 Main result

Recall that $1 \le p \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $V = W_0^{1,p}(\Omega)$, the partition $\cup_{\boldsymbol{a}\in\mathcal{V}_h}\omega_{\boldsymbol{a}}$ covers the domain Ω with maximal overlap N_{ov}, the patches $\omega_{\boldsymbol{a}}$ are indexed by the vertices $\boldsymbol{a}$ where $\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}$ lies inside Ω and $\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{ext}}$ on the boundary of Ω, and that the constant $C_{\mathrm{cont,PF}}$ from (2.6) is supposed uniformly bounded for different partitions.

Our localization result is:

Theorem 3.7 (Localization of dual norms of functionals with $\psi_{\boldsymbol{a}}$-orthogonality)**.** *Let $\mathcal{R}\in V'$ be arbitrary. Let*

$$\langle \mathcal{R}, \psi_{\boldsymbol{a}}\rangle_{V',V} = 0 \qquad \forall \boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}. \tag{3.20}$$

Then, when $1<p\le\infty$,

$$\|\mathcal{R}\|_{V'} \le N_{\mathrm{ov}} C_{\mathrm{cont,PF}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} \right\}^{\frac{1}{q}}, \tag{3.21a}$$

$$\left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} \right\}^{\frac{1}{q}} \le \|\mathcal{R}\|_{V'}, \tag{3.21b}$$

and, when $p=1$,

$$\|\mathcal{R}\|_{V'} \le N_{\mathrm{ov}} C_{\mathrm{cont,PF}} \max_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}, \tag{3.22a}$$

$$\max_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \le \|\mathcal{R}\|_{V'}. \tag{3.22b}$$

Condition (3.20) is actually only needed in (3.21a) and (3.22a).

Proof. We first show (3.21a) and (3.22a). Let $v\in V$ with $\|\nabla v\|_p = 1$ be fixed. The partition of unity (2.2), the linearity of $\mathcal{R}$, definition (3.3), and the orthogonality requirement (3.20) give

$$\begin{aligned}\langle \mathcal{R}, v\rangle_{V',V} &= \sum_{\boldsymbol{a}\in\mathcal{V}_h} \langle \mathcal{R}, \psi_{\boldsymbol{a}} v\rangle_{V',V} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \langle \mathcal{R}, \psi_{\boldsymbol{a}} v\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}} \\ &= \sum_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} \langle \mathcal{R}, \psi_{\boldsymbol{a}}\left(v-\Pi_{0,\omega_{\boldsymbol{a}}} v\right)\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}} + \sum_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{ext}}} \langle \mathcal{R}, \psi_{\boldsymbol{a}} v\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}},\end{aligned} \tag{3.23}$$

where $\Pi_{0,\omega_{\boldsymbol{a}}} v$ is the mean value of the test function v on the patch $\omega_{\boldsymbol{a}}$. There holds $\left(v-\Pi_{0,\omega_{\boldsymbol{a}}} v\right)|_{\omega_{\boldsymbol{a}}} \in W_*^{1,p}(\omega_{\boldsymbol{a}})$, where $W_*^{1,p}(\omega_{\boldsymbol{a}})$ is defined by (2.4), and $\left(\psi_{\boldsymbol{a}}\left(v-\Pi_{0,\omega_{\boldsymbol{a}}} v\right)\right)|_{\omega_{\boldsymbol{a}}} \in V^{\boldsymbol{a}}$ for $\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}$. Thus, using (3.4) and Lemma 2.4 yields, for $\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}$,

$$\begin{aligned}\langle \mathcal{R}, \psi_{\boldsymbol{a}}\left(v-\Pi_{0,\omega_{\boldsymbol{a}}} v\right)\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}} &\le \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \|\nabla\left(\psi_{\boldsymbol{a}}\left(v-\Pi_{0,\omega_{\boldsymbol{a}}} v\right)\right)\|_{p,\omega_{\boldsymbol{a}}} \\ &\le C_{\mathrm{cont,PF}} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \|\nabla\left(v-\Pi_{0,\omega_{\boldsymbol{a}}} v\right)\|_{p,\omega_{\boldsymbol{a}}} \\ &= C_{\mathrm{cont,PF}} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \|\nabla v\|_{p,\omega_{\boldsymbol{a}}}.\end{aligned}$$

For $\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{ext}}$, there holds $v|_{\omega_{\boldsymbol{a}}} \in W_*^{1,p}(\omega_{\boldsymbol{a}})$ and $(\psi_{\boldsymbol{a}} v)|_{\omega_{\boldsymbol{a}}} \in V^{\boldsymbol{a}}$. Hence, similarly, we obtain

$$\langle \mathcal{R}, \psi_{\boldsymbol{a}} v\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}} \le C_{\mathrm{cont,PF}} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \|\nabla v\|_{p,\omega_{\boldsymbol{a}}}.$$

Thus, the Hölder inequality gives, for $1<p<\infty$,

$$\langle \mathcal{R}, v\rangle_{V',V} \le N_{\mathrm{ov}} C_{\mathrm{cont,PF}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} \right\}^{\frac{1}{q}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla v\|^p_{p,\omega_{\boldsymbol{a}}} \right\}^{\frac{1}{p}}.$$

Combining (2.3) used for ∇v with (3.2) now implies the result if $1<p<\infty$. Cases $p=1$ and $p=\infty$ are obvious modifications.

We now pass to (3.21b) and (3.22b). First assume that $1 < p \le \infty$. From (3.4) we deduce that for any $\boldsymbol{a} \in \mathcal{V}_h$

$$\|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} = \sup_{v \in V^{\boldsymbol{a}};\, \|\nabla v\|_{p,\omega_{\boldsymbol{a}}} = \|\mathcal{R}\|^{q-1}_{(V^{\boldsymbol{a}})'}} \langle \mathcal{R}, v\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}}.$$

For a fixed $\boldsymbol{a} \in \mathcal{V}_h$, we can characterize the supremum by a sequence $\{\mathscr{v}^{\boldsymbol{a}}_j\}_{j=1}^\infty \subset V^{\boldsymbol{a}}$ with

$$\|\nabla \mathscr{v}^{\boldsymbol{a}}_j\|_{p,\omega_{\boldsymbol{a}}} = \|\mathcal{R}\|^{q-1}_{(V^{\boldsymbol{a}})'} \qquad \text{(with convention } 0^0 = 1) \tag{3.24}$$

and

$$\|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} = \lim_{j\to\infty} \langle \mathcal{R}, \mathscr{v}^{\boldsymbol{a}}_j\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}}. \tag{3.25}$$

After summing over $\boldsymbol{a} \in \mathcal{V}_h$, dividing by N_{ov}, and using (3.3) together with the linearity of $\mathcal{R}$, we can estimate

$$\frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} = \lim_{j\to\infty} \left\langle \mathcal{R}, \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \mathscr{v}^{\boldsymbol{a}}_j \right\rangle_{V',V} \le \lim_{j\to\infty} \|\mathcal{R}\|_{V'} \left\| \frac{1}{N_{\mathrm{ov}}} \nabla \sum_{\boldsymbol{a}\in\mathcal{V}_h} \mathscr{v}^{\boldsymbol{a}}_j \right\|_p.$$

Using Lemma 2.5 and (3.24) we get

$$\frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} \le \begin{cases} \|\mathcal{R}\|_{V'} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} \right\}^{\frac{1}{p}} & 1 < p < \infty, \\ \|\mathcal{R}\|_{V'} & p = \infty, \end{cases}$$

which proves (3.21b). The case (3.22b) follows easily by (3.2)–(3.4). □

3.5 Remarks

We collect here a couple of remarks associated with Theorem 3.7.

Remark 3.8 (Expressing local norms using p-Laplace local liftings)**.** *Let $1 < p < \infty$. In this case $V^{\boldsymbol{a}}$ is reflexive, so for the sequence $\{\mathscr{v}^{\boldsymbol{a}}_j\}_{j=1}^\infty$ from the proof of Theorem 3.7, there is a subsequence which converges weakly to some $\mathscr{v}^{\boldsymbol{a}} \in V^{\boldsymbol{a}}$ with $\|\nabla \mathscr{v}^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}} \le \liminf_{j\to\infty} \|\nabla \mathscr{v}^{\boldsymbol{a}}_j\|_{p,\omega_{\boldsymbol{a}}} = \|\mathcal{R}\|^{q-1}_{(V^{\boldsymbol{a}})'}$ thanks to weak lower semi-continuity of norm and (3.24). On the other hand, from (3.25) and the weak convergence, we conclude that*

$$\|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} = \langle \mathcal{R}, \mathscr{v}^{\boldsymbol{a}}\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}}, \tag{3.26}$$

which implies that $\|\mathcal{R}\|^{q-1}_{(V^{\boldsymbol{a}})'} \le \|\nabla \mathscr{v}^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}}$. Hence, altogether we have that

$$\|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} = \|\nabla \mathscr{v}^{\boldsymbol{a}}\|^p_{p,\omega_{\boldsymbol{a}}}. \tag{3.27}$$

Moreover, as $V^{\boldsymbol{a}}$ (or, equivalently, $\|\nabla\cdot\|^p_{p,\omega_{\boldsymbol{a}}}$) is a strictly convex (in fact uniformly convex) space, when $1 < p < \infty$, $\mathscr{v}^{\boldsymbol{a}} \in V^{\boldsymbol{a}}$ with properties (3.26), (3.27) is unique. For proof assume that $\mathcal{R} \neq 0$ on $V^{\boldsymbol{a}}$ (the case $\mathcal{R} = 0$ on $V^{\boldsymbol{a}}$ is trivial) and that there is $\mathscr{p}^{\boldsymbol{a}} \neq \alpha \mathscr{v}^{\boldsymbol{a}}$ and $\mathscr{p}^{\boldsymbol{a}}$ satisfies (3.26) and (3.27) with $\mathscr{p}^{\boldsymbol{a}}$ in place of $\mathscr{v}^{\boldsymbol{a}}$. Define $\mathscr{x}^{\boldsymbol{a}} := \frac{\mathscr{v}^{\boldsymbol{a}}+\mathscr{p}^{\boldsymbol{a}}}{\|\nabla(\mathscr{v}^{\boldsymbol{a}}+\mathscr{p}^{\boldsymbol{a}})\|_{p,\omega_{\boldsymbol{a}}}} \in V^{\boldsymbol{a}}$ with $\|\nabla \mathscr{x}^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}} = 1$ and observe using (3.26), (3.27) and the strict convexity $\|\nabla(\mathscr{v}^{\boldsymbol{a}} + \mathscr{p}^{\boldsymbol{a}})\|_{p,\omega_{\boldsymbol{a}}} < \|\nabla \mathscr{v}^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}} + \|\nabla \mathscr{p}^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}}$ that $\langle \mathcal{R}, \mathscr{x}^{\boldsymbol{a}}\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}} > \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}$, which is a contradiction with (3.4).

