Adaptive regularization, linearization, and algebraic solution in numerical discretizations

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joint work with D. A. Di Pietro, A. Ern, M. F. Wheeler, and S. Yousef

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Outline

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Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A}:\mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*
$$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$
3.
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do 1 algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*
   $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$
   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i+1$, back to 3.2.*
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Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error)

- *How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?*

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Previous results

Inexact Newton method

- Eisenstat and Walker (1990's)
- Moret (1989)

Stopping criteria

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid st. crit.
- Arioli (2000's)

A posteriori error estimates for nonlinear problems

- Ladevèze (since 1990's), guaranteed upper bound
- Han (1994), general framework
- Verfürth (1994), residual estimates
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- Chaillou and Suri (2006, 2007), distinguishing discretization and linearization errors
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Quasi-linear elliptic problem

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$$\begin{aligned} -\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(u,\nabla u) = \underline{\mathbf{A}}(u)\nabla u$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(u,\nabla u) = \underline{\mathbf{A}}(\nabla u)\nabla u$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\nabla u) = |\nabla u|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

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Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla \varphi\|_p = 1} (\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}), \nabla \varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \| [\![u - u_h^{k,i}]\!] \|_{q,e}^q \right\}^{1/q}$$

- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **difference** of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \leq \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

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A posteriori error estimate

Assumption A (Total flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and an algebraic remainder* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla \cdot \mathbf{t}_h^{k,i} = f - \rho_h^{k,i}.$$

Theorem (A posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

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- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

Distinguishing error components

Assumption B (Discretization, linearization, and algebraic errors)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i} = \mathbf{t}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

(iii) *as the nonlinear solver converges,* $\|\mathbf{l}_h^{k,i}\|_q \to 0$.

Comments

- $\mathbf{d}_h^{k,i}$: *discretization flux reconstruction*
- $\mathbf{l}_h^{k,i}$: *linearization error flux reconstruction*
- $\mathbf{a}_h^{k,i}$: *algebraic error flux reconstruction*

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Estimate distinguishing different error components

Theorem (Estimate distinguishing different error components)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumptions A and B hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i} := \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i}.$$

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Estimators

- *discretization estimator*

$$\eta_{\mathrm{disc},K}^{k,i} := 2^{1/p}\left(\|\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}) + \mathbf{d}_h^{k,i}\|_{q,K} + \left\{ \sum_{e\in\mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}} \right)$$

- *linearization estimator*

$$\eta_{\mathrm{lin},K}^{k,i} := \|\mathbf{l}_h^{k,i}\|_{q,K}$$

- *algebraic estimator*

$$\eta_{\mathrm{alg},K}^{k,i} := \|\mathbf{a}_h^{k,i}\|_{q,K}$$

- *algebraic remainder estimator*

$$\eta_{\mathrm{rem},K}^{k,i} := h_\Omega \|\rho_h^{k,i}\|_{q,K}$$

- $\eta_{\cdot}^{k,i} := \left\{ \sum_{K\in\mathcal{T}_h} \left(\eta_{\cdot,K}^{k,i}\right)^q \right\}^{1/q}$

Outline

Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{alg}}^{k,i} \leq \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},$$
$$\eta_{\mathrm{lin}}^{k,i} \leq \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

- $\gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$

Local stopping criteria

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$$\eta_{\mathrm{alg},K}^{k,i} \leq \gamma_{\mathrm{alg},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{lin},K}^{k,i} \leq \gamma_{\mathrm{lin},K} \eta_{\mathrm{disc},K}^{k,i} \qquad \forall K \in \mathcal{T}_h$$

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Assumption for efficiency

Assumption C (Approximation property)

For all $K \in \mathcal{T}_h$, there holds

$$\|\overline{\boldsymbol{\sigma}}_h^{k,i} + \mathbf{d}_h^{k,i}\|_{q,K} \lesssim \eta_{\sharp,\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i},$$

where

$$\eta_{\sharp,\mathfrak{T}_K}^{k,i} := \left\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^q \|f_h + \nabla\cdot\overline{\boldsymbol{\sigma}}_h^{k,i}\|_{q,K'}^q + \sum_{e \in \mathcal{E}_K^{\mathrm{int}}} h_e \|[\![\overline{\boldsymbol{\sigma}}_h^{k,i}\cdot\mathbf{n}_e]\!]\|_{q,e}^q \right.$$
$$\left. + \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}}.$$

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}),$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

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for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

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Outline

Algebraic error flux and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^k U^{k,i} = F^k - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^k U^{k,i+\nu} = F^k - R^{k,i+\nu}.$$

- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

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Example: nonconforming finite elements for the p-Laplacian

Discretization

Find $u_h \in V_h$ such that

$$(\boldsymbol{\sigma}(\nabla u_h), \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $\sigma(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- V_h the Crouzeix–Raviart space
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

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Linearization

Linearization

Find $u_h^k \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f, \psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}.$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2}\boldsymbol{\xi}$$

- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2}\boldsymbol{\xi} + (p-2)|\nabla u_h^{k-1}|^{p-4} (\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\boldsymbol{\xi} - \nabla u_h^{k-1})$$

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Find $u_h^{k,i} \in V_h$ such that

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- algebraic residual vector $R^{k,i} = \{R_e^{k,i}\}_{e \in \mathcal{E}_h^{\mathrm{int}}}$
- discrete system

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Flux reconstructions

Definition (Construction of $\mathbf{d}_h^{k,i}$)

For all $K \in \mathcal{T}_h$,

$$\mathbf{d}_h^{k,i}|_K := -\boldsymbol{\sigma}(\nabla u_h^{k,i})|_K + \frac{f|_K}{d}(\mathbf{x} - \mathbf{x}_K) - \sum_{e \in \mathcal{E}_K} \frac{\bar{R}_e^{k,i}}{d|D_e|}(\mathbf{x} - \mathbf{x}_K)|_{K_e},$$

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Verification of the assumptions

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.

Lemma (Assumption C)

Assumption C holds.

Comments

- $\mathbf{d}_h^{k,i}$ close to $\sigma(\nabla u_h^{k,i})$
- approximation properties of finite element spaces

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Summary

Discretization methods

- nonconforming finite elements
- discontinuous Galerkin
- finite elements
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

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Outline

Numerical experiment I

Model problem

- p-Laplacian

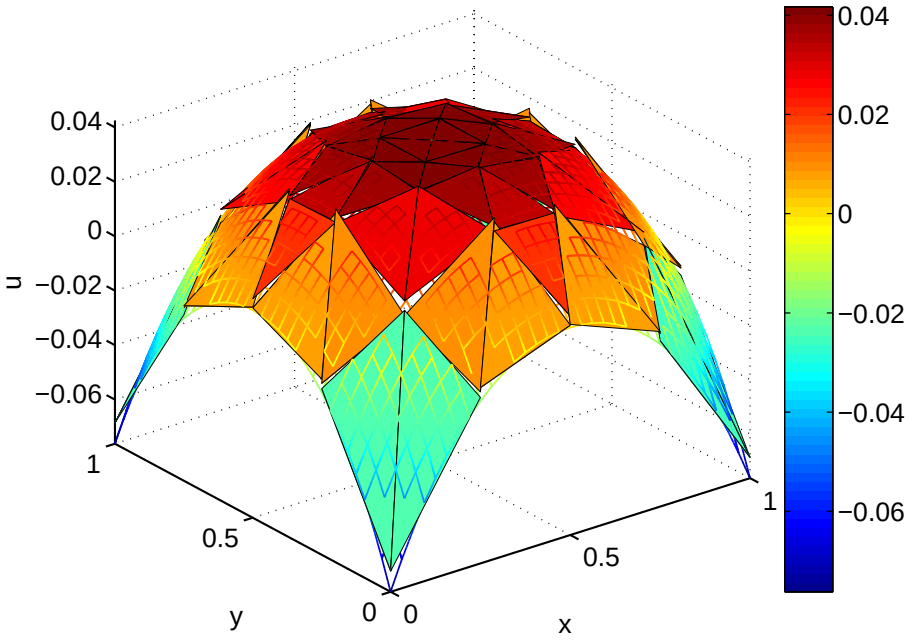
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = u_0 \qquad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

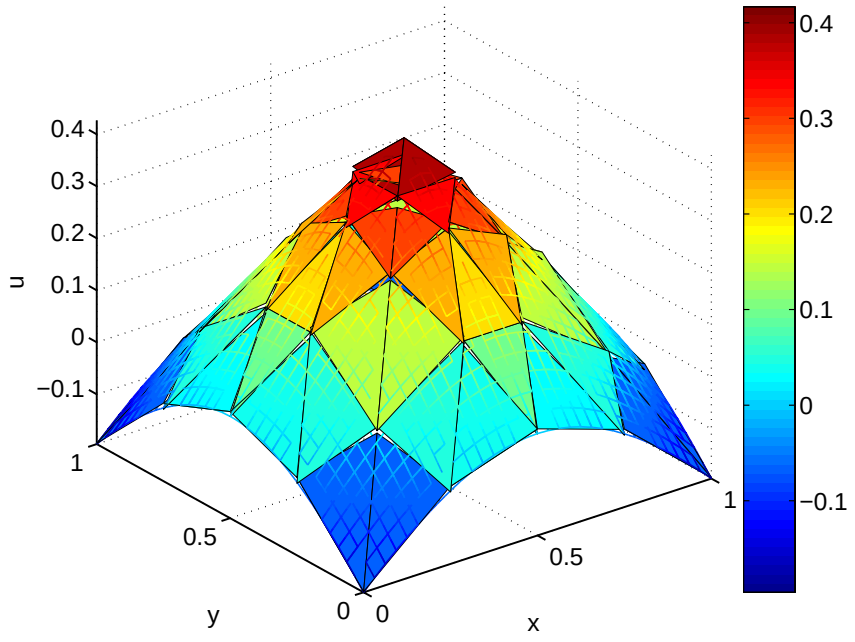
$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested value $p = 10$
- nonconforming finite elements

