A stable local commuting projector and optimal hp approximation estimates in $H(\mathrm{curl})$

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Abstract

We design an operator from the infinite-dimensional Sobolev space $\boldsymbol{H}(\mathrm{curl})$ to its finite-dimensional subspace formed by the Nédélec piecewise polynomials on a tetrahedral mesh that has the following properties: 1) it is defined over the entire $\boldsymbol{H}(\mathrm{curl})$, including boundary conditions imposed on a part of the boundary; 2) it is defined locally in a neighborhood of each mesh element; 3) it is based on simple piecewise polynomial projections; 4) it is stable in the $\boldsymbol{L}^2$-norm, up to data oscillation; 5) it has optimal (local-best) approximation properties; 6) it satisfies the commuting property with its sibling operator on $\boldsymbol{H}(\mathrm{div})$; 7) it is a projector, i.e., it leaves intact objects that are already in the Nédélec piecewise polynomial space. This operator can be used in various parts of numerical analysis related to the $\boldsymbol{H}(\mathrm{curl})$ space. We in particular employ it here to establish the two following results: i) equivalence of global-best, tangential-trace- and curl-constrained, and local-best, unconstrained approximations in $\boldsymbol{H}(\mathrm{curl})$ including data oscillation terms; and ii) fully h- and p- (mesh-size- and polynomial-degree-) optimal approximation bounds valid under the minimal Sobolev regularity only requested elementwise. As a result of independent interest, we also prove a p-robust equivalence of curl-constrained and unconstrained best-approximations on a single tetrahedron in the $\boldsymbol{H}(\mathrm{curl})$-setting, including hp data oscillation terms.

Key words: Sobolev space $\boldsymbol{H}(\mathrm{curl})$, minimal regularity, commuting projector, piecewise polynomial, Nédélec space, hp approximation, error bound, constrained–unconstrained equivalence

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz polyhedral domain (open, bounded, and connected set) and Γ_{N} a Lipschitz polygonal relatively open subset of its boundary $\partial\Omega$ (details on setting and notation are given in Section 2 below). A central concept in numerical analysis of partial differential equations including the grad, curl, and div operators, connected with the Sobolev spaces $H^1_{0,\mathrm{N}}(\Omega)$, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$, and $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, is the following commuting de Rham complex:

$$
\begin{array}{ccccccc}
H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & L^2_*(\Omega) \\
\Big\downarrow P_h^{p,\mathrm{grad}} & & \Big\downarrow \boldsymbol{P}_h^{p,\mathrm{curl}} & & \Big\downarrow \boldsymbol{P}_h^{p,\mathrm{div}} & & \Big\downarrow \Pi_h^p \\
\mathcal{P}_p(\mathcal{T}_h)\cap H^1_{0,\mathrm{N}}(\Omega) & \xrightarrow{\nabla} & \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) & \xrightarrow{\nabla\times} & \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) & \xrightarrow{\nabla\cdot} & \mathcal{P}_p(\mathcal{T}_h)\cap L^2_*(\Omega).
\end{array}
\tag{1.1}
$$

Assuming for simplicity in the introduction that Ω is simply connected and that Γ_{N} is connected, the first line of (1.1) is the well-known exact sequence on the continuous, infinite-dimensional, level, see, e.g., Arnold *et al.* [3] and the references therein. It in particular states that 1) each function from the $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ space whose weak curl vanishes is a weak gradient of a function from $H^1_{0,\mathrm{N}}(\Omega)$; 2) each function from the $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ space whose weak divergence vanishes is a weak curl of a function from $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$; 3) each function from the $L^2_*(\Omega)$ space is a weak divergence of a function from $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$.

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Similarly, the second line is the counterpart of the first one on the discrete, finite-dimensional, piecewise polynomial, level, see e.g., Boffi *et al.* [9] and the references therein. The passage between the first and the second line is then the key interest in this contribution, where the three operators $P_h^{p,\mathrm{grad}}$, $\boldsymbol{P}_h^{p,\mathrm{curl}}$, and $\boldsymbol{P}_h^{p,\mathrm{div}}$ should:

1. be defined over the *entire infinite-dimensional spaces* $H^1_{0,\mathrm{N}}(\Omega)$, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$, and $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$;
2. be defined *locally*, in a neighborhood of mesh elements at most;
3. be based on *simple* piecewise polynomial projections;
4. be *stable* in $\boldsymbol{L}^2(\Omega)$ for $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ and $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ and in $\boldsymbol{L}^2(\Omega)$ of the weak gradient for $H^1_{0,\mathrm{N}}(\Omega)$ (up to data oscillation);
5. have *optimal approximation properties*, i.e., that of *local-best* unconstrained $\boldsymbol{L}^2$-orthogonal projectors;
6. satisfy the *commuting properties* expressed by the arrows in (1.1);
7. be *projectors*, i.e., leave intact objects that are already in the piecewise polynomial spaces.

There is an immense literature devoted to (1.1). A first consideration for the operators $P_h^{p,\mathrm{grad}}$, $\boldsymbol{P}_h^{p,\mathrm{curl}}$, and $\boldsymbol{P}_h^{p,\mathrm{div}}$ is given by the canonical projectors, see Ciarlet [18], Nédélec [39], and Raviart and Thomas [41], respectively. These satisfy many of the properties above, but, unfortunately, not property 1, since their action is not defined on all objects from the entire infinite-dimensional spaces $H^1_{0,\mathrm{N}}(\Omega)$, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$, and $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$. The commuting diagram (1.1) has been addressed at the abstract level of the finite element exterior calculus in, e.g., Christiansen and Winther [17], still leading to the loss of some of the desirable properties, namely the locality. Simultaneous definition on the entire Sobolev spaces, locality, commutativity, and the projection property have been achieved in Falk and Winther [30], though the stability in the $\boldsymbol{L}^2(\Omega)$ norms (up to data oscillation) and the local-best unconstrained approximation properties have not been addressed. This has been recently addressed in [2]. Two different sets of projectors, satisfying together (but not individually) all properties 1–7, were then designed in Ern and Guermond [27, 28]. Finally an operator $\boldsymbol{P}_h^{p,\mathrm{div}}$ satisfying the integrality of the requested properties has been recently devised in Ern *et al.* [26, Section 3.1].

The first goal of the present contribution is to design an operator $\boldsymbol{P}_h^{p,\mathrm{curl}}$ satisfying the *integrality of the requested properties* 1–7. Definition 3.3 is designed to this purpose, relying on (a slight modification of) $\boldsymbol{P}_h^{p,\mathrm{div}}$ from [26], see Definition 3.1, and using similar building principles as in [26]. The main result here is Theorem 3.6. The central technical tool allowing to achieve the commuting property is related to equilibration in $\boldsymbol{H}(\mathrm{curl},\Omega)$. A first contribution in this direction is that of Braess and Schöberl [11]. Recent extensions to higher polynomial degrees are developed in Gedicke *et al.* [32, 33] as well as [16], which we use here.

Our contribution stands apart from the existing literature namely in the satisfaction of property 5. This leads to the result of equivalence of global-best (tangential-trace- and curl-constrained) and local-best (unconstrained) approximations in $\boldsymbol{H}(\mathrm{curl},\Omega)$, see Theorem 3.8. This result, not taking into account data oscillation, has been recently established in [14], building on the seminal contribution by Veeser [43] in the $H^1(\Omega)$-setting and on [26] in the $\boldsymbol{H}(\mathrm{div},\Omega)$-setting, see also the references therein. Here, we present a direct proof. We take into account data oscillation, which actually turns out quite demanding.

Yet a separate, and involved, question in numerical analysis is that of deriving *hp*-approximation estimates. This has been addressed in the $\boldsymbol{H}(\mathrm{div},\Omega)$- and $\boldsymbol{H}(\mathrm{curl},\Omega)$-settings in particular in Suri [42], Monk [38], Demkowicz and Buffa [22], and Demkowicz [21], see also the references therein. These references feature a slight suboptimality in the polynomial degree p on tetrahedral meshes (presence of a logarithmic factor), which has been removed in Bespalov and Heuer [7, 8] and recently in Melenk and Rojik [37]. Unfortunately, none of these references allows for minimal Sobolev regularity. The result in [26, Theorem 3.6] is equally fully h- and p-optimal, and this, moreover, under the minimal Sobolev regularity, only requested elementwise. Deriving such estimates in the $\boldsymbol{H}(\mathrm{curl},\Omega)$-setting is the last goal of the present contribution. Theorem 3.10 (possibly under a technical restriction of convex patches) in particular presents a fully h- and p-optimal approximation bound valid under the minimal Sobolev regularity that is only requested separately in each mesh element. The key ingredients are here again linked to (polynomial-degree-robust) flux equilibration in $\boldsymbol{H}(\mathrm{curl},\Omega)$ of [16], with the cornerstones being the results in a single tetrahedron: on the right inverses by Costabel and McIntosh [20], and on the polynomial extension operators by Demkowicz *et al.* [23, 24, 25].

This contribution is organized as follows: Section 2 fixes the setting and notation. The above-described main results are collected in Section 3. The well-posedness of the central definition of our stable local commuting projector is verified in Section 4, and the proofs of the three principal theorems are then presented respectively in Sections 5–7. Finally, a result of independent interest, stipulating a polynomial-degree-robust equivalence of constrained and unconstrained best-approximation on a single tetrahedron in the $\boldsymbol{H}(\mathrm{curl},\Omega)$-setting, including hp data oscillation terms, is presented in Appendix A.

2 Setting and notation

Let $\omega,\Omega \subset \mathbb{R}^3$ be open, Lipschitz polyhedral domains; Ω will be used to denote the computational domain, while we reserve the notation $\omega \subseteq \Omega$ for its simply connected subsets. Notice that we do not require Ω to be simply connected. We will use the notation $a \lesssim b$ when there exists a positive constant C such that $a \leq Cb$; we will always specify the dependencies of C.

2.1 Sobolev spaces H^1, $\boldsymbol{H}(\mathrm{curl})$, and $\boldsymbol{H}(\mathrm{div})$

We let $L^2(\omega)$ be the space of scalar-valued square-integrable functions defined on ω; we use the notation $\boldsymbol{L}^2(\omega) := [L^2(\omega)]^3$ for vector-valued functions with each component in $L^2(\omega)$. We denote by $\|{\cdot}\|_\omega$ the $L^2(\omega)$ or $\boldsymbol{L}^2(\omega)$ norm and by $(\cdot,\cdot)_\omega$ the corresponding scalar product; we drop the index when $\omega = \Omega$. We will extensively work with the following three Sobolev spaces: 1) $H^1(\omega)$, the space of scalar-valued $L^2(\omega)$ functions with weak gradients in $\boldsymbol{L}^2(\omega)$, $H^1(\omega) := \{v \in L^2(\omega);\, \nabla v \in \boldsymbol{L}^2(\omega)\}$; 2) $\boldsymbol{H}(\mathrm{curl},\omega)$, the space of vector-valued $\boldsymbol{L}^2(\omega)$ functions with weak curls in $\boldsymbol{L}^2(\omega)$, $\boldsymbol{H}(\mathrm{curl},\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\, \nabla{\times}\boldsymbol{v} \in \boldsymbol{L}^2(\omega)\}$; and 3) $\boldsymbol{H}(\mathrm{div},\omega)$, the space of vector-valued $\boldsymbol{L}^2(\omega)$ functions with weak divergences in $L^2(\omega)$, $\boldsymbol{H}(\mathrm{div},\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\, \nabla{\cdot}\boldsymbol{v} \in L^2(\omega)\}$. We refer the reader to Adams [1] and Girault and Raviart [34] for an in-depth description of these spaces. Moreover, component-wise $H^1(\omega)$ functions will be denoted by $\boldsymbol{H}^1(\omega) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega);\, \boldsymbol{v}_i \in H^1(\omega),\, i = 1,\ldots,3\}$. We will employ the notation $\langle\cdot,\cdot\rangle_S$ for the integral product on boundary (sub)sets $S \subset \partial\omega$.

2.2 Tetrahedral mesh, patches of elements, and the hat functions

Let $\mathcal{T}_h$ be a simplicial mesh of the domain Ω, i.e., $\cup_{K\in\mathcal{T}_h} K = \overline{\Omega}$, where any element $K \in \mathcal{T}_h$ is a closed tetrahedron with nonzero measure, and where the intersection of two different tetrahedra is either empty or their common vertex, edge, or face. The shape-regularity parameter of the mesh $\mathcal{T}_h$ is the positive real number $\kappa_{\mathcal{T}_h} := \max_{K\in\mathcal{T}_h} h_K/\rho_K$, where h_K is the diameter of the tetrahedron K and ρ_K is the diameter of the largest ball contained in K. These assumptions are standard, and allow for strongly graded meshes with local refinements, though not for anisotropic elements.

We denote the set of vertices of the mesh $\mathcal{T}_h$ by $\mathcal{V}_h$; it is composed of interior vertices lying in Ω and of vertices lying on the boundary $\partial\Omega$. For an element $K \in \mathcal{T}_h$, $\mathcal{F}_K$ denotes the set of its faces and $\mathcal{V}_K$ the set of its vertices. Conversely, for a vertex $\boldsymbol{a} \in \mathcal{V}_h$, $\mathcal{T}_{\boldsymbol{a}}$ denotes the patch of the elements of $\mathcal{T}_h$ that share $\boldsymbol{a}$, and $\omega_{\boldsymbol{a}}$ is the corresponding open subdomain with diameter $h_{\omega_{\boldsymbol{a}}}$. A particular role below will be played by the continuous, piecewise affine "hat" function $\psi^{\boldsymbol{a}}$ which takes value 1 at the vertex $\boldsymbol{a}$ and zero at the other vertices. We note that $\omega_{\boldsymbol{a}}$ corresponds to the support of $\psi^{\boldsymbol{a}}$ and that the functions $\psi^{\boldsymbol{a}}$ form the partition of unity

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \psi^{\boldsymbol{a}} = 1. \tag{2.1}$$

By $[\![v]\!]$, we denote the jump of the function v on a face F, i.e., the difference of the traces of v from the two elements sharing F along an arbitrary but fixed normal.

2.3 Sobolev spaces with partially vanishing traces on Ω and $\omega_{\boldsymbol{a}}$

Let Γ_{D}, Γ_{N} be two disjoint, relatively open, and possibly empty subsets of the computational domain boundary $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma_{\mathrm{D}}} \cup \overline{\Gamma_{\mathrm{N}}}$. We also require that Γ_{D} and Γ_{N} have polygonal Lipschitz boundaries, and we assume that each boundary face of the mesh $\mathcal{T}_h$ lies entirely either in $\overline{\Gamma_{\mathrm{D}}}$ or in $\overline{\Gamma_{\mathrm{N}}}$. Then $L^2_*(\Omega)$ is the subspace of $L^2(\Omega)$ functions of mean value 0 if $\Gamma_{\mathrm{N}} = \partial\Omega$, whereas $H^1_{0,\mathrm{D}}(\Omega)$ is the subspace of $H^1(\Omega)$ formed by functions vanishing on Γ_{D} in the sense of traces, $H^1_{0,\mathrm{D}}(\Omega) := \{v \in H^1(\Omega);\, v = 0 \text{ on } \Gamma_{\mathrm{D}}\}$. Let $\boldsymbol{n}_\Omega$ be the unit normal vector on $\partial\Omega$, outward to Ω. Then $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ is the subspace of $\boldsymbol{H}(\mathrm{curl},\Omega)$ formed by functions with vanishing tangential trace on Γ_{N}, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) :=$

$\{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_\mathrm{N}\}$, where $\boldsymbol{v}\times\boldsymbol{n}_\Omega = 0$ on Γ_N means that $(\nabla\times\boldsymbol{v},\boldsymbol{\varphi}) - (\boldsymbol{v},\nabla\times\boldsymbol{\varphi}) = 0$ for all functions $\boldsymbol{\varphi} \in \boldsymbol{H}^1(\Omega)$ such that $\boldsymbol{\varphi}\times\boldsymbol{n}_\Omega = \boldsymbol{0}$ on Γ_D. We will also employ the similar notation $\boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_\mathrm{D}\}$. Finally, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ is the subspace of $\boldsymbol{H}(\mathrm{div},\Omega)$ formed by functions with vanishing normal trace on Γ_N, $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0 \text{ on } \Gamma_\mathrm{N}\}$, where $\boldsymbol{v}\cdot\boldsymbol{n}_\Omega = 0$ on Γ_N means that $(\boldsymbol{v},\nabla\varphi) + (\nabla\cdot\boldsymbol{v},\varphi) = 0$ for all functions $\varphi \in H^1_{0,\mathrm{D}}(\Omega)$. Fernandes and Gilardi [31] present a rigorous characterization of tangential (resp. normal) traces of $\boldsymbol{H}(\mathrm{curl},\Omega)$ (resp. $\boldsymbol{H}(\mathrm{div},\Omega)$) on a part of the boundary $\partial\Omega$.

We will also need local spaces on the patch subdomains $\omega_{\boldsymbol{a}}$. Let first $\boldsymbol{a} \in \mathcal{V}_h$ be an interior vertex. Then we set 1) $H^1_*(\omega_{\boldsymbol{a}}) := \{v \in H^1(\omega_{\boldsymbol{a}});\ (v,1)_{\omega_{\boldsymbol{a}}} = 0\}$, so that $H^1_*(\omega_{\boldsymbol{a}})$ is the subspace of those $H^1(\omega_{\boldsymbol{a}})$ functions whose mean value vanishes; 2) $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}}\}$, where the tangential trace is understood as above; and, similarly, 3) $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}}\}$. We will also need 4) $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}}) := \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}})$. The situation is more subtle for boundary vertices. As a first possibility, if $\boldsymbol{a} \in \Gamma_\mathrm{N}$ (i.e., $\boldsymbol{a} \in \mathcal{V}_h$ is a boundary vertex such that all the boundary faces sharing the vertex $\boldsymbol{a}$ lie in Γ_N), then the spaces $H^1_*(\omega_{\boldsymbol{a}})$, $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})$, $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, and $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}})$ are defined as above. Secondly, when $\boldsymbol{a} \in \overline{\Gamma_\mathrm{D}}$, then at least one of the faces sharing the vertex $\boldsymbol{a}$ lies in $\overline{\Gamma_\mathrm{D}}$, and we denote by γ_D the subset of Γ_D corresponding to all such faces. In this situation, we let 1) $H^1_*(\omega_{\boldsymbol{a}}) := \{v \in H^1(\omega_{\boldsymbol{a}});\ v = 0 \text{ on } \gamma_\mathrm{D}\}$; 2) $\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}} \setminus \gamma_\mathrm{D}\}$; 3) $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\cdot\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \partial\omega_{\boldsymbol{a}} \setminus \gamma_\mathrm{D}\}$; and 4) $\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl},\omega_{\boldsymbol{a}});\ \boldsymbol{v}\times\boldsymbol{n}_{\omega_{\boldsymbol{a}}} = 0 \text{ on } \gamma_\mathrm{D}\}$. In all cases, component-wise $H^1_*(\omega_{\boldsymbol{a}})$ functions are denoted by $\boldsymbol{H}^1_*(\omega_{\boldsymbol{a}})$.

2.4 Piecewise polynomial spaces

Let $q \geq 0$ be an integer. For a single tetrahedron $K \in \mathcal{T}_h$, we denote by $\mathcal{P}_q(K)$ the space of scalar-valued polynomials on K of total degree at most q, and by $[\mathcal{P}_q(K)]^3$ the space of vector-valued polynomials on K with each component in $\mathcal{P}_q(K)$. The Nédélec [9, 39] space of degree q on K is then given by

$$\boldsymbol{\mathcal{N}}_q(K) := [\mathcal{P}_q(K)]^3 + \boldsymbol{x}\times[\mathcal{P}_q(K)]^3. \tag{2.2}$$

Similarly, the Raviart–Thomas [9, 41] space of degree q on K is given by

$$\boldsymbol{\mathcal{RT}}_q(K) := [\mathcal{P}_q(K)]^3 + \mathcal{P}_q(K)\boldsymbol{x}. \tag{2.3}$$

We note that (2.2) and (2.3) are equivalent to the writing with a direct sum and only homogeneous polynomials in the second terms. The second term in (2.2) is also equivalently given by homogeneous $(q+1)$-degree polynomials $\boldsymbol{v}_h$ such that $\boldsymbol{x}\cdot\boldsymbol{v}_h(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in K$.