It is easy to check that the unique solution $\mathscr{v}^{\boldsymbol{a}} \in V^{\boldsymbol{a}}$ of (3.26), (3.27) is in fact the solution of p-Laplacian solve on the patch $\omega_{\boldsymbol{a}}$:

$$\left(|\nabla \mathscr{v}^{\boldsymbol{a}}|^{p-2} \nabla \mathscr{v}^{\boldsymbol{a}}, \nabla v\right)_{\omega_{\boldsymbol{a}}} = \langle \mathcal{R}, v\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}} \qquad \forall v \in V^{\boldsymbol{a}}. \tag{3.28}$$

Note that the above reasoning about the existence and uniqueness of representation (3.28), which in its generality referred only to reflexivity and strict convexity of $V^{\boldsymbol{a}}$, applies also to global representation of $\mathcal{R}$ on V, as defined by (3.17); see also footnote 4 on page 11.

Remark 3.9 (Localization based on weighted Poincaré–Friedrichs inequalities)**.** *Poincaré–Friedrichs inequalities can be derived for the weighted $L^p(\Omega)$-norm of v on $\omega_{\boldsymbol{a}}$, $\|\psi_{\boldsymbol{a}}^{\frac{1}{p}} v\|_{p,\omega_{\boldsymbol{a}}}$ in place of $\|v\|_{p,\omega_{\boldsymbol{a}}}$ in (2.5), see Chua and Wheeden [25] and Veeser and Verfürth [50]. Then, in the spirit of Carstensen and Funken [22] and Veeser and Verfürth [49], weighted equivalents of Lemma 2.4 and Theorem 3.7 could be given. This might reduce the size of the constants in (3.21)–(3.22), originating from overlapping of the supports of the test functions $\psi_{\boldsymbol{a}}$, at the price of making the formulas a little more involved.*

We finally show that inequality (3.21b) can be split into local contributions when passing from dual norms of the functional $\mathcal{R}$ to its liftings.

Remark 3.10 (Splitting (3.21b) into local contributions using lifted norms)**.** *Let $1 < p < \infty$ and let $\mathcal{R} \in V'$ and $\boldsymbol{a} \in \mathcal{V}_h$ be given. Define the* global lifting *$\mathscr{r} \in V$ of the functional $\mathcal{R}$ by (3.17) and the* local lifting *$\mathscr{r}^{\boldsymbol{a}} \in V^{\boldsymbol{a}}$ by (3.28). Then it holds*

$$\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} = \|\nabla \mathscr{r}^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}}^{p-1} \leq \|\nabla \mathscr{r}\|_{p,\omega_{\boldsymbol{a}}}^{p-1}. \tag{3.29}$$

Indeed, the equality has been shown in equation (3.27) and the inequality follows using definition (3.4), definition of the global lifting (3.17), and the Hölder inequality

$$\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} = \sup_{v \in V^{\boldsymbol{a}};\, \|\nabla v\|_{p,\omega_{\boldsymbol{a}}}=1} \langle \mathcal{R}, v\rangle_{V',V} = \sup_{v \in V^{\boldsymbol{a}};\, \|\nabla v\|_{p,\omega_{\boldsymbol{a}}}=1} \left(|\nabla \mathscr{r}|^{p-2}\nabla \mathscr{r}, \nabla v\right)_{\omega_{\boldsymbol{a}}} \leq \|\nabla \mathscr{r}\|_{p,\omega_{\boldsymbol{a}}}^{p-1}.$$

Note that summing (3.29) in q-th power over all vertices $\mathcal{V}_h$ and using (2.3a) and (3.18) one gets (3.21b) as a trivial consequence.

4 Extensions

This section collects various extensions of the main result of Theorem 3.7.

4.1 Localization without the orthogonality condition

We begin by a simple generalization of Theorem 3.7 to the case without orthogonality (3.20) to the partition of unity functions $\psi_{\boldsymbol{a}}$.

Theorem 4.1 (Simple localization of dual norms of functionals without $\psi_{\boldsymbol{a}}$-orthogonality)**.** *Let $\mathcal{R} \in V'$ be arbitrary and define*

$$r^{\boldsymbol{a}} := \frac{h_\Omega C_{\mathrm{PF},p,\Omega}}{|\omega_{\boldsymbol{a}}|^{\frac{1}{p}}} |\langle \mathcal{R}, \psi_{\boldsymbol{a}}\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}}|. \tag{4.1}$$

Then, when $1 < p \leq \infty$,

$$\|\mathcal{R}\|_{V'} \leq N_{\mathrm{ov}} C_{\mathrm{cont,PF}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}^q \right\}^{\frac{1}{q}} + N_{\mathrm{ov}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} (r^{\boldsymbol{a}})^q \right\}^{\frac{1}{q}}, \tag{4.2a}$$

$$\left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}^q \right\}^{\frac{1}{q}} \leq \|\mathcal{R}\|_{V'}, \tag{4.2b}$$

and, when $p = 1$,

$$\|\mathcal{R}\|_{V'} \leq N_{\mathrm{ov}} C_{\mathrm{cont,PF}} \max_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} + N_{\mathrm{ov}} \max_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} r^{\boldsymbol{a}}, \tag{4.3a}$$

$$\max_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \leq \|\mathcal{R}\|_{V'}. \tag{4.3b}$$

Proof. Estimates (4.2b) and (4.3b) have been proven in Theorem 3.7. Estimates (4.2a) and (4.3a) are proven along the lines of Theorem 3.7, counting for the additional nonzero term

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} (\Pi_{0,\omega_{\boldsymbol{a}}} v)\, \langle \mathcal{R}, \psi_{\boldsymbol{a}}\rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}} \tag{4.4}$$

in (3.23). For each $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$, the Hölder inequality gives

$$|\omega_{\boldsymbol{a}}|^{\frac{1}{p}} \left(\Pi_{0,\omega_{\boldsymbol{a}}} v\right) = |\omega_{\boldsymbol{a}}|^{\frac{1}{p}} \left(v, 1\right)_{\omega_{\boldsymbol{a}}} |\omega_{\boldsymbol{a}}|^{-1} \leq |\omega_{\boldsymbol{a}}|^{\frac{1}{p}} \|v\|_{p,\omega_{\boldsymbol{a}}} |\omega_{\boldsymbol{a}}|^{\frac{1}{q}} |\omega_{\boldsymbol{a}}|^{-1} = \|v\|_{p,\omega_{\boldsymbol{a}}}.$$

Thus, the Hölder inequality, the Poincaré–Friedrichs inequality (2.5) used in the entire domain Ω on the space V, and (2.3) lead to, for $1 < p < \infty$,

$$\begin{aligned}
&\sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \left(\Pi_{0,\omega_{\boldsymbol{a}}} v\right) \langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}} = \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} |\omega_{\boldsymbol{a}}|^{-\frac{1}{p}} \langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}} |\omega_{\boldsymbol{a}}|^{\frac{1}{p}} \left(\Pi_{0,\omega_{\boldsymbol{a}}} v\right) \\
&\leq N_{\mathrm{ov}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \left(|\omega_{\boldsymbol{a}}|^{-\frac{1}{p}} |\langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}}| \right)^q \right\}^{\frac{1}{q}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \|v\|_{p,\omega_{\boldsymbol{a}}}^p \right\}^{\frac{1}{p}} \\
&\leq N_{\mathrm{ov}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} (r^{\boldsymbol{a}})^q \right\}^{\frac{1}{q}} \|\nabla v\|_p,
\end{aligned}$$

and (3.2) gives the assertion. Cases $p = 1$ and $p = \infty$ are proved with obvious modifications. □

This result implies the following remark:

Remark 4.2 (h-unstable localization of dual norms of functionals)**.** *Observe that in* (4.2a) *and* (4.3a)*, we can estimate using* $|\langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}}| \leq \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \|\nabla \psi_{\boldsymbol{a}}\|_p$ *and the Hölder inequality in order to arrive at*

$$\|\mathcal{R}\|_{V'} \leq N_{\mathrm{ov}} C_h^{\ddagger} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}^q \right\}^{\frac{1}{q}}, \qquad 1 < p \leq \infty, \tag{4.5}$$

$$\|\mathcal{R}\|_{V'} \leq N_{\mathrm{ov}} C_h^{\ddagger} \max_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}, \qquad p = 1, \tag{4.6}$$

with

$$C_h^{\ddagger} := \left(C_{\mathrm{cont,PF}} + h_\Omega C_{\mathrm{PF},p,\Omega} \max_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \right). \tag{4.7}$$

Whereas h_Ω *and* $C_{\mathrm{PF},p,\Omega}$ *do not depend on the partition and* $C_{\mathrm{cont,PF}}$ *is uniformly bounded for regular partitions, there typically holds* $\max_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \approx h^{-1}$*, so that* $C_h^{\ddagger}$ *explodes for small patches* $\omega_{\boldsymbol{a}}$*,* $\boldsymbol{a} \in \mathcal{V}_h$*. We note that one can actually estimate a little more sharply with* $C_h^{\ddagger} = 1 + h_\Omega C_{\mathrm{PF},p,\Omega} \max_{\boldsymbol{a} \in \mathcal{V}_h} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}$*.*

Estimates (4.2a) and (4.3a) of Theorem 4.1 take a simple form but, unfortunately, as Example 4.6 below shows, the second term in (4.2a) may severely overestimate $\|\mathcal{R}\|_{V'}$. Correspondingly, (4.5) and (4.6) of Remark 4.2 blow up with mesh refinement due to presence of $\max_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}$ in (4.7). We discuss in Example 4.6 that it is related to l^2-norm estimates of the algebraic residual vector from numerical linear algebra; in both cases, the local contributions are first taken in *absolute value* in (4.1) and then the size of the *resulting algebraic vector* is measured in the second term in (4.2a). The following estimate, obtained while employing the ideas of [38, Section 7.3] and [43], removes this deficiency, while *first summing* the local contributions and then constructing a *discrete* $\boldsymbol{H}^q(\mathrm{div}, \Omega)$*-lifting.*