Analytical and approximate solutions

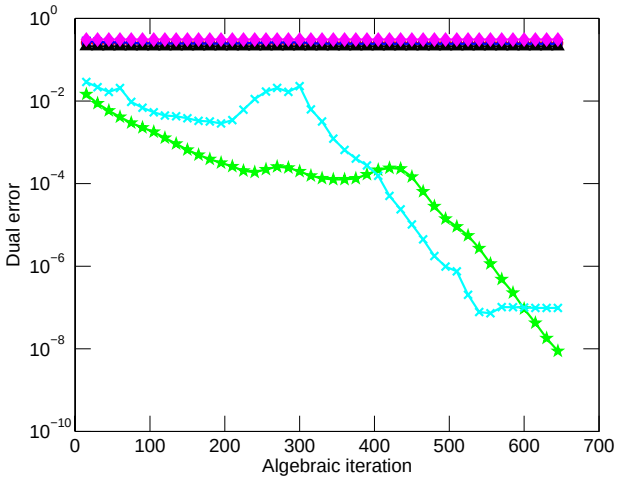


Case $p = 1.5$

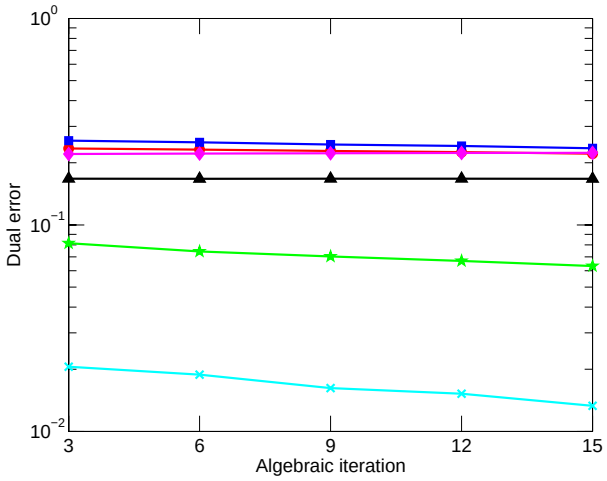


Case $p = 10$

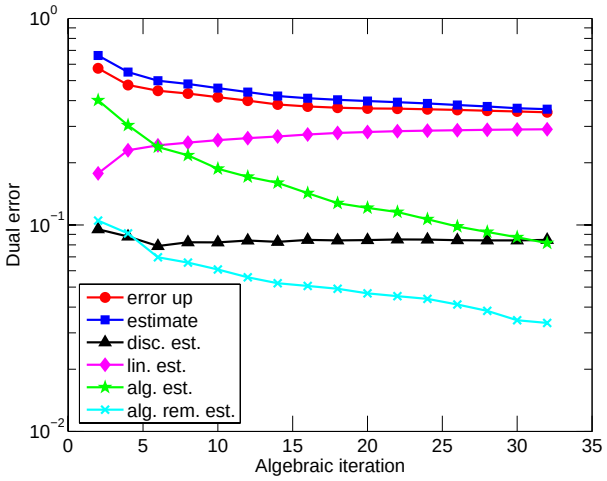
Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.



Newton

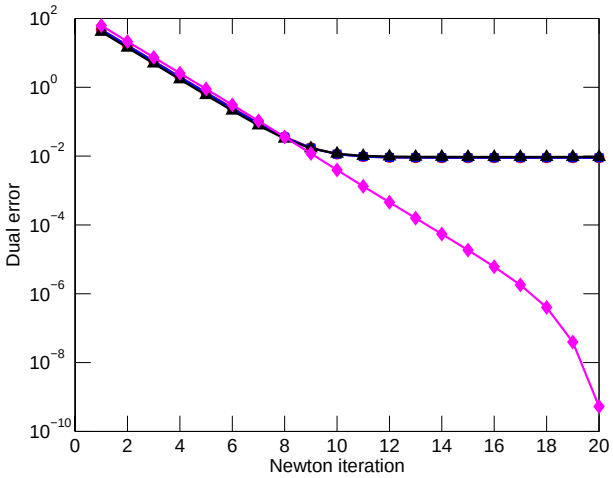


inexact Newton

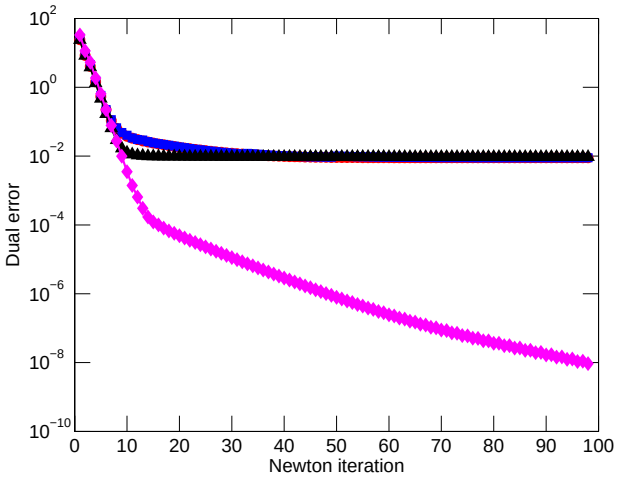


ad. inexact Newton

Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh



Newton



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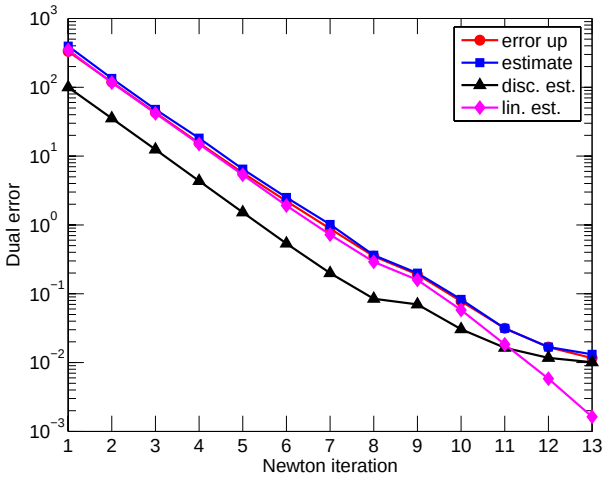
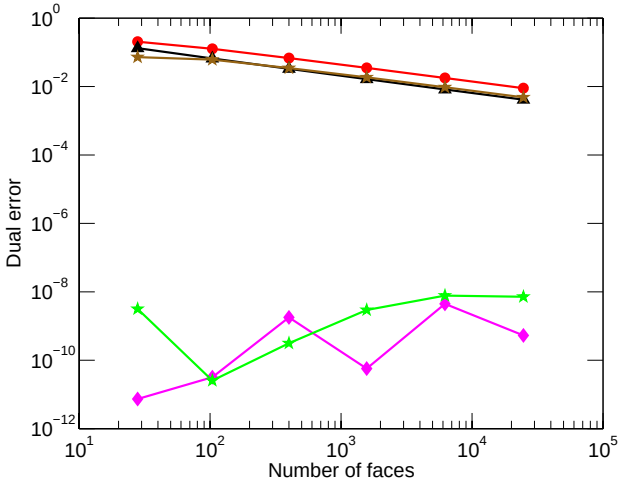
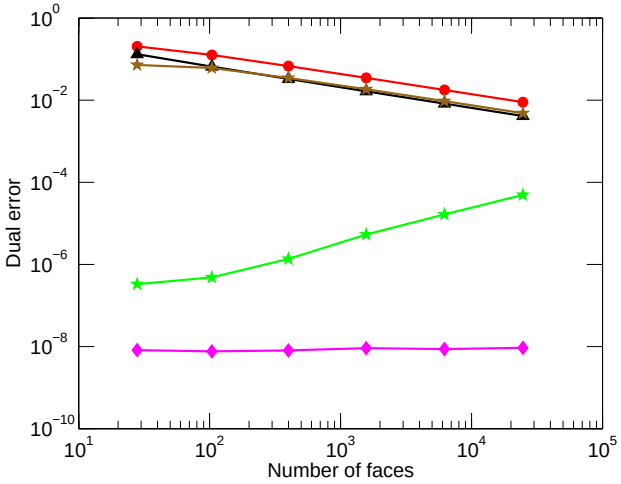