We will below extensively use the broken, piecewise polynomial spaces formed from the above element spaces

$$\begin{aligned}
\mathcal{P}_q(\mathcal{T}_h) &:= \{v_h \in L^2(\Omega);\ v_h|_K \in \mathcal{P}_q(K) \quad \forall K \in \mathcal{T}_h\},\\
\boldsymbol{\mathcal{N}}_q(\mathcal{T}_h) &:= \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{N}}_q(K) \quad \forall K \in \mathcal{T}_h\},\\
\boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h) &:= \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega);\ \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{RT}}_q(K) \quad \forall K \in \mathcal{T}_h\}.
\end{aligned}$$

To form the usual finite-dimensional Sobolev subspaces, we will write $\mathcal{P}_q(\mathcal{T}_h) \cap H^1(\Omega)$ (for $q \geq 1$), $\boldsymbol{\mathcal{N}}_q(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl},\Omega)$, $\boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div},\Omega)$ (both for $q \geq 0$), and similarly for the subspaces reflecting the different boundary conditions. The same notation will also be used on the patches $\mathcal{T}_{\boldsymbol{a}}$.

2.5 L^2-orthogonal projectors and elementwise canonical interpolators

For $q \geq 0$, let Π_h^q denote the $L^2(K)$-orthogonal projector onto $\mathcal{P}_q(K)$ or the elementwise $L^2(\Omega)$-orthogonal projector onto $\mathcal{P}_q(\mathcal{T}_h)$, i.e., for $v \in L^2(\Omega)$, $\Pi_h^q(v) \in \mathcal{P}_q(\mathcal{T}_h)$ is, separately for all $K \in \mathcal{T}_h$, given by

$$(\Pi_h^q(v), v_h)_K = (v, v_h)_K \qquad \forall v_h \in \mathcal{P}_q(K). \tag{2.4}$$

Then, $\boldsymbol{\Pi}_h^q$ is given componentwise by Π_h^q. We will also use the $L^2(\Omega)$-orthogonal projector $\boldsymbol{\Pi}^q_{\boldsymbol{\mathcal{RT}}}$ onto $\boldsymbol{\mathcal{RT}}_q(\mathcal{T}_h)$, given for $\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)$ also elementwise as: for all $K \in \mathcal{T}_h$, $\boldsymbol{\Pi}^q_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{v})|_K \in \boldsymbol{\mathcal{RT}}_q(K)$ is such that

$$(\boldsymbol{\Pi}^q_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{v}), \boldsymbol{v}_h)_K = (\boldsymbol{v}, \boldsymbol{v}_h)_K \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_q(K), \tag{2.5a}$$

or, equivalently,

$$\boldsymbol{\Pi}^q_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{v})|_K := \arg \min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_q(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K. \tag{2.5b}$$

Let $K \in \mathcal{T}_h$ be a mesh tetrahedron and $\boldsymbol{v} \in [C^1(K)]^3$ be given. Following [9, 41], the canonical q-degree Raviart–Thomas interpolate $\boldsymbol{I}^q_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{v}) \in \boldsymbol{\mathcal{RT}}_q(K)$, $q \geq 0$, is given by

$$\langle \boldsymbol{I}^q_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{v}) \cdot \boldsymbol{n}_K, r_h \rangle_F = \langle \boldsymbol{v} \cdot \boldsymbol{n}_K, r_h \rangle_F \qquad \forall r_h \in \mathcal{P}_q(F), \quad \forall F \in \mathcal{F}_K, \tag{2.6a}$$

$$(\boldsymbol{I}^q_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{v}), \boldsymbol{r}_h)_K = (\boldsymbol{v}, \boldsymbol{r}_h)_K \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_{q-1}(K)]^3, \tag{2.6b}$$

where $\boldsymbol{n}_K$ is the unit outer normal vector of the element K. Following [9, 39], canonical q-degree Nédélec interpolate $\boldsymbol{I}^q_{\boldsymbol{\mathcal{N}}}(\boldsymbol{v}) \in \boldsymbol{\mathcal{N}}_q(K)$, $q \geq 0$, is given by

$$\langle \boldsymbol{I}^q_{\boldsymbol{\mathcal{N}}}(\boldsymbol{v}) \cdot \boldsymbol{\tau}_e, r_h \rangle_e = \langle \boldsymbol{v} \cdot \boldsymbol{\tau}_e, r_h \rangle_e \qquad \forall r_h \in \mathcal{P}_q(e), \quad \forall e \in \mathcal{E}_K, \tag{2.7a}$$

$$\langle \boldsymbol{I}^q_{\boldsymbol{\mathcal{N}}}(\boldsymbol{v}) \times \boldsymbol{n}_K, \boldsymbol{r}_h \rangle_F = \langle \boldsymbol{v} \times \boldsymbol{n}_K, \boldsymbol{r}_h \rangle_F \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_{q-1}(F)]^2, \quad \forall F \in \mathcal{F}_K, \tag{2.7b}$$

$$(\boldsymbol{I}^q_{\boldsymbol{\mathcal{N}}}(\boldsymbol{v}), \boldsymbol{r}_h)_K = (\boldsymbol{v}, \boldsymbol{r}_h)_K \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_{q-2}(K)]^3, \tag{2.7c}$$

where $\mathcal{E}_K$ stands for the set of edges of K, $\boldsymbol{\tau}_e$ and $\langle \cdot, \cdot \rangle_e$ respectively denote a (unit) tangential vector (the orientation does not matter) and the $L^2(e)$ scalar product of the edge $e \in \mathcal{E}_K$, and where we implicitly understand $\boldsymbol{I}^q_{\boldsymbol{\mathcal{N}}}(\boldsymbol{v}) \times \boldsymbol{n}_K$ and $\boldsymbol{v} \times \boldsymbol{n}_K$ as two-dimensional vectors in the face F in (2.7b). Less regular functions can be used in (2.6)–(2.7), but $\boldsymbol{v} \in [C^1(K)]^3$ will be sufficient for our purposes; we will actually only use polynomial $\boldsymbol{v}$. These interpolators crucially satisfy, on the tetrahedron K and for all $\boldsymbol{v} \in [C^1(K)]^3$, the commuting properties

$$\nabla \cdot \boldsymbol{I}^q_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{v}) = \Pi^q_h(\nabla \cdot \boldsymbol{v}), \tag{2.8a}$$

$$\nabla \times (\boldsymbol{I}^q_{\boldsymbol{\mathcal{N}}}(\boldsymbol{v})) = \boldsymbol{I}^q_{\boldsymbol{\mathcal{RT}}}(\nabla \times \boldsymbol{v}). \tag{2.8b}$$

2.6 Functional inequalities

We will need the four following functional inequalities.

First, from [19, Theorems 3.4 and 3.5], [36, Theorem 2.1], and the discussion in [13, Section 3.2.1], it follows that for a polyhedral domain $\omega \subset \Omega$ with a Dirichlet boundary $\gamma_{\rm D}$ given by some of its faces, there exists a constant $C_{{\rm L},\omega}$ such that for all $\boldsymbol{v} \in \boldsymbol{H}_{0,{\rm D}}(\mathrm{curl}, \omega)$, there exists $\boldsymbol{w} \in \boldsymbol{H}^1(\omega) \cap \boldsymbol{H}_{0,{\rm D}}(\mathrm{curl}, \omega)$ such that $\nabla \times \boldsymbol{w} = \nabla \times \boldsymbol{v}$ and

$$\|\nabla \boldsymbol{w}\|_\omega \leq C_{{\rm L},\omega} \|\nabla \times \boldsymbol{v}\|_\omega. \tag{2.9}$$

When the $\gamma_{\rm D}$ has either a zero measure (in which case $\boldsymbol{w}$ can additionally be taken of mean value zero componentwise) or coincides with $\partial\omega$ and if ω is convex, one can take $C_{{\rm L},\omega} = 1$, see [19] together with [34, Theorem 3.7] for Dirichlet boundary conditions and [34, Theorem 3.9] for Neumann boundary conditions. Actually, $\boldsymbol{w}$ can even be taken in $\boldsymbol{H}^1_{0,{\rm D}}(\omega)$, though the above characterizations $C_{{\rm L},\omega} = 1$ may be lost. We typically employ (2.9) with ω taken as the vertex patch $\omega_{\boldsymbol{a}}$, where $C_{{\rm L},\omega_{\boldsymbol{a}}}$ only depends on the shape-regularity parameter of the mesh $\kappa_{\mathcal{T}_h}$.

Second, for any $\boldsymbol{v} \in \boldsymbol{H}^1(\omega)$ of componentwise mean value zero on ω or with the trace equal to zero on $\gamma_{\rm D} \subset \partial\omega$ which consists of at least one face of ω, the Poincaré–Friedrichs inequality gives

$$\|\boldsymbol{v}\|_\omega \leq C_{{\rm PF},\omega} h_\omega \|\nabla \boldsymbol{v}\|_\omega. \tag{2.10}$$

In the first case, $C_{{\rm PF},\omega} \leq 1/\pi$ for convex ω, see [40, 5], whereas in the second case, $C_{{\rm PF},\omega} \leq 1$ when there exists a unit vector $\boldsymbol{m}$ such that the straight semi-line of direction $\boldsymbol{m}$ originating at any point in ω hits the boundary $\partial\omega$ in the subset $\gamma_{\rm D}$, see [44]. As above, we will employ (2.10) with $\omega = \omega_{\boldsymbol{a}}$, where $C_{{\rm PF},\omega_{\boldsymbol{a}}}$ only depends on the shape-regularity parameter of the mesh $\kappa_{\mathcal{T}_h}$.

Third, for any mesh element $K \in \mathcal{T}_h$, $\boldsymbol{v} \in \boldsymbol{H}^1(K)$, and $q \geq 0$, there holds the hp approximation/Poincaré inequality

$$\|\boldsymbol{v} - \boldsymbol{\Pi}^q_h(\boldsymbol{v})\|_K \leq C_{\rm hp} \frac{h_K}{q+1} \|\nabla \boldsymbol{v}\|_K \tag{2.11}$$

for a generic constant $C_{\rm hp}$ only depending on κ_K [4].

Finally, under the additional assumption that $\gamma_{\rm D}$ is connected, the Poincaré–Friedrichs–Weber inequality, see [31, Proposition 7.4] and more precisely [13, Theorem A.1] for the form of the constant, will be useful: for all vertices $\boldsymbol{a} \in \mathcal{V}_h$ and all vector-valued functions $\boldsymbol{v} \in \boldsymbol{H}^\dagger(\mathrm{curl}, \omega_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla \cdot \boldsymbol{v} = 0$, we have

$$\|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} \leq C_{\rm PFW} h_{\omega_{\boldsymbol{a}}} \|\nabla \times \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}, \tag{2.12}$$

where $C_{\rm PFW}$ only depends on the mesh shape-regularity $\kappa_{\mathcal{T}_h}$. Strictly speaking, the inequality is established in [13, Theorem A.1] for edge patches, but the proof can be easily extended to vertex patches.

3 Main results

We present here our main results.

3.1 A stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$

We need to first recall the projector $\boldsymbol{P}_h^{p,\mathrm{div}}$ from [26, Definition 3.1]:

Definition 3.1 (A stable local commuting projector in $\boldsymbol{H}(\mathrm{div})$)**.** *Let $\boldsymbol{w} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ be given.*

1. *Define a broken Raviart–Thomas polynomial $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)$, on each mesh element, via*

$$\boldsymbol{\tau}_h|_K := \arg \min_{\substack{\boldsymbol{w}_h \in \boldsymbol{\mathcal{RT}}_p(K) \\ \nabla\cdot\boldsymbol{w}_h = \Pi_h^p(\nabla\cdot\boldsymbol{w})}} \|\boldsymbol{w}-\boldsymbol{w}_h\|_K \qquad \forall K \in \mathcal{T}_h. \tag{3.1}$$

2. *Define a Raviart–Thomas polynomial $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, on each vertex patch, via*

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{w}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{w}_h = \Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{w} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{w})}} \left\|\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{w}_h\right\|_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{3.2}$$

3. *Extending $\boldsymbol{\sigma}_h^{\boldsymbol{a}}$ by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, define $\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{w}) \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ via*

$$\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{w}) := \boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}}. \tag{3.3}$$

The above definition works on the entire space $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, in contrast to the canonical projector $\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p$ from (2.6). In Step 1, we simply project $\boldsymbol{w}$ to the $\boldsymbol{\mathcal{RT}}_p$ space elementwise, under the divergence constraint. The intermediate field $\boldsymbol{\tau}_h$ is close to $\boldsymbol{w}$ but does not in general lie in $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$. This is corrected patchwise in Step 2: a cut-off of $\boldsymbol{\tau}_h$ is made by the hat function $\psi^{\boldsymbol{a}}$, the canonical elementwise projector $\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p$ from (2.6) is applied in order not to increase the polynomial degree by one, and a suitable divergence constraint given by the elementwise L^2 projection (2.4) of $\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{w})$ onto piecewise p-degree polynomials is applied to crucially obtain the commuting property after Step 3:

$$\nabla\cdot\boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{w}) = \Pi_h^p(\nabla\cdot\boldsymbol{w}). \tag{3.4}$$

Remark 3.2 (Divergence constraint in (3.2))**.** *The second term in the divergence constraint in (3.2) is modified with respect to [26, Definition 3.1], where $\Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{w} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\tau}_h)$ is used instead. With this modification, a supplementary term arises on the right-hand side of equation (4.8) in the proof of Lemma 4.6 in [26]. This term writes and can be bounded as*

$$(\nabla\psi^{\boldsymbol{a}}\cdot(\boldsymbol{v}-\boldsymbol{\tau}_{\mathcal{T}}), \Pi_{\mathcal{T}}^p\varphi)_{\omega_{\boldsymbol{a}}} \le C\|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}} \|\boldsymbol{v}-\boldsymbol{\tau}_{\mathcal{T}}\|_{\omega_{\boldsymbol{a}}} \|\nabla\varphi\|_{\omega_{\boldsymbol{a}}} \le C\|\boldsymbol{v}-\boldsymbol{\tau}_{\mathcal{T}}\|_{\omega_{\boldsymbol{a}}}, \tag{3.5}$$

where the constant C only depends on the shape regularity parameter $\kappa_{\mathcal{T}_h}$, so that Lemma 4.6 and all results in [26] still hold. This subtle switch from $\boldsymbol{\tau}_h$ to $\boldsymbol{w}$ in the second term in the divergence constraint in (3.2), however, turns out crucial for our developments here, insuring in particular the well-posedness of Definition 3.3 below in that it makes (4.6) in Lemma 4.4 hold true.

3.2 A stable local commuting projector in $\boldsymbol{H}(\mathrm{curl})$

We define here our stable local commuting projector in $\boldsymbol{H}(\mathrm{curl})$.

Definition 3.3 (A locally defined mapping from $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ to $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$)**.** *Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ be given.*

1. *Set $\boldsymbol{w} := \nabla\times\boldsymbol{v}$, so that $\boldsymbol{w} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{w} = 0$, and define $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)$ by (3.1) and $\boldsymbol{\sigma}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ by (3.2) from Definition 3.1.*
2. *Define a broken Nédélec polynomial $\boldsymbol{\iota}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)$, on each mesh element, via*

$$\boldsymbol{\iota}_h|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(K) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\tau}_h}} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K \qquad \forall K \in \mathcal{T}_h. \tag{3.6}$$

3. *Define a Raviart–Thomas polynomial $\boldsymbol{\theta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$, on each vertex patch, via*

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi_h^{p+1}(-\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v})) \\ (\boldsymbol{v}_h,\boldsymbol{r}_h)_K = (\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h,\boldsymbol{r}_h)_K \quad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3, \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{3.7}$$

4. Extending $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, define

$$\boldsymbol{\delta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}_h^{\boldsymbol{a}}, \tag{3.8}$$

which gives

$$\boldsymbol{\delta}_h \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \quad \text{and} \quad \nabla{\cdot}\boldsymbol{\delta}_h = 0. \tag{3.9}$$

5. For all tetrahedra $K \in \mathcal{T}_h$, consider the $(p+1)$-degree Raviart–Thomas elementwise minimizations

$$\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{RT}}_{p+1}(K)\\ \nabla\cdot\boldsymbol{v}_h=0\\ \boldsymbol{v}_h\cdot\boldsymbol{n}_K=\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{p+1}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h)\cdot\boldsymbol{n}_K \text{ on } \partial K}} \left\|\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{p+1}(\psi^{\boldsymbol{a}}\boldsymbol{\delta}_h) - \boldsymbol{v}_h\right\|_K \qquad \forall \boldsymbol{a}\in\mathcal{V}_K, \tag{3.10}$$

which gives

$$\boldsymbol{\delta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \quad \text{and} \quad \nabla{\cdot}\boldsymbol{\delta}_h^{\boldsymbol{a}} = 0 \qquad \forall \boldsymbol{a}\in\mathcal{V}_h, \tag{3.11a}$$

$$\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\delta}_h^{\boldsymbol{a}}. \tag{3.11b}$$

6. Define a Nédélec polynomial $\boldsymbol{h}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})$, on each vertex patch, via

$$\boldsymbol{h}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}_h=\boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{p}(\boldsymbol{\theta}_h^{\boldsymbol{a}}-\boldsymbol{\delta}_h^{\boldsymbol{a}})}} \left\|\boldsymbol{I}_{\boldsymbol{\mathcal{N}}}^{p}(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h) - \boldsymbol{v}_h\right\|_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a}\in\mathcal{V}_h. \tag{3.12}$$

7. Extending $\boldsymbol{h}_h^{\boldsymbol{a}}$ by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, define

$$\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) := \boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}_h^{\boldsymbol{a}}, \tag{3.13}$$

which gives

$$\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega). \tag{3.14}$$

It is not obvious from Definition 3.3 that the operator $\boldsymbol{P}_h^{p,\mathrm{curl}}$ is well defined. This is justified in Section 4 below. We, on the other hand, immediately see:

Remark 3.4 (Definition 3.3)**.** *The operator $\boldsymbol{P}_h^{p,\mathrm{curl}}$ is defined over the entire infinite-dimensional space $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$, is composed of simple piecewise polynomial projections, and is defined locally, in a neighborhood of each mesh vertex.*

The purpose of Step 1 is to prepare suitable piecewise polynomial data approximating $\nabla{\times}\boldsymbol{v}$ following Definition 3.1, which will also later ensure the commuting property with the operator $\boldsymbol{P}_h^{p,\mathrm{div}}$.

The purpose of Step 2 is to bring all further considerations from the infinite-dimensional level of the given function $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ to the finite-dimensional level of $\boldsymbol{\iota}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)$. In particular, all subsequent steps only involve $\boldsymbol{\iota}_h$ and other piecewise polynomials, which in particular justifies the use of the canonical Raviart–Thomas and Nédélec interpolators $\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}$ and $\boldsymbol{I}_{\boldsymbol{\mathcal{N}}}$ in Steps 5 and 6.

At the continuous level, $\nabla\psi^{\boldsymbol{a}}{\times}\boldsymbol{v}$ would belong to the $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ space, with the divergence being equal to $-\nabla\psi^{\boldsymbol{a}}{\cdot}(\nabla{\times}\boldsymbol{v})$, cf. [16, equations (4.4)–(4.5)]. In Step 3, we mimic this at the discrete level with $\boldsymbol{\theta}_h^{\boldsymbol{a}}$. Contrarily to the continuous level, where $\sum_{\boldsymbol{a}\in\mathcal{V}_h}(\nabla\psi^{\boldsymbol{a}}{\times}\boldsymbol{v}) = \boldsymbol{0}$, on Step 4, $\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h}\boldsymbol{\theta}_h^{\boldsymbol{a}} \neq \boldsymbol{0}$, though one may hope that $\boldsymbol{\delta}_h \approx \boldsymbol{0}$. One, in turn, immediately sees that $\nabla{\cdot}\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h}\nabla{\cdot}\boldsymbol{\theta}_h^{\boldsymbol{a}} = 0$ by the partition of unity (2.1), so that $\boldsymbol{\delta}_h$ is a divergence-free Raviart–Thomas polynomial.