Theorem 4.3 (Improved localization of dual norms of functionals without $\psi_{\boldsymbol{a}}$-orthogonality)**.** *Let* $\mathcal{R} \in V'$ *be arbitrary and define* $r_h \in \mathbb{P}_0(\mathcal{T}_h)$ *to be the piecewise constant function with respect to the partition* $\mathcal{T}_h$ *given by*

$$r_h|_K := \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}} \cap \mathcal{V}_K} \frac{1}{|\omega_{\boldsymbol{a}}|} \langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}} \qquad \forall K \in \mathcal{T}_h. \tag{4.8}$$

Let $\boldsymbol{\sigma}_{h,\mathrm{alg}} \in \boldsymbol{H}^q(\mathrm{div}, \Omega) := \{\boldsymbol{v} \in [L^q(\Omega)]^d; \mathrm{div}\, \boldsymbol{v} \in L^q(\Omega)\}$ *be arbitrary but such that*

$$\mathrm{div}\, \boldsymbol{\sigma}_{h,\mathrm{alg}} = r_h. \tag{4.9}$$

Then, when $1 < p \leq \infty$,

$$\|\mathcal{R}\|_{V'} \leq N_{\mathrm{ov}} C_{\mathrm{cont,PF}} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} \right\}^{\frac{1}{q}} + \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q, \tag{4.10a}$$

$$\left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} \right\}^{\frac{1}{q}} \leq \|\mathcal{R}\|_{V'}, \tag{4.10b}$$

and, when $p = 1$,

$$\|\mathcal{R}\|_{V'} \leq N_{\mathrm{ov}} C_{\mathrm{cont,PF}} \max_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} + \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_\infty, \tag{4.11a}$$

$$\max_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \leq \|\mathcal{R}\|_{V'}. \tag{4.11b}$$

Proof. The proof consists in finding an alternative, sharper bound on the term (4.4) above. Let $v \in V$ with $\|\nabla v\|_p = 1$ be fixed. Note that, for each interior vertex $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$,

$$(\Pi_{0,\omega_{\boldsymbol{a}}} v) \langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}} = \frac{1}{|\omega_{\boldsymbol{a}}|} \left(\langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}}, v \right)_{\omega_{\boldsymbol{a}}}.$$

Hence, considering $\frac{1}{|\omega_{\boldsymbol{a}}|} \langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}}$ as constant on $\omega_{\boldsymbol{a}}$ and zero elsewhere and using definition (4.8),

$$\begin{aligned} \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} (\Pi_{0,\omega_{\boldsymbol{a}}} v) \langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}} &= \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} \left(\frac{1}{|\omega_{\boldsymbol{a}}|} \langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}}, v \right) = (r_h, v) \\ &= (\operatorname{div} \boldsymbol{\sigma}_{h,\mathrm{alg}}, v) = -(\boldsymbol{\sigma}_{h,\mathrm{alg}}, \nabla v) \leq \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q \|\nabla v\|_p = \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q, \end{aligned}$$

where we have also applied the requirement (4.9), the Green theorem, and the Hölder inequality. Actually, generalizing [38, Theorem 5.5] to the present setting, it follows that, at least for $1 < p < \infty$,

$$\sup_{v \in V;\, \|\nabla v\|_p = 1} \sum_{\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}} (\Pi_{0,\omega_{\boldsymbol{a}}} v) \langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})', V^{\boldsymbol{a}}} = \min_{\boldsymbol{\sigma}_{h,\mathrm{alg}} \in \boldsymbol{H}^q(\mathrm{div},\Omega);\, \operatorname{div} \boldsymbol{\sigma}_{h,\mathrm{alg}} = r_h} \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q,$$

so that this estimate is as sharp as possible.

□

Example 4.4 (Construction of $\boldsymbol{\sigma}_{h,\mathrm{alg}}$)**.** *Several practical constructions of* $\boldsymbol{\sigma}_{h,\mathrm{alg}}$ *in finite-dimensional subspaces of* $\boldsymbol{H}^q(\mathrm{div},\Omega)$ *in the context of simplicial or parallelepipedal meshes of Remark 2.1 are possible, employing the lowest-order Raviart–Thomas–Nédélec (*$\mathbf{RTN}_0$*) space, cf. [16] and the references therein. A construction with a cost linear in terms of the number of mesh elements of* $\mathcal{T}_h$ *has been proposed in [38, Section 7.3]. It consists in a (sequential) sweep through all mesh elements in a proper order. Numerically often much sharper construction has been proposed in [43, Definition 6.3]. It needs a hierarchy of meshes of whose* $\mathcal{T}_h$ *is a refinement, in the multigrid spirit, and consists in an exact solve on the coarsest mesh and a (parallel) sweep through all mesh vertices on all mesh levels. This latter construction can be shown to be an optimal estimate (giving both upper and lower (up to a constant) bounds)(work in progress). Note that although references [38, 43] consider the Hilbertian setting* $p = 2$*, there is no structural loss in passing to* $p \neq 2$*, see [30, 32] and the references therein.*

Remark 4.5 (Localization of $\|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q$)**.** *Note that from (2.3), one has* $\|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q^q \leq \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_{q,\omega_{\boldsymbol{a}}}^q \leq N_{\mathrm{ov}} \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q^q$*. In the context of Remark 2.1, actually* $\|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q = \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_{q,\omega_{\boldsymbol{a}}}^q \right\}^{\frac{1}{q}}$*. Thus, the second terms on the right-hand sides of (4.10a) and (4.11a) are fully localizable.*

Example 4.6 (Link of estimate of Theorem 4.1 to the l^2-norm of the algebraic residual vector when $p=2$ and their deficiency[5]). *Consider $p=2$, $d=1$, $\Omega=(0,1)$, and the following situation: $-\Delta u=-u''=2$ in Ω and $u=0$ on $\partial\Omega$, so that the solution of this PDE is $u(x)=x(1-x)$. In the context of (3.11) of Example 3.2, let $V=W_0^{1,2}(\Omega)$, and, for any $u_H\in W_0^{1,2}(\Omega)$, let $\mathcal{R}\in V'$ be defined by*

$$\langle\mathcal{R},v\rangle_{V',V}:=(2,v)-(\nabla u_H,\nabla v)=\int_0^1(2v-u_H'v')\,\mathrm{d}x,\qquad v\in V, \tag{4.12}$$

leading to

$$\|\mathcal{R}\|_{V'}=\|\nabla(u-u_H)\|_2=\left\{\int_0^1[(u-u_H)']^2\,\mathrm{d}x\right\}^{\frac{1}{2}}.$$

Let us consider an even integer $N>0$, define $h:=1/N$, and introduce a mesh $\mathcal{T}_h$ of Ω given by the vertices $\boldsymbol{a}_i:=ih$, $i=0,\ldots,N$, forming the set $\mathcal{V}_h$ and the elements (intervals) $K_i:=[\boldsymbol{a}_i,\boldsymbol{a}_{i+1}]$, $i=0,\ldots,N-1$. We also consider the twice coarser mesh $\mathcal{T}_H$ given similarly by the points $\boldsymbol{a}_{2i}=2ih$, $i=0,\ldots,N/2$. Let now u_H be piecewise affine with respect to $\mathcal{T}_H$, $C^0(\overline{\Omega})$-continuous, taking the values of the exact solution u in the vertices $\boldsymbol{a}_i=2ih$, $i=0,\ldots,N/2$, see Figure 1, left. This u_H is the finite element solution on the mesh $\mathcal{T}_H$, or, equivalently, the Lagrange interpolate of u on the mesh $\mathcal{T}_H$ (with mesh size $2h$). Consequently,

$$\|\mathcal{R}\|_{V'}=\|\nabla(u-u_H)\|_2=\mathcal{O}(2h)=\mathcal{O}(h), \tag{4.13}$$

where $g(h)=\mathcal{O}(h)$ when there exist two positive constants c,C independent of h such that $ch\le g(h)\le Ch$ for all $h>0$. The residual $\mathcal{R}$ generated by the function u_H by (4.12), though, does not satisfy the orthogonality condition (3.20) on $\mathcal{T}_h$. A simple calculation gives

$$\langle\mathcal{R},\psi_{\boldsymbol{a}}\rangle_{V',V}=(2,\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}-(\nabla u_H,\nabla\psi_{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}}=\begin{cases}|\omega_{\boldsymbol{a}}|=2h & \boldsymbol{a}=\boldsymbol{a}_{2i+1}\in\mathcal{V}_h^{\mathrm{int}}\ \text{odd}, i=0,\ldots,N/2-1,\\ -|\omega_{\boldsymbol{a}}|=-2h & \boldsymbol{a}=\boldsymbol{a}_{2i}\in\mathcal{V}_h^{\mathrm{int}}\ \text{even}, i=1,\ldots,N/2-1.\end{cases} \tag{4.14}$$

Consequently, as $h_\Omega=1$ and $C_{\mathrm{PF},2,\Omega}=1/\pi$, $r^{\boldsymbol{a}}$ given by (4.1) take the values $r^{\boldsymbol{a}}=(2h)^{\frac{1}{2}}/\pi$. Thus, since $N_{\mathrm{ov}}=2$ and $q=2$,

$$N_{\mathrm{ov}}\left\{\frac{1}{N_{\mathrm{ov}}}\sum_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}(r^{\boldsymbol{a}})^q\right\}^{\frac{1}{q}}=2^{\frac{1}{2}}\left\{\sum_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}(r^{\boldsymbol{a}})^2\right\}^{\frac{1}{2}}=\frac{2h^{\frac{1}{2}}}{\pi}(N-1)^{\frac{1}{2}}=\mathcal{O}(1). \tag{4.15}$$

Thus by comparison with (4.13), the second term on the right-hand side of (4.2a) critically overestimates $\|\mathcal{R}\|_{V'}$. The same holds for estimate (4.5) with (4.7). Indeed, $C_h^{\clubsuit}=C_{\mathrm{cont,PF}}+\mathcal{O}(h^{-1})$ and consequently (4.2b) and (4.13) give that the right-hand side of (4.5) behaves as $\mathcal{O}(h)+\mathcal{O}(1)$.

