Adaptivita pro lineární a nelineární řešiče a výběr časového kroku a prostorové sítě v numerických diskretizacích

Martin Vohralík

INRIA Paris-Rocquencourt

Seminář numerické analýzy, Rožnov pod Radhoštěm,
21.–25. ledna 2013

Outline

Outline

Numerical approximation of a nonlinear, unsteady PDE

Exact and approximate solution

- let u be the weak solution of $A(u) = f$, A **nonlinear**, **unsteady** partial differential equation (PDE) on $\Omega \times (0, T)$
- let $u_{h\tau}$ be its approximate numerical solution, $\mathcal{A}_{h\tau}(u_{h\tau}) = F_{h\tau}$

Solution algorithm

- introduce a temporal mesh of $(0, T)$ given by t^n, $0 \leq n \leq N$
- introduce a spatial mesh $\mathcal{T}_h^n$ of Ω on each t^n
- on each t^n and $\mathcal{T}_h^n$, solve a system of nonlinear algebraic equations $\mathcal{A}_h^n(u_h^n) = F_h^n$

Numerical approximation of a nonlinear, unsteady PDE

Exact and approximate solution

- let u be the weak solution of $A(u) = f$, A **nonlinear**, **unsteady** partial differential equation (PDE) on $\Omega \times (0, T)$
- let $u_{h\tau}$ be its approximate numerical solution, $\mathcal{A}_{h\tau}(u_{h\tau}) = F_{h\tau}$

Solution algorithm

- introduce a temporal mesh of $(0, T)$ given by t^n, $0 \leq n \leq N$
- introduce a spatial mesh $\mathcal{T}_h^n$ of Ω on each t^n
- on each t^n and $\mathcal{T}_h^n$, solve a system of nonlinear algebraic equations $\mathcal{A}_h^n(u_h^n) = F_h^n$

Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
- **when do we stop?**

Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
- **when do we stop?**

Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
- **when do we stop?**

Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
- **when do we stop?**

Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
- **when do we stop?**

Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
- **when do we stop?**

Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
- **when do we stop?**

Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

Iterative solvers and space and time steps choice

Iterative linearization of $\mathcal{A}_h^n(u_h^n) = F_h^n$ on each t^n

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$: discrete iterative linearization (Newton, fixed-point)
- loop in k
- **when do we stop?**

Iterative algebraic solver on each t^n and for each k

- $\mathbb{A}_h^{n,k-1} u_h^{n,k} = F_h^{n,k-1}$ is a linear algebraic system
- we only solve it inexactly by some iterative algebraic solver: loop in i
- **when do we stop?**

Temporal mesh

- choice of the discrete times t^n?

Spatial mesh

- choice of the meshes $\mathcal{T}_h^n$?

Optimal solution strategy

Optimal solution strategy

- give a **guaranteed** and **robust** upper bound on the overall error $\|u - u_{h\tau}^{k,i}\|_{\Omega\times(0,T)}$, as tight as possible
- **distinguish** the algebraic, linearization, temporal, and spatial error **components**
- **stop** the **iterative solvers** whenever the corresponding errors do not affect the overall error significantly
- **refine/derefine adaptively** the time and space **meshes** and **equilibrate** the space and time **errors**

Benefits

- ***optimal computable overall error bound***
- ***important computational savings***
- ***improvement of approximation precision***

Optimal solution strategy

Optimal solution strategy

- give a **guaranteed** and **robust** upper bound on the overall error $\|u - u_{h\tau}^{k,i}\|_{\Omega\times(0,T)}$, as tight as possible
- **distinguish** the algebraic, linearization, temporal, and spatial error **components**
- **stop** the **iterative solvers** whenever the corresponding errors do not affect the overall error significantly
- **refine/derefine adaptively** the time and space **meshes** and **equilibrate** the space and time **errors**

Benefits

- **optimal computable overall error bound**
- **important computational savings**
- **improvement of approximation precision**

Optimal solution strategy

Optimal solution strategy

- give a **guaranteed** and **robust** upper bound on the overall error $\|u - u_{h\tau}^{k,i}\|_{\Omega\times(0,T)}$, as tight as possible
- **distinguish** the algebraic, linearization, temporal, and spatial error **components**
- **stop** the **iterative solvers** whenever the corresponding errors do not affect the overall error significantly
- **refine/derefine adaptively** the time and space **meshes** and **equilibrate** the space and time **errors**

Benefits

- **optimal computable overall error bound**
- **important computational savings**
- **improvement of approximation precision**