The purpose of Step 5 is to achieve the decomposition of $\boldsymbol{\delta}_h$ by the divergence-free local contributions $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ as per (3.11). This procedure is taken from [16, Section 5.1] and is crucial here. Its key ingredient is the (non-traditional) additional constraint in (3.7) on orthogonality with respect to piecewise vector-valued constants.

Finally, as in the $\boldsymbol{H}(\mathrm{curl})$-equilibration of [16, Section 5.2], all this preparatory work allows us to pose the local problem (3.12) in Step 6; in particular, the requested curl given by $\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^{p}(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})$ has to belong to $\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ and be divergence-free. In other words, Steps 3–5 serve to prepare this crucial datum for problem (3.12); as for Steps 2, 6, and 7, these fully mimic Steps 1–3 from Definition 3.1. In particular, the definition of $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})$ can finally be finished in Step 7.

Remark 3.5 (Constraints in (3.1) and (3.6))**.** *The divergence and curl constraints in respectively (3.1) and (3.6) are not essential for the projectors $\boldsymbol{P}_h^{p,\mathrm{div}}$ and $\boldsymbol{P}_h^{p,\mathrm{curl}}$ and could be removed. Then, actually, the modification of the divergence constraint in (3.2) of Remark 3.2 would not be necessary, since then the use of $\boldsymbol{\tau}_h$ or $\boldsymbol{w}$ in (3.2) would be equivalent. However, the use of constraints in (3.1) and (3.6) leads to a slightly sharper constants in (3.17)–(3.19) and is a key for the hp-approximation result of Theorem 3.10.*

Recall the $L^2(\Omega)$-orthogonal projector $\boldsymbol{\Pi}^q_{\mathcal{RT}}$ from (2.5a). Our first main result is:

Theorem 3.6 (Commutativity, projection, approximation, and stability of $\boldsymbol{P}_h^{p,\mathrm{curl}}$)**.** *Let a mesh $\mathcal{T}_h$ of Ω and a polynomial degree $p \geq 0$ be fixed. Then, the operator $\boldsymbol{P}_h^{p,\mathrm{curl}}$ from Definition 3.3 is a commuting projector since*

$$\nabla\times\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega), \tag{3.15}$$

$$\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) = \boldsymbol{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega). \tag{3.16}$$

Moreover, $\boldsymbol{P}_h^{p,\mathrm{curl}}$ has optimal approximation properties of an elementwise unconstrained $\boldsymbol{L}^2$-orthogonal projector and is stable in that for any function $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ and any mesh element $K \in \mathcal{T}_h$, there holds

$$\begin{aligned}
&\big\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big\|_K^2 \\
&+\left(\frac{h_K}{p+1}\big\|\nabla\times\big(\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big)\big\|_K\right)^2 \lesssim \sum_{K'\in\mathcal{T}_K}\Bigg\{\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K')}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{K'}^2 \\
&\qquad\qquad + \left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\Bigg\},
\end{aligned} \tag{3.17}$$

$$\big\|\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big\|_K^2 \lesssim \sum_{K'\in\mathcal{T}_K}\left\{\|\boldsymbol{v}\|_{K'}^2 + \left(\frac{h_{K'}}{p+1}\big\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\big\|_{K'}\right)^2\right\}, \tag{3.18}$$

$$\big\|\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big\|_K^2 + h_\Omega^2\big\|\nabla\times\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big\|_K^2 \lesssim \sum_{K'\in\mathcal{T}_K}\big\{\|\boldsymbol{v}\|_{K'}^2 + h_\Omega^2\|\nabla\times\boldsymbol{v}\|_{K'}^2\big\}, \tag{3.19}$$

where $\mathcal{T}_K$ collects the elements K' of $\mathcal{T}_h$ sharing a vertex with K or its neighbor and h_Ω denotes the diameter of Ω. The constant hidden in $\lesssim$ only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ of the mesh $\mathcal{T}_h$ and the polynomial degree p.

We will prove Theorem 3.6 in Section 5 below. As stated, it shows that $\boldsymbol{P}_h^{p,\mathrm{curl}}$ satisfies all the properties 1–7 of Section 1:

Remark 3.7 (Theorem 3.6)**.** *We remark that (3.15) is precisely the commuting property desired in the middle column of (1.1), whereas (3.16) is the projector property. Moreover, (3.17) is the optimal approximation property: the first term on the right-hand side of (3.17) is the local-best (elementwise) approximation error, that of the $\boldsymbol{L}^2$-orthogonal projector onto $\boldsymbol{\mathcal{N}}_p(K')$, without any constraint on the curl, whereas the second term on the right-hand side of (3.17) is an hp data oscillation term, which in particular disappears when $\nabla\times\boldsymbol{v}$ is a piecewise polynomial. Finally, (3.18) is stability in the $\boldsymbol{L}^2(\Omega)$-norm, up to hp data oscillation, and (3.19) is stability in the $\boldsymbol{H}(\mathrm{curl},\Omega)$-norm, where the (physical) scaling by h_Ω is chosen for the curl term (other scalings, by at least the local mesh sizes h_K, could be chosen as well).*

3.3 Equivalence of local-best and global-best approximations in $\boldsymbol{H}(\mathrm{curl})$ including data oscillation

The results of Theorem 3.6 immediately lead to the following extensions of [14, Theorem 2]:

Theorem 3.8 (Equivalence of local-best and global-best approximations in $\boldsymbol{H}(\mathrm{curl})$ including data oscillation)**.** *Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$, a mesh $\mathcal{T}_h$ of Ω, and a polynomial degree $p \geq 0$ be fixed. Then*

$$\begin{aligned}
&\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)\\ \nabla\times\boldsymbol{v}_h = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})}}\|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\|_K\right)^2 \\
&\approx \sum_{K\in\mathcal{T}_h}\left[\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\|_K\right)^2\right],
\end{aligned} \tag{3.20a}$$

and

$$\begin{aligned} &\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)}\left[\|\boldsymbol{v}-\boldsymbol{v}_h\|^2+\sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p+1}\|\nabla\times(\boldsymbol{v}-\boldsymbol{v}_h)\|_K\right)^2\right] \\ &\approx\sum_{K\in\mathcal{T}_h}\left[\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2+\left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_K\right)^2\right],\end{aligned}\tag{3.20b}$$

where the hidden constants only depend on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ of the mesh $\mathcal{T}_h$ and the polynomial degree p.

The constraint in (3.20a) uses the projector $\boldsymbol{P}_h^{p,\mathrm{div}}$ from Definition 3.1. By the projection property of $\boldsymbol{P}_h^{p,\mathrm{div}}$ on $\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, there immediately follows that $\boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})=\nabla\times\boldsymbol{v}$ when $\nabla\times\boldsymbol{v}\in[\mathcal{P}_p(\mathcal{T}_h)]^3$, since in this case $\nabla\times\boldsymbol{v}\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$. Thus (3.20a) indeed extends [14, Theorem 2] to the case when the datum $\boldsymbol{v}\in\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ has a non-polynomial curl, $\nabla\times\boldsymbol{v}\notin[\mathcal{P}_p(\mathcal{T}_h)]^3$. Remarkably, both the tangential trace continuity and the curl constraint from the left-hand side of (3.20a) are removed in the right-hand side of (3.20a). Finally, in (3.20b), only the tangential trace continuity constraint is removed, but the simultaneous approximation of the curl is the best-possible.

Remark 3.9 (Mixed finite element discretization)**.** *The result (3.20a) includes the projector $\boldsymbol{P}_h^{p,\mathrm{div}}$ in its constraint, which may appear as an arbitrary choice. In contrast, the following estimates may have a more direct application: if $p\geq1$, Ω is convex, either $\Gamma_{\mathrm{D}}=\emptyset$ or $\Gamma_{\mathrm{N}}=\emptyset$, and the mesh $\mathcal{T}_h$ is quasi-uniform in that $h_K\approx h:=\max_{K'\in\mathcal{T}_h}h_{K'}$ for all $K\in\mathcal{T}_h$, we have*

$$\begin{aligned} &\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)\\ \nabla\times\boldsymbol{v}_h=\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})}}\|\boldsymbol{v}-\boldsymbol{v}_h\|^2+\sum_{K\in\mathcal{T}_h}\left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_K\right)^2 \\ &\approx\sum_{K\in\mathcal{T}_h}\left[\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2+\left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_K\right)^2\right],\end{aligned}\tag{3.21}$$

where the hidden constant only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ of the mesh $\mathcal{T}_h$ and the polynomial degree p and where

$$\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}):=\arg\min_{\substack{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)\\ \nabla\cdot\boldsymbol{w}_h=0}}\left\|\nabla\times\boldsymbol{v}-\boldsymbol{w}_h\right\|^2\tag{3.22}$$

is the global $\boldsymbol{L}^2(\Omega)$-orthogonal projection of $\nabla\times\boldsymbol{v}$ onto the divergence-free subspace of the Raviart–Thomas space $\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$. Indeed, the constrained minimization problem (3.22) naturally arises in the mixed finite element discretization of the curl–curl problem, consisting in finding

$$\boldsymbol{v}_h:=\arg\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)\\ \nabla\times\boldsymbol{v}_h=\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})}}\|\boldsymbol{v}-\boldsymbol{v}_h\|^2,$$

i.e., $\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$, $\boldsymbol{p}_h\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$, and $q_h\in\mathcal{P}_p(\mathcal{T}_h)$, of mean value zero when $\Gamma_{\mathrm{D}}=\emptyset$, such that

$$\left\{\begin{array}{lllllll}(\boldsymbol{v}_h,\boldsymbol{w}_h) & + & (\boldsymbol{p}_h,\nabla\times\boldsymbol{w}_h) & & = & (\boldsymbol{v},\boldsymbol{w}_h) & \forall\boldsymbol{w}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega),\\ (\nabla\times\boldsymbol{v}_h,\boldsymbol{r}_h) & & & +\ (q_h,\nabla\cdot\boldsymbol{r}_h) & = & (\nabla\times\boldsymbol{v},\boldsymbol{r}_h) & \forall\boldsymbol{r}_h\in\boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega),\\ & & (\nabla\cdot\boldsymbol{p}_h,s_h) & & = & 0 & \forall s_h\in\mathcal{P}_p(\mathcal{T}_h).\end{array}\right.$$

Unfortunately, our current proof of (3.21) only holds under the above fairly restrictive assumptions on the domain and the mesh, and it remains an open question whether (3.21) holds in more general settings. We also remark that (3.20a) and (3.21) are equivalents of [26, Theorem 3.3] in the $\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ context.

We will prove Theorem 3.8 and property (3.21) in Section 6 below.

3.4 Optimal hp approximation estimates in $\boldsymbol{H}(\mathrm{curl})$

We finally introduce an hp approximation estimate, an equivalent of [26, Theorem 3.6] in the $\boldsymbol{H}(\mathrm{curl})$-setting. It is optimal with respect to both the mesh size h and the polynomial degree p for arbitrary Sobolev regularity indices $s\geq0$ for the approximated function and $t\geq0$ for its curl, where, importantly, these additional regularities are only requested elementwise.

Theorem 3.10 (hp-optimal approximation estimates in $\boldsymbol{H}(\mathrm{curl})$ under minimal Sobolev regularity)**.** *Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$, a mesh $\mathcal{T}_h$ of Ω, and a polynomial degree $p \geq 0$ be fixed. If $\nabla{\times}\boldsymbol{v} \notin [\mathcal{P}_{p-1}(\mathcal{T}_h)]^3$ (for $p \geq 1$), assume in addition that $\Gamma_{\mathrm{D}} = \emptyset$ and that the patch subdomains $\omega_{\boldsymbol{a}}$ are convex for all vertices $\boldsymbol{a} \in \mathcal{V}_h$. Let*

$$\boldsymbol{v}|_K \in \boldsymbol{H}^s(K) \qquad \forall K \in \mathcal{T}_h, \tag{3.23a}$$
$$(\nabla{\times}\boldsymbol{v})|_K \in \boldsymbol{H}^t(K) \qquad \forall K \in \mathcal{T}_h \tag{3.23b}$$

for fixed regularity exponents $s \geq 0$ and t such that $s \geq t \geq \max\{0, s-1\}$. Then

$$\begin{aligned}
&\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)} \left[\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla{\times}(\boldsymbol{v} - \boldsymbol{v}_h)\|_K \right)^2 \right] \\
&\lesssim \sum_{K \in \mathcal{T}_h} \left[\left(\frac{h_K^{\min\{p+1,s\}}}{(p+1)^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(K)} \right)^2 + \left(\frac{h_K}{p+1} \frac{h_K^{\min\{p+1,t\}}}{(p+1)^t} \|\nabla{\times}\boldsymbol{v}\|_{\boldsymbol{H}^t(K)} \right)^2 \right],
\end{aligned} \tag{3.24}$$

where the hidden constant only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ of the mesh $\mathcal{T}_h$ and the regularity exponents s and t.

Remark 3.11 ($\Gamma_{\mathrm{D}} = \emptyset$ and convex patch subdomains $\omega_{\boldsymbol{a}}$)**.** *The above assumptions on $\Gamma_{\mathrm{D}} = \emptyset$ and on the convex patch subdomains $\omega_{\boldsymbol{a}}$ are only needed to treat the terms $h_K/(p+1)\|\nabla{\times}(\boldsymbol{v} - \boldsymbol{v}_h)\|_K$ and the associated data-oscillation terms in the form $h_K/(p+1)\|\nabla{\times}\boldsymbol{v} - \boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla{\times}\boldsymbol{v})\|_K$ and $h_K/p\|\nabla{\times}\boldsymbol{v} - \boldsymbol{\Pi}^{p-1}_{\boldsymbol{\mathcal{RT}}}(\nabla{\times}\boldsymbol{v})\|_K$, see the Proof of Theorem 3.10 in Section 7 below. Shall the terms $h_K/(p+1)$ in the second terms on the left- and right-hand side of (3.24) be replaced by h_K, then these assumptions are not necessary.*

4 Well-posedness of Definition 3.3 of $\boldsymbol{P}_h^{p,\mathrm{curl}}$

In this section, we establish that the operator $\boldsymbol{P}_h^{p,\mathrm{curl}}$ from Definition 3.3 is well defined. Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ be fixed.

Lemma 4.1 (Problem (3.6))**.** *For each mesh element $K \in \mathcal{T}_h$, problem (3.6) for $\boldsymbol{\iota}_h|_K$ is well defined. It can equivalently be stated by its Euler–Lagrange optimality conditions which read: find $\boldsymbol{\iota}_h \in \boldsymbol{\mathcal{N}}_p(K)$ with $\nabla{\times}\boldsymbol{\iota}_h = \boldsymbol{\tau}_h$ such that*

$$(\boldsymbol{\iota}_h, \boldsymbol{v}_h)_K = (\boldsymbol{v}, \boldsymbol{v}_h)_K \qquad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(K) \text{ with } \nabla{\times}\boldsymbol{v}_h = \boldsymbol{0}. \tag{4.1}$$

Proof. As we take $\boldsymbol{w} := \nabla{\times}\boldsymbol{v}$ in (3.1), we have $\nabla{\cdot}\boldsymbol{w} = \nabla{\cdot}(\nabla{\times}\boldsymbol{v}) = 0$, so that $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{RT}}_p(K)$ from (3.1) is divergence-free. Consequently, the minimization set in (3.6) is nonempty, cf., e.g., [9, equation (2.3.62)] which gives

$$\nabla{\times}\boldsymbol{\mathcal{N}}_p(K) = \{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(K);\ \nabla{\cdot}\boldsymbol{v}_h = 0\}. \tag{4.2}$$

Consequently, the convex minimization (3.6) is well posed and its Euler–Lagrange optimality conditions read as (4.1). □

Lemma 4.2 (Problem (3.7))**.** *For each mesh vertex $\boldsymbol{a} \in \mathcal{V}_h$, problem (3.7) for $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ is well defined. Moreover, (3.9) holds.*

Proof. Problem (3.7) is, on a first sight, over-constrained; the orthogonality with respect to piecewise vector-valued constants $\boldsymbol{r}_h$ adds additional constraints to a well-posed problem without it. It, however, by construction fits the framework of [16, Theorem A.2], which ensures existence and uniqueness of $\boldsymbol{\theta}_h^{\boldsymbol{a}}$. Indeed, let $q := q' := p+1$, $g^{\boldsymbol{a}} := (-\nabla\psi^{\boldsymbol{a}}{\cdot}(\nabla{\times}\boldsymbol{v}))$, and $\boldsymbol{\tau}_h^{\boldsymbol{a}} := \nabla\psi^{\boldsymbol{a}} {\times} \boldsymbol{\iota}_h$. Then, we only need to verify Assumption A.1 of [16], which requests

$$g^{\boldsymbol{a}} \in L^2(\omega_{\boldsymbol{a}}) \quad \text{and} \quad \boldsymbol{\tau}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_{\boldsymbol{a}}), \tag{4.3a}$$
$$(g^{\boldsymbol{a}}, 1)_{\omega_{\boldsymbol{a}}} = 0 \quad \text{when } \boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}, \tag{4.3b}$$
$$(\boldsymbol{\tau}_h^{\boldsymbol{a}}, \nabla q_h)_{\omega_{\boldsymbol{a}}} + (g^{\boldsymbol{a}}, q_h)_{\omega_{\boldsymbol{a}}} = 0 \qquad \forall q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}}). \tag{4.3c}$$

Requirement (4.3a) is obvious. As for (4.3b), we have

$$(-\nabla\psi^{\boldsymbol{a}}{\cdot}(\nabla{\times}\boldsymbol{v}), 1)_{\omega_{\boldsymbol{a}}} = -(\nabla{\times}\boldsymbol{v}, \nabla\psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} = (\underbrace{\nabla{\cdot}(\nabla{\times}\boldsymbol{v})}_{=0}, \psi^{\boldsymbol{a}})_{\omega_{\boldsymbol{a}}} - \underbrace{\langle(\nabla{\times}\boldsymbol{v}){\cdot}\boldsymbol{n}, \psi^{\boldsymbol{a}}\rangle_{\partial\omega_{\boldsymbol{a}}}}_{=0}$$

by the Green theorem for a vertex $\boldsymbol{a}$ not being part of $\overline{\Gamma_{\rm D}}$. Finally, let $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$. Then, again by the Green theorem,

$$-(\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v}), q_h)_{\omega_{\boldsymbol{a}}} = (\nabla\cdot(\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{v}), q_h)_{\omega_{\boldsymbol{a}}} = -(\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{v}, \nabla q_h)_{\omega_{\boldsymbol{a}}}.$$

Consequently, since $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})$ for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$,

$$(\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h, \nabla q_h)_{\omega_{\boldsymbol{a}}} - (\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v}), q_h)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\iota}_h - \boldsymbol{v}, \underbrace{\nabla q_h\times\nabla\psi^{\boldsymbol{a}}}_{\in\boldsymbol{\mathcal{N}}_p(K) \text{ with } \nabla\times(\nabla q_h\times\nabla\psi^{\boldsymbol{a}})=\boldsymbol{0}})_K \overset{(4.1)}{=} 0$$

from the Euler–Lagrange conditions (4.1).