Let now u_h be piecewise affine with respect to the mesh $\mathcal{T}_h$, $C^0(\overline{\Omega})$-continuous, taking the values of the exact solution u in the points $\boldsymbol{a}_i=ih$, $i=0,\ldots,N$. The function u_h is the finite element solution on the mesh $\mathcal{T}_h$, or the Lagrange interpolate of u on the mesh $\mathcal{T}_h$, see Figure 1, left. (If the residual $\mathcal{R}$ would be defined from u_h and not by (4.12), it would satisfy the orthogonality condition (3.20)). The triangle inequality gives

$$\|\mathcal{R}\|_{V'}=\|\nabla(u-u_H)\|_2\le\|\nabla(u-u_h)\|_2+\|\nabla(u_h-u_H)\|_2. \tag{4.16}$$

Immediately, $\|\nabla(u-u_h)\|_2=\mathcal{O}(h)$ and also $\|\nabla(u_h-u_H)\|_2=\mathcal{O}(h)$, so there is no structural loss in this inequality. Viewing u_H as an approximate solution to u_h, $u_H=\sum_{i=1}^{N-1}u_H(\boldsymbol{a}_i)\psi_{\boldsymbol{a}_i}$, $\mathrm{U}_H\in\mathbb{R}^{N-1}$, $(\mathrm{U}_H)_i=u_H(\boldsymbol{a}_i)$, $i=1,\ldots,N-1$, where only u_H is supposed to be known explicitly but not u_h, we now consider the most commonly used estimate on the "algebraic" error

$$\|\nabla(u_h-u_H)\|_2^2=\left(\mathbb{A}_h^{-1}\mathrm{R}_h\right)\cdot\mathrm{R}_h\le\left|\mathbb{A}_h^{-1}\right|_2|\mathrm{R}_h|_2^2,$$

[5] We would like to thank the anonymous referee for suggesting this illustrative example.

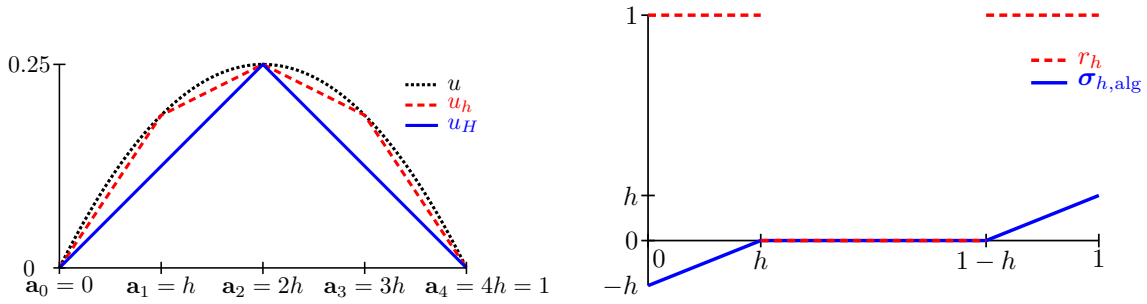

Figure 1: Example 4.6. Setting and exact solution u, approximation u_h on the mesh $\mathcal{T}_h$, and approximation u_H on the twice coarser mesh $\mathcal{T}_H$ (left); r_h from (4.8) and optimal $\boldsymbol{\sigma}_{h,\mathrm{alg}}$ from $\mathbf{RTN}_0$ (right).

cf. [44, Section 3.1] and the references therein; here

$$\mathbb{A}_h := \frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 2 \end{pmatrix}, \qquad \mathrm{F}_h := \begin{pmatrix} 2h \\ 2h \\ \vdots \\ \vdots \\ 2h \end{pmatrix}$$

are respectively the finite element matrix and the right-hand side vector and

$$\mathrm{R}_h := \mathrm{F}_h - \mathbb{A}_h \mathrm{U}_H = \begin{pmatrix} 2h \\ -2h \\ 2h \\ \vdots \\ 2h \end{pmatrix}$$

is the algebraic residual vector; note that $(\mathrm{R}_h)_i = \langle \mathcal{R}, \psi_{\boldsymbol{a}_i} \rangle_{V',V} = (-1)^{i+1}\, 2h$, $i = 1, \ldots, N-1$, *using* (4.14). *Now, cf. [38, Section 7.1] or [44, Section 5.2] and the references therein for similar developments,*

$$|\mathrm{R}_h|_2 = \left\{ \sum_{i=1}^{N-1} |\,(\mathrm{R}_h)_i\,|^2 \right\}^{\frac{1}{2}} = 2h\,(N-1)^{\frac{1}{2}} = \mathcal{O}\left(h^{\frac{1}{2}}\right)$$

and

$$\left|\mathbb{A}_h^{-1}\right|_2 = \lambda_{\max}\left(\mathbb{A}_h^{-1}\right) = \frac{1}{\lambda_{\min}\left(\mathbb{A}_h\right)} = \mathcal{O}\left(h^{-1}\right),$$

where the characterization of the smallest eigenvalue $\lambda_{\min}\left(\mathbb{A}_h\right) = \mathcal{O}\left(h\right)$ *of the matrix* $\mathbb{A}_h$ *in one space dimension is standard, see, e.g., [31, Example 9.15]. Altogether,*

$$\|\nabla\left(u_h - u_H\right)\|_2 \le \left|\mathbb{A}_h^{-1}\right|_2^{\frac{1}{2}} |\mathrm{R}_h|_2 = \mathcal{O}\left(1\right). \tag{4.17}$$

We conclude that the simple estimate of Theorem 4.1 has in this case the same quality as the commonly used l^2*-norm estimate of the algebraic residual vector from numerical linear algebra, and that both are greatly imprecise.*

Example 4.7 (Optimality of estimate of Theorem 4.3)**.** *We now investigate, for the same setting as in Example 4.6, the quality of the upper bound* (4.10a) *of Theorem 4.3. Following* (4.14)*, the quantities* $\frac{1}{|\omega_{\boldsymbol{a}}|}\langle \mathcal{R}, \psi_{\boldsymbol{a}} \rangle_{(V^{\boldsymbol{a}})',V^{\boldsymbol{a}}}$ *in* (4.8) *take here the value* 1 *for odd vertices and* -1 *for even vertices. Thus, the elementwise constant function* r_h *actually vanishes in all the elements except for* K_1 *and* K_N*, where it takes*

the value 1*, see Figure 1, right. Then, it is easy to check that the best-available* $\boldsymbol{\sigma}_{h,\mathrm{alg}}$ *from* $\mathbf{RTN}_0$ *such that* $\operatorname{div}\boldsymbol{\sigma}_{h,\mathrm{alg}} = r_h$ *is the function vanishing on all the elements except for* K_1 *and* K_N*, depicted in Figure 1, right. This leads to*

$$\|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_2 = \mathcal{O}\left(h^{\frac{3}{2}}\right),$$

The construction of $\boldsymbol{\sigma}_{h,\mathrm{alg}}$ *from [38, Section 7.3] then still leads to* $\|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_2 = \mathcal{O}\left(h^{\frac{3}{2}}\right)$*, whereas that from [43, Definition 6.3] yields* $\|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_2 = \mathcal{O}(h)$*. Consequently, in both practical constructions of Example 4.4, the second term on the right-hand side of* (4.10a) *does not spoil the quality of the estimate, in contrast to* (4.2a) *with* (4.15) *and* (4.16) *with* (4.17)*.*

Having identified the additional terms in inequalities (4.2a) and (4.10a), one typically controls adaptively their size of with respect to the principal contribution $\left\{\frac{1}{N_{\mathrm{ov}}}\sum_{\boldsymbol{a}\in\mathcal{V}_h}\|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'}\right\}^{\frac{1}{q}}$ (and similarly for $\max_{\boldsymbol{a}\in\mathcal{V}_h}\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}$ if $p=1$), see, e.g., [38, equation (6.1)] or [32, equation (3.10)]. The following corollary shows that *localization of* $\|\mathcal{R}\|_{V'}$ can be *restored* in this way. It, however, follows from Examples 4.6 and 4.7 that it may be excessively costly to satisfy the balance condition (4.18) in the case of Theorem 4.1, in contrast to the case of Theorem 4.3:

Corollary 4.8 (Localization of dual norms of functionals with controlled loss of orthogonality)**.** *Let* $\mathcal{R}\in V'$ *be arbitrary, and consider either the context of Theorem 4.1 with*

$$r_{\mathrm{res}} := \left\{\frac{1}{N_{\mathrm{ov}}}\sum_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}}(r^{\boldsymbol{a}})^q\right\}^{\frac{1}{q}} \quad \text{if } 1<p\le\infty, \qquad r_{\mathrm{res}} := \max_{\boldsymbol{a}\in\mathcal{V}_h^{\mathrm{int}}} r^{\boldsymbol{a}} \quad \text{if } p=1,$$

or the context of Theorem 4.3 with

$$r_{\mathrm{res}} := \frac{1}{N_{\mathrm{ov}}}\|\boldsymbol{\sigma}_{h,\mathrm{alg}}\|_q.$$

Assume moreover that

$$r_{\mathrm{res}} \le \gamma_{\mathrm{res}} C_{\mathrm{cont,PF}}\left\{\frac{1}{N_{\mathrm{ov}}}\sum_{\boldsymbol{a}\in\mathcal{V}_h}\|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'}\right\}^{\frac{1}{q}}, \qquad 1<p\le\infty, \tag{4.18a}$$

$$r_{\mathrm{res}} \le \gamma_{\mathrm{res}} C_{\mathrm{cont,PF}}\max_{\boldsymbol{a}\in\mathcal{V}_h}\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}, \qquad p=1 \tag{4.18b}$$

for some parameter $\gamma_{\mathrm{res}}\ge 0$*. Then, when* $1<p\le\infty$*,*

$$\|\mathcal{R}\|_{V'} \le (1+\gamma_{\mathrm{res}})\, N_{\mathrm{ov}} C_{\mathrm{cont,PF}}\left\{\frac{1}{N_{\mathrm{ov}}}\sum_{\boldsymbol{a}\in\mathcal{V}_h}\|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'}\right\}^{\frac{1}{q}},$$

$$\left\{\frac{1}{N_{\mathrm{ov}}}\sum_{\boldsymbol{a}\in\mathcal{V}_h}\|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'}\right\}^{\frac{1}{q}} \le \|\mathcal{R}\|_{V'},$$

and, when $p=1$*,*

$$\|\mathcal{R}\|_{V'} \le (1+\gamma_{\mathrm{res}})\, N_{\mathrm{ov}} C_{\mathrm{cont,PF}}\max_{\boldsymbol{a}\in\mathcal{V}_h}\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'},$$

$$\max_{\boldsymbol{a}\in\mathcal{V}_h}\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \le \|\mathcal{R}\|_{V'}.$$

4.2 Localization in vectorial setting

We now finally present a vectorial variant of Theorem 3.7, with typical applications in Stokes-type fluid flows, cf. [13]. We only make a concise presentation, as the extension from the scalar case is rather straightforward.