ad. inexact Newton

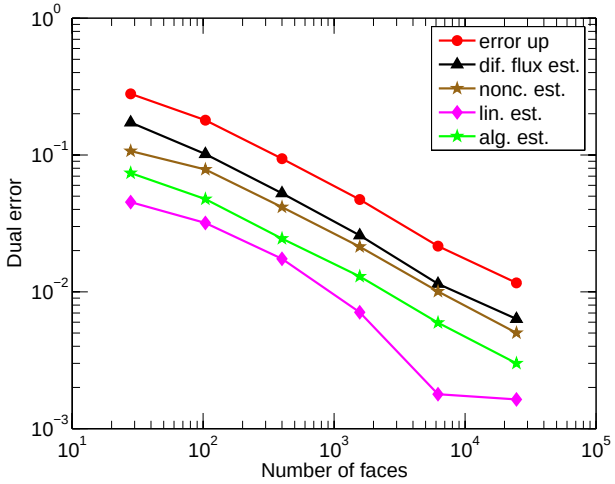
Error and estimators, $p = 10$



Newton

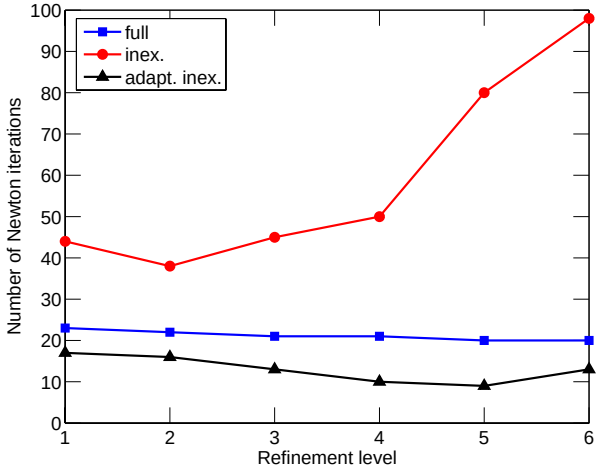


inexact Newton

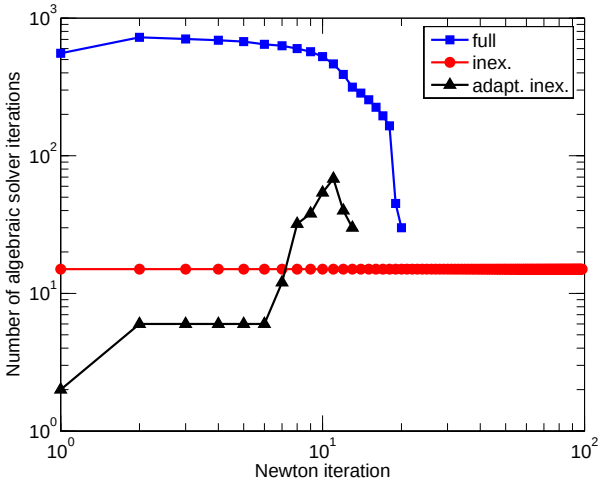


ad. inexact Newton

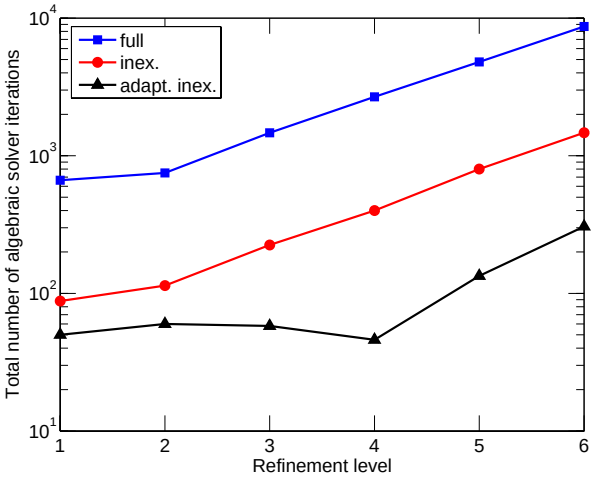
Newton and algebraic iterations, $p = 10$



Newton it. / refinement



alg. it. / Newton step



alg. it. / refinement

Numerical experiment II

Model problem

- p-Laplacian

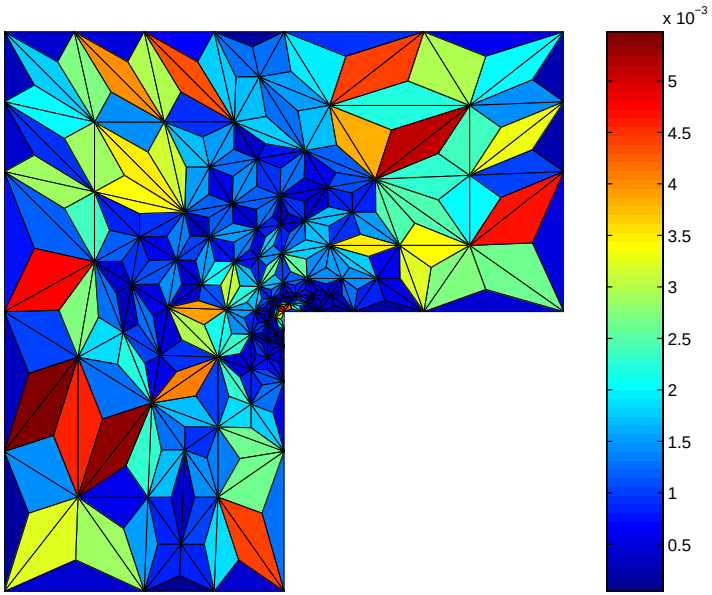
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = u_0 \qquad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

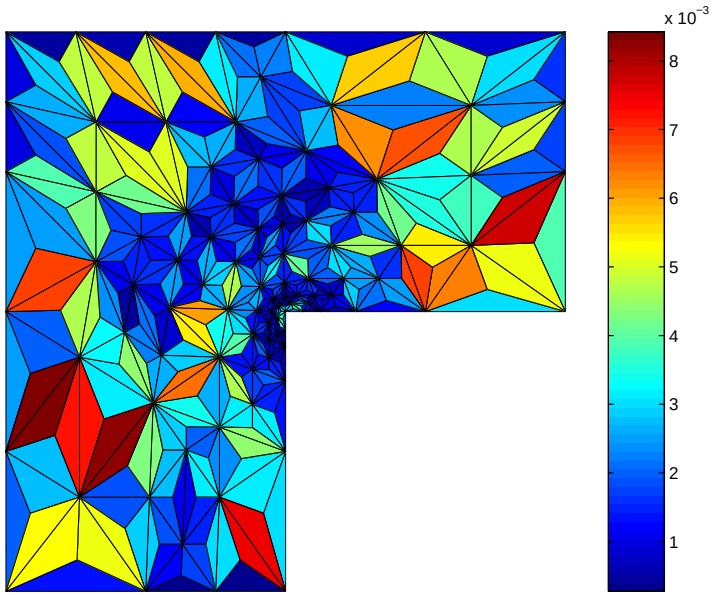
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta\tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- nonconforming finite elements

Error distribution on an adaptively refined mesh

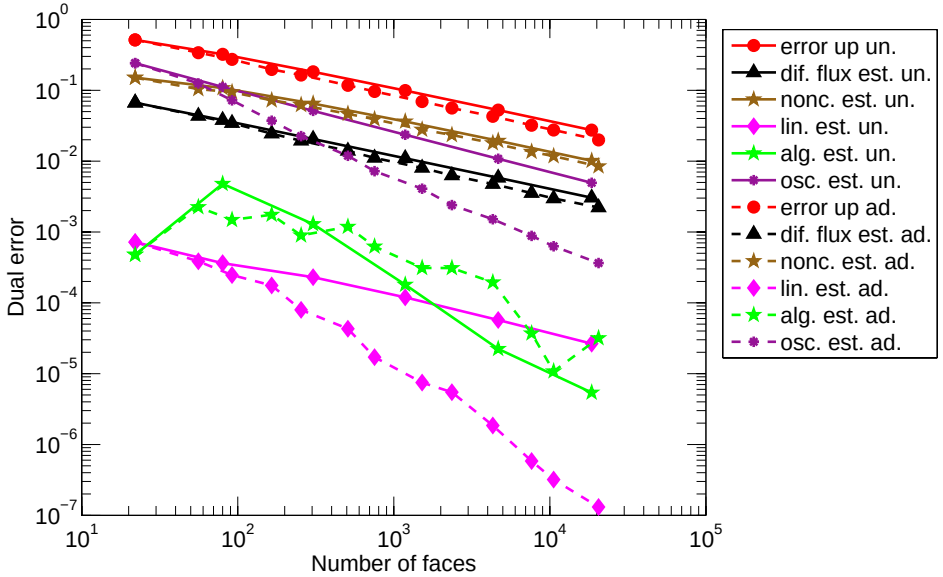


Estimated error distribution

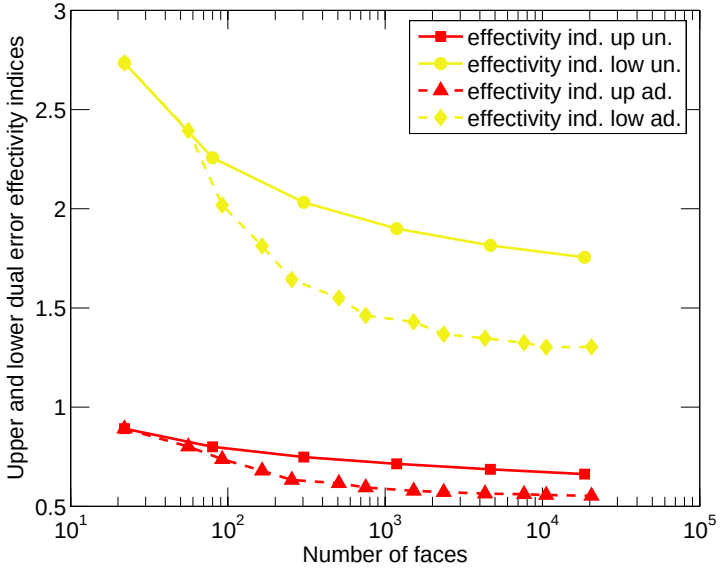


Exact error distribution

Estimated and actual errors and the effectivity index

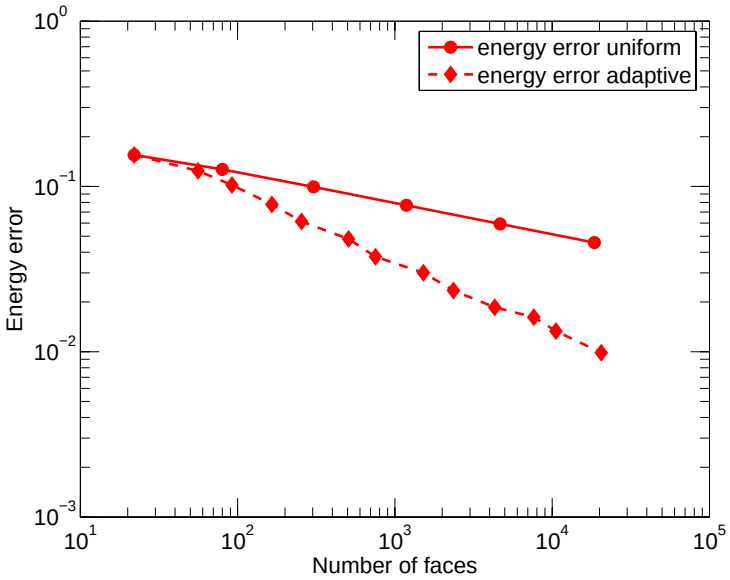


Estimated and actual errors

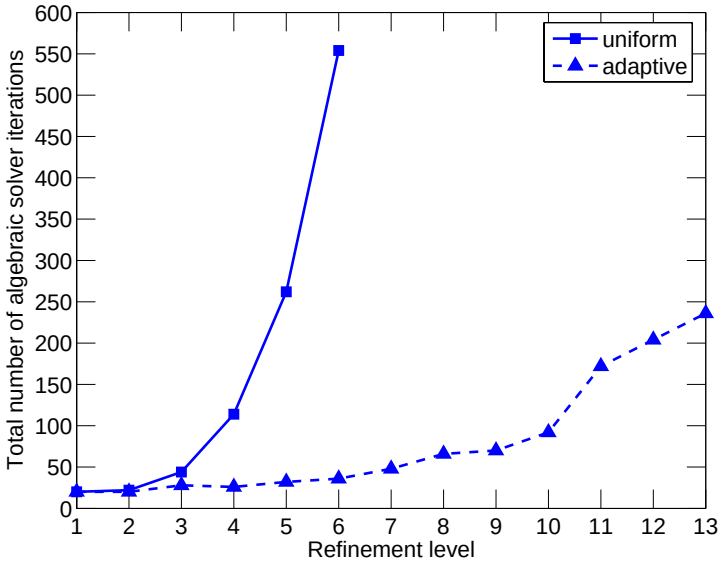


Effectivity index

Energy error and overall performance



Energy error



Overall performance

Outline

The Stefan problem

The Stefan problem

$$\begin{aligned} \partial_t u - \Delta \beta(u) &= f && \text{in } \Omega \times (0,T), \\ u(\cdot,0) &= u_0 && \text{in } \Omega, \\ \beta(u) &= 0 && \text{on } \partial\Omega \times (0,T) \end{aligned}$$