As for (3.9), $\boldsymbol{\delta}_h \in \boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\rm N}(\mathrm{div}, \Omega)$ is obvious since the contributions $\boldsymbol{\theta}_h^{\boldsymbol{a}}$, after extensions by zero outside of the patch subdomains $\omega_{\boldsymbol{a}}$, belong to $\boldsymbol{\mathcal{RT}}_{p+1}(\mathcal{T}_h)$ and are $\boldsymbol{H}_{0,\rm N}(\mathrm{div}, \Omega)$-conforming. The divergence-free property is then immediately seen from the partition of unity (2.1) which gives

$$\nabla\cdot\boldsymbol{\delta}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} [\Pi_h^{p+1}(-\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v}))] = 0.$$

□

Lemma 4.3 (Problem (3.10))**.** *For all tetrahedra $K \in \mathcal{T}_h$ and all vertices $\boldsymbol{a} \in \mathcal{V}_K$, problem (3.10) for $\boldsymbol{\delta}_h^{\boldsymbol{a}}|_K$ is well defined. Moreover, properties (3.11) hold.*

Proof. Theorem B.1 from [16] yields all claims for $q = q' = p+1$, provided that the assumption

$$(\boldsymbol{\delta}_h, \boldsymbol{r}_h)_K = 0 \qquad \forall \boldsymbol{r}_h \in [\mathcal{P}_0(K)]^3,\ \forall K \in \mathcal{T}_h \tag{4.4}$$

is satisfied. Let $K \in \mathcal{T}_h$ be fixed. Property (4.4) is a simple consequence of the orthogonality assumption in (3.7), of definition (3.8), and of the partition of unity (2.1), which altogether give

$$(\boldsymbol{\delta}_h, \boldsymbol{r}_h)_K \overset{(3.8)}{=} \sum_{\boldsymbol{a}\in\mathcal{V}_K} (\boldsymbol{\theta}_h^{\boldsymbol{a}}, \boldsymbol{r}_h)_K \overset{(2.1)}{=} \sum_{\boldsymbol{a}\in\mathcal{V}_K} (\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h, \boldsymbol{r}_h)_K \overset{(3.7)}{=} 0.$$

□

Lemma 4.4 (Problem (3.12))**.** *For each mesh vertex $\boldsymbol{a} \in \mathcal{V}_h$, problem (3.12) is well defined. Moreover, (3.14) holds.*

Proof. The convex minimization (3.12) is well posed if the minimization set in (3.12) is nonempty. Importantly, here, $\omega_{\boldsymbol{a}}$ is simply connected and the essential (no tangential flow) boundary in $\boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}})$ is connected; indeed, following the definition in Section 2.3, the no tangential flow is imposed either on the whole $\partial\omega_{\boldsymbol{a}}$, or on all faces not sharing the vertex $\boldsymbol{a}$ plus possibly some more faces, which forms a connected set. Consequently, we have

$$\nabla\times(\boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}})) = \{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}});\ \nabla\cdot\boldsymbol{v}_h = 0\}, \tag{4.5}$$

so that the minimization set in (3.12) is nonempty when

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}) \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \quad \text{and} \quad \nabla\cdot(\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})) = 0. \tag{4.6}$$

The first requirement in (4.6) is immediate from (3.2), (3.7), (3.11a), and (2.6); please note the crucial use of the canonical elementwise Raviart–Thomas interpolate $\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p$ which reduces the order $(p+1)$ down to p, while preserving the homogeneous Neumann boundary condition. We also stress that we crucially only employ $\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p$ to the piecewise polynomials $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ and $\boldsymbol{\delta}_h^{\boldsymbol{a}}$. As for the divergence constraint in the second requirement in (4.6), we have from (3.2)

$$\nabla\cdot\boldsymbol{\sigma}_h^{\boldsymbol{a}} = \Pi_h^p(\psi^{\boldsymbol{a}}\nabla\cdot(\nabla\times\boldsymbol{v}) + \nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v})) = \Pi_h^p(\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v})).$$

We have, in turn, from the commuting property (2.8a) and (3.7)

$$\nabla\cdot\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\boldsymbol{\theta}_h^{\boldsymbol{a}}) = \Pi_h^p(\nabla\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}}) = \Pi_h^p(\Pi_h^{p+1}(-\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v}))) = \Pi_h^p(-\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v})),$$

whereas $\boldsymbol{\delta}_h^{\boldsymbol{a}}$, and consequently $\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\boldsymbol{\delta}_h^{\boldsymbol{a}})$, are divergence-free from (3.11a).

Finally, (3.14) is an immediate consequence of $\boldsymbol{h}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}})$ and of the definition (3.13). □

5 Proof of Theorem 3.6

We prove the claims of Theorem 3.6 separately. Let the assumptions of Theorem 3.6 be satisfied.

Lemma 5.1 (Commuting property (3.15))**.** *The commuting property* (3.15) *holds true.*

Proof. We have

$$\nabla\times\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) \overset{(3.13)}{=} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\times\boldsymbol{h}_h^{\boldsymbol{a}} \overset{(3.12)}{=} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \{\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\mathcal{RT}}^p(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})\} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\mathcal{RT}}^p\Bigg(\sum_{\boldsymbol{a}\in\mathcal{V}_h}(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})\Bigg)$$
$$\overset{(3.8)}{\underset{(3.11b)}{=}} \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\mathcal{RT}}^p(\boldsymbol{\delta}_h - \boldsymbol{\delta}_h) = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}} \overset{(3.3)}{=} \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}),$$

where we also have crucially used the linearity of the canonical elementwise Raviart–Thomas interpolate $\boldsymbol{I}_{\mathcal{RT}}^p$ from (2.6) in the third step.

□

Lemma 5.2 (Approximation property (3.17))**.** *The approximation property* (3.17) *holds true.*

Proof. Let a mesh element $K \in \mathcal{T}_h$ be fixed. We proceed in two steps.

Step 1. We first bound the term $\frac{h_K}{p+1}\big\|\nabla\times\big(\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big)\big\|_K$. From the commuting property (3.15), we have $\nabla\times\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})$, so that

$$\begin{aligned}\frac{h_K}{p+1}\big\|\nabla\times\big(\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big)\big\|_K &= \frac{h_K}{p+1}\big\|\nabla\times\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})\big\|_K \\ &\lesssim \left\{\sum_{K'\in\mathcal{T}_K}\left(\frac{h_{K'}}{p+1}\big\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\big\|_{K'}\right)^2\right\}^{1/2}\end{aligned} \tag{5.1}$$

from [26, Theorem 3.2, bound (3.6)], also using that $\nabla\cdot(\nabla\times\boldsymbol{v}) = 0$, the shape-regularity of the mesh yielding $h_K \approx h_{K'}$, and the characterization (2.5b).

Step 2. We bound $\big\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big\|_K$. Consider for this purpose $\boldsymbol{\iota}_h$ defined in (3.6). Using the linearity and projection property of the elementwise canonical Nédélec interpolate $\boldsymbol{I}_{\mathcal{N}}^p$ from (2.7), the partition of unity (2.1), and definition (3.13)

$$\begin{aligned}\big\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\big\|_K &= \Bigg\|\boldsymbol{v} - \boldsymbol{\iota}_h + \sum_{\boldsymbol{a}\in\mathcal{V}_K}\{\boldsymbol{I}_{\mathcal{N}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h) - \boldsymbol{h}_h^{\boldsymbol{a}}\}\Bigg\|_K \\ &\le \|\boldsymbol{v} - \boldsymbol{\iota}_h\|_K + \sum_{\boldsymbol{a}\in\mathcal{V}_K}\big\|\boldsymbol{I}_{\mathcal{N}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h) - \boldsymbol{h}_h^{\boldsymbol{a}}\big\|_{\omega_{\boldsymbol{a}}}.\end{aligned}$$

We now bound the two terms separately.

Step 2a. We bound $\|\boldsymbol{v} - \boldsymbol{\iota}_h\|_K$. Using the second inequality in (A.2) of Lemma A.1 below, we have

$$\|\boldsymbol{v} - \boldsymbol{\iota}_h\|_K = \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)\\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\tau}_h}}\|\boldsymbol{v} - \boldsymbol{v}_h\|_K \lesssim \Big(\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)}\|\boldsymbol{v} - \boldsymbol{v}_h\|_K + \frac{h_K}{p+1}\big\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^p(\nabla\times\boldsymbol{v})\big\|_K\Big),$$

for a constant actually only depending on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$.

Step 2b. Let a vertex $\boldsymbol{a} \in \mathcal{V}_K$ be fixed. We bound $\big\|\boldsymbol{I}_{\mathcal{N}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h) - \boldsymbol{h}_h^{\boldsymbol{a}}\big\|_{\omega_{\boldsymbol{a}}}$. Theorem 3.3 and Corollary 4.3 of [15] give

$$\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\mathcal{RT}}^p(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})}}\big\|\boldsymbol{I}_{\mathcal{N}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h) - \boldsymbol{v}_h\big\|_{\omega_{\boldsymbol{a}}} \lesssim \min_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{\varphi} = \boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\mathcal{RT}}^p(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})}}\big\|\boldsymbol{I}_{\mathcal{N}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h) - \boldsymbol{\varphi}\big\|_{\omega_{\boldsymbol{a}}}, \tag{5.2}$$

where again the constant beyond $\lesssim$ only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$. This result builds on [24, 20] and [12] and extends [10, 29, 13] to vertex patches in $\boldsymbol{H}(\mathrm{curl})$.

By the characterization (3.12) for the above left-hand side and by a primal–dual equivalence for the above right-hand side as in, e.g., [13, Lemma 5.5], we obtain

$$\big\|\boldsymbol{I}_{\mathcal{N}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h) - \boldsymbol{h}_h^{\boldsymbol{a}}\big\|_{\omega_{\boldsymbol{a}}} \lesssim \sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}^\dagger(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{\varphi}\|_{\omega_{\boldsymbol{a}}}=1}}\big\{(\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}_{\mathcal{RT}}^p(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}), \boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}} - (\boldsymbol{I}_{\mathcal{N}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h), \nabla\times\boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}}\big\}. \tag{5.3}$$

Fix now $\boldsymbol{\varphi} \in \boldsymbol{H}^\dagger(\mathrm{curl}, \omega_{\boldsymbol{a}})$ with $\|\nabla\times\boldsymbol{\varphi}\|_{\omega_{\boldsymbol{a}}} = 1$. From (2.9), there exists $\boldsymbol{\psi} \in \boldsymbol{H}^1_*(\omega_{\boldsymbol{a}})$ (of mean value zero componentwise when $\boldsymbol{a}$ is an interior vertex or when $\boldsymbol{a} \in \Gamma_{\mathrm{N}}$, and zero on γ_{D} when $\boldsymbol{a} \in \overline{\Gamma_{\mathrm{D}}}$), such that $\nabla\times\boldsymbol{\psi} = \nabla\times\boldsymbol{\varphi}$ and

$$\|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{L},\omega_{\boldsymbol{a}}}\|\nabla\times\boldsymbol{\varphi}\|_{\omega_{\boldsymbol{a}}} \lesssim 1. \tag{5.4}$$

Moreover, (2.10) also gives

$$\|\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{PF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}}\|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}. \tag{5.5}$$

In the second term on the right-hand side of (5.3), $\boldsymbol{\varphi}$ can be exchanged for $\boldsymbol{\psi}$. This is also true for the first one: from (4.6) and (4.5), there exists $\boldsymbol{w}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}})$ such that $\nabla\times\boldsymbol{w}_h = \boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}})$, so that

$$\begin{aligned}(\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}), \boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}} &= (\nabla\times\boldsymbol{w}_h, \boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}} = (\boldsymbol{w}_h, \nabla\times\boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}} \\ &= (\boldsymbol{w}_h, \nabla\times\boldsymbol{\psi})_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}}), \boldsymbol{\psi})_{\omega_{\boldsymbol{a}}}.\end{aligned} \tag{5.6}$$

Integrating by parts, the second term on the right-hand side of (5.3) becomes

$$\begin{aligned}-(\boldsymbol{I}^p_{\boldsymbol{\mathcal{N}}}(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h), \nabla\times\boldsymbol{\psi})_{\omega_{\boldsymbol{a}}} &= -\sum_{K'\in\mathcal{T}_{\boldsymbol{a}}} (\boldsymbol{I}^p_{\boldsymbol{\mathcal{N}}}(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h), \nabla\times\boldsymbol{\psi})_{K'} \\ &= -\sum_{K'\in\mathcal{T}_{\boldsymbol{a}}} (\nabla\times(\boldsymbol{I}^p_{\boldsymbol{\mathcal{N}}}(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h)), \boldsymbol{\psi})_{K'} + \sum_{F\in\mathcal{F}^{\mathrm{int}}_{\boldsymbol{a}}} \langle [\![\boldsymbol{I}^p_{\boldsymbol{\mathcal{N}}}(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h)\times\boldsymbol{n}_F]\!], \boldsymbol{\psi}\rangle_F,\end{aligned} \tag{5.7}$$

where $\mathcal{F}^{\mathrm{int}}_{\boldsymbol{a}}$ denotes the interior faces of $\mathcal{T}_{\boldsymbol{a}}$ plus, for boundary patches, the faces in $\overline{\Gamma_{\mathrm{N}}}$; the other faces do not appear because of the presence of the hat function $\psi^{\boldsymbol{a}}$ and/or due to the zero trace in $\boldsymbol{H}^1_*(\omega_{\boldsymbol{a}})$ on γ_{D}.

Step 2bi. We address the jump term in (5.7). The trace inequality $\|\boldsymbol{\psi}\|_F^2 \lesssim \|\nabla\boldsymbol{\psi}\|_{K'}\|\boldsymbol{\psi}\|_{K'} + h_{K'}^{-1}\|\boldsymbol{\psi}\|_{K'}^2$ for any $K' \in \mathcal{T}_h$ sharing F, the Cauchy–Schwarz inequality, and the Poincaré–Friedrichs inequality (2.10) lead to

$$\left\{\sum_{F\in\mathcal{F}^{\mathrm{int}}_{\boldsymbol{a}}} \|\boldsymbol{\psi}\|_F^2\right\}^{1/2} \lesssim h_{\omega_{\boldsymbol{a}}}^{1/2}\|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}.$$

The stability of the elementwise Nédélec interpolate and the fact that $\|\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}} = 1$ give, for $F \in \mathcal{F}^{\mathrm{int}}_{\boldsymbol{a}}$,

$$\|[\![\boldsymbol{I}^p_{\boldsymbol{\mathcal{N}}}(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h)\times\boldsymbol{n}_F]\!]\|_F \lesssim \|[\![\boldsymbol{\iota}_h\times\boldsymbol{n}_F]\!]\|_F,$$

where henceforth $\lesssim$ adds a dependence on the polynomial degree p. The above right-hand side is now in a form of a standard residual-based estimator. Thus, introducing the face bubble function ψ^F as a product of the three $\psi^{\boldsymbol{a}}$ of the vertices of F (supported on ω_F corresponding to the (two or one) tetrahedra sharing F), denoting $\boldsymbol{\zeta}_F := [\![\boldsymbol{\iota}_h\times\boldsymbol{n}_F]\!]$ together with its constant extension to ω_F, noting that

$$(\nabla\times\boldsymbol{v}, \boldsymbol{\zeta}_F\psi^F)_{\omega_F} - (\boldsymbol{v}, \nabla\times(\boldsymbol{\zeta}_F\psi^F))_{\omega_F} = 0$$

since $\boldsymbol{\zeta}_F\psi^F \in \boldsymbol{H}^1(\omega_F)$, $\boldsymbol{\zeta}_F\psi^F$ is zero on $\partial\omega_F \setminus F$, and $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$, and using standard scaling arguments as in, e.g., [6, page 172], leads to

$$\begin{aligned}\|[\![\boldsymbol{\iota}_h\times\boldsymbol{n}_F]\!]\|_F^2 = \|\boldsymbol{\zeta}_F\|_F^2 &\lesssim \langle[\![\boldsymbol{\iota}_h\times\boldsymbol{n}_F]\!], \boldsymbol{\zeta}_F\psi^F\rangle_F \\ &= (\boldsymbol{v} - \boldsymbol{\iota}_h, \nabla\times(\boldsymbol{\zeta}_F\psi^F))_{\omega_F} - (\nabla\times(\boldsymbol{v} - \boldsymbol{\iota}_h), \boldsymbol{\zeta}_F\psi^F)_{\omega_F} \\ &\leq \|\boldsymbol{v} - \boldsymbol{\iota}_h\|_{\omega_F}\|\nabla\times(\boldsymbol{\zeta}_F\psi^F)\|_{\omega_F} + (\boldsymbol{\tau}_h - \nabla\times\boldsymbol{v}, \boldsymbol{\zeta}_F\psi^F)_{\omega_F},\end{aligned}$$

where we have also used that $\nabla\times\boldsymbol{\iota}_h = \boldsymbol{\tau}_h$ from (3.6). The second term above is treated by Lemma A.3 below with $\boldsymbol{\psi} = \boldsymbol{\zeta}_F\psi^F$ on each element K in the face patch subdomain ω_F. This yields

$$(\boldsymbol{\tau}_h - \nabla\times\boldsymbol{v}, \boldsymbol{\zeta}_F\psi^F)_{\omega_F} \lesssim \frac{h_{\omega_F}}{p+1}\left\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\right\|_{\omega_F}\|\nabla(\boldsymbol{\zeta}_F\psi^F)\|_{\omega_F}.$$

By the inverse inequality,

$$\|\nabla\times(\boldsymbol{\zeta}_F\psi^F)\|_{\omega_F} \lesssim h_{\omega_F}^{-1}\|\boldsymbol{\zeta}_F\psi^F\|_{\omega_F}, \quad \|\nabla(\boldsymbol{\zeta}_F\psi^F)\|_{\omega_F} \lesssim h_{\omega_F}^{-1}\|\boldsymbol{\zeta}_F\psi^F\|_{\omega_F},$$

whereas the constant extension of $\boldsymbol{\zeta}_F$ from F to ω_F and $\|\psi^F\|_{\infty,\omega_F} \leq 1$ give

$$\|\boldsymbol{\zeta}_F\psi^F\|_{\omega_F} \lesssim h_{\omega_F}^{1/2}\|\boldsymbol{\zeta}_F\|_F.$$

Thus, altogether,

$$h_{\omega_F}^{1/2}\|\boldsymbol{\zeta}_F\|_F \lesssim \|\boldsymbol{v}-\boldsymbol{\iota}_h\|_{\omega_F} + \frac{h_{\omega_F}}{p+1}\big\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\big\|_{\omega_F}.$$

This yields the desired components in (3.17), using (5.4), the mesh shape regularity, and the definition (3.6) of $\boldsymbol{\iota}_h$ together with the constrained–unconstrained equivalence result of Lemma A.1 below, which allows to pass from $\|\boldsymbol{v}-\boldsymbol{\iota}_h\|_{K'}$ to $\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K')}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{K'}$ plus the oscillation terms on each simplex K'.

Step 2bii. We now address the volume term in (5.7) together with (5.6). On each $K'\in\mathcal{T}_{\boldsymbol{a}}$, using the commuting property of the canonical interpolators (2.8b),

$$\nabla\times(\boldsymbol{I}^p_{\boldsymbol{\mathcal{N}}}(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h)) = \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h)) = \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h + \psi^{\boldsymbol{a}}\nabla\times\boldsymbol{\iota}_h).$$

Regrouping, since elementwise $\nabla\times\boldsymbol{\iota}_h = \boldsymbol{\tau}_h$ from (3.6), and using the Cauchy–Schwarz inequality together with (5.5), (5.4), and the mesh shape regularity,

$$\begin{aligned}
(\boldsymbol{\sigma}_h^{\boldsymbol{a}} - \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\psi^{\boldsymbol{a}}\nabla\times\boldsymbol{\iota}_h),\boldsymbol{\psi})_{\omega_{\boldsymbol{a}}} &= (\boldsymbol{\sigma}_h^{\boldsymbol{a}} - \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h),\boldsymbol{\psi})_{\omega_{\boldsymbol{a}}}\\
&\le \|\boldsymbol{\sigma}_h^{\boldsymbol{a}} - \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h)\|_{\omega_{\boldsymbol{a}}}\|\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}\\
&\lesssim \left\{\sum_{K'\in\mathcal{T}_{\boldsymbol{a}}}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|^2_{K'}\right\}^{1/2} h_{\omega_{\boldsymbol{a}}}\|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}\\
&\lesssim \left\{\sum_{K'\in\mathcal{T}_{\boldsymbol{a}}}\left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\}^{1/2},
\end{aligned}$$

where we have also used $1\lesssim 1/(p+1)$ and, crucially, recalling (2.5b), (3.2) where $\boldsymbol{w}=\nabla\times\boldsymbol{v}$, and Remark 3.2, the bound

$$\|\boldsymbol{\sigma}_h^{\boldsymbol{a}} - \boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h)\|_{\omega_{\boldsymbol{a}}} \lesssim \left\{\sum_{K'\in\mathcal{T}_{\boldsymbol{a}}}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|^2_{K'}\right\}^{1/2} \tag{5.8}$$

from [26, Lemma 4.6], where we note that the data oscillation terms disappears since $\nabla\cdot(\nabla\times\boldsymbol{v})=0$.