Let $\nabla \boldsymbol{v}$ for $\boldsymbol{v} \in [W^{1,p}(\omega)]^d$ be the matrix with lines given by $\nabla \boldsymbol{v}_i$, $1 \le i \le d$; in accordance with the notation of Section 2.1, $\|\nabla \boldsymbol{v}\|_{p,\omega} := \left(\int_\omega \left(\sum_{i=1}^d \sum_{j=1}^d |\partial_{\boldsymbol{x}_j} \boldsymbol{v}_i(\boldsymbol{x})|^2 \right)^{\frac{p}{2}} \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}}$. For vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$, $\boldsymbol{u} \otimes \boldsymbol{v}$ defines a tensor $\mathbb{t} \in \mathbb{R}^{d \times d}$ such that $\mathbb{t}_{i,j} := \boldsymbol{u}_i \boldsymbol{v}_j$. Then the vectorial variant of Lemma 2.4 is:

Lemma 4.9 (Cut-off estimate in vectorial setting)**.** *There exists a constant $C_{\mathrm{cont,PF},d} > 0$, only depending on the space dimension d and on the constant $C_{\mathrm{cont,PF}}$ from (2.6), such that for all $\boldsymbol{a} \in \mathcal{V}_h$, there holds*

$$\|\nabla(\psi_{\boldsymbol{a}} \boldsymbol{v})\|_{p,\omega_{\boldsymbol{a}}} \le C_{\mathrm{cont,PF},d} \|\nabla \boldsymbol{v}\|_{p,\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{v} \in [W^{1,p}_*(\omega_{\boldsymbol{a}})]^d.$$

Proof. Assume first $1 \le p < \infty$. Using the scalar Poincaré–Friedrichs inequality (2.5) and the norm equivalence (2.1),

$$\begin{aligned}
\|\boldsymbol{v}\|_{p,\omega_{\boldsymbol{a}}} &= \left(\int_{\omega_{\boldsymbol{a}}} \left(\sum_{i=1}^d |\boldsymbol{v}_i(\boldsymbol{x})|^2 \right)^{\frac{p}{2}} \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}} \le \underline{C}_{p,d} \left(\sum_{i=1}^d \int_{\omega_{\boldsymbol{a}}} |\boldsymbol{v}_i(\boldsymbol{x})|^p \, \mathrm{d}\boldsymbol{x} \right)^{\frac{1}{p}} \\
&\le \underline{C}_{p,d} C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \left(\sum_{i=1}^d \|\nabla \boldsymbol{v}_i\|^p_{p,\omega_{\boldsymbol{a}}} \right)^{\frac{1}{p}} \le \underline{C}_{p,d} \overline{C}_{p,d} C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla \boldsymbol{v}\|_{p,\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{v} \in [W^{1,p}_*(\omega_{\boldsymbol{a}})]^d,
\end{aligned}$$

where

$$\underline{C}_{p,d} := \begin{cases} 1 & \text{if } p \le 2, \\ d^{\frac{1}{2}-\frac{1}{p}} & \text{if } p \ge 2 \end{cases}$$

and

$$\overline{C}_{p,d} := \begin{cases} d^{\frac{1}{p}-\frac{1}{2}} & \text{if } p \le 2, \\ 1 & \text{if } p \ge 2. \end{cases}$$

Denote $C_{p,d} := \underline{C}_{p,d} \overline{C}_{p,d} = d^{\left|\frac{1}{2}-\frac{1}{p}\right|}$ and notice that $1 \le C_{p,d} < \sqrt{d}$. Then, we readily arrive at

$$\begin{aligned}
\|\nabla(\psi_{\boldsymbol{a}} \boldsymbol{v})\|_{p,\omega_{\boldsymbol{a}}} &= \|\boldsymbol{v} \otimes \nabla \psi_{\boldsymbol{a}} + \psi_{\boldsymbol{a}} \nabla \boldsymbol{v}\|_{p,\omega_{\boldsymbol{a}}} \\
&\le \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \|\boldsymbol{v}\|_{p,\omega_{\boldsymbol{a}}} + \|\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} \|\nabla \boldsymbol{v}\|_{p,\omega_{\boldsymbol{a}}} \\
&\le \left(1 + C_{p,d} C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\right) \|\nabla \boldsymbol{v}\|_{p,\omega_{\boldsymbol{a}}},
\end{aligned}$$

and the assertion follows with $C_{\mathrm{cont,PF},d} := \max_{\boldsymbol{a} \in \mathcal{V}_h} \{1 + C_{p,d} C_{\mathrm{PF},p,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla \psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\}$. Case $p = \infty$ is an obvious modification. □

Denote

$$\begin{aligned}
\boldsymbol{V} &:= [W^{1,p}_0(\Omega)]^d, \\
\mathcal{R} &\in \boldsymbol{V}', \\
\|\mathcal{R}\|_{\boldsymbol{V}'} &:= \sup_{\boldsymbol{v} \in \boldsymbol{V};\, \|\nabla \boldsymbol{v}\|_p = 1} \langle \mathcal{R}, \boldsymbol{v} \rangle_{\boldsymbol{V}',\boldsymbol{V}}.
\end{aligned}$$

For a vertex $\boldsymbol{a} \in \mathcal{V}_h$, let the local setting be

$$\begin{aligned}
\boldsymbol{V}^{\boldsymbol{a}} &:= [W^{1,p}_0(\omega_{\boldsymbol{a}})]^d, \\
\langle \mathcal{R}, \boldsymbol{v} \rangle_{(\boldsymbol{V}^{\boldsymbol{a}})',\boldsymbol{V}^{\boldsymbol{a}}} &:= \langle \mathcal{R}, \boldsymbol{v} \rangle_{\boldsymbol{V}',\boldsymbol{V}} \qquad \boldsymbol{v} \in \boldsymbol{V}^{\boldsymbol{a}}, \\
\|\mathcal{R}\|_{(\boldsymbol{V}^{\boldsymbol{a}})'} &:= \sup_{\boldsymbol{v} \in \boldsymbol{V}^{\boldsymbol{a}};\, \|\nabla \boldsymbol{v}\|_{p,\omega_{\boldsymbol{a}}} = 1} \langle \mathcal{R}, \boldsymbol{v} \rangle_{(\boldsymbol{V}^{\boldsymbol{a}})',\boldsymbol{V}^{\boldsymbol{a}}}.
\end{aligned}$$

Define $\boldsymbol{\psi}_{\boldsymbol{a},m}$, $1 \le m \le d$, as the vectorial variant of the partition of unity functions $\psi_{\boldsymbol{a}}$ such that $(\boldsymbol{\psi}_{\boldsymbol{a},m})_m = \psi_{\boldsymbol{a}}$ and $(\boldsymbol{\psi}_{\boldsymbol{a},m})_n = 0$ for $1 \le n \le d$, $n \ne m$. The following is a generalization of Theorem 3.7 to vectorial setting:

Theorem 4.10 (Localization of dual norms of functionals in vectorial case)**.** *Let $\mathcal{R} \in \boldsymbol{V}'$ be arbitrary and let*

$$\langle \mathcal{R}, \boldsymbol{\psi}_{\boldsymbol{a},m} \rangle_{\boldsymbol{V}',\boldsymbol{V}} = 0 \qquad \forall 1 \le m \le d,\ \forall \boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

Then, when $1 < p \le \infty$,

$$\|\mathcal{R}\|_{\boldsymbol{V}'} \le N_{\mathrm{ov}} C_{\mathrm{cont,PF},d} \left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(\boldsymbol{V}^{\boldsymbol{a}})'} \right\}^{\frac{1}{q}},$$

$$\left\{ \frac{1}{N_{\mathrm{ov}}} \sum_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(\boldsymbol{V}^{\boldsymbol{a}})'} \right\}^{\frac{1}{q}} \le \|\mathcal{R}\|_{\boldsymbol{V}'},$$

and, when $p = 1$,

$$\|\mathcal{R}\|_{\boldsymbol{V}'} \le N_{\mathrm{ov}} C_{\mathrm{cont,PF},d} \max_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(\boldsymbol{V}^{\boldsymbol{a}})'},$$

$$\max_{\boldsymbol{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(\boldsymbol{V}^{\boldsymbol{a}})'} \le \|\mathcal{R}\|_{\boldsymbol{V}'}.$$

The orthogonality condition is actually again only needed in the first inequalities.

Proof. Along the lines of proof of Theorem 3.7, using Lemma 4.9 instead of Lemma 2.4. □

Extension of Remark 3.10, Theorems 4.1 and 4.3, and of Corollary 4.8 to vectorial case is straightforward.

5 Numerical illustration

We now numerically demonstrate the validity of Theorem 3.7 in the setting $1 < p < \infty$. The experiments were implemented using `dolfin-tape` [10] package built on top of the FEniCS Project [2]. The complete supporting code for reproducing the experiments can be obtained at [11].