Optimal solution strategy

Optimal solution strategy

- give a **guaranteed** and **robust** upper bound on the overall error $\|u - u_{h\tau}^{k,i}\|_{\Omega\times(0,T)}$, as tight as possible
- **distinguish** the algebraic, linearization, temporal, and spatial error **components**
- **stop** the **iterative solvers** whenever the corresponding errors do not affect the overall error significantly
- **refine/derefine adaptively** the time and space **meshes** and **equilibrate** the space and time **errors**

Benefits

- **optimal computable overall error bound**
- **important computational savings**
- **improvement of approximation precision**

Optimal solution strategy

Optimal solution strategy

- give a **guaranteed** and **robust** upper bound on the overall error $\|u - u_{h\tau}^{k,i}\|_{\Omega\times(0,T)}$, as tight as possible
- **distinguish** the algebraic, linearization, temporal, and spatial error **components**
- **stop** the **iterative solvers** whenever the corresponding errors do not affect the overall error significantly
- **refine/derefine adaptively** the time and space **meshes** and **equilibrate** the space and time **errors**

Benefits

- **optimal computable overall error bound**
- **important computational savings**
- **improvement of approximation precision**

Optimal solution strategy

Optimal solution strategy

- give a **guaranteed** and **robust** upper bound on the overall error $\|u - u_{h\tau}^{k,i}\|_{\Omega\times(0,T)}$, as tight as possible
- **distinguish** the algebraic, linearization, temporal, and spatial error **components**
- **stop** the **iterative solvers** whenever the corresponding errors do not affect the overall error significantly
- **refine/derefine adaptively** the time and space **meshes** and **equilibrate** the space and time **errors**

Benefits

- **optimal computable overall error bound**
- **important computational savings**
- **improvement of approximation precision**

Optimal solution strategy

Optimal solution strategy

- give a **guaranteed** and **robust** upper bound on the overall error $\|u - u_{h\tau}^{k,i}\|_{\Omega\times(0,T)}$, as tight as possible
- **distinguish** the algebraic, linearization, temporal, and spatial error **components**
- **stop** the **iterative solvers** whenever the corresponding errors do not affect the overall error significantly
- **refine/derefine adaptively** the time and space **meshes** and **equilibrate** the space and time **errors**

Benefits

- ***optimal computable overall error bound***
- ***important computational savings***
- ***improvement of approximation precision***

Previous results

Steady problems

- Babuška and Rheinboldt (1978), introduction of a posteriori estimates
- Ladevèze and Leguillon (1983), equilibrated fluxes estimates (equality of Prager and Synge (1947))
- Verfürth (1996), residual-based estimates
- Ainsworth (2005), nonconforming methods

Unsteady problems

- Bieterman and Babuška (1982), introduction
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction

Previous results

Steady problems

- Babuška and Rheinboldt (1978), introduction of a posteriori estimates
- Ladevèze and Leguillon (1983), equilibrated fluxes estimates (equality of Prager and Synge (1947))
- Verfürth (1996), residual-based estimates
- Ainsworth (2005), nonconforming methods

Unsteady problems

- Bieterman and Babuška (1982), introduction
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction

Previous results

Nonlinear problems

- Han (1994), general framework
- Verfürth (1994), residual estimates

Stopping criteria

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid st. crit.
- Arioli (2000's), general linear solver st. crit.
- Chaillou and Suri (2006, 2007), distinguishing discretization and linearization errors

Previous results

Nonlinear problems

- Han (1994), general framework
- Verfürth (1994), residual estimates

Stopping criteria

- engineering literature, since 1950's
- Becker, Johnson, and Rannacher (1995), multigrid st. crit.
- Arioli (2000's), general linear solver st. crit.
- Chaillou and Suri (2006, 2007), distinguishing discretization and linearization errors

Outline

Model diffusion problem

Model diffusion problem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(\underline{\mathbf{K}}\nabla u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where

- $\underline{\mathbf{K}} : \Omega \to \mathbb{R}^{d\times d}$ is a diffusion tensor,
- $f : \Omega \to \mathbb{R}$ is a source term.

Form in 1D

Let $\Omega =]a, b[$, $a < b$. Let $k :]a, b[\to \mathbb{R}$ and $f :]a, b[\to \mathbb{R}$ be two given functions. Find $u :]a, b[\to \mathbb{R}$ such that

$$\begin{aligned} -(ku')' &= f, \\ u(a) = u(b) &= 0. \end{aligned}$$

Weak formulation

Find $u \in V := H_0^1(\Omega)$ such that

$$(\underline{\mathbf{K}}\nabla u, \nabla v) = (f, v) \qquad \forall v \in V.$$

Model diffusion problem

Model diffusion problem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(\underline{\mathbf{K}}\nabla u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where

- $\underline{\mathbf{K}} : \Omega \to \mathbb{R}^{d\times d}$ is a diffusion tensor,
- $f : \Omega \to \mathbb{R}$ is a source term.