We now start to estimate the remaining term. By the Cauchy–Schwarz inequality together with (5.5) and (5.4), we have

$$\begin{aligned}
(\boldsymbol{I}^p_{\boldsymbol{\mathcal{RT}}}(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \boldsymbol{\delta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h),\boldsymbol{\psi})_{\omega_{\boldsymbol{a}}} &\lesssim [\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} + \|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}]\|\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}\\
&\lesssim h_{\omega_{\boldsymbol{a}}}[\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} + \|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}],
\end{aligned} \tag{5.9}$$

where we have also employed the stability of the elementwise Raviart–Thomas interpolate. We now employ definition (3.7) and, crucially, [16, Theorem A.2] (cf. also Lemma 7.3 therein for the rewriting of the data oscillation term). We have already verified in the proof of Lemma 4.2 that our setting fits the assumptions of [16, Theorem A.2]. We additionally note that $\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{v}$ lies in $\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\nabla\cdot(\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{v}) = -\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v})$. This leads to

$$\begin{aligned}
\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} &\lesssim \|\nabla\psi^{\boldsymbol{a}}\times(\boldsymbol{v}-\boldsymbol{\iota}_h)\|_{\omega_{\boldsymbol{a}}} + h_{\omega_{\boldsymbol{a}}}^{-1}\left\{\sum_{K'\in\mathcal{T}_{\boldsymbol{a}}}\left(\frac{h_{K'}}{\pi}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^{p+1}_h(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\}^{1/2}\\
&\lesssim h_{\omega_{\boldsymbol{a}}}^{-1}\|\boldsymbol{v}-\boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} + h_{\omega_{\boldsymbol{a}}}^{-1}\left\{\sum_{K'\in\mathcal{T}_{\boldsymbol{a}}}\left(\frac{h_{K'}}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^{p}_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_{K'}\right)^2\right\}^{1/2}.
\end{aligned} \tag{5.10}$$

Thus, definition (3.6) of $\boldsymbol{\iota}_h$ together with the constrained–unconstrained equivalence result of Lemma A.1 below yields the desired bound for the first term in (5.9).

Finally, we bound $\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$. The construction of $\boldsymbol{\delta}_h$ and $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ from steps 4 and 5 of Definition 3.3 fits the framework of [16, Theorem B.1] with $q=q'=p+1$, as we have already verified in the proof of Lemma 4.3. Thus, using [16, estimate (B.6b)], (3.8), and (2.1), we have, for any $K'\in\mathcal{T}_{\boldsymbol{a}}$,

$$\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_{K'} \lesssim \|\boldsymbol{\delta}_h\|_{K'} = \left\|\sum_{\boldsymbol{b}\in\mathcal{V}_{K'}}(\boldsymbol{\theta}_h^{\boldsymbol{b}} - \nabla\psi^{\boldsymbol{b}}\times\boldsymbol{\iota}_h)\right\|_{K'},$$

so that the bound on $\|\boldsymbol{\delta}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}$ follows from that on $\|\boldsymbol{\theta}_h^{\boldsymbol{b}} - \nabla\psi^{\boldsymbol{b}}\times\boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{b}}}$ (this final estimate extends the bound (3.17) from neighbors of K to the neighbors of neighbors of K).

□

Lemma 5.3 (Projection property (3.16)). *The projection property* (3.16) *holds true.*

Proof. When $\boldsymbol{v} \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ is a Nédélec piecewise polynomial, the right-hand side in the approximation property (3.17) is zero, so that immediately $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) = \boldsymbol{v}$ follows. □

Lemma 5.4 (Stability properties (3.18) and (3.19)). *The stability properties* (3.18) *and* (3.19) *hold true.*

Proof. The triangle inequality gives

$$\left\|\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\right\|_K \leq \left\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\right\|_K + \|\boldsymbol{v}\|_K.$$

Thus (3.18) follows from (3.17) and the orthogonal projection stability

$$\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(K')} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{K'} \leq \|\boldsymbol{v}\|_{K'}. \tag{5.11}$$

As for (3.19), we only need to treat the second term, since (3.18) bounds the first one as $h_{K'}/(p+1) \leq h_\Omega$. The triangle inequality gives

$$h_\Omega \left\|\nabla \times \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\right\|_K \leq h_\Omega \left\|\nabla \times (\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}))\right\|_K + h_\Omega \left\|\nabla \times \boldsymbol{v}\right\|_K,$$

and the first term above is estimated as in (5.1), with the weight h_Ω in place of $h_K/(p+1)$, and, as in (5.11), $\left\|\nabla \times \boldsymbol{v} - \boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla \times \boldsymbol{v})\right\|_{K'} \leq \|\nabla \times \boldsymbol{v}\|_{K'}$. □

6 Proof of Theorem 3.8 and of (3.21)

The proof of properties (3.20a) and (3.20b) follows straightforwardly from Theorem 3.6, whereas the proof (3.21) will turn out slightly more involved.

Proof of Theorem 3.8. Step 1. We first show (3.20a). Since the second terms are identically present on both sides of the equivalence (cf. [26, Remark 3.4]), we only need to consider the first ones. As the inequality trivially holds with the $\geq$ sign, we only need to bound the first term of the left hand side of (3.20a) from above. From (3.15), we immediately have

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla \times \boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\|^2 \leq \left\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\right\|^2, \tag{6.1}$$

so that the result follows from the local approximation property (3.17) by summing over all mesh elements $K \in \mathcal{T}_h$ and invoking the mesh shape-regularity.

Step 2. We now show (3.20b). This again trivially holds with $\geq$, since

$$\nabla \times (\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)) \subset \{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega);\ \nabla \cdot \boldsymbol{v}_h = 0\}; \tag{6.2}$$

thus both terms on the right-hand side of (3.20b) are the elementwise versions of those on the left. For the converse estimate, we bound the left-hand side of (3.20b) by

$$\|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla \times (\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}))\|_K\right)^2$$

and again invoke (3.17). □

Proof of (3.21). *Step 1.* We recall that here, $p \geq 1$, $\mathcal{T}_h$ is quasi-uniform, and Ω is convex with either $\Gamma_{\mathrm{D}} = \emptyset$ or $\Gamma_{\mathrm{N}} = \emptyset$. The second term on the left-hand side of (3.21) is again the same as on the right-hand side, so we only need to treat the first one. Let

$$\boldsymbol{\zeta}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla \times \boldsymbol{v}) - \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla \times \boldsymbol{v})}} \|\boldsymbol{v}_h\|^2.$$

Note that $\boldsymbol{\zeta}_h$ is only nonzero when the datum $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ has a non-polynomial curl, $\nabla \times \boldsymbol{v} \notin [\mathcal{P}_p(\mathcal{T}_h)]^3$ (or $\nabla \times \boldsymbol{v} \notin \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$). Since $\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) + \boldsymbol{\zeta}_h$ lies in $\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ and satisfies $\nabla \times (\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v}) + \boldsymbol{\zeta}_h) = \boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla \times \boldsymbol{v})$, it is clear that

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega) \\ \nabla \times \boldsymbol{v}_h = \boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla \times \boldsymbol{v})}} \|\boldsymbol{v} - \boldsymbol{v}_h\| \leq \|\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{v})\| + \|\boldsymbol{\zeta}_h\|.$$

The first term on the right-hand side above is as in (6.1), so we only treat the second one.

Let

$$\boldsymbol{\zeta} := \arg \min_{\substack{\boldsymbol{v}\in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)\\ \nabla\times\boldsymbol{v}=\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})-\boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})}} \|\boldsymbol{v}\|^2;$$

$\boldsymbol{\zeta}$ is again only nonzero when the datum $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ has a non-polynomial curl. Since the curl constraint in the above minimization problem is given by a Raviart–Thomas piecewise polynomial, we infer from the definition of $\boldsymbol{\zeta}_h$, the commuting property (3.15), and the stability (3.18) that

$$\|\boldsymbol{\zeta}_h\| \leq \|\boldsymbol{P}_h^{p,\mathrm{curl}}(\boldsymbol{\zeta})\| \lesssim \|\boldsymbol{\zeta}\|,$$

where the constant hidden in $\lesssim$ only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ of the mesh $\mathcal{T}_h$ and the polynomial degree p. By a primal–dual equivalence, as in, e.g., [13, Lemma 5.5], we have

$$\|\boldsymbol{\zeta}\| = \sup_{\substack{\boldsymbol{\varphi}\in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)\\ \|\nabla\times\boldsymbol{\varphi}\|=1}} \left\{ \big(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) - \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}), \boldsymbol{\varphi}\big) \right\}.$$

Let now $\boldsymbol{\varphi} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ with $\|\nabla\times\boldsymbol{\varphi}\| = 1$ be fixed. Following (2.9) (recall that Ω is convex with either $\Gamma_{\mathrm{D}} = \emptyset$ or $\Gamma_{\mathrm{N}} = \emptyset$), there exists $\boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega)$ such that $\nabla\times\boldsymbol{\psi} = \nabla\times\boldsymbol{\varphi}$ and $\|\nabla\boldsymbol{\psi}\| \leq C_{\mathrm{L},\Omega}\|\nabla\times\boldsymbol{\varphi}\| = \|\nabla\times\boldsymbol{\varphi}\| = 1$. Since $\nabla\times\boldsymbol{\zeta}_h = \boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) - \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v})$, we can exchange $\boldsymbol{\varphi}$ for $\boldsymbol{\psi}$, so that

$$\begin{aligned}\big(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) - \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}), \boldsymbol{\varphi}\big) &= \big(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) - \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}), \boldsymbol{\psi}\big)\\ &= \big(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) - \nabla\times\boldsymbol{v}, \boldsymbol{\psi}\big) + \big(\nabla\times\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}), \boldsymbol{\psi}\big).\end{aligned}$$

We will treat the two arising terms separately.

Step 2. Let now $q \in H^1_{0,\mathrm{D}}(\Omega)$ be such that

$$(\nabla q, \nabla v) = (\boldsymbol{\psi}, \nabla v) \qquad \forall v \in H^1_{0,\mathrm{D}}(\Omega). \tag{6.3}$$

Defining $\boldsymbol{\chi} := \boldsymbol{\psi} - \nabla q$, we have, recalling the notation from Section 2.3,

$$\boldsymbol{\chi} \in \boldsymbol{H}_{0,\mathrm{D}}(\mathrm{curl},\Omega) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \text{ with } \nabla\cdot\boldsymbol{\chi} = 0, \tag{6.4}$$

together with $\nabla\times\boldsymbol{\chi} = \nabla\times\boldsymbol{\psi} = \nabla\times\boldsymbol{\varphi}$. Moreover, by the Green theorem, the right-hand side in (6.3) equivalently writes as $(\boldsymbol{\psi}, \nabla v)_{\omega_{\boldsymbol{a}}} = -(\nabla\cdot\boldsymbol{\psi}, v)_{\omega_{\boldsymbol{a}}}$. Thus, since Ω is convex with either $\Gamma_{\mathrm{D}} = \emptyset$ or $\Gamma_{\mathrm{N}} = \emptyset$, the elliptic regularity shift, see, e.g., [35, Theorem 4.3.1.4], gives $q \in H^2(\Omega)$ with

$$\|\nabla(\nabla q)\| \lesssim \|\nabla\cdot\boldsymbol{\psi}\| \leq \|\nabla\boldsymbol{\psi}\|.$$

Thus, in addition to (6.4), there also holds $\boldsymbol{\chi} \in \boldsymbol{H}^1(\Omega)$ with

$$\|\nabla\boldsymbol{\chi}\| \leq \|\nabla\boldsymbol{\psi}\| + \|\nabla(\nabla q)\| \lesssim \|\nabla\boldsymbol{\psi}\|, \tag{6.5}$$

and we have

$$\big(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) - \nabla\times\boldsymbol{v}, \boldsymbol{\psi}\big) = \big(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) - \nabla\times\boldsymbol{v}, \boldsymbol{\chi}\big).$$

Denote henceforth $\boldsymbol{w} := \nabla\times\boldsymbol{v}$. From the definition (3.22) of the projector $\boldsymbol{\Pi}_h^{p,\mathrm{div}}$, we can subtract any $\boldsymbol{\chi}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{\chi}_h = 0$; we choose $\boldsymbol{\chi}_h := \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{\chi})$ following Definition 3.1, yielding

$$\big(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\boldsymbol{w}) - \boldsymbol{w}, \boldsymbol{\chi}\big) = \big(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\boldsymbol{w}) - \boldsymbol{w}, \boldsymbol{\chi} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{\chi})\big) \leq \big\|\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\boldsymbol{w}) - \boldsymbol{w}\big\|\big\|\boldsymbol{\chi} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{\chi})\big\|.$$

Now the equivalence of global-best and local-best approximations from [26, Theorem 3.3] gives, using that $\nabla\cdot\boldsymbol{w} = 0$,

$$\big\|\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\boldsymbol{w}) - \boldsymbol{w}\big\| \lesssim \left\{ \sum_{K\in\mathcal{T}_h} \|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_K^2 \right\}^{1/2},$$

where again the constant of $\lesssim$ only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ of the mesh $\mathcal{T}_h$ and the polynomial degree p. In particular, it seems impossible to estimate $\big\|\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\boldsymbol{w}) - \boldsymbol{w}\big\|_K$, locally on each $K \in \mathcal{T}_h$. Similarly, the approximation properties of the projector $\boldsymbol{P}_h^{p,\mathrm{div}}$ from equation (3.6) of [26, Theorem 3.2] lead to

$$\|\boldsymbol{\chi} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{\chi})\| \lesssim \left\{ \sum_{K\in\mathcal{T}_h} \min_{\boldsymbol{w}_h \in \boldsymbol{\mathcal{RT}}_p(K)} \|\boldsymbol{\chi} - \boldsymbol{w}_h\|_K^2 \right\}^{1/2} \lesssim \left\{ \sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla\boldsymbol{\chi}\|_K^2 \right\}^{1/2} \le h \underbrace{\|\nabla\boldsymbol{\chi}\|}_{\lesssim 1},$$

where, recall, $h := \max_{K\in\mathcal{T}_h} h_K$. Thus, since $\mathcal{T}_h$ is supposed here quasi-uniform, so that $h \lesssim h_K$ for all $K \in \mathcal{T}_h$, and using $1 \lesssim 1/(p+1)$ as in the proof of Lemma 5.2, we conclude

$$\left(\boldsymbol{\Pi}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}) - \nabla\times\boldsymbol{v}, \boldsymbol{\chi}\right) \lesssim \left\{ \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^p(\nabla\times\boldsymbol{v})\|_K\right)^2 \right\}^{1/2}.$$

Step 3. Denote $\boldsymbol{w} := \nabla\times\boldsymbol{v}$ as above. Consider $\boldsymbol{\tau}_h$ given by (3.1). Using the partition of unity (2.1), the linearity and the projection property of the elementwise Raviart–Thomas interpolate $\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p$ of (2.6), we have

$$\boldsymbol{\tau}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} (\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) = \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\boldsymbol{\tau}_h) = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \left(\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h)\right).$$

Thus, also using (3.3),

$$\left(\boldsymbol{w} - \boldsymbol{P}_h^{p,\mathrm{div}}(\boldsymbol{w}), \boldsymbol{\psi}\right) = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \left(\psi^{\boldsymbol{a}}\boldsymbol{w} - \psi^{\boldsymbol{a}}\boldsymbol{\tau}_h + \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{\psi}\right)_{\omega_{\boldsymbol{a}}}.$$

Consider now any $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$ for a fixed $\boldsymbol{a} \in \mathcal{V}_h$. Since $\boldsymbol{w} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega)$ with $\nabla\cdot\boldsymbol{w} = 0$ and by the constraint in (3.2), we see, employing the Green theorem,

$$\left(\psi^{\boldsymbol{a}}\boldsymbol{w} - \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \nabla q_h\right)_{\omega_{\boldsymbol{a}}} = -\left(\nabla\cdot(\psi^{\boldsymbol{a}}\boldsymbol{w} - \boldsymbol{\sigma}_h^{\boldsymbol{a}}), q_h\right)_{\omega_{\boldsymbol{a}}} = -\left(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{w} - \Pi_h^p(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{w})), q_h\right)_{\omega_{\boldsymbol{a}}} = 0.$$

On the other side, on each simplex in the patch $K \in \mathcal{T}_{\boldsymbol{a}}$, the Green theorem, (2.6), and the commuting property (2.8a) of $\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p$ give

$$\begin{aligned}\left(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h), \nabla q_h\right)_K &= -\left(\nabla\cdot\left(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h)\right), q_h\right)_K + \left\langle\left(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h)\right)\cdot\boldsymbol{n}_K, q_h\right\rangle_{\partial K},\\ &= -\left(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\tau}_h - \Pi_h^p(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\tau}_h), q_h\right)_K = 0.\end{aligned}$$

For the last two properties, we have also employed the assumption $p \ge 1$.

We now choose $q_h \in \mathcal{P}_1(\mathcal{T}_{\boldsymbol{a}}) \cap H^1_*(\omega_{\boldsymbol{a}})$ such that $\nabla q_h = \overline{\boldsymbol{\psi}}$, the componentwise mean value of $\boldsymbol{\psi}$ on the patch subdomain $\omega_{\boldsymbol{a}}$. Thus, the Cauchy–Schwarz inequality and the Poincaré–Friedrichs inequality (2.10) give

$$\begin{aligned}\left(\psi^{\boldsymbol{a}}\boldsymbol{w} - \psi^{\boldsymbol{a}}\boldsymbol{\tau}_h + \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{\psi}\right)_{\omega_{\boldsymbol{a}}} &= \left(\psi^{\boldsymbol{a}}\boldsymbol{w} - \psi^{\boldsymbol{a}}\boldsymbol{\tau}_h + \boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{\psi} - \overline{\boldsymbol{\psi}}\right)_{\omega_{\boldsymbol{a}}}\\ &\lesssim h_{\omega_{\boldsymbol{a}}}\left(\|\psi^{\boldsymbol{a}}\boldsymbol{w} - \psi^{\boldsymbol{a}}\boldsymbol{\tau}_h\|_{\omega_{\boldsymbol{a}}} + \left\|\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{\sigma}_h^{\boldsymbol{a}}\right\|_{\omega_{\boldsymbol{a}}}\right)\|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}.\end{aligned}$$

Now, from the definition (3.1) of $\boldsymbol{\tau}_h$ as a local constrained minimizer, from the constrained–unconstrained equivalence of [26, Lemma A.1] (recall that $\nabla\cdot\boldsymbol{w} = 0$), and redeveloping $\boldsymbol{w} = \nabla\times\boldsymbol{v}$, we see

$$\|\psi^{\boldsymbol{a}}\boldsymbol{w} - \psi^{\boldsymbol{a}}\boldsymbol{\tau}_h\|_{\omega_{\boldsymbol{a}}} \le \|\boldsymbol{w} - \boldsymbol{\tau}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \left\{ \sum_{K'\in\mathcal{T}_{\boldsymbol{a}}} \|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^p(\nabla\times\boldsymbol{v})\|_{K'}^2 \right\}^{1/2}.$$

Similarly, it follows from (5.8) that

$$\left\|\boldsymbol{I}_{\boldsymbol{\mathcal{RT}}}^p(\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h) - \boldsymbol{\sigma}_h^{\boldsymbol{a}}\right\|_{\omega_{\boldsymbol{a}}} \lesssim \left\{ \sum_{K'\in\mathcal{T}_{\boldsymbol{a}}} \|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^p(\nabla\times\boldsymbol{v})\|_{K'}^2 \right\}^{1/2}.$$

Hence, the shape-regularity of the mesh yielding finite overlap between the patches, the Cauchy–Schwarz inequality, and using $1 \lesssim 1/(p+1)$ as in the proof of Lemma 5.2, we see

$$\left(\nabla\times\boldsymbol{v} - \boldsymbol{P}_h^{p,\mathrm{div}}(\nabla\times\boldsymbol{v}), \boldsymbol{\psi}\right) \lesssim \left\{ \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p+1}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^p(\nabla\times\boldsymbol{v})\|_K\right)^2 \right\}^{1/2} \underbrace{\|\nabla\boldsymbol{\psi}\|}_{\le 1}.$$

This finishes the proof. $\square$

7 Proof of Theorem 3.10

We proceed in the spirit of [26, Section 5]. The main point is to derive an estimate of the form of $\lesssim$ in (3.20b) which is p-robust, i.e., where the hidden constant is independent of the polynomial degree p, for the price of lowering the approximation polynomial degree on the right from p to $p-1$. For this purpose, we only consider $p \geq 1$ and present alternatives of Definitions 3.1 and 3.3 where all instances of the elementwise Raviart–Thomas and Nédélec interpolators $\boldsymbol{I}^p_{\mathcal{RT}}$, $\boldsymbol{I}^{p+1}_{\mathcal{RT}}$, and $\boldsymbol{I}^p_{\mathcal{N}}$ are removed. In addition, we simplify Definition 3.3, as commuting will not be needed here.