Let $V_h := \mathbb{P}_1(\mathcal{T}_h) \cap W^{1,p}(\Omega)$ be the space of continuous, piecewise first-order polynomials with respect to a matching triangular mesh $\mathcal{T}_h$ of the domain $\Omega \subset \mathbb{R}^2$, see Remark 2.1. Let $V_h^0 := V_h \cap W_0^{1,p}(\Omega)$ be its zero-trace subspace and let u_h be a finite element approximation to the p-Laplace problem (3.11) of Example 3.2, i.e.,

$$u_h - u_h^{\mathrm{D}} \in V_h^0, \tag{5.1a}$$

$$\left(|\nabla u_h|^{p-2} \nabla u_h, \nabla v_h\right) = (f_h, v_h) \qquad \forall v_h \in V_h^0, \tag{5.1b}$$

where $u_h^{\mathrm{D}} \in V_h$ is a $\mathbb{P}_1$-nodal interpolant of $u^{\mathrm{D}} \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ (approximation error of u^{D} by u_h^{D} is neglected) and $(f_h, \cdot)$ approximates $(f, \cdot)$ by a six-node quadrature rule with fourth-order precision from [46, p. 184, Table 4.1]. We consider $\mathcal{R} \in V'$, the residual of u_h with respect to equation (3.11b) (with $\boldsymbol{\sigma}(\nabla u) = |\nabla u|^{p-2} \nabla u$) given by (3.12). Taking $v_h = \psi_{\boldsymbol{a}}$ in (5.1b) immediately gives the orthogonality property (3.20) for all interior vertices $\boldsymbol{a} \in \mathcal{V}_h^{\mathrm{int}}$. Computationally, regularization and linearization of the degenerate p-Laplace operator is employed to approximately solve (5.1). The arising errors are secured to be small by error-distinguishing a posteriori estimation techniques of [32], thus ensuring sufficiently approximate fulfillment of the Galerkin orthogonality (3.20).

The evaluation of the norms $\|\mathcal{R}\|_{V'}$ and $\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}$ in (3.21)–(3.22) is equivalent to solving respectively for the *global lifting* $\mathscr{r}$ on Ω defined by (3.17) and for the *local liftings* $\mathscr{r}^{\boldsymbol{a}}$ on every patch $\omega_{\boldsymbol{a}}$ defined by (3.28). Again, only approximations $\mathscr{r}_h \in V$ and $\mathscr{r}_h^{\boldsymbol{a}} \in V^{\boldsymbol{a}}$ are available, where the evaluation error $\mathcal{E}_h \in V'$ is given by

$$\langle \mathcal{E}_h, v \rangle_{V',V} := \left(|\nabla \mathscr{r}_h|^{p-2} \nabla \mathscr{r}_h, \nabla v\right) - \langle \mathcal{R}, v \rangle_{V',V} \qquad v \in V.$$

Since, simultaneously,

$$\|\mathcal{R}\|_{V'} \le \|\mathcal{E}_h\|_{V'} + \|\nabla \mathscr{r}_h\|_p^{p-1},$$

$$\|\nabla \mathscr{r}_h\|_p^{p-1} \le \|\mathcal{R}\|_{V'} + \|\mathcal{E}_h\|_{V'},$$

we obtain

$$\frac{\left|\|\nabla \mathscr{z}_h\|_p^{p-1} - \|\mathcal{R}\|_{V'}\right|}{\|\nabla \mathscr{z}_h\|_p^{p-1}} \leq \frac{\|\mathcal{E}_h\|_{V'}}{\|\nabla \mathscr{z}_h\|_p^{p-1}}.$$

Consequently, using a posteriori techniques from [32], the approximation

$$\|\mathcal{R}\|_{V'} \approx \|\nabla \mathscr{z}_h\|_p^{p-1}$$

is guaranteed to hold with a given relative accuracy that we set to 10^{-2}. Similarly, we secure the relative accuracy of the approximation

$$\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \approx \|\nabla \mathscr{z}_h^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}}^{p-1}$$

to 10^{-2}. For clarity of notation, we drop the subscript h in what follows.

In order to plot local distributions, we find it natural to define two non-negative functions from $\mathbb{P}_1(\mathcal{T}_h)$

$$\epsilon_{\text{glob}}^q := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla \mathscr{z}\|_{p,\omega_{\boldsymbol{a}}}^p \frac{\psi_{\boldsymbol{a}}}{|\omega_{\boldsymbol{a}}|}, \tag{5.2a}$$

$$\epsilon_{\text{loc}}^q := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla \mathscr{z}^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}}^p \frac{\psi_{\boldsymbol{a}}}{|\omega_{\boldsymbol{a}}|}. \tag{5.2b}$$

The employed normalization gives on simplicial meshes $|\omega_{\boldsymbol{a}}|^{-1}(\psi_{\boldsymbol{a}}, 1)_{\omega_{\boldsymbol{a}}} = N_{\text{ov}}^{-1}$ (with $N_{\text{ov}} = d+1$) and together with (2.7a) ensures that

$$\|\epsilon_{\text{glob}}\|_q^q = \|\nabla \mathscr{z}\|_p^p \overset{(3.18)}{=} \|\mathcal{R}\|_{V'}^q,$$
$$\|\epsilon_{\text{loc}}\|_q^q = \frac{1}{N_{\text{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla \mathscr{z}^{\boldsymbol{a}}\|_{p,\omega_{\boldsymbol{a}}}^p \overset{(3.27)}{=} \frac{1}{N_{\text{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}^q.$$

Consequently, Theorem 3.7 can be rephrased as

$$\|\epsilon_{\text{glob}}\|_q \leq N_{\text{ov}} C_{\text{cont,PF}} \|\epsilon_{\text{loc}}\|_q,$$
$$\|\epsilon_{\text{loc}}\|_q \leq \|\epsilon_{\text{glob}}\|_q.$$

Moreover, the second inequality above can be split into local contributions using Remark 3.10, so that

$$\epsilon_{\text{loc}} \leq \epsilon_{\text{glob}}. \tag{5.3}$$

Let us also introduce the effectivity index of an inequality (`ineq`)

$$\text{Eff}_{(\texttt{ineq})} := \frac{\text{rhs of } (\texttt{ineq})}{\text{lhs of } (\texttt{ineq})} \geq 1.$$

For testing, we choose

- *Chaillou–Suri* [24, 32], $\Omega = (0,1)^2$, $p \in \{1.5, 10\}$, $u^{\text{D}}(\mathbf{x}) = q^{-1}(0.5^q - |\mathbf{x} - (0.5, 0.5)|^q)$, $f = -\Delta_p u^{\text{D}} = 2$,
- *Carstensen–Klose* [23, Example 3], $\Omega = (-1,1)^2 \setminus [0,1] \times [-1,0]$, $p = 4$, $u^{\text{D}}(r,\theta) = r^{\frac{7}{8}} \sin\left(\frac{7}{8}\theta\right)$, $f = -\Delta_p u^{\text{D}}$.

As we have the exact solution $u = u^{\text{D}}$ in our hands, we can also check the distribution of the flux error (3.14) and of the $W_0^{1,p}(\Omega)$-norm error (3.15). Therefore, as above, we define non-negative functions from $\mathbb{P}_1(\mathcal{T}_h)$

$$\epsilon_{\text{flux}}^q := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_{q,\omega_{\boldsymbol{a}}}^q \frac{\psi_{\boldsymbol{a}}}{|\omega_{\boldsymbol{a}}|}, \tag{5.4}$$

$$\epsilon_{\text{en}}^p := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\nabla(u - u_h)\|_{p,\omega_{\boldsymbol{a}}}^p \frac{\psi_{\boldsymbol{a}}}{|\omega_{\boldsymbol{a}}|} \tag{5.5}$$

Case	#cells	$C_{\text{cont,PF}}$	$\|\epsilon_{\text{glob}}\|_q$	$\|\epsilon_{\text{loc}}\|_q$	$\|\epsilon_{\text{flux}}\|_q$	$\text{Eff}_{(3.21a)}$	$\text{Eff}_{(5.7)}$	$\text{Eff}_{(3.21b)}$	$\text{Eff}_{(3.13a)}$
	100	5.670	0.0502	0.0431	0.0546	14.6	13.8	1.17	1.09
Chaillou–Suri	400	5.670	0.0259	0.0220	0.0274	14.4	14.1	1.18	1.06
$p = 1.5$, $N_{\text{ov}} = 3$	900	5.670	0.0174	0.0147	0.0183	14.4	14.2	1.18	1.05
	1600	5.670	0.0131	0.0111	0.0137	14.4	14.2	1.18	1.04
	100	7.645	0.0604	0.0484	0.1043	18.4	16.6	1.25	1.73
Chaillou–Suri	400	7.645	0.0312	0.0255	0.0501	18.8	17.8	1.22	1.61
$p = 10.0$, $N_{\text{ov}} = 3$	900	7.645	0.0214	0.0175	0.0343	18.8	18.1	1.22	1.60
	1600	7.645	0.0161	0.0132	0.0255	18.8	18.4	1.22	1.58
	40	9.706	0.1611	0.1236	0.1889	22.3	16.3	1.30	1.17
Carstensen–Klose	189	13.844	0.0930	0.0753	0.1029	33.6	19.0	1.23	1.11
$p = 4.0$, $N_{\text{ov}} = 3$	428	12.981	0.0635	0.0518	0.0701	31.8	19.4	1.23	1.10
	739	12.801	0.0471	0.0383	0.0527	31.2	19.9	1.23	1.12

Table 1: Computed quantities of localization inequalities (3.21), (5.7), and of estimate (3.13a) for the chosen model problems. Recall that $\|\epsilon_{\text{glob}}\|_q = \|\mathcal{R}\|_{V'}$, $\|\epsilon_{\text{loc}}\|_q = \left\{ \sum_{\boldsymbol{a}\in\mathcal{V}_h} \frac{1}{N_{\text{ov}}} \|\mathcal{R}\|^q_{(V^{\boldsymbol{a}})'} \right\}^{\frac{1}{q}}$, and $\|\epsilon_{\text{flux}}\|_q = \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_q$.

having properties

$$\|\epsilon_{\text{flux}}\|_q = \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_q,$$
$$\|\epsilon_{\text{en}}\|_p = \|\nabla(u - u_h)\|_p.$$

Estimates (3.13) translate to

$$\|\epsilon_{\text{glob}}\|_q \le \|\epsilon_{\text{flux}}\|_q, \tag{5.6a}$$
$$\epsilon_{\text{loc}} \le \epsilon_{\text{flux}}. \tag{5.6b}$$

The results of numerical experiments are shown in Table 1 and Figures 2–5. Effectivity indices in Table 1 show that the reverse bound (3.21b) is quite tight but the forward bound (3.21a) suffers by a larger, though still reasonable and predictable, overestimation. This overestimation decreases a little when improving (3.21a) to

$$\|\mathcal{R}\|_{V'} \le N_{\text{ov}} \left\{ \frac{1}{N_{\text{ov}}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \left(C_{\text{cont,PF},\omega_{\boldsymbol{a}}} \|\mathcal{R}\|_{(V^{\boldsymbol{a}})'} \right)^q \right\}^{\frac{1}{q}}, \tag{5.7}$$

where $C_{\text{cont,PF},\omega_{\boldsymbol{a}}} := 1 + C_{\text{PF},p,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\nabla\psi_{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}$ is the continuity constant of each patch; this improvement is much more significant for the case with singularity (*Carstensen–Klose*), see Table 1; we conjecture that the improvement would lose its significance if the residuals $\mathcal{R}$ were obtained on a sequence of adaptively refined meshes.