Form in 1D

Let $\Omega =]a, b[$, $a < b$. Let $k :]a, b[\to \mathbb{R}$ and $f :]a, b[\to \mathbb{R}$ be two given functions. Find $u :]a, b[\to \mathbb{R}$ such that

$$\begin{aligned} -(ku')' &= f, \\ u(a) = u(b) &= 0. \end{aligned}$$

Weak formulation

Find $u \in V := H_0^1(\Omega)$ such that

$$(\underline{\mathbf{K}}\nabla u, \nabla v) = (f, v) \qquad \forall v \in V.$$

Model diffusion problem

Model diffusion problem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\nabla\cdot(\underline{\mathbf{K}}\nabla u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where

- $\underline{\mathbf{K}} : \Omega \to \mathbb{R}^{d\times d}$ is a diffusion tensor,
- $f : \Omega \to \mathbb{R}$ is a source term.

Form in 1D

Let $\Omega =]a, b[$, $a < b$. Let $k :]a, b[\to \mathbb{R}$ and $f :]a, b[\to \mathbb{R}$ be two given functions. Find $u :]a, b[\to \mathbb{R}$ such that

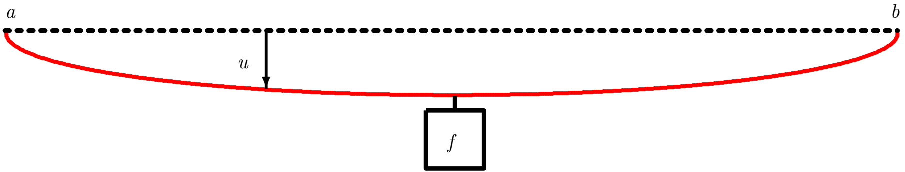
$$\begin{aligned} -(ku')' &= f, \\ u(a) = u(b) &= 0. \end{aligned}$$

Weak formulation

Find $u \in V := H_0^1(\Omega)$ such that

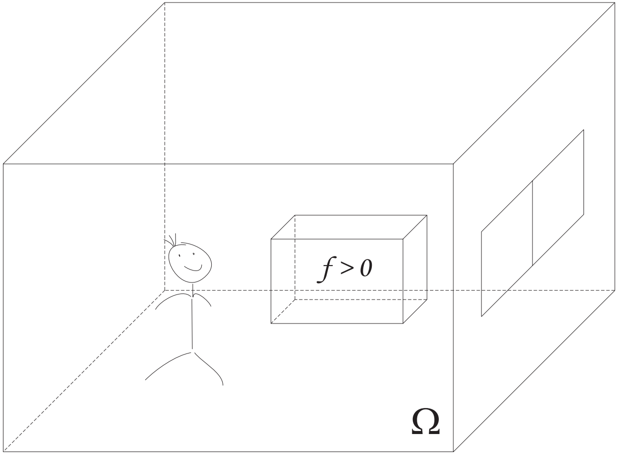
$$(\underline{\mathbf{K}}\nabla u, \nabla v) = (f, v) \qquad \forall v \in V.$$

Example: elastic string



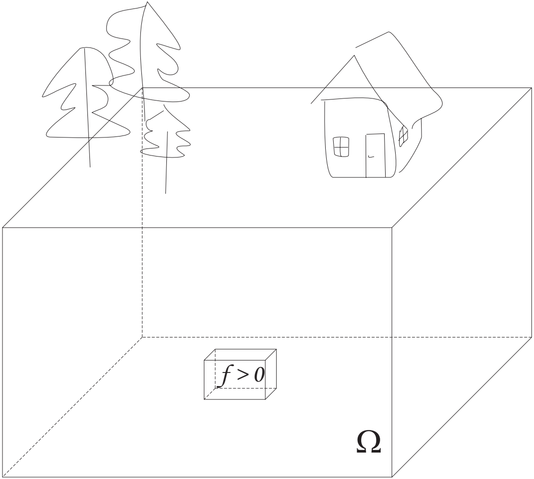
Elastic string with displacement u and weight f