We start with reworking Definition 3.1:

Definition 7.1 (Alternative of Definition 3.1)**.** *Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ be given. For $\boldsymbol{w} := \nabla\times\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla\cdot\boldsymbol{w} = 0$ and polynomial degree $p \geq 1$:*

1. *Define a broken Raviart–Thomas polynomial $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{RT}}_{p-1}(\mathcal{T}_h)$, on each mesh element, via*

$$\boldsymbol{\tau}_h|_K := \arg \min_{\substack{\boldsymbol{w}_h \in \boldsymbol{\mathcal{RT}}_{p-1}(K) \\ \nabla\cdot\boldsymbol{w}_h = 0}} \|\boldsymbol{w} - \boldsymbol{w}_h\|_K \qquad \forall K \in \mathcal{T}_h. \tag{7.1}$$

2. *Define a Raviart–Thomas polynomial $\boldsymbol{\sigma}^{\boldsymbol{a}}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$, on each vertex patch, via*

$$\boldsymbol{\sigma}^{\boldsymbol{a}}_h := \arg \min_{\substack{\boldsymbol{w}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{w}_h = \Pi^p_h(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{w})}} \left\|\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{w}_h\right\|_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{7.2}$$

3. *Extending $\boldsymbol{\sigma}^{\boldsymbol{a}}_h$ by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, define $\boldsymbol{\sigma}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ via*

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\sigma}^{\boldsymbol{a}}_h. \tag{7.3}$$

Definition 7.1 takes the same form as [26, Definition 5.2], up to the fact that the divergence constraint in (7.2) is modified in that we use $\Pi^p_h(\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{w})$ in place of $\nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\tau}_h$. Similarly to Remark 3.2, we now inspect the (beginning of) the proof of [26, Lemma 5.3]. Here a term $(g_{\boldsymbol{a}}, \varphi)_{\omega_{\boldsymbol{a}}} = (\Pi^p_{\mathcal{T}}(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v}) + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{\tau}_{\mathcal{T}})_{\omega_{\boldsymbol{a}}}$ appears (notation from the proof of [26, Lemma 5.3]), which has to be replaced by $(\Pi^p_{\mathcal{T}}(\psi^{\boldsymbol{a}}\nabla\cdot\boldsymbol{v} + \nabla\psi^{\boldsymbol{a}}\cdot\boldsymbol{v}), \varphi)_{\omega_{\boldsymbol{a}}}$. This makes appear the same supplementary term and its bound as in (3.5). Thus, [26, Lemma 5.3], recalling (2.5b) and $\nabla\cdot\boldsymbol{w} = \nabla\cdot(\nabla\times\boldsymbol{v}) = 0$ (current notation), so that there are no oscillation terms, lead to

$$\left\|\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h - \boldsymbol{\sigma}^{\boldsymbol{a}}_h\right\|_{\omega_{\boldsymbol{a}}} \lesssim \left\{ \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^{p-1}_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\|^2_K \right\}^{1/2}, \tag{7.4}$$

where crucially the hidden constant only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$.

We now rework Definition 3.3.

Definition 7.2 (Alternative of Definition 3.3)**.** *Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$ be given. For $p \geq 1$:*

1. *For $\boldsymbol{w} := \nabla\times\boldsymbol{v}$ yielding $\boldsymbol{w} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div}, \Omega)$ with $\nabla\cdot\boldsymbol{w} = 0$, define $\boldsymbol{\tau}_h \in \boldsymbol{\mathcal{RT}}_{p-1}(\mathcal{T}_h)$ by (7.1) and $\boldsymbol{\sigma}^{\boldsymbol{a}}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ by (7.2) from Definition 7.1.*
2. *Define a broken Nédélec polynomial $\boldsymbol{\iota}_h \in \boldsymbol{\mathcal{N}}_{p-1}(\mathcal{T}_h)$, on each mesh element, via*

$$\boldsymbol{\iota}_h|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p-1}(K) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\tau}_h}} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K \qquad \forall K \in \mathcal{T}_h. \tag{7.5}$$

3. *Define a Raviart–Thomas polynomial $\boldsymbol{\theta}^{\boldsymbol{a}}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$, on each vertex patch, via*

$$\boldsymbol{\theta}^{\boldsymbol{a}}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}_h = \Pi^p_h(-\nabla\psi^{\boldsymbol{a}}\cdot(\nabla\times\boldsymbol{v}))}} \|\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h - \boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{7.6}$$

4. *Define a Nédélec polynomial $\boldsymbol{h}^{\boldsymbol{a}}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl}, \omega_{\boldsymbol{a}})$, on each vertex patch, via*

$$\boldsymbol{h}^{\boldsymbol{a}}_h := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}}) \\ \nabla\times\boldsymbol{v}_h = \boldsymbol{\sigma}^{\boldsymbol{a}}_h + \boldsymbol{\theta}^{\boldsymbol{a}}_h}} \left\|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h - \boldsymbol{v}_h\right\|_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{a} \in \mathcal{V}_h. \tag{7.7}$$

5. *Extending $\boldsymbol{h}^{\boldsymbol{a}}_h$ by zero outside of the patch subdomain $\omega_{\boldsymbol{a}}$, define*

$$\boldsymbol{h}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{h}^{\boldsymbol{a}}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega). \tag{7.8}$$

Compared to Definition 3.3, Definition 7.2 avoids the construction of $\boldsymbol{\delta}_h$ on Step 4 and of the correction terms $\boldsymbol{\delta}_h^{\boldsymbol{a}}$ on Step 5. As in (3.8), we can still set

$$\boldsymbol{\delta}_h := \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{div},\Omega) \quad \text{with} \quad \nabla\cdot\boldsymbol{\delta}_h = 0, \tag{7.9}$$

but the commuting property $\nabla\times\boldsymbol{h}_h = \boldsymbol{\sigma}_h$ from (3.15) is here lost in that from (7.8), (7.7), and (7.3),

$$\nabla\times\boldsymbol{h}_h = \sum_{\boldsymbol{a}\in\mathcal{V}_h} \nabla\times\boldsymbol{h}_h^{\boldsymbol{a}} = \sum_{\boldsymbol{a}\in\mathcal{V}_h} (\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{\theta}_h^{\boldsymbol{a}}) = \boldsymbol{\sigma}_h + \boldsymbol{\delta}_h. \tag{7.10}$$

This will be sufficient here, as in Theorem 3.10, the minimization does not have a constrained curl. It is also to be noted that the minimizations in (7.6) avoid the additional elementwise orthogonality constraint of (3.7). Commuting could be achieved by further lowering the degree $p-1$ to $p-2$ and constructing an appropriate $\boldsymbol{\delta}_h^{\boldsymbol{a}}$, but the above adjustments will also be helpful for us in deriving p-robust bounds.

Definition 7.2 is well-posed, which can be easily verified as in Section 4; namely, $\boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{\theta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})\cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ with $\nabla\cdot(\boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{\theta}_h^{\boldsymbol{a}}) = 0$. We now modify the reasoning of Step 2b from the proof of Lemma 5.2, extending to the $\omega_{\boldsymbol{a}}$-patch setting the hp-reasoning of Lemmas A.2 and A.3 from Appendix A below.

Lemma 7.3 (p-robust patch estimate)**.** *Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)$ be given. Consider a vertex $\boldsymbol{a} \in \mathcal{V}_h$ with $\boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}$ and assume that the associated patch subdomain $\omega_{\boldsymbol{a}}$ is convex. Then, if $\boldsymbol{\sigma}_h^{\boldsymbol{a}}$ and $\boldsymbol{h}_h^{\boldsymbol{a}}$ are respectively given by Definitions 7.1 and 7.2, we have*

$$\|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h - \boldsymbol{h}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}^2 \lesssim \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \left[\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_{p-1}(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^{p-1}(\nabla\times\boldsymbol{v})\|_K\right)^2\right],$$

where the hidden constant only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ of the mesh $\mathcal{T}_h$.

Proof. Taking into account definition (7.5) and the constrained–unconstrained equivalence established in Lemma A.1 (employed with $p-1$ in place of p), we only need to show

$$\|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h - \boldsymbol{h}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{v}-\boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} + \left\{\sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{p}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^{p-1}(\nabla\times\boldsymbol{v})\|_K\right)^2\right\}^{1/2}.$$

From the definition (7.7), as in (5.2), [15, Theorem 3.3] gives

$$\begin{aligned}
\|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h - \boldsymbol{h}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} &\lesssim \min_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \nabla\times\boldsymbol{\varphi}=\boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{\theta}_h^{\boldsymbol{a}}}} \|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h - \boldsymbol{\varphi}\|_{\omega_{\boldsymbol{a}}}\\
&= \sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})\\ \|\nabla\times\boldsymbol{\varphi}\|_{\omega_{\boldsymbol{a}}}=1}} \left\{(\boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{\theta}_h^{\boldsymbol{a}},\boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h,\nabla\times\boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}}\right\},
\end{aligned}$$

where we have, for the equality, employed the primal–dual equivalence as in (5.3). Fix now $\boldsymbol{\varphi} \in \boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})$ with $\|\nabla\times\boldsymbol{\varphi}\|_{\omega_{\boldsymbol{a}}} = 1$. We invoke $\boldsymbol{\psi} \in \boldsymbol{H}_*^1(\omega_{\boldsymbol{a}}) \cap \boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}})$ satisfying $\nabla\times\boldsymbol{\psi} = \nabla\times\boldsymbol{\varphi}$ and

$$\|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{L},\omega_{\boldsymbol{a}}}\|\nabla\times\boldsymbol{\varphi}\|_{\omega_{\boldsymbol{a}}} \lesssim 1, \tag{7.11a}$$

$$\|\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{PF},\omega_{\boldsymbol{a}}} h_{\omega_{\boldsymbol{a}}}\|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}, \tag{7.11b}$$

as in (5.4)–(5.5). Also, as in (5.6), since $\boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{\theta}_h^{\boldsymbol{a}} \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}})$ is divergence-free, there exists $\boldsymbol{w}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{curl},\omega_{\boldsymbol{a}})$ such that $\nabla\times\boldsymbol{w}_h = \boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{\theta}_h^{\boldsymbol{a}}$, and we can exchange $\boldsymbol{\varphi}$ for $\boldsymbol{\psi}$, i.e.,

$$T := (\boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{\theta}_h^{\boldsymbol{a}},\boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h,\nabla\times\boldsymbol{\varphi})_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\sigma}_h^{\boldsymbol{a}}+\boldsymbol{\theta}_h^{\boldsymbol{a}},\boldsymbol{\psi})_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h,\nabla\times\boldsymbol{\psi})_{\omega_{\boldsymbol{a}}}.$$

Let now $q \in H_*^1(\omega_{\boldsymbol{a}})$ be such that

$$(\nabla q,\nabla v)_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\psi},\nabla v)_{\omega_{\boldsymbol{a}}} \qquad \forall v \in H_*^1(\omega_{\boldsymbol{a}}). \tag{7.12}$$

Defining $\boldsymbol{\chi} := \boldsymbol{\psi} - \nabla q$, we have, recalling the notation from Section 2.3,

$$\boldsymbol{\chi} \in \boldsymbol{H}^{\dagger}(\mathrm{curl},\omega_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \text{ with } \nabla\cdot\boldsymbol{\chi} = 0, \tag{7.13}$$

together with $\nabla\times\boldsymbol{\chi} = \nabla\times\boldsymbol{\psi} = \nabla\times\boldsymbol{\varphi}$. Moreover, by the Green theorem, the right-hand side in (7.12) equivalently writes as $(\boldsymbol{\psi}, \nabla v)_{\omega_{\boldsymbol{a}}} = -(\nabla\cdot\boldsymbol{\psi}, v)_{\omega_{\boldsymbol{a}}}$. Thus, since we suppose $\omega_{\boldsymbol{a}}$ convex and $\boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}$, so that $\gamma_{\mathrm{D}} = \emptyset$ and only the Neumann boundary condition is prescribed on the whole $\partial\omega_{\boldsymbol{a}}$, see Section 2.3, the elliptic regularity shift, see, e.g., [35, Theorem 4.3.1.4], gives $q \in H^2(\omega_{\boldsymbol{a}})$ with

$$\|\nabla(\nabla q)\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla\cdot\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}} \leq \|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}.$$

Thus, in addition to (7.13), there also holds $\boldsymbol{\chi} \in \boldsymbol{H}^1(\omega_{\boldsymbol{a}})$ with

$$\|\nabla\boldsymbol{\chi}\|_{\omega_{\boldsymbol{a}}} \leq \|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}} + \|\nabla(\nabla q)\|_{\omega_{\boldsymbol{a}}} \lesssim \|\nabla\boldsymbol{\psi}\|_{\omega_{\boldsymbol{a}}}. \tag{7.14}$$

Finally, since $\nabla\times\boldsymbol{\chi} = \nabla\times\boldsymbol{\psi}$ and, by the Green theorem (recall the complementarity of the boundary conditions of $H^1_*(\omega_{\boldsymbol{a}})$ and $\boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$), $(\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{\theta}_h^{\boldsymbol{a}}, \nabla q)_{\omega_{\boldsymbol{a}}} = 0$, we can further equivalently write

$$T = (\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{\theta}_h^{\boldsymbol{a}}, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} - (\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h, \nabla\times\boldsymbol{\chi})_{\omega_{\boldsymbol{a}}}.$$

We will now work on this term.

Using the chain rule

$$\nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\chi}) = \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\chi} + \psi^{\boldsymbol{a}}\nabla\times\boldsymbol{\chi}, \tag{7.15}$$

and since $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})$ and $\boldsymbol{c}\times\boldsymbol{a} = -\boldsymbol{a}\times\boldsymbol{c}$ for vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$, we arrive at

$$\begin{aligned} T &= (\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{\theta}_h^{\boldsymbol{a}}, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\iota}_h, \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} - (\boldsymbol{\iota}_h, \nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\chi}))_{\omega_{\boldsymbol{a}}} \\ &= (\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} - (\boldsymbol{\iota}_h, \nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\chi}))_{\omega_{\boldsymbol{a}}}. \end{aligned}$$

Moreover, the Green theorem gives

$$(\nabla\times\boldsymbol{v}, (\psi^{\boldsymbol{a}}\boldsymbol{\chi}))_{\omega_{\boldsymbol{a}}} - (\boldsymbol{v}, \nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\chi}))_{\omega_{\boldsymbol{a}}} = 0,$$

so that altogether, also adding and subtracting $\psi^{\boldsymbol{a}}\boldsymbol{\tau}_h$,

$$\begin{aligned} T &= (\boldsymbol{\sigma}_h^{\boldsymbol{a}} + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h - \psi^{\boldsymbol{a}}\nabla\times\boldsymbol{v}, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} - (\boldsymbol{\iota}_h - \boldsymbol{v}, \nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\chi}))_{\omega_{\boldsymbol{a}}} \\ &= (\boldsymbol{\sigma}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}}\boldsymbol{\tau}_h + \boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} + (\boldsymbol{\tau}_h - \nabla\times\boldsymbol{v}, \psi^{\boldsymbol{a}}\boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} - (\boldsymbol{\iota}_h - \boldsymbol{v}, \nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\chi}))_{\omega_{\boldsymbol{a}}}. \end{aligned}$$

In the remainder of the proof, we estimate the above terms separately, in four steps.

Step 1. The chain rule (7.15), the Cauchy–Schwarz inequality, mesh shape-regularity, and the Poincaré–Friedrichs–Weber inequality (2.12) (recalling (7.13)) give

$$\|\nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\chi}))\|_{\omega_{\boldsymbol{a}}} = \|\nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\chi} + \psi^{\boldsymbol{a}}\nabla\times\boldsymbol{\chi}\|_{\omega_{\boldsymbol{a}}} \leq \|\nabla\psi^{\boldsymbol{a}}\|_{\infty,\omega_{\boldsymbol{a}}}\|\boldsymbol{\chi}\|_{\omega_{\boldsymbol{a}}} + \|\nabla\times\boldsymbol{\chi}\|_{\omega_{\boldsymbol{a}}} \lesssim 1, \tag{7.16}$$

so that the Cauchy–Schwarz inequality yields

$$-(\boldsymbol{\iota}_h - \boldsymbol{v}, \nabla\times(\psi^{\boldsymbol{a}}\boldsymbol{\chi}))_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{\iota}_h - \boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Step 2. As for (7.16), using (7.14) and (7.11a),

$$\|\nabla(\psi^{\boldsymbol{a}}\boldsymbol{\chi}))\|_{\omega_{\boldsymbol{a}}} \lesssim 1.$$

Thus, employing Lemma A.3 for each $K \in \mathcal{T}_{\boldsymbol{a}}$ and the Cauchy–Schwarz inequality, we infer

$$\begin{aligned} (\boldsymbol{\tau}_h - \nabla\times\boldsymbol{v}, \psi^{\boldsymbol{a}}\boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} &= \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} (\boldsymbol{\tau}_h - \nabla\times\boldsymbol{v}, \psi^{\boldsymbol{a}}\boldsymbol{\chi})_K \lesssim \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{p}\left\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^{p-1}_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\right\|_K \|\nabla(\psi^{\boldsymbol{a}}\boldsymbol{\chi})\|_K\right) \\ &\lesssim \left\{\sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{p}\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^{p-1}_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\|_K\right)^2\right\}^{1/2}. \end{aligned}$$

Step 3. The Euler–Lagrange conditions of (7.6), where it is important that no additional elementwise orthogonality constraints as in (3.7) appear, give

$$(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} = (\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h, \boldsymbol{\chi} - \boldsymbol{\chi}_h)_{\omega_{\boldsymbol{a}}}$$

for

$$\boldsymbol{\chi}_h := \arg \min_{\substack{\boldsymbol{w}_h \in \boldsymbol{\mathcal{RT}}_p(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{w}_h = 0}} \|\boldsymbol{\chi} - \boldsymbol{w}_h\|_{\omega_{\boldsymbol{a}}}. \tag{7.17}$$

We now crucially rely on the hp-approximation estimate of [26, Theorem 3.6] for $\Omega = \omega_{\boldsymbol{a}}$, recalling that $\boldsymbol{\chi} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ has been prepared such that $\nabla{\cdot}\boldsymbol{\chi} = 0$ and $\boldsymbol{\chi} \in \boldsymbol{H}^1(\omega_{\boldsymbol{a}})$, so that we can take $s = 1$ in [26]. This gives

$$\|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{\omega_{\boldsymbol{a}}} \lesssim \left\{ \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{p+1} \left(\|\boldsymbol{\chi}\|_K^2 + \|\nabla \boldsymbol{\chi}\|_K^2 \right)^{1/2} \right)^2 \right\}^{1/2} \lesssim \frac{h_{\omega_{\boldsymbol{a}}}}{p} \left(\|\boldsymbol{\chi}\|_{\omega_{\boldsymbol{a}}}^2 + \|\nabla \boldsymbol{\chi}\|_{\omega_{\boldsymbol{a}}}^2 \right)^{1/2} \lesssim \frac{h_{\omega_{\boldsymbol{a}}}}{p}, \quad (7.18)$$

where we have concluded by (2.12), $\nabla{\times}\boldsymbol{\chi} = \nabla{\times}\boldsymbol{\varphi}$, (7.14), and (7.11a). Thus,

$$(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla \psi^{\boldsymbol{a}} {\times} \boldsymbol{\iota}_h, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla \psi^{\boldsymbol{a}} {\times} \boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} \frac{h_{\omega_{\boldsymbol{a}}}}{p}.$$

Finally, relying on [16, Lemma A.3], as in (5.10), we have

$$\|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla \psi^{\boldsymbol{a}} {\times} \boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} \lesssim h_{\omega_{\boldsymbol{a}}}^{-1} \|\boldsymbol{v} - \boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} + h_{\omega_{\boldsymbol{a}}}^{-1} \left\{ \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} \left(h_K \|\nabla{\times}\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^{p-1}(\nabla{\times}\boldsymbol{v})\|_K \right)^2 \right\}^{1/2}; \quad (7.19)$$

note that this bound is indeed p-robust, as the above oscillation term is not of hp type. Now, since $p \geq 1$, we conclude

$$(\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla \psi^{\boldsymbol{a}} {\times} \boldsymbol{\iota}_h, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} \lesssim \|\boldsymbol{v} - \boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}} + \left\{ \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{p} \|\nabla{\times}\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^{p-1}(\nabla{\times}\boldsymbol{v})\|_K \right)^2 \right\}^{1/2}.$$

Step 4. As above in Step 3, we employ $\boldsymbol{\chi}_h$ from (7.17) and (7.18). Together with the Euler–Lagrange conditions of (7.2) and the Cauchy–Schwarz inequality, this gives

$$\begin{aligned}(\boldsymbol{\sigma}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}} \boldsymbol{\tau}_h, \boldsymbol{\chi})_{\omega_{\boldsymbol{a}}} &= (\boldsymbol{\sigma}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}} \boldsymbol{\tau}_h, \boldsymbol{\chi} - \boldsymbol{\chi}_h)_{\omega_{\boldsymbol{a}}} \leq \|\boldsymbol{\sigma}_h^{\boldsymbol{a}} - \psi^{\boldsymbol{a}} \boldsymbol{\tau}_h\|_{\omega_{\boldsymbol{a}}} \|\boldsymbol{\chi} - \boldsymbol{\chi}_h\|_{\omega_{\boldsymbol{a}}} \\ &\lesssim \left\{ \sum_{K \in \mathcal{T}_{\boldsymbol{a}}} \left(\frac{h_K}{p} \|\nabla{\times}\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^{p-1}(\nabla{\times}\boldsymbol{v})\|_K \right)^2 \right\}^{1/2},\end{aligned}$$

where we have employed (7.4) and mesh shape-regularity in the last bound. □

We are now ready to derive a p-robust variant of the inequality $\lesssim$ from (3.20b) with a lowered polynomial degree on the right-hand side.