Nomenclature

- u enthalpy, $\beta(u)$ temperature
- β: L_β-Lipschitz continuous, $\beta(s) = 0$ in $(0,1)$, strictly increasing otherwise
- phase change, degenerate parabolic problem
- $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$

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Residual and its dual norm

Functional spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

Weak formulation

$u \in Z$ with $\beta(u) \in X$

$u(\cdot,0) = u_0$ in Ω

$$\langle \partial_t u, \varphi\rangle(t) + (\nabla\beta(u), \nabla\varphi)(t) = (f,\varphi)(t) \quad \forall \varphi \in H_0^1(\Omega) \quad \text{a.e. } t \in (0,T)$$

Residual for $u_{h\tau} \in Z$ such that $\beta(u_{h\tau}) \in X$

$$\langle \mathcal{R}(u_{h\tau}), \varphi\rangle_{X',X} = \int_0^T \{\langle \partial_t(u - u_{h\tau}), \varphi\rangle + (\nabla\beta(u) - \nabla\beta(u_{h\tau}), \nabla\varphi)\}(t)\,\mathrm{d}t,$$

$$\varphi \in X$$

Dual norm of the residual

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi\in X,\, \|\varphi\|_X = 1} \langle \mathcal{R}(u_{h\tau}), \varphi\rangle_{X',X}$$

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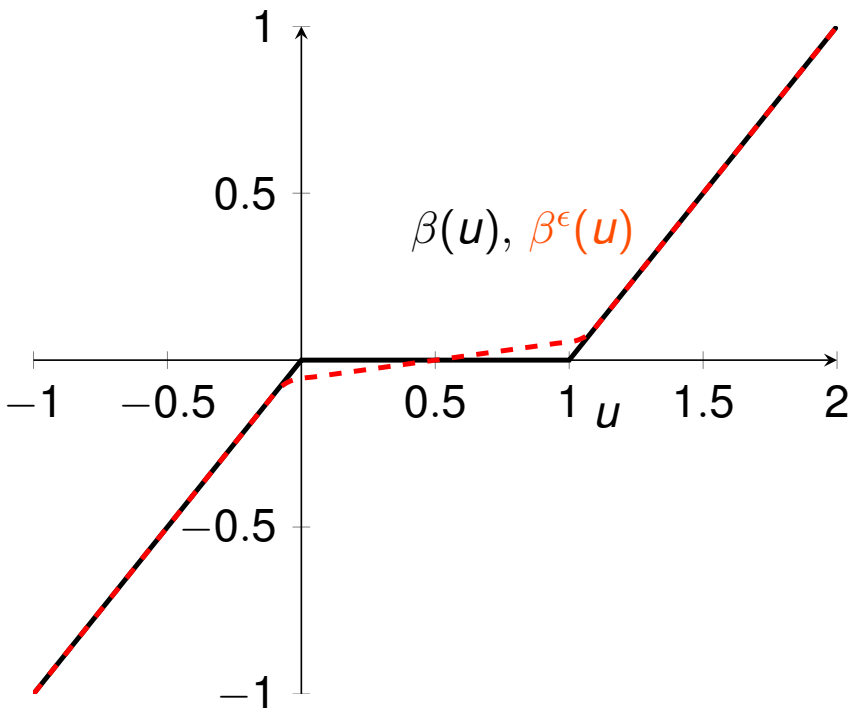
Outline

Practice: regularization

Regularization with a parameter ϵ



Practice: questions

Discretization

- ...

Question (Stopping and balancing criteria)

- *What is a good choice of the*
  - *regularization parameter ϵ?*
  - *time step?*
  - *space mesh?*
- *What is a good stopping criterion for the*
  - *nonlinear solver?*
  - *linear solver?*

Question (Error)

- *How big is the error $\|u|_{I_n} - u_{h\tau}^{n,\epsilon,k,i}\|$ on time step n, space mesh $\mathcal{T}_h^n$, for the regularization parameter ϵ, Newton step k, and algebraic solver step i? How big are the individual components? How is error distributed in time and space?*

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Outline

A posteriori estimate and its efficiency

Theorem (Estimate and its efficiency)

There holds

$$\|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)}$$
$$\leq \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} \left(\eta_{\mathrm{R},K}^n + \eta_{\mathrm{F},K}^n\right)^2 \right\}^{\frac{1}{2}} + \eta_{\mathrm{IC}}$$
$$\lesssim \|\mathcal{R}(u_{h\tau})\|_{X'} + \|u_0 - u_{h\tau}(0)\|_{H^{-1}(\Omega)},$$

with

$$\eta_{\mathrm{R},K}^n := C_{\mathrm{P},K} h_K \|f^n - \partial_t u_{h\tau} - \nabla \cdot \mathbf{t}_h^n\|_K,$$
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Theorem (An estimate distinguishing the error components)

For time n, linearization k, and regularization ϵ, there holds

$$\|\mathcal{R}(u_{h\tau}^{n,\epsilon,k})\|_{X',I_n} \leq \eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{lin}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k}.$$

A posteriori estimate and its efficiency

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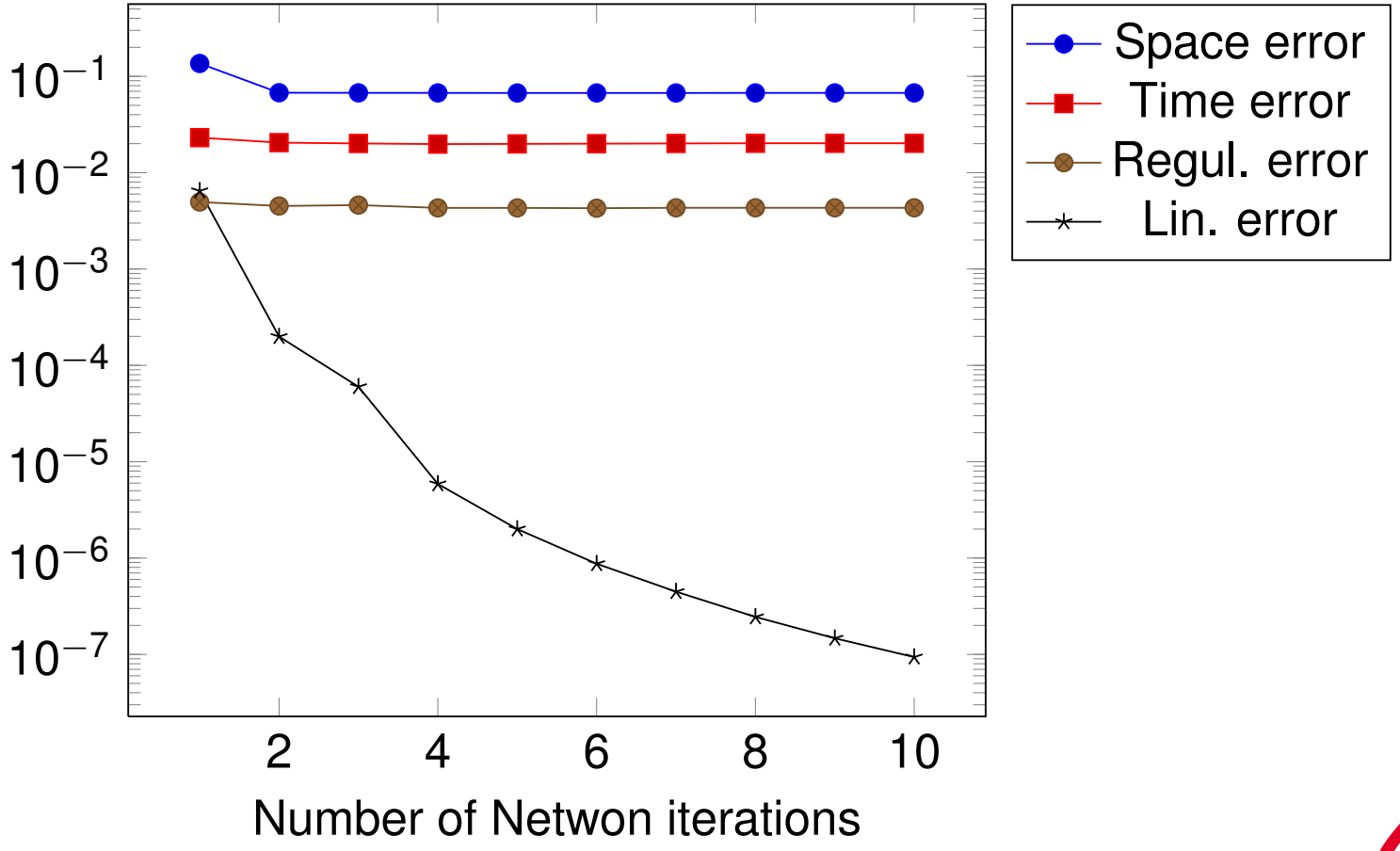
Outline