Figures 2, 3, 4, and 5 nicely demonstrate the local inequalities (3.29) and (3.13b) as expressed by (5.3) and (5.6b), respectively. The figures also show that there is no hope of locally comparing the $W^{1,p}_0(\Omega)$-norm error $\|\nabla(u - u_h)\|^p_p$ (expressed here by ϵ^p_{en} of (5.5)) and the lifted residual error $\|\nabla \mathscr{R}\|^p_p$ (expressed here by ϵ^q_{glob} of (5.2a)). The colorbars systematically present the minimal and maximal values, taken at the vertices $\boldsymbol{a} \in \mathcal{V}_h$. In the plots, there is one color per triangle, corresponding to the mean value over its vertices.

In conclusion, the main result, Theorem 3.7, as well as Remark 3.10, are well supported by the performed numerical experiments.

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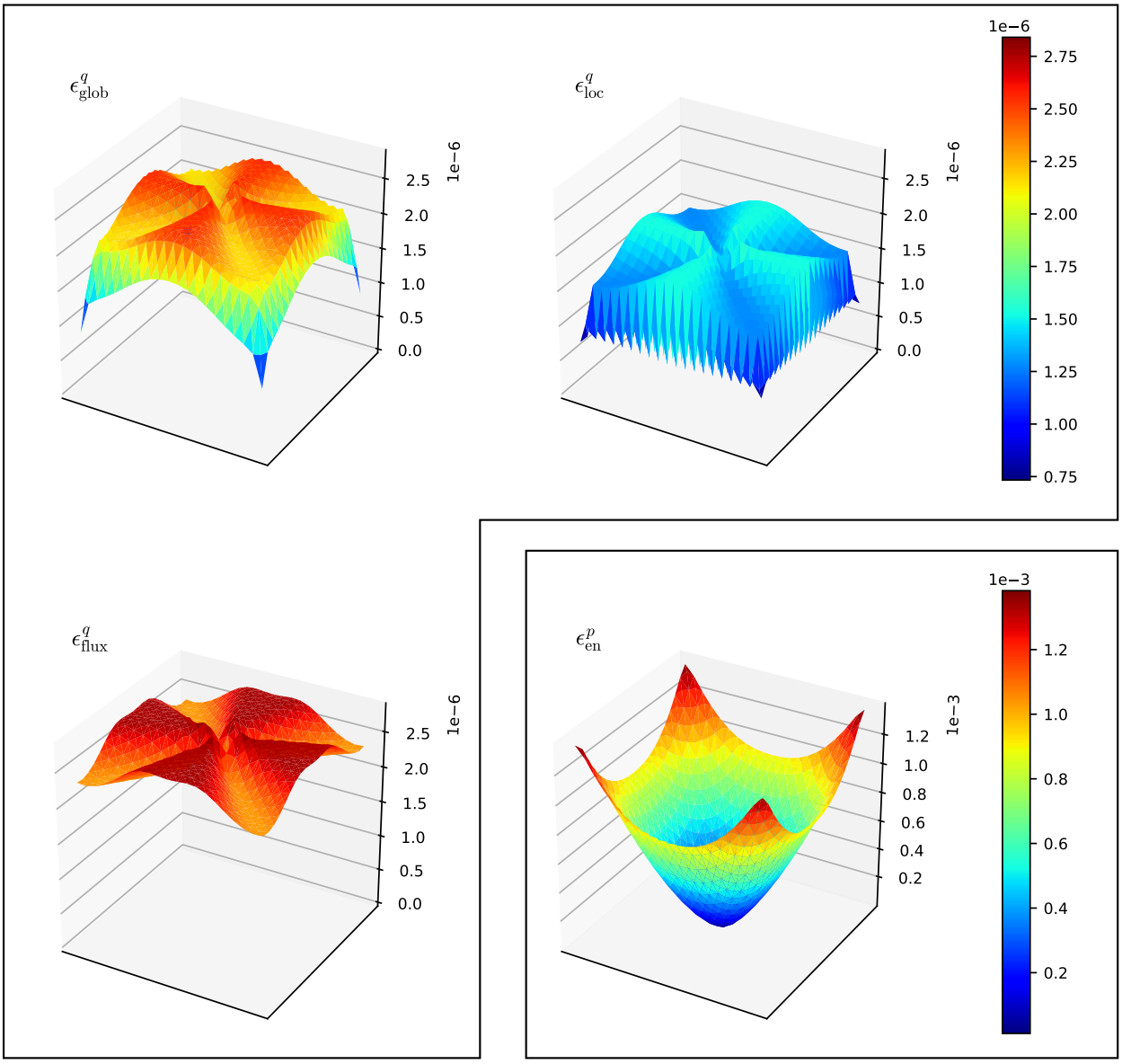

Figure 2: Distribution of $\|\epsilon_{\text{glob}}\|_q^q = \|\mathcal{R}\|_{V'}^q$ (top left), $\|\epsilon_{\text{loc}}\|_q^q = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \frac{1}{N}\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}^q$ (top right), $\|\epsilon_{\text{flux}}\|_q^q = \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_q^q$ (bottom left), and $\|\epsilon_{\text{en}}\|_p^p = \|\nabla(u - u_h)\|_p^p$ (bottom right) for the case *Chaillou–Suri*, $p = 1.5$, #cells=1600

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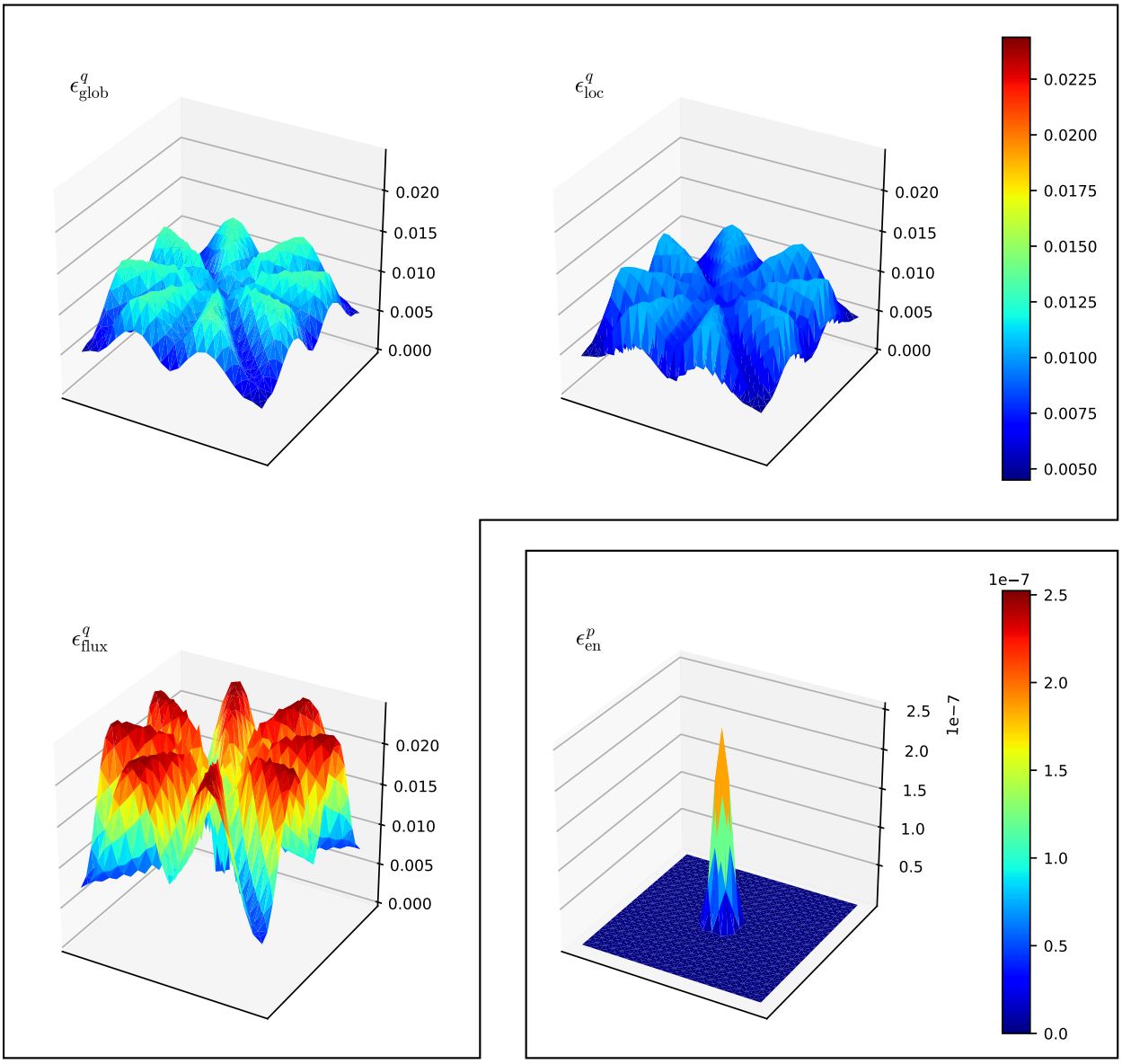