Proposition 7.4 (p-robust one-sided bound)**.** *Let $\boldsymbol{v} \in \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl}, \Omega)$, a mesh $\mathcal{T}_h$ of Ω, and a polynomial degree $p \geq 1$ be fixed. If $\nabla{\times}\boldsymbol{v} \notin [\mathcal{P}_{p-1}(\mathcal{T}_h)]^3$, assume in addition that $\Gamma_{\mathrm{D}} = \emptyset$ and that the patch subdomains $\omega_{\boldsymbol{a}}$ are convex for all vertices $\boldsymbol{a} \in \mathcal{V}_h$. Then*

$$\begin{aligned}&\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_p(\mathcal{T}_h) \cap \boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)} \left[\|\boldsymbol{v} - \boldsymbol{v}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla{\times}(\boldsymbol{v} - \boldsymbol{v}_h)\|_K \right)^2 \right] \\ &\lesssim \sum_{K \in \mathcal{T}_h} \left[\min_{\boldsymbol{v}_h \in \boldsymbol{\mathcal{N}}_{p-1}(K)} \|\boldsymbol{v} - \boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p} \|\nabla{\times}\boldsymbol{v} - \boldsymbol{\Pi}_{\mathcal{RT}}^{p-1}(\nabla{\times}\boldsymbol{v})\|_K \right)^2 \right],\end{aligned} \quad (7.20)$$

where the hidden constant only depends on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ of the mesh $\mathcal{T}_h$.

Proof. We bound the left-hand side of (7.20) by

$$\left[\|\boldsymbol{v} - \boldsymbol{h}_h\|^2 + \sum_{K \in \mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla{\times}(\boldsymbol{v} - \boldsymbol{h}_h)\|_K \right)^2 \right],$$

using $\boldsymbol{h}_h$ from Definition 7.2. We now estimate the two above terms separately, in two steps.

Step 1. The triangle inequality gives, for $\boldsymbol{\iota}_h$ given by (7.5),

$$\|\boldsymbol{v} - \boldsymbol{h}_h\| \leq \|\boldsymbol{v} - \boldsymbol{\iota}_h\| + \|\boldsymbol{\iota}_h - \boldsymbol{h}_h\|.$$

Thus, the constrained–unconstrained equivalence from Lemma A.1 (employed with $p-1$ in place of p) give the desired result for the first term above. As for the second one, definition (7.8), the partition of unity (2.1), and an overlap estimate imply

$$\|\boldsymbol{\iota}_h - \boldsymbol{h}_h\|^2 = \sum_{K\in\mathcal{T}_h} \left\| \sum_{\boldsymbol{a}\in\mathcal{V}_K} (\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h - \boldsymbol{h}_h^{\boldsymbol{a}}) \right\|_K^2 \leq 4 \sum_{\boldsymbol{a}\in\mathcal{V}_h} \|\psi^{\boldsymbol{a}}\boldsymbol{\iota}_h - \boldsymbol{h}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}}^2,$$

so that the desired bound follows from Lemma 7.3 when $\nabla\times\boldsymbol{v} \notin [\mathcal{P}_{p-1}(\mathcal{T}_h)]^3$. The assumption $\Gamma_{\mathrm{D}} = \emptyset$, which implies $\boldsymbol{a} \notin \overline{\Gamma_{\mathrm{D}}}$ when $\boldsymbol{a}$ lies on the boundary $\partial\Omega$, together with the assumption that the patch subdomains $\omega_{\boldsymbol{a}}$ are convex for all vertices $\boldsymbol{a} \in \mathcal{V}_h$ is only used in Lemma 7.3 to gain the factors h_K/p in place of h_K in front of $\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^{p-1}(\nabla\times\boldsymbol{v})\|_K$. These terms, however, simply disappear when $\nabla\times\boldsymbol{v} \in [\mathcal{P}_{p-1}(\mathcal{T}_h)]^3$, so that (7.20) holds in this case as well.

Step 2. From (7.10), the partition of unity (2.1), (7.3), and adding and subtracting $\psi^{\boldsymbol{a}}\nabla\times\boldsymbol{\iota}_h$,

$$\begin{aligned}\|\nabla\times(\boldsymbol{v}-\boldsymbol{h}_h)\|_K = \|\nabla\times\boldsymbol{v} - \boldsymbol{\sigma}_h - \boldsymbol{\delta}_h\|_K &= \left\| \sum_{\boldsymbol{a}\in\mathcal{V}_K} \psi^{\boldsymbol{a}}(\nabla\times(\boldsymbol{v}-\boldsymbol{\iota}_h)) + \sum_{\boldsymbol{a}\in\mathcal{V}_K} (\psi^{\boldsymbol{a}}\nabla\times\boldsymbol{\iota}_h - \boldsymbol{\sigma}_h^{\boldsymbol{a}}) - \boldsymbol{\delta}_h \right\|_K \\ &\leq \sum_{\boldsymbol{a}\in\mathcal{V}_K} \|\psi^{\boldsymbol{a}}(\nabla\times(\boldsymbol{v}-\boldsymbol{\iota}_h))\|_{\omega_{\boldsymbol{a}}} + \sum_{\boldsymbol{a}\in\mathcal{V}_K} \|\psi^{\boldsymbol{a}}\nabla\times\boldsymbol{\iota}_h - \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} + \|\boldsymbol{\delta}_h\|_K.\end{aligned}$$

Recalling that the constraint in (7.5) gives $\nabla\times\boldsymbol{\iota}_h = \boldsymbol{\tau}_h$, we estimate the three terms separately.

For the first term, we estimate

$$\|\psi^{\boldsymbol{a}}(\nabla\times(\boldsymbol{v}-\boldsymbol{\iota}_h))\|_{\omega_{\boldsymbol{a}}} \leq \|\boldsymbol{\tau}_h - \nabla\times\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}} \lesssim \left\{ \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^{p-1}(\nabla\times\boldsymbol{v})\|_K^2 \right\}^{1/2},$$

relying on (7.1) and the p-robust constrained–unconstrained equivalence of [26, Lemma A.1] (recall that $\nabla\cdot(\nabla\times\boldsymbol{v}) = 0$.) For the second one, we use (7.4) to see that

$$\|\psi^{\boldsymbol{a}}\nabla\times\boldsymbol{\iota}_h - \boldsymbol{\sigma}_h^{\boldsymbol{a}}\|_{\omega_{\boldsymbol{a}}} \lesssim \left\{ \sum_{K\in\mathcal{T}_{\boldsymbol{a}}} \|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}_{\boldsymbol{\mathcal{RT}}}^{p-1}(\nabla\times\boldsymbol{v})\|_K^2 \right\}^{1/2}.$$

For the third one, we infer

$$\|\boldsymbol{\delta}_h\|_K = \left\| \sum_{\boldsymbol{a}\in\mathcal{V}_K} (\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h) \right\|_K \leq \sum_{\boldsymbol{a}\in\mathcal{V}_K} \|\boldsymbol{\theta}_h^{\boldsymbol{a}} - \nabla\psi^{\boldsymbol{a}}\times\boldsymbol{\iota}_h\|_{\omega_{\boldsymbol{a}}},$$

where we have used (7.9) and (2.1). For the arising term, we can now use (7.19). Thus, multiplying $\|\nabla\times(\boldsymbol{v}-\boldsymbol{h}_h)\|_K$ by $h_K/(p+1)$, using that $h_K/(p+1) \leq h_K/p$ as well as $1/(p+1) \leq 1$, and invoking the mesh shape-regularity, the proof is finished. □

Denote

$$m^2 := \min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(\mathcal{T}_h)\cap\boldsymbol{H}_{0,\mathrm{N}}(\mathrm{curl},\Omega)} \left[\|\boldsymbol{v}-\boldsymbol{v}_h\|^2 + \sum_{K\in\mathcal{T}_h} \left(\frac{h_K}{p+1} \|\nabla\times(\boldsymbol{v}-\boldsymbol{v}_h)\|_K \right)^2 \right] \tag{7.21}$$

and

$$[v_{K,q,s,t}(\boldsymbol{v})]^2 := \left(\frac{h_K^{\min\{q,s\}}}{q^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s(K)} \right)^2 + \left(\frac{h_K}{q} \frac{h_K^{\min\{q,t\}}}{q^t} \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{H}^t(K)} \right)^2. \tag{7.22}$$

From (3.20b) in Theorem 3.8 and the characterization (2.5b), we have

$$m^2 \lesssim \sum_{K\in\mathcal{T}_h} \left[\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1} \min_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_p(K)} \|\nabla\times\boldsymbol{v} - \boldsymbol{w}_h\|_K \right)^2 \right], \tag{7.23}$$

where the hidden constant can depend on $\kappa_{\mathcal{T}_h}$ and the polynomial degree p. From Proposition 7.4, in turn,

$$m^2 \lesssim \sum_{K\in\mathcal{T}_h} \left[\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_{p-1}(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p} \min_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_{p-1}(K)} \|\nabla\times\boldsymbol{v} - \boldsymbol{w}_h\|_K \right)^2 \right], \tag{7.24}$$

where p has to be greater or equal to 1 but the hidden constant is independent of p. These are the two ingredients necessary for the proof of Theorem 3.10 in the spirit of the proof of Theorem 3.6 in [26, Section 5.2]:

Proof of Theorem 3.10. Recalling the notations (7.21)–(7.22), we need to show that

$$m^2 \lesssim \sum_{K\in\mathcal{T}_h} [v_{K,p+1,s,t}(\boldsymbol{v})]^2,$$

where the hidden constant only depends on $\kappa_{\mathcal{T}_h}$, s, and t. We distinguish two cases that we later combine together.

Case 1. Suppose $p \le s$. On each tetrahedron $K \in \mathcal{T}_h$, we have from (2.2) and (2.3) that $[\mathcal{P}_p(K)]^3 \subset \boldsymbol{\mathcal{N}}_p(K)$ and $[\mathcal{P}_p(K)]^3 \subset \boldsymbol{\mathcal{RT}}_p(K)$. Thus the elementwise regularity assumptions (3.23) and [4, Lemma 4.1] give

$$\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p+1}\min_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_p(K)}\|\nabla\times\boldsymbol{v}-\boldsymbol{w}_h\|_K\right)^2 \lesssim [v_{K,p+1,s,t}(\boldsymbol{v})]^2,$$

where the hidden constant only depends on $\kappa_{\mathcal{T}_h}$, s, and t. Thus, from (7.23), there exists a constant $C_{\kappa_{\mathcal{T}_h},p,s,t}$ only depending on $\kappa_{\mathcal{T}_h}$, p, s, and t such that

$$m^2 \le C_{\kappa_{\mathcal{T}_h},p,s,t} \sum_{K\in\mathcal{T}_h} [v_{K,p+1,s,t}(\boldsymbol{v})]^2.$$

Defining $C^\star_{\kappa_{\mathcal{T}_h},s,t} := \max_{0\le p\le s} C_{\kappa_{\mathcal{T}_h},p,s,t}$, only depending on $\kappa_{\mathcal{T}_h}$, s, and t, there holds, for all $0 \le p \le s$,

$$m^2 \le C^\star_{\kappa_{\mathcal{T}_h},s,t} \sum_{K\in\mathcal{T}_h} [v_{K,p+1,s,t}(\boldsymbol{v})]^2. \tag{7.25}$$

Case 2. Suppose $p > s$. Since p is an integer, this implies that $p \ge 1$, so that we will be able to apply (7.24). As above, reducing $p+1$ to p on the right-hand side,

$$\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_{p-1}(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p}\min_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_{p-1}(K)}\|\nabla\times\boldsymbol{v}-\boldsymbol{w}_h\|_K\right)^2 \lesssim [v_{K,p,s,t}(\boldsymbol{v})]^2,$$

where the hidden constant only depends on $\kappa_{\mathcal{T}_h}$, s, and t. Now, actually, $\min\{p,s\} = s = \min\{p+1,s\}$ and $p+1 \le 2p$, so that $1/p^s \le 2^s/(p+1)^s$. Similarly, due to the assumption $s \ge t$, $\min\{p,t\} = t = \min\{p+1,t\}$ and $1/p^t \le 2^t/(p+1)^t$. As a consequence, we can rise p back to $p+1$ on the right-hand side,

$$\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_{p-1}(K)} \|\boldsymbol{v}-\boldsymbol{v}_h\|_K^2 + \left(\frac{h_K}{p}\min_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_{p-1}(K)}\|\nabla\times\boldsymbol{v}-\boldsymbol{w}_h\|_K\right)^2 \lesssim [v_{K,p+1,s,t}(\boldsymbol{v})]^2,$$

where the constant beyond $\lesssim$ still only depends on $\kappa_{\mathcal{T}_h}$, s, and t. Thus, using (7.24),

$$m^2 \le C^\sharp_{\kappa_{\mathcal{T}_h},s,t} \sum_{K\in\mathcal{T}_h} [v_{K,p+1,s,t}(\boldsymbol{v})]^2 \tag{7.26}$$

for $C^\sharp_{\kappa_{\mathcal{T}_h},s,t}$ only depending on $\kappa_{\mathcal{T}_h}$, s, and t for all $p > s$.

Conclusion. The claim (3.24) follows from (7.25) and (7.26) for the constant $\max\{C^\star_{\kappa_{\mathcal{T}_h},s,t}, C^\sharp_{\kappa_{\mathcal{T}_h},s,t}\}$ only depending on $\kappa_{\mathcal{T}_h}$, s, and t. □

A p-robust equivalence of constrained and unconstrained best-approximation in $\boldsymbol{H}(\mathrm{curl})$ on a tetrahedron

In this Appendix, we extend [14, Lemma 1] to functions $\boldsymbol{v}$ with nonpolynomial curl, in the spirit of [26, Lemma A.1]. This is a consequence of the breakthrough result of Costabel and McIntosh in [20, Proposition 4.2]. Interestingly enough, the constant hidden in the inequality is here independent of the polynomial degree p.

Lemma A.1 (Equivalence of constrained and unconstrained best-approximation on a tetrahedron)**.** *Let a polynomial degree $p \ge 0$, a tetrahedron K, and an arbitrary $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}, K)$ be fixed. Let*

$$\boldsymbol{\tau}_h := \arg\min_{\substack{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_p(K)\\ \nabla\cdot\boldsymbol{w}_h=0}} \|\nabla\times\boldsymbol{v}-\boldsymbol{w}_h\|_K. \tag{A.1}$$

Then

$$\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K \le \min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)\\ \nabla\times\boldsymbol{v}_h=\boldsymbol{\tau}_h}}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K \lesssim \Big(\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K + \frac{h_K}{p+1}\big\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\big\|_K\Big), \tag{A.2}$$

where the hidden constant only depends on the shape-regularity $\kappa_K := h_K/\rho_K$ *of* K*.*

Proof. The first inequality is obvious, since the second minimization set has an additional curl constraint. In order to show the second one, denote respectively

$$\boldsymbol{\iota}_h := \arg\min_{\substack{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)\\ \nabla\times\boldsymbol{v}_h=\boldsymbol{\tau}_h}}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K \tag{A.3}$$

and

$$\tilde{\boldsymbol{\iota}}_h := \arg\min_{\boldsymbol{v}_h\in\boldsymbol{\mathcal{N}}_p(K)}\|\boldsymbol{v}-\boldsymbol{v}_h\|_K \tag{A.4}$$

the constrained and unconstrained minimizers. We then need to show

$$\|\boldsymbol{v}-\boldsymbol{\iota}_h\|_K \lesssim \|\boldsymbol{v}-\tilde{\boldsymbol{\iota}}_h\|_K + \frac{h_K}{p+1}\big\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\big\|_K, \tag{A.5}$$

where $\lesssim$ means inequality up to a constant only depending on the shape-regularity κ_K.

Using (A.1) and (4.2), $\boldsymbol{\tau}_h - \nabla\times\tilde{\boldsymbol{\iota}}_h \in \{\boldsymbol{w}_h \in \boldsymbol{\mathcal{RT}}_p(K);\ \nabla\cdot\boldsymbol{w}_h = 0\}$. Thus, we can use [20, Proposition 4.2], cf. also the reformulation in [12, Theorem 2], stipulating the existence of $\boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{N}}_p(K)$ with $\nabla\times\boldsymbol{\varphi}_h = \boldsymbol{\tau}_h - \nabla\times\tilde{\boldsymbol{\iota}}_h$ such that

$$\|\boldsymbol{\varphi}_h\|_K \lesssim \min_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}(\mathrm{curl},K)\\ \nabla\times\boldsymbol{\varphi}=\boldsymbol{\tau}_h-\nabla\times\tilde{\boldsymbol{\iota}}_h}}\|\boldsymbol{\varphi}\|_K. \tag{A.6}$$

Shifting now the right-hand side of (A.6) by $\tilde{\boldsymbol{\iota}}_h$, we arrive at

$$\|\boldsymbol{\varphi}_h\|_K \lesssim \min_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}(\mathrm{curl},K)\\ \nabla\times\boldsymbol{\varphi}=\boldsymbol{\tau}_h}}\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\iota}}_h\|_K.$$

A primal–dual equivalence as in, e.g., [13, Lemma 5.5] implies (as in Section 2.3, $\boldsymbol{H}_0(\mathrm{curl},K)$ is composed of those $\boldsymbol{\varphi}\in\boldsymbol{H}(\mathrm{curl},K)$ that verify $\boldsymbol{\varphi}\times\boldsymbol{n}_K = 0$ on ∂K in appropriate sense)

$$\begin{aligned}
\min_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}(\mathrm{curl},K)\\ \nabla\times\boldsymbol{\varphi}=\boldsymbol{\tau}_h}}\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\iota}}_h\|_K &= \sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}_0(\mathrm{curl},K)\\ \|\nabla\times\boldsymbol{\varphi}\|_K=1}}\big\{(\boldsymbol{\tau}_h,\boldsymbol{\varphi})_K - (\tilde{\boldsymbol{\iota}}_h,\nabla\times\boldsymbol{\varphi})_K\big\}\\
&\le \sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}_0(\mathrm{curl},K)\\ \|\nabla\times\boldsymbol{\varphi}\|_K=1}}\big\{(\nabla\times\boldsymbol{v},\boldsymbol{\varphi})_K - (\tilde{\boldsymbol{\iota}}_h,\nabla\times\boldsymbol{\varphi})_K\big\} + \sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}_0(\mathrm{curl},K)\\ \|\nabla\times\boldsymbol{\varphi}\|_K=1}}(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\varphi})_K\\
&\lesssim \min_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}(\mathrm{curl},K)\\ \nabla\times\boldsymbol{\varphi}=\nabla\times\boldsymbol{v}}}\|\boldsymbol{\varphi}-\tilde{\boldsymbol{\iota}}_h\|_K + \frac{h_K}{p+1}\big\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\big\|_K,
\end{aligned}$$

where, to estimate the second term on the middle line, we have used the technical result of Lemma A.2 below.