Linearization stopping criterion

Linearization stopping criterion

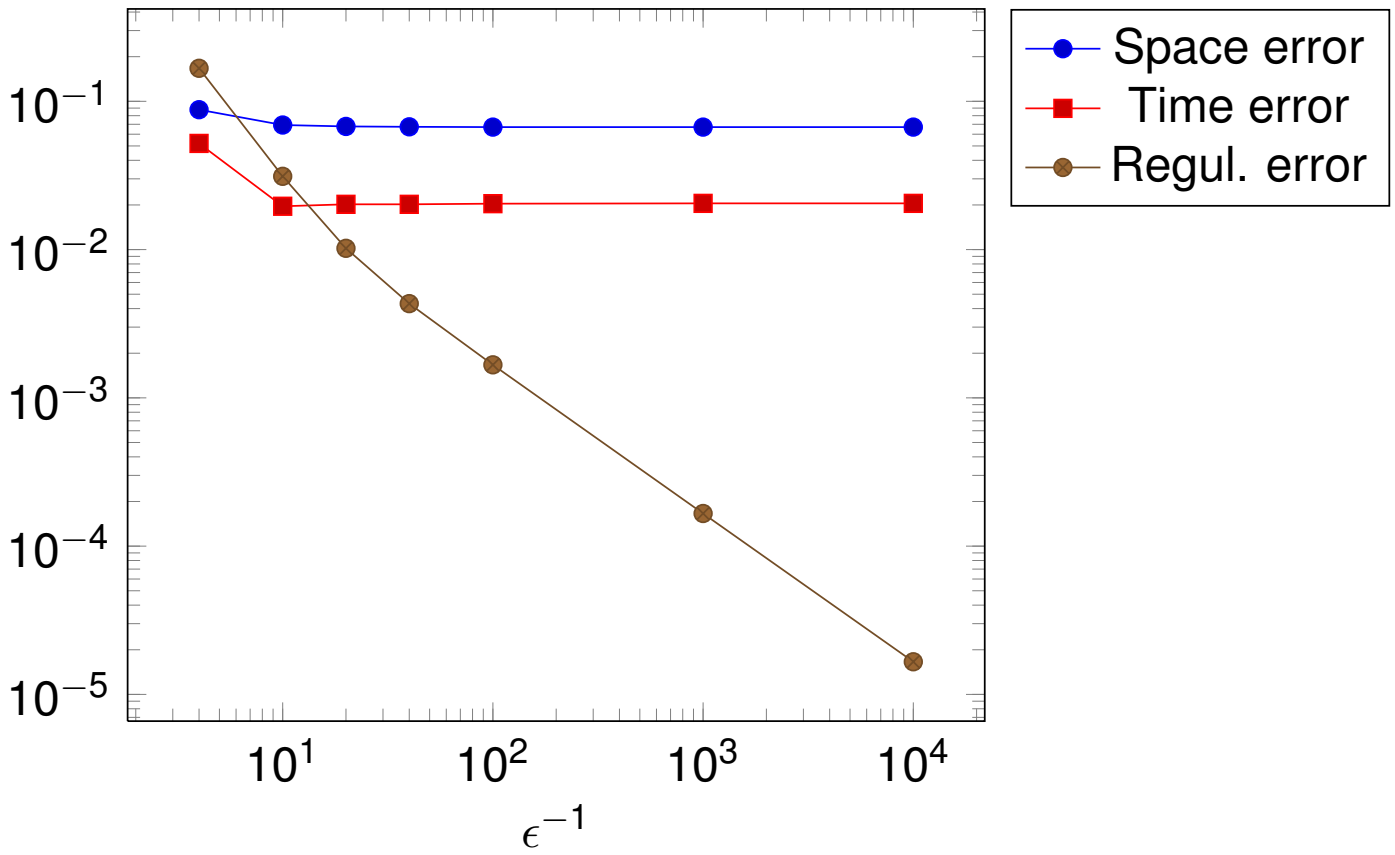
$$\eta_{\mathrm{lin}}^{n,\epsilon,k} \leq \gamma_{\mathrm{lin}} \left(\eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k} \right)$$



Regularization stopping criterion

Regularization stopping criterion

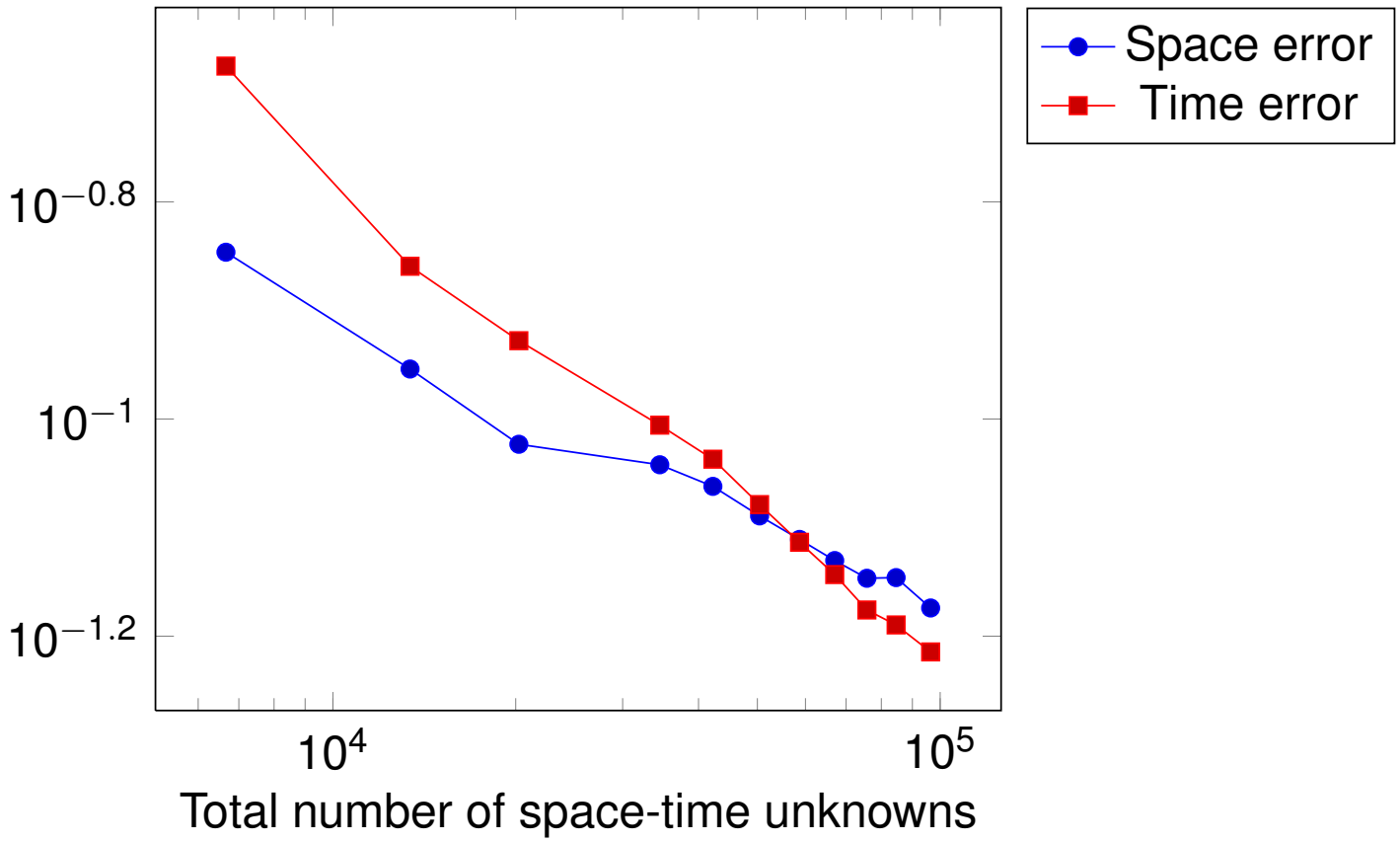
$$\eta_{\mathrm{reg}}^{n,\epsilon,k} \leq \gamma_{\mathrm{reg}} \left(\eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} \right)$$