Figure 3: Distribution of $\|\epsilon_{\text{glob}}\|_q^q = \|\mathcal{R}\|_{V'}^q$ (top left), $\|\epsilon_{\text{loc}}\|_q^q = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \frac{1}{N}\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}^q$ (top right), $\|\epsilon_{\text{flux}}\|_q^q = \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_q^q$ (bottom left), and $\|\epsilon_{\text{en}}\|_p^p = \|\nabla(u - u_h)\|_p^p$ (bottom right) for the case *Chaillou–Suri*, $p = 10$, #cells=1600

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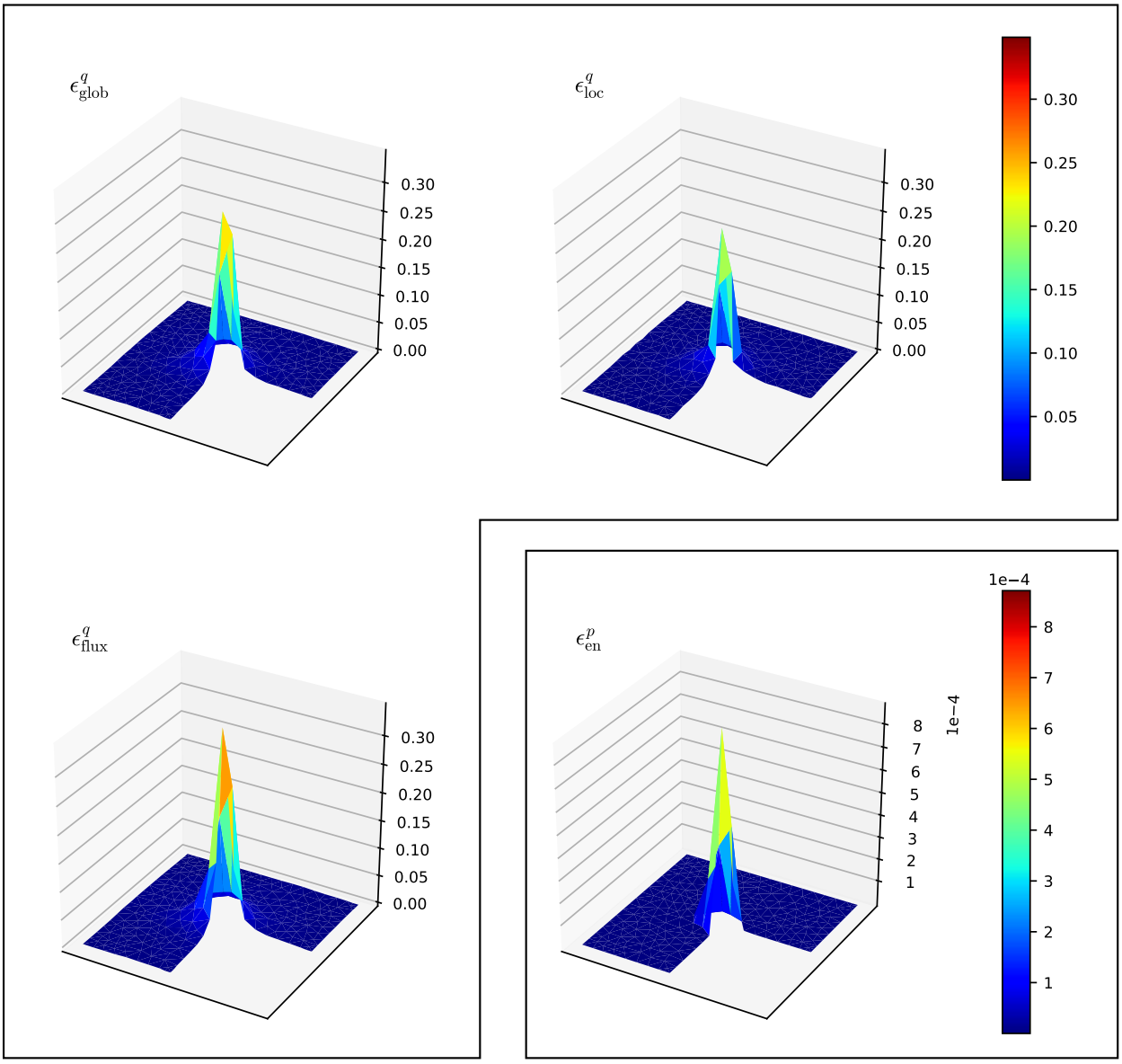

Figure 4: Distribution of $\|\epsilon_{\text{glob}}\|_q^q = \|\mathcal{R}\|_{V'}^q$ (top left), $\|\epsilon_{\text{loc}}\|_q^q = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \frac{1}{N}\|\mathcal{R}\|_{(V^{\boldsymbol{a}})'}^q$ (top right), $\|\epsilon_{\text{flux}}\|_q^q = \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|_q^q$ (bottom left), and $\|\epsilon_{\text{en}}\|_p^p = \|\nabla(u - u_h)\|_p^p$ (bottom right) for the case *Carstensen–Klose*, $p = 4$, #cells=428

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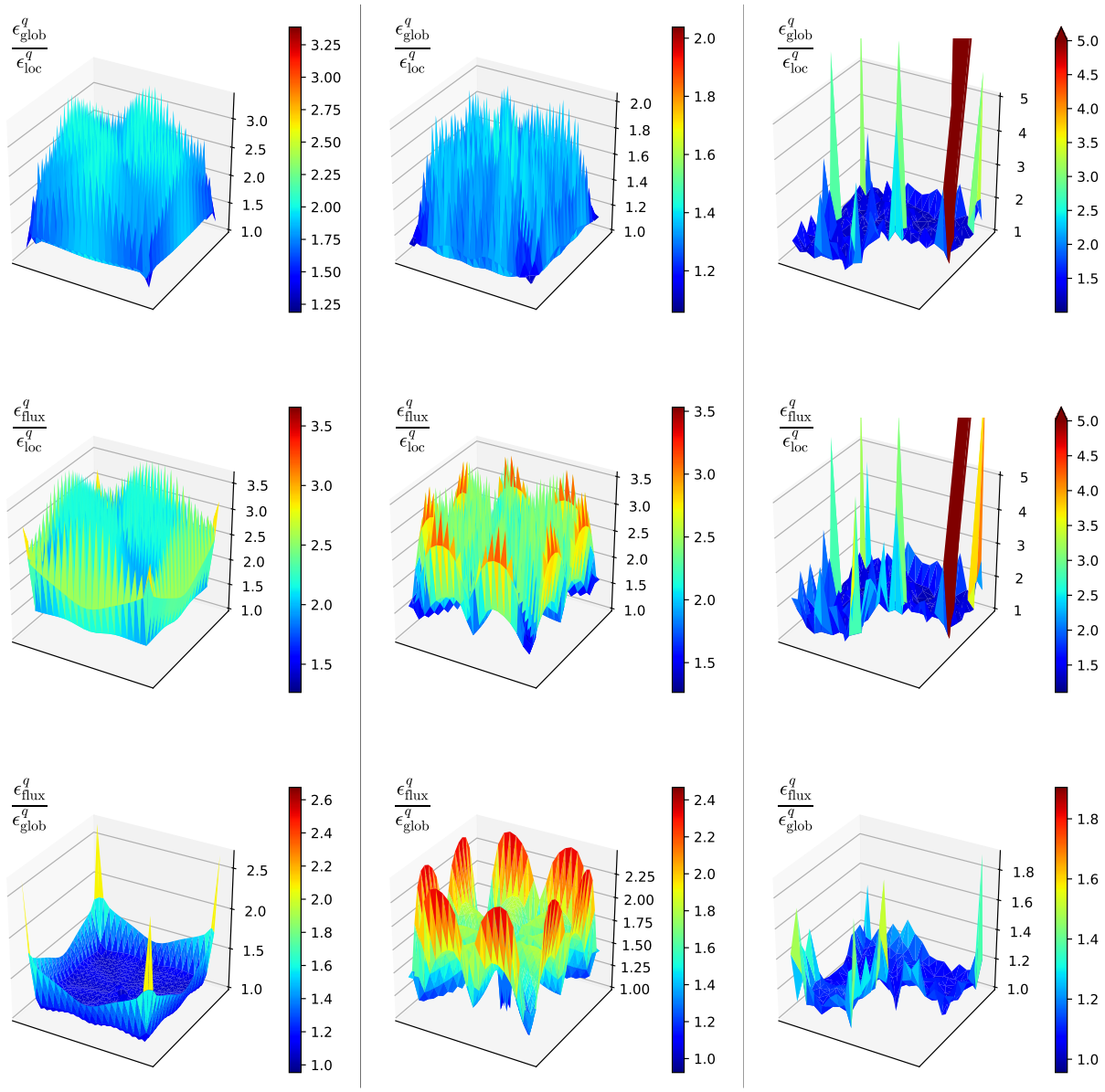

Figure 5: Local ratios of error distributions ϵ_{glob}, ϵ_{loc}, and ϵ_{flux} as functions $\sum_{\boldsymbol{a}\in\mathcal{V}_h} \alpha_{\boldsymbol{a}} \frac{\psi_{\boldsymbol{a}}}{|\omega_{\boldsymbol{a}}|}$ from $\mathbb{P}_1(\mathcal{T}_h)$ with respectively $\alpha_{\boldsymbol{a}} = \|\nabla \mathscr{z}\|^p_{p,\omega_{\boldsymbol{a}}} / \|\nabla \mathscr{z}^{\boldsymbol{a}}\|^p_{p,\omega_{\boldsymbol{a}}}$, $\alpha_{\boldsymbol{a}} = \|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|^q_{q,\omega_{\boldsymbol{a}}} / \|\nabla \mathscr{z}^{\boldsymbol{a}}\|^p_{p,\omega_{\boldsymbol{a}}}$, and $\|\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h)\|^q_{q,\omega_{\boldsymbol{a}}} / \|\nabla \mathscr{z}\|^p_{p,\omega_{\boldsymbol{a}}}$ for cases *Chaillou–Suri*, $p = 1.5$, #cells=1600 (left), *Chaillou–Suri*, $p = 10$, #cells=1600 (middle), *Carstensen–Klose*, $p = 4$, #cells=428 (right). The top and middle row express effectivity of inequalities (5.3) and (5.6b) respectively, hence the quantities are bounded from below by one; the bottom quantity is not known to be bounded from below by one and did not turn out to be bounded in the experiments.

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