Consequently, since $\boldsymbol{v}\in\boldsymbol{H}(\mathrm{curl},K)$ satisfies the curl constraint above,

$$\|\boldsymbol{\varphi}_h\|_K \lesssim \|\boldsymbol{v}-\tilde{\boldsymbol{\iota}}_h\|_K + \frac{h_K}{p+1}\big\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\big\|_K. \tag{A.7}$$

Now note that $(\boldsymbol{\varphi}_h+\tilde{\boldsymbol{\iota}}_h)\in\boldsymbol{\mathcal{N}}_p(K)$ with $\nabla\times(\boldsymbol{\varphi}_h+\tilde{\boldsymbol{\iota}}_h) = \boldsymbol{\tau}_h$. Thus, $\boldsymbol{\varphi}_h+\tilde{\boldsymbol{\iota}}_h$ belongs to the minimization set in (A.3), and the minimization property (A.3) of $\boldsymbol{\iota}_h$ implies $\|\boldsymbol{v}-\boldsymbol{\iota}_h\|_K \le \|\boldsymbol{v}-(\boldsymbol{\varphi}_h+\tilde{\boldsymbol{\iota}}_h)\|_K$. Thus, by virtue of the triangle inequality and using (A.7), we altogether infer

$$\|\boldsymbol{v}-\boldsymbol{\iota}_h\|_K \le \|\boldsymbol{v}-(\boldsymbol{\varphi}_h+\tilde{\boldsymbol{\iota}}_h)\|_K \le \|\boldsymbol{v}-\tilde{\boldsymbol{\iota}}_h\|_K + \|\boldsymbol{\varphi}_h\|_K \lesssim \|\boldsymbol{v}-\tilde{\boldsymbol{\iota}}_h\|_K + \frac{h_K}{p+1}\big\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\big\|_K,$$

i.e., (A.5), and the proof is finished.

□

Lemma A.2 (*hp* data oscillation)**.** *Let the assumptions of Lemma A.1 be verified. Then*

$$\sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{H}_0(\mathrm{curl},K)\\ \|\nabla\times\boldsymbol{\varphi}\|_K=1}} (\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\varphi})_K \lesssim \frac{h_K}{p+1}\left\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\right\|_K,$$

where the hidden constant only depends on the shape-regularity $\kappa_K := h_K/\rho_K$ of K.

Proof. Fix $\boldsymbol{\varphi}\in\boldsymbol{H}_0(\mathrm{curl},K)$ with $\|\nabla\times\boldsymbol{\varphi}\|_K=1$. From (2.9), there exists $\boldsymbol{\psi}\in\boldsymbol{H}^1(K)\cap\boldsymbol{H}_0(\mathrm{curl},K)$ such that $\nabla\times\boldsymbol{\psi}=\nabla\times\boldsymbol{\varphi}$ and

$$\|\nabla\boldsymbol{\psi}\|_K\le\|\nabla\times\boldsymbol{\varphi}\|_K=1.$$

Since $\boldsymbol{\tau}_h\in\boldsymbol{\mathcal{RT}}_p(K)$ with $\nabla\cdot\boldsymbol{\tau}_h=0$, there exists $\boldsymbol{w}_h\in\boldsymbol{\mathcal{N}}_p(K)$ such that $\nabla\times\boldsymbol{w}_h=\boldsymbol{\tau}_h$. Thus, by the Green theorem,

$$(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\varphi})_K=(\nabla\times(\boldsymbol{w}_h-\boldsymbol{v}),\boldsymbol{\varphi})_K=(\boldsymbol{w}_h-\boldsymbol{v},\nabla\times\boldsymbol{\varphi})_K=(\boldsymbol{w}_h-\boldsymbol{v},\nabla\times\boldsymbol{\psi})_K=(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\psi})_K,$$

and we conclude by the following Lemma A.3. □

Lemma A.3 (*hp* data estimate)**.** *Let the assumptions of Lemma A.1 be verified. Let $\boldsymbol{\psi}\in\boldsymbol{H}^1(K)$. Then*

$$|(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\psi})_K|\lesssim\frac{h_K}{p+1}\left\|\nabla\times\boldsymbol{v}-\boldsymbol{\Pi}^p_{\boldsymbol{\mathcal{RT}}}(\nabla\times\boldsymbol{v})\right\|_K\|\nabla\boldsymbol{\psi}\|_K,$$

where the hidden constant only depends on the shape-regularity $\kappa_K := h_K/\rho_K$ of K.

Proof. Let $q\in H^1_0(K)$ be such that

$$(\nabla q,\nabla v)_K=(\boldsymbol{\psi},\nabla v)_K\qquad\forall v\in H^1_0(K). \tag{A.8}$$

Defining $\boldsymbol{\chi}:=\boldsymbol{\psi}-\nabla q$, we have

$$\boldsymbol{\chi}\in\boldsymbol{H}(\mathrm{div},K)\text{ with }\nabla\cdot\boldsymbol{\chi}=0. \tag{A.9}$$

Moreover, by the Green theorem, the right-hand side in (A.8) can be equivalently written as $(\boldsymbol{\psi},\nabla v)_K=-(\nabla\cdot\boldsymbol{\psi},v)_K$. Thus, since K is convex, the elliptic regularity shift, see, e.g., [35, Theorem 4.3.1.4], gives $q\in H^2(K)$ with

$$\|\nabla(\nabla q)\|_K\lesssim\|\nabla\cdot\boldsymbol{\psi}\|_K\le\|\nabla\boldsymbol{\psi}\|_K,$$

where $\lesssim$ means inequality up to a constant only depending on the shape-regularity κ_K. Thus, in addition to (A.9), there also holds $\boldsymbol{\chi}\in\boldsymbol{H}^1(K)$ with

$$\|\nabla\boldsymbol{\chi}\|_K\le\|\nabla\boldsymbol{\psi}\|_K+\|\nabla(\nabla q)\|_K\lesssim\|\nabla\boldsymbol{\psi}\|_K. \tag{A.10}$$

Now, since from (A.1) $\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div},K)$ with $\nabla\cdot(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v})=0$, there follows by the Green theorem

$$(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\nabla q)_K=0.$$

Consequently,

$$(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\psi})_K=(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\chi})_K. \tag{A.11}$$

Let respectively

$$\boldsymbol{\chi}_h:=\arg\min_{\substack{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_p(K)\\ \nabla\cdot\boldsymbol{w}_h=0}}\|\boldsymbol{\chi}-\boldsymbol{w}_h\|_K$$

and

$$\tilde{\boldsymbol{\chi}}_h:=\arg\min_{\boldsymbol{w}_h\in\boldsymbol{\mathcal{RT}}_p(K)}\|\boldsymbol{\chi}-\boldsymbol{w}_h\|_K$$

be the constrained and unconstrained Raviart–Thomas approximations of $\boldsymbol{\chi}$. The Euler–Lagrange conditions of (A.1) allow us to subtract $\boldsymbol{\chi}_h$ (but not $\tilde{\boldsymbol{\chi}}_h$) in (A.11), so that the Cauchy–Schwarz inequality and, crucially, the p-robust constrained–unconstrained $\boldsymbol{H}(\mathrm{div})$ equivalence of [26, Lemma A.1] (note that $\nabla\cdot\boldsymbol{\chi}=0$) lead to

$$(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\chi})_K=(\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v},\boldsymbol{\chi}-\boldsymbol{\chi}_h)_K\le\|\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v}\|_K\|\boldsymbol{\chi}-\boldsymbol{\chi}_h\|_K\lesssim\|\boldsymbol{\tau}_h-\nabla\times\boldsymbol{v}\|_K\|\boldsymbol{\chi}-\tilde{\boldsymbol{\chi}}_h\|_K. \tag{A.12}$$

Now, since $\mathcal{RT}_p(K)$ contains by (2.3) polynomials up to degree p in each component and since the minimization for $\tilde{\boldsymbol{\chi}}_h$ is unconstrained, the hp approximation bound (2.11) gives

$$\|\boldsymbol{\chi} - \tilde{\boldsymbol{\chi}}_h\|_K \lesssim \frac{h_K}{p+1}\|\nabla\boldsymbol{\chi}\|_K. \tag{A.13}$$

Finally, in addition to (A.1), let

$$\tilde{\boldsymbol{\tau}}_h := \arg \min_{\boldsymbol{w}_h \in \mathcal{RT}_p(K)} \|\nabla\times\boldsymbol{v} - \boldsymbol{w}_h\|_K,$$

i.e., from (2.5b), $\tilde{\boldsymbol{\tau}}_h = \boldsymbol{\Pi}^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})$. It follows once again from the p-robust constrained–unconstrained $\boldsymbol{H}(\mathrm{div})$ equivalence of [26, Lemma A.1] (note that $\nabla\cdot(\nabla\times\boldsymbol{v}) = 0$) that

$$\|\boldsymbol{\tau}_h - \nabla\times\boldsymbol{v}\|_K \lesssim \|\tilde{\boldsymbol{\tau}}_h - \nabla\times\boldsymbol{v}\|_K = \left\|\nabla\times\boldsymbol{v} - \boldsymbol{\Pi}^p_{\mathcal{RT}}(\nabla\times\boldsymbol{v})\right\|_K. \tag{A.14}$$

Thus, the desired result is a combination of (A.11), (A.12), (A.13), and (A.14) together with (A.10). $\square$

References

[1] Adams, R. A. Pure and Applied Mathematics, Vol. 65. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975.

[2] Arnold, D., and Guzmán, J. Local L^2-bounded commuting projections in FEEC. *ESAIM Math. Model. Numer. Anal.* **55** (2021), 2169–2184. https://doi.org/10.1051/m2an/2021054.

[3] Arnold, D. N., Falk, R. S., and Winther, R. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.* **15** (2006), 1–155. https://doi.org/10.1017/S0962492906210018.

[4] Babuška, I., and Suri, M. The h-p version of the finite element method with quasi-uniform meshes. *RAIRO Modél. Math. Anal. Numér.* **21** (1987), 199–238. http://dx.doi.org/10.1051/m2an/1987210201991.

[5] Bebendorf, M. A note on the Poincaré inequality for convex domains. *Z. Anal. Anwendungen* **22** (2003), 751–756. http://dx.doi.org/10.4171/ZAA/1170.

[6] Beck, R., Hiptmair, R., Hoppe, R. H. W., and Wohlmuth, B. Residual based a posteriori error estimators for eddy current computation. *M2AN Math. Model. Numer. Anal.* **34** (2000), 159–182. https://doi.org/10.1051/m2an:2000136.

[7] Bespalov, A., and Heuer, N. Optimal error estimation for H(curl)-conforming p-interpolation in two dimensions. *SIAM J. Numer. Anal.* **47** (2009), 3977–3989. https://doi-org.ezproxy.is.cuni.cz/10.1137/090753802.

[8] Bespalov, A., and Heuer, N. A new $\mathbf{H}(\mathrm{div})$-conforming p-interpolation operator in two dimensions. *ESAIM Math. Model. Numer. Anal.* **45** (2011), 255–275. https://doi.org/10.1051/m2an/2010039.

[9] Boffi, D., Brezzi, F., and Fortin, M. *Mixed finite element methods and applications*, vol. **44** of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013. https://doi.org/10.1007/978-3-642-36519-5.

[10] Braess, D., Pillwein, V., and Schöberl, J. Equilibrated residual error estimates are p-robust. *Comput. Methods Appl. Mech. Engrg.* **198** (2009), 1189–1197. http://dx.doi.org/10.1016/j.cma.2008.12.010.

[11] Braess, D., and Schöberl, J. Equilibrated residual error estimator for edge elements. *Math. Comp.* **77** (2008), 651–672. http://dx.doi.org/10.1090/S0025-5718-07-02080-7.

[12] Chaumont-Frelet, T., Ern, A., and Vohralík, M. Polynomial-degree-robust $\boldsymbol{H}(\mathrm{curl})$-stability of discrete minimization in a tetrahedron. *C. R. Math. Acad. Sci. Paris* **358** (2020), 1101–1110. https://doi.org/10.5802/crmath.133.

[13] Chaumont-Frelet, T., Ern, A., and Vohralík, M. Stable broken $\boldsymbol{H}(\mathbf{curl})$ polynomial extensions and p-robust a posteriori error estimates by broken patchwise equilibration for the curl–curl problem. *Math. Comp.* **91** (2022), 37–74. https://doi.org/10.1090/mcom/3673.

[14] Chaumont-Frelet, T., and Vohralík, M. Equivalence of local-best and global-best approximations in $\boldsymbol{H}(\mathrm{curl})$. *Calcolo* **58** (2021), 53. https://doi.org/10.1007/s10092-021-00430-9.

[15] Chaumont-Frelet, T., and Vohralík, M. Constrained and unconstrained stable discrete minimizations for p-robust local reconstructions in vertex patches in the de Rham complex. HAL Preprint 03749682, submitted for publication, https://hal.inria.fr/hal-03749682, 2022.

[16] Chaumont-Frelet, T., and Vohralík, M. p-robust equilibrated flux reconstruction in $\boldsymbol{H}(\mathrm{curl})$ based on local minimizations. Application to a posteriori analysis of the curl–curl problem. HAL Preprint 03227570, submitted for publication, https://hal.inria.fr/hal-03227570, 2022.

[17] Christiansen, S. H., and Winther, R. Smoothed projections in finite element exterior calculus. *Math. Comp.* **77** (2008), 813–829. http://dx.doi.org/10.1090/S0025-5718-07-02081-9.

[18] Ciarlet, P. G. *The Finite Element Method for Elliptic Problems*, vol. **4** of *Studies in Mathematics and its Applications*. North-Holland, Amsterdam, 1978.

[19] Costabel, M., Dauge, M., and Nicaise, S. Singularities of Maxwell interface problems. *M2AN Math. Model. Numer. Anal.* **33** (1999), 627–649. https://doi.org/10.1051/m2an:1999155.

[20] Costabel, M., and McIntosh, A. On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Math. Z.* **265** (2010), 297–320. http://dx.doi.org/10.1007/s00209-009-0517-8.

[21] Demkowicz, L. Polynomial exact sequences and projection-based interpolation with application to Maxwell equations. In *Mixed finite elements, compatibility conditions, and applications*, D. Boffi, F. Brezzi, L. F. Demkowicz, R. G. Durán, R. S. Falk, and M. Fortin, Eds., vol. **1939** of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin; Fondazione C.I.M.E., Florence, 2008, pp. 101–158. Lectures given at the C.I.M.E. Summer School held in Cetraro, June 26–July 1, 2006, Edited by D. Boffi and L. Gastaldi, https://doi.org/10.1007/978-3-540-78319-0_3.

[22] Demkowicz, L., and Buffa, A. H^1, $H(\mathrm{curl})$ and $H(\mathrm{div})$-conforming projection-based interpolation in three dimensions. Quasi-optimal p-interpolation estimates. *Comput. Methods Appl. Mech. Engrg.* **194** (2005), 267–296. https://doi.org/10.1016/j.cma.2004.07.007.

[23] Demkowicz, L., Gopalakrishnan, J., and Schöberl, J. Polynomial extension operators. Part I. *SIAM J. Numer. Anal.* **46** (2008), 3006–3031. http://dx.doi.org/10.1137/070698786.

[24] Demkowicz, L., Gopalakrishnan, J., and Schöberl, J. Polynomial extension operators. Part II. *SIAM J. Numer. Anal.* **47** (2009), 3293–3324. http://dx.doi.org/10.1137/070698798.

[25] Demkowicz, L., Gopalakrishnan, J., and Schöberl, J. Polynomial extension operators. Part III. *Math. Comp.* **81** (2012), 1289–1326. http://dx.doi.org/10.1090/S0025-5718-2011-02536-6.

[26] Ern, A., Gudi, T., Smears, I., and Vohralík, M. Equivalence of local- and global-best approximations, a simple stable local commuting projector, and optimal hp approximation estimates in $\boldsymbol{H}(\mathrm{div})$. *IMA J. Numer. Anal.* **42** (2022), 1023–1049. http://dx.doi.org/10.1093/imanum/draa103.

[27] Ern, A., and Guermond, J.-L. Mollification in strongly Lipschitz domains with application to continuous and discrete de Rham complexes. *Comput. Methods Appl. Math.* **16** (2016), 51–75. https://doi.org/10.1515/cmam-2015-0034.

[28] Ern, A., and Guermond, J.-L. Finite element quasi-interpolation and best approximation. *ESAIM Math. Model. Numer. Anal.* **51** (2017), 1367–1385. https://doi.org/10.1051/m2an/2016066.

[29] Ern, A., and Vohralík, M. Stable broken H^1 and $\boldsymbol{H}(\mathrm{div})$ polynomial extensions for polynomial-degree-robust potential and flux reconstruction in three space dimensions. *Math. Comp.* **89** (2020), 551–594. http://dx.doi.org/10.1090/mcom/3482.

[30] Falk, R. S., and Winther, R. Local bounded cochain projections. *Math. Comp.* **83** (2014), 2631–2656. http://dx.doi.org/10.1090/S0025-5718-2014-02827-5.

[31] Fernandes, P., and Gilardi, G. Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Math. Models Methods Appl. Sci.* **7** (1997), 957–991. https://doi.org/10.1142/S0218202597000487.

[32] Gedicke, J., Geevers, S., and Perugia, I. An equilibrated a posteriori error estimator for arbitrary-order Nédélec elements for magnetostatic problems. *J. Sci. Comput.* **83** (2020), Paper No. 58, 23. https://doi.org/10.1007/s10915-020-01224-x.

[33] Gedicke, J., Geevers, S., Perugia, I., and Schöberl, J. A polynomial-degree-robust a posteriori error estimator for Nédélec discretizations of magnetostatic problems. *SIAM J. Numer. Anal.* **59** (2021), 2237–2253. https://doi-org.ezproxy.is.cuni.cz/10.1137/20M1333365.

[34] Girault, V., and Raviart, P.-A. *Finite element methods for Navier-Stokes equations*, vol. **5** of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986.

[35] Grisvard, P. *Elliptic problems in nonsmooth domains*, vol. **24** of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[36] Hiptmair, R., and Pechstein, C. Discrete regular decompositions of tetrahedral discrete 1-forms. In *Maxwell's Equations*, U. Langer, D. Pauly, and S. Repin, Eds. De Gruyter, 2019, ch. 7, pp. 199–258. https://doi.org/10.1515/9783110543612-007.

[37] Melenk, J. M., and Rojik, C. On commuting p-version projection-based interpolation on tetrahedra. *Math. Comp.* **89** (2020), 45–87. https://doi.org/10.1090/mcom/3454.

[38] Monk, P. On the p- and hp-extension of Nédélec's curl-conforming elements. *J. Comput. Appl. Math.* **53** (1994), 117–137. https://doi-org.ezproxy.is.cuni.cz/10.1016/0377-0427(92)00127-U.

[39] Nédélec, J.-C. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* **35** (1980), 315–341.

[40] Payne, L. E., and Weinberger, H. F. An optimal Poincaré inequality for convex domains. *Arch. Rational Mech. Anal.* **5** (1960), 286–292.

[41] Raviart, P.-A., and Thomas, J.-M. A mixed finite element method for 2nd order elliptic problems. In *Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975)*. Springer, Berlin, 1977, pp. 292–315. Lecture Notes in Math., Vol. 606.

[42] Suri, M. On the stability and convergence of higher-order mixed finite element methods for second-order elliptic problems. *Math. Comp.* **54** (1990), 1–19. http://dx.doi.org/10.2307/2008679.

[43] Veeser, A. Approximating gradients with continuous piecewise polynomial functions. *Found. Comput. Math.* **16** (2016), 723–750. http://dx.doi.org/10.1007/s10208-015-9262-z.

[44] Vohralík, M. On the discrete Poincaré–Friedrichs inequalities for nonconforming approximations of the Sobolev space H^1. *Numer. Funct. Anal. Optim.* **26** (2005), 925–952. http://dx.doi.org/10.1080/01630560500444533.