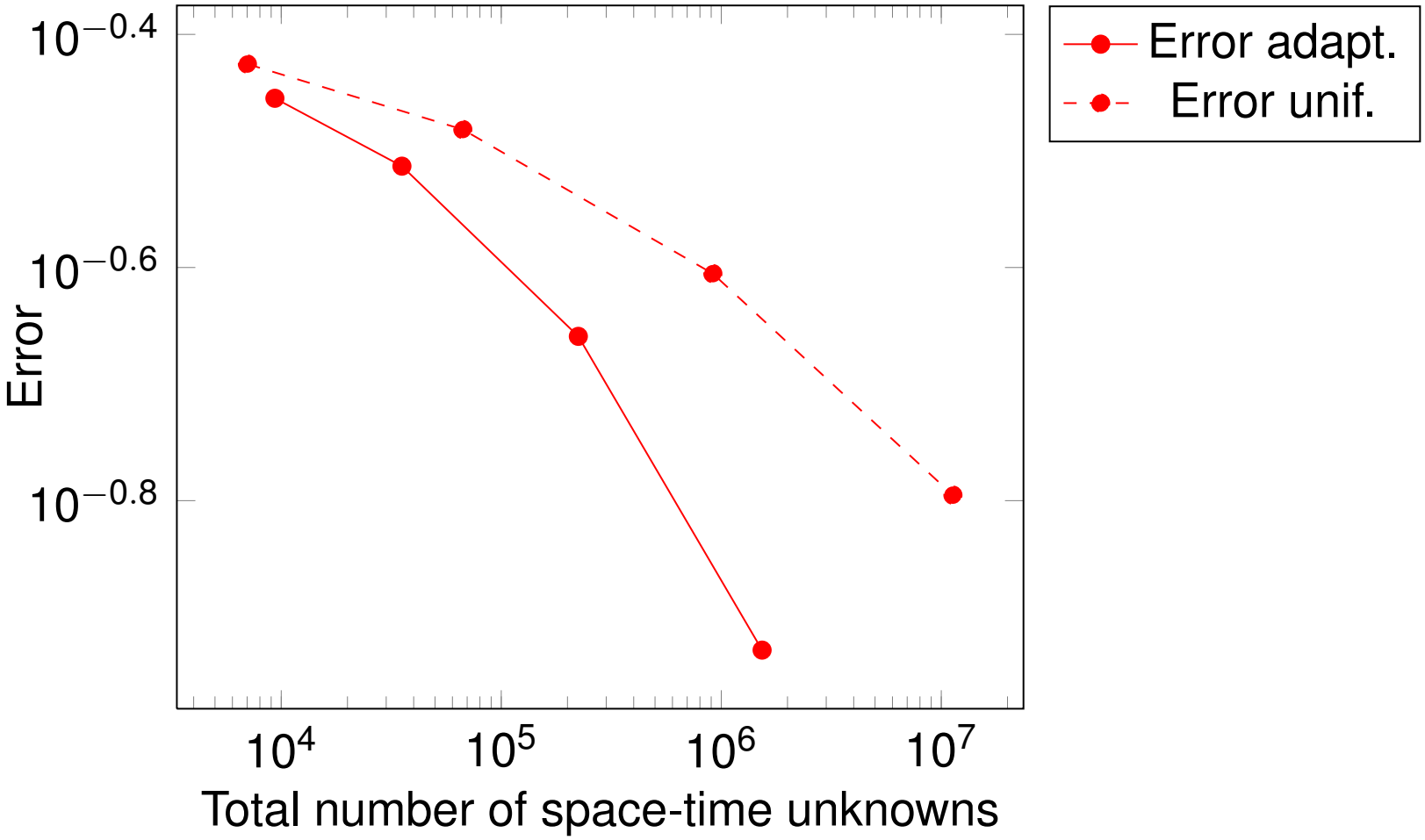
Equilibrating time and space errors

Equilibrating time and space errors

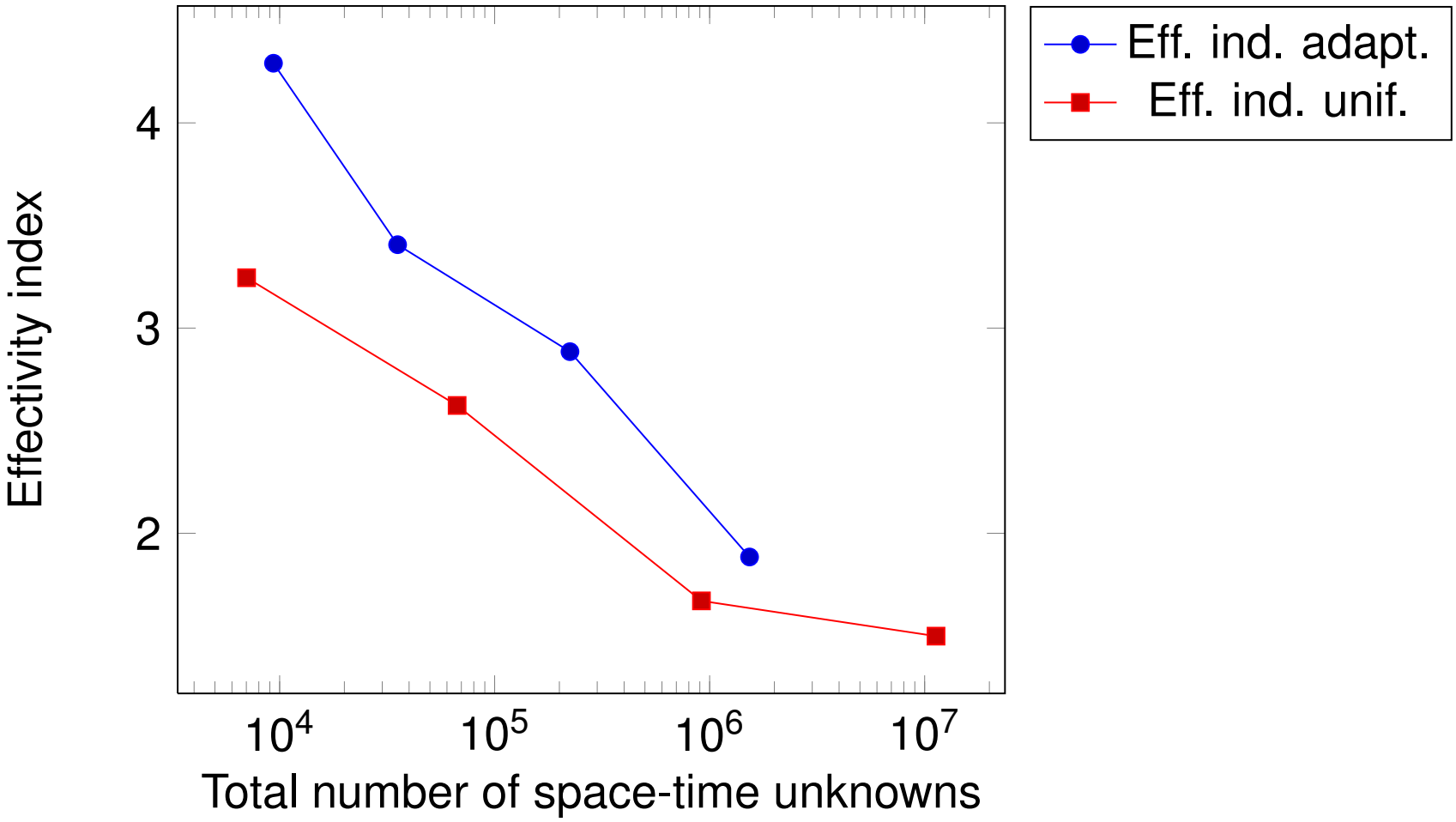
$$\gamma_{\min}\eta_{\mathrm{sp}}^{\epsilon,n,k} \leq \eta_{\mathrm{tm}}^{\epsilon,n,k} \leq \gamma_{\max}\eta_{\mathrm{sp}}^{\epsilon,n,k}$$



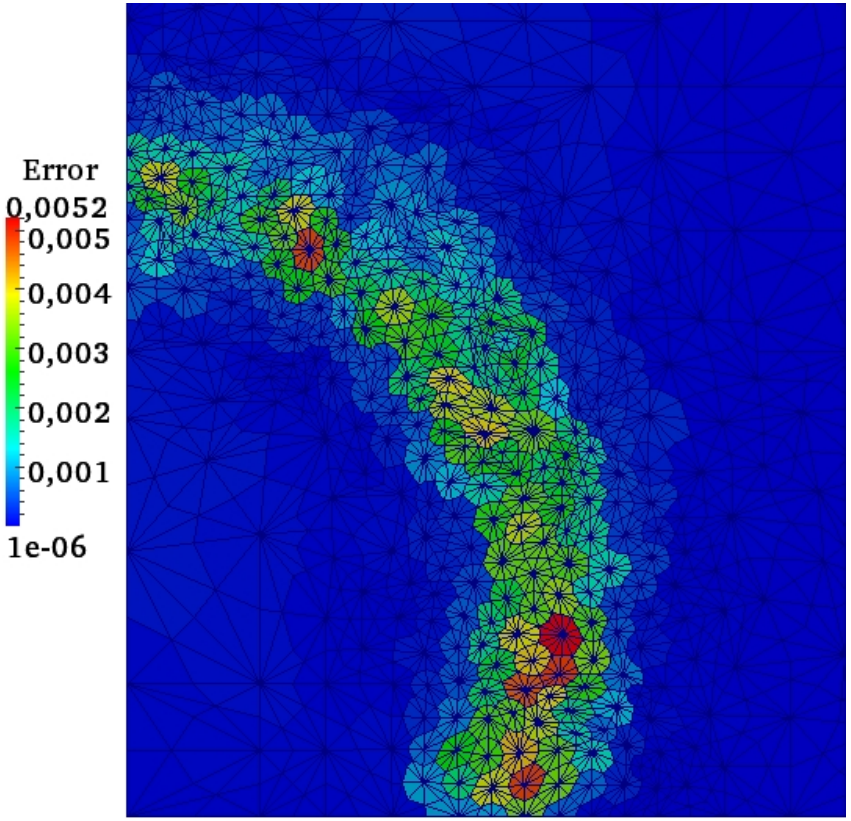
Error and estimate

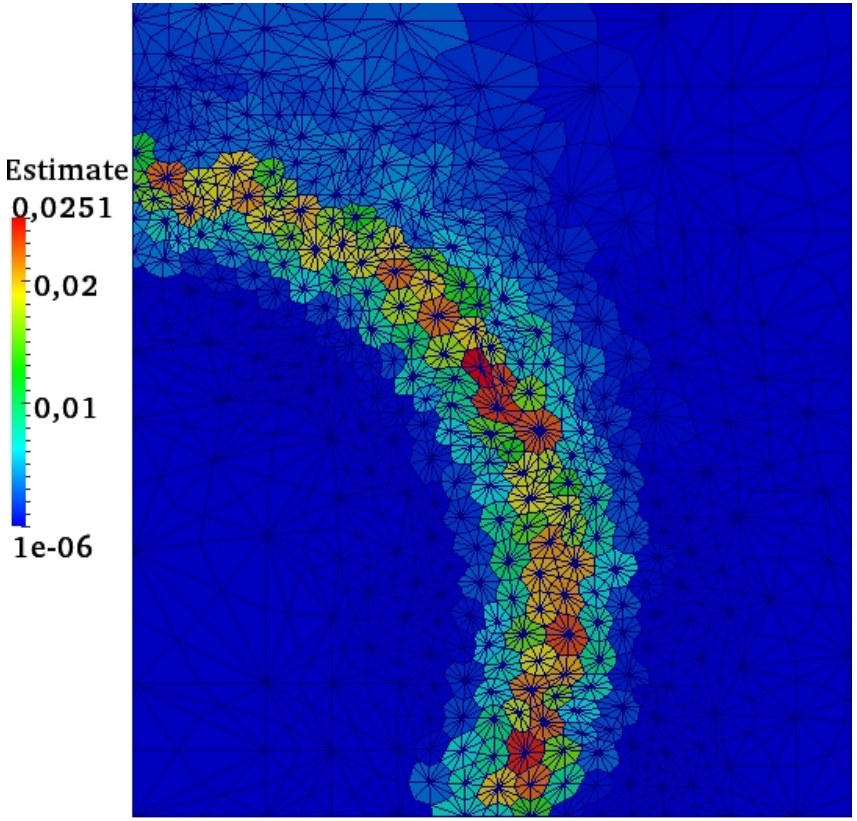


Effectivity indices



Actual and estimated error distribution





Computational efficiency

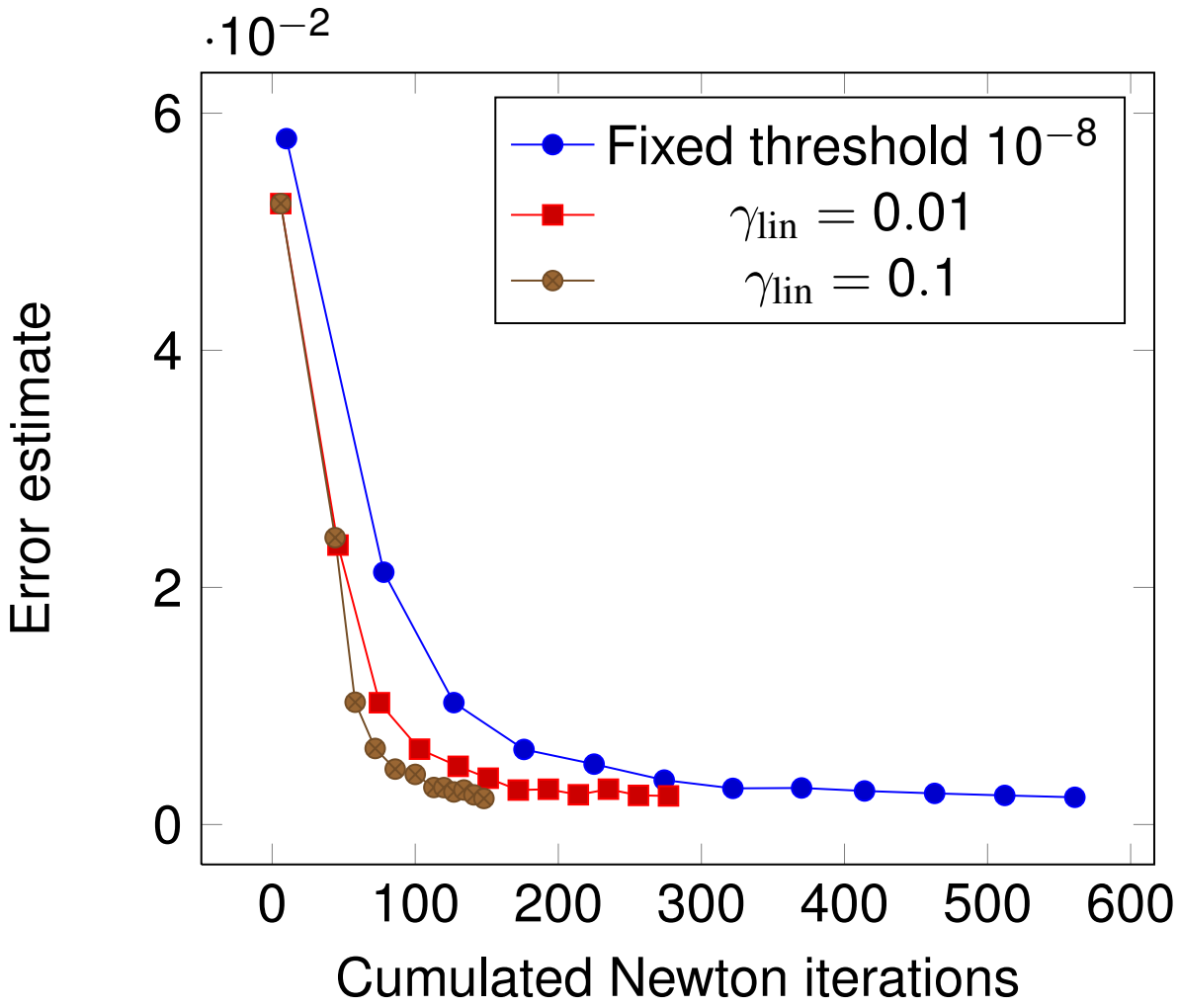


Figure: Number of cumulated Newton iterations vs. error estimate

Outline

Two-phase flow in porous media

Two-phase flow in porous media

$$\partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha = q_\alpha, \qquad \alpha \in \{\mathrm{n}, \mathrm{w}\},$$

$$-\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z) = \mathbf{u}_\alpha, \qquad \alpha \in \{\mathrm{n}, \mathrm{w}\},$$

$$s_\mathrm{n} + s_\mathrm{w} = 1,$$

$$p_\mathrm{n} - p_\mathrm{w} = p_c(s_\mathrm{w})$$

Mathematical issues

- coupled system
- unsteady, nonlinear
- elliptic–parabolic degenerate type
- dominant advection

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Outline

Two-phase flow in porous media

Theorem (A posteriori error estimate distinguishing the error components)

Let

- *n be the time step,*
- *k be the linearization step,*
- *i be the algebraic solver step,*

with the approximations $(s^{k,i}_{\alpha,h\tau}, p^{k,i}_{\alpha,h\tau})$. Then

$$|||(s_\alpha - s^{k,i}_{\alpha,h\tau}, p_\alpha - p^{k,i}_{\alpha,h\tau})|||_{I_n} \leq \eta^{n,k,i}_{\mathrm{sp},\alpha} + \eta^{n,k,i}_{\mathrm{tm},\alpha} + \eta^{n,k,i}_{\mathrm{lin},\alpha} + \eta^{n,k,i}_{\mathrm{alg},\alpha}.$$

Error components

- $\eta^{n,k,i}_{\mathrm{sp},\alpha}$: spatial discretization
- $\eta^{n,k,i}_{\mathrm{tm},\alpha}$: temporal discretization
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Outline

Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \leq n \leq N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

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where the fluxes are given by

$$F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,w}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,w}}(s^n_{\mathrm{w},K'})}{2} |\underline{\mathbf{K}}| \frac{\bar{p}^n_{\mathrm{w},K'} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_{K'}|} |e_{KK'}|,$$

$$F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K'})}{2} |\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},K'} + \pi(s^n_{\mathrm{w},K'}) - (\bar{p}^n_{\mathrm{n},K} + \pi(s^n_{\mathrm{n},K}))}{|\mathbf{x}_K - \mathbf{x}_{K'}|} |e_{KK'}|$$

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Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$\begin{aligned} F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) := & F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \\ & + \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) \\ & + \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}). \end{aligned}$$

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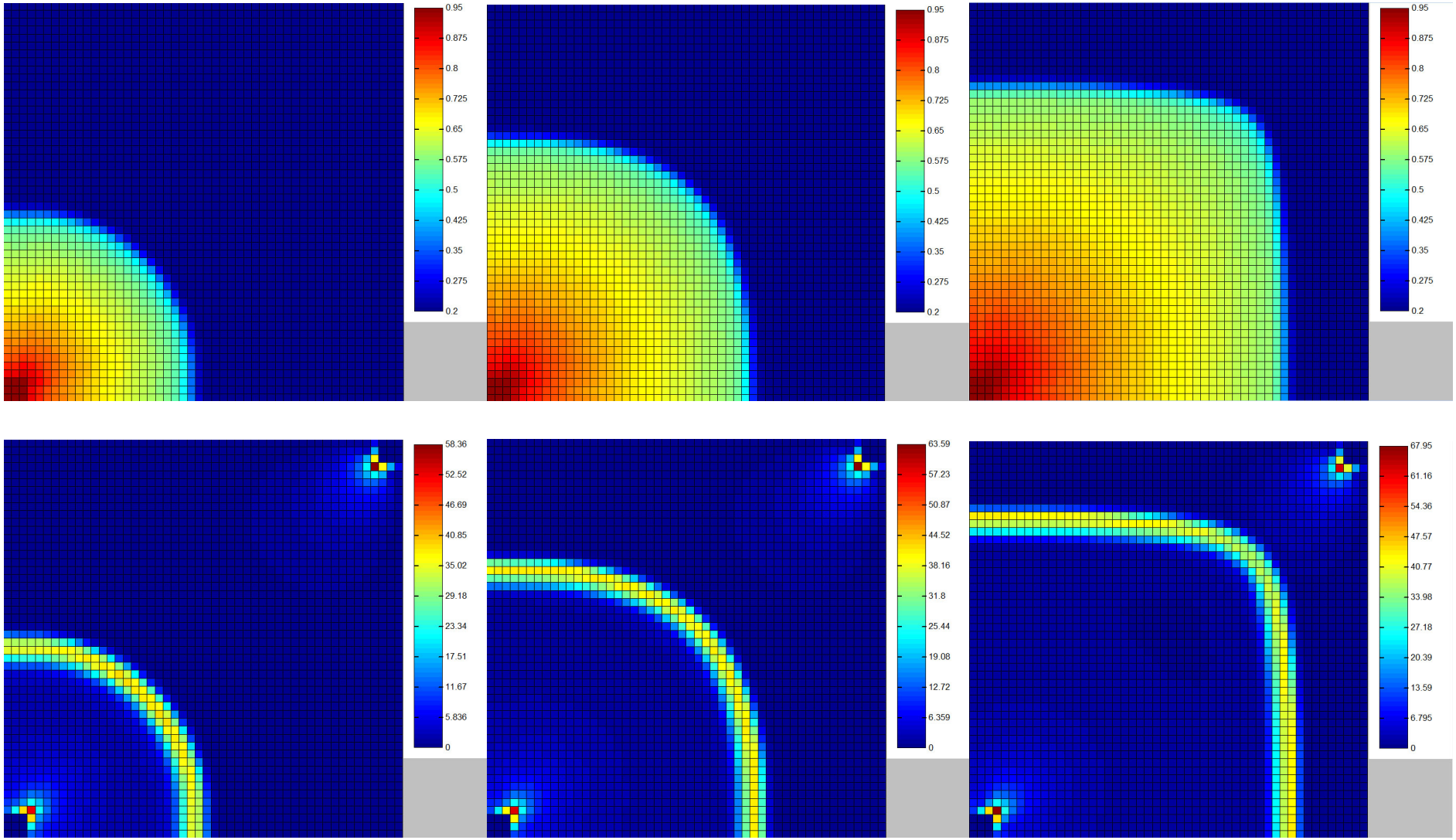
Fluxes reconstructions and pressure postprocessing

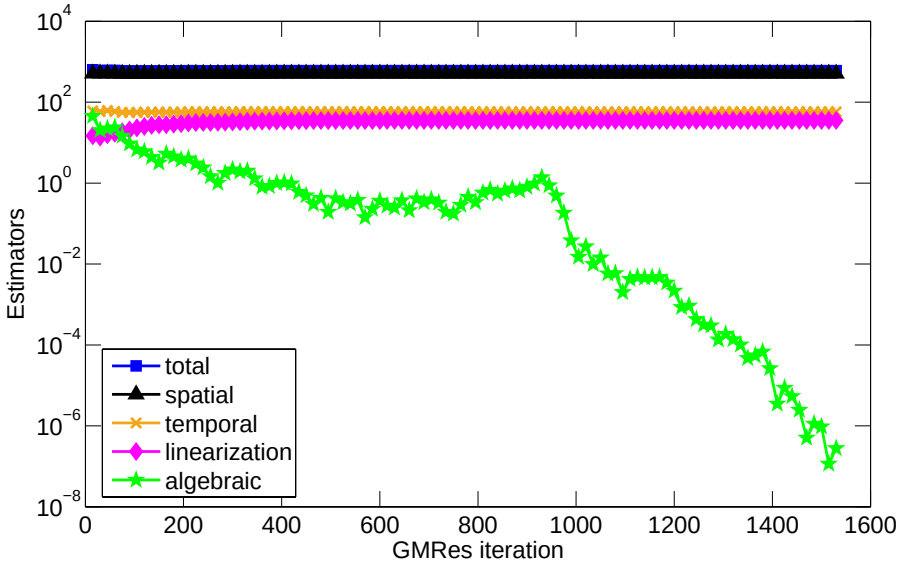
Fluxes reconstructions

$$
\begin{aligned}
(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K,1)_{e_{KK'}} &:= F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),\\
((\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K,1)_{e_{KK'}} &:= F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),\\
\mathbf{a}_{\alpha,h}^{n,k,i} &:= \mathbf{d}_{\alpha,h}^{n,k,i+\nu}+\mathbf{l}_{\alpha,h}^{n,k,i+\nu}-(\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})
\end{aligned}
$$

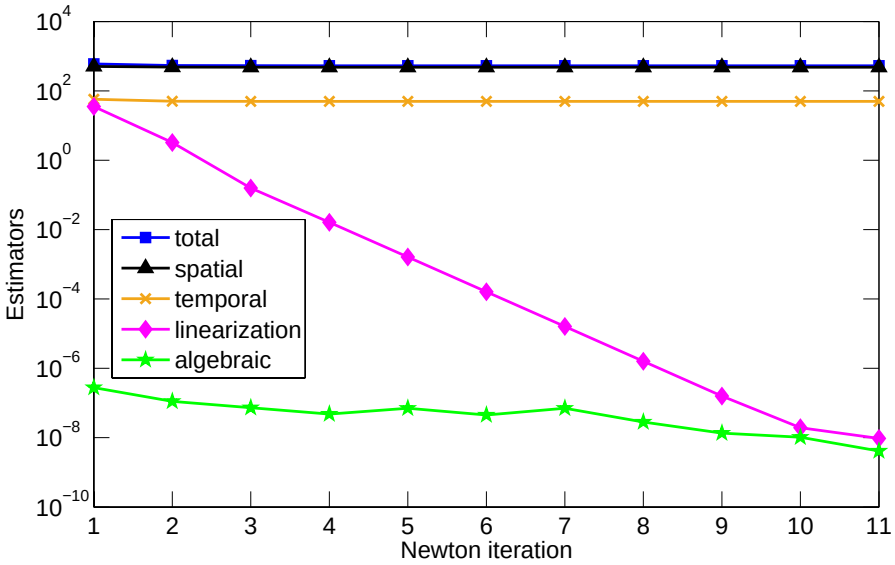
Water saturation/estimators evolution



Estimators and stopping criteria

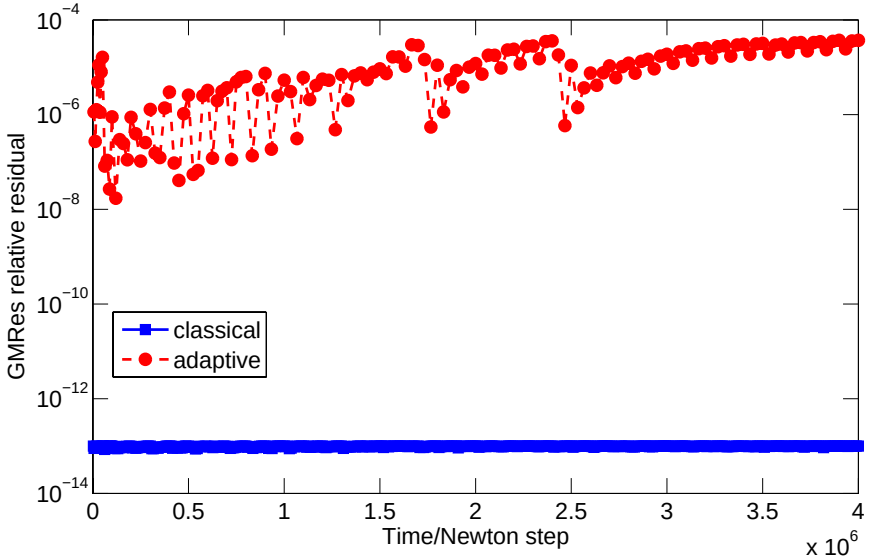


Estimators in function of GMRes iterations

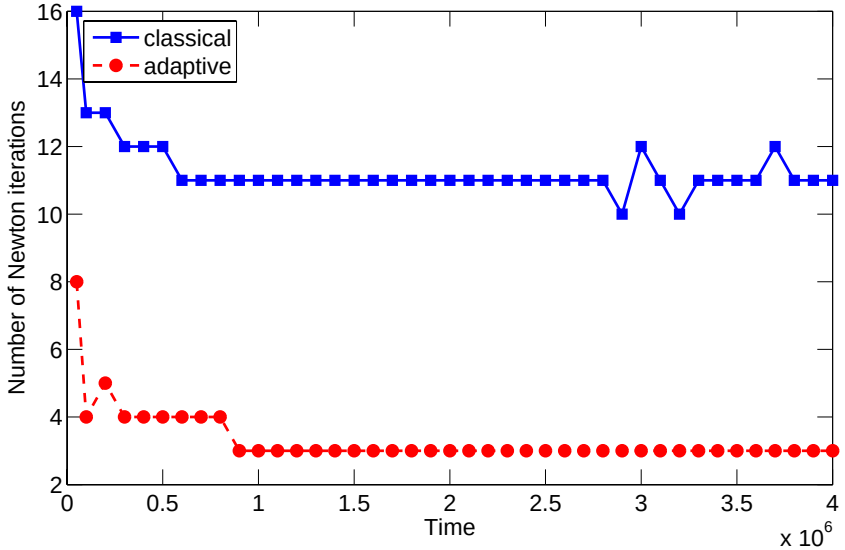


Estimators in function of Newton iterations

GMRes relative residual/Newton iterations

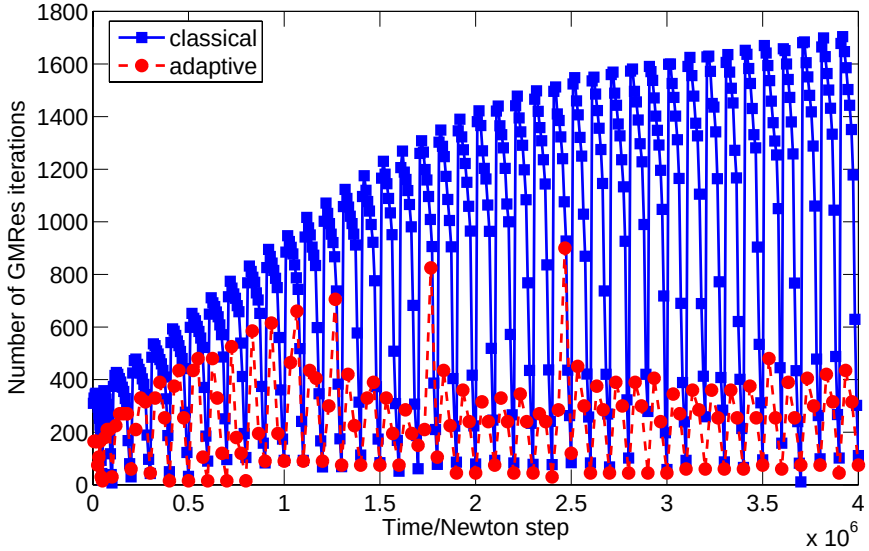


GMRes relative residual

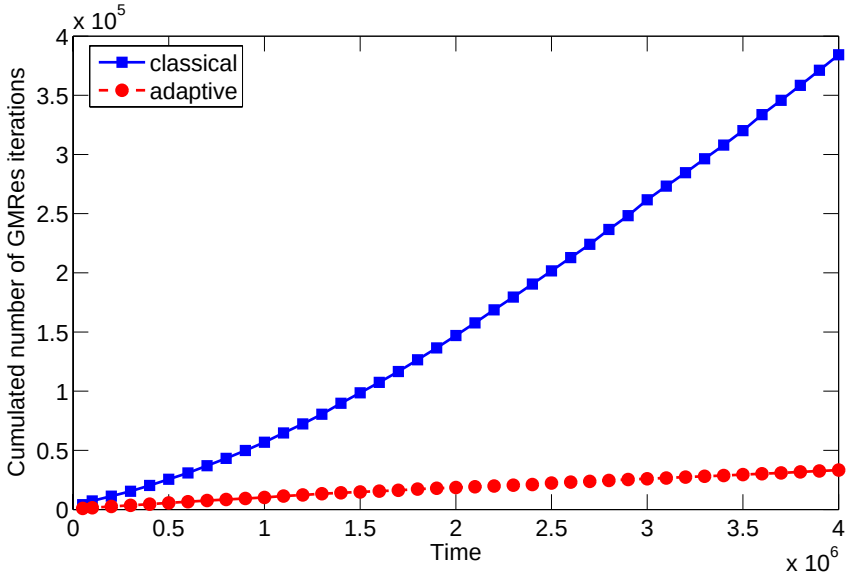


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Outline

Conclusions

Complete adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- **optimal** choice of the **regularization parameter**
- space-time mesh adaptivity
- important **computational savings**
- guaranteed and robust upper bound via **a posteriori error estimates**

Future directions

- other coupled nonlinear systems
- convergence and optimality

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Thank you for your attention!