Example: heat flow



A room with a heater of $f > 0$ and temperature u

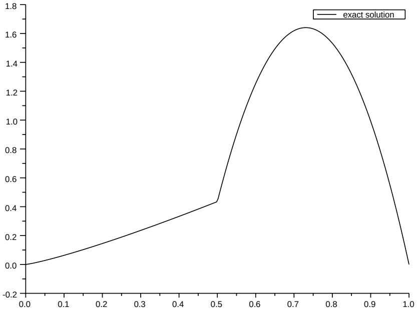
Example: underground water flow

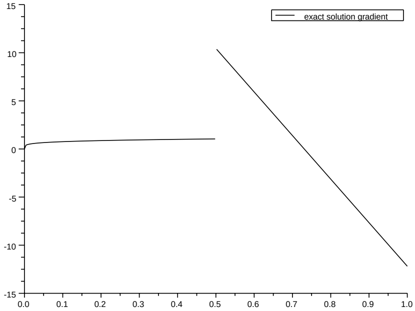


Underground with a water well of $f > 0$ and pressure head u

Outline

Properties of the exact solution

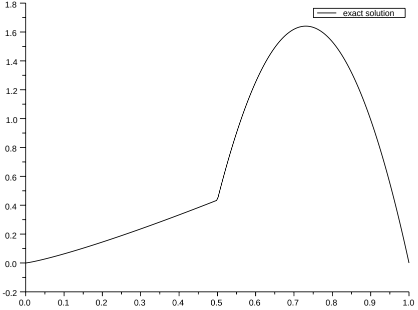




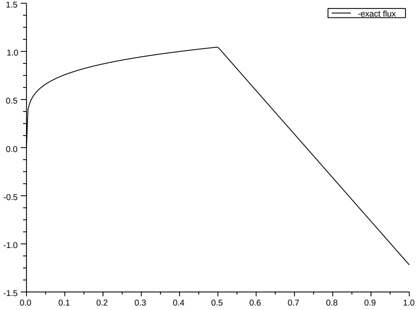
Solution u (displacement, temperature, pressure …) is continuous

Solution gradient ∇u (derivative u' in 1D) is not necessarily continuous

Properties of the exact solution

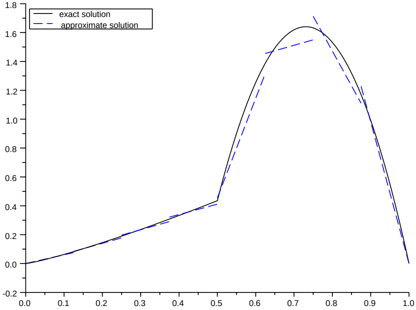


Solution u is continuous

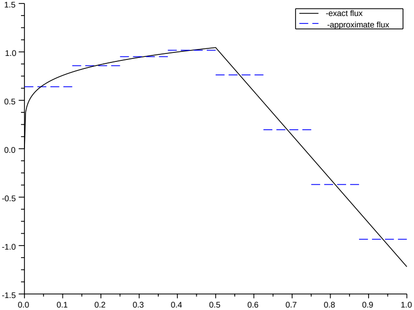


Flux $\mathbf{t} := -\underline{\mathbf{K}}\nabla u$ (or $-ku'$ in 1D) is continuous

Approximate solution and approximate flux

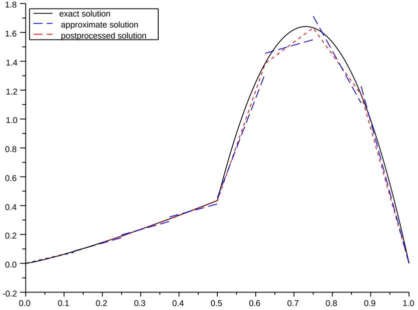


Approximate solution u_h is not necessarily continuous

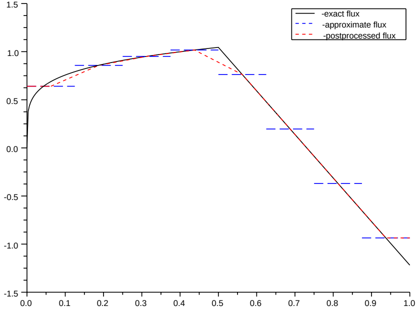


Approximate flux $-\underline{\mathbf{K}}\nabla u_h$ $(-ku_h')$ is not necessarily continuous

Potential and flux reconstructions



Potential reconstruction

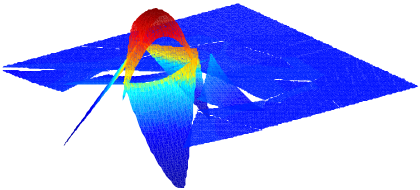


Flux reconstruction

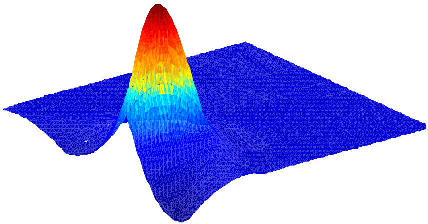
Potential reconstruction in multiple space dimensions

Assumption A (Potential reconstruction)

There exists a **potential reconstruction** $s_h \in V$.



Potential u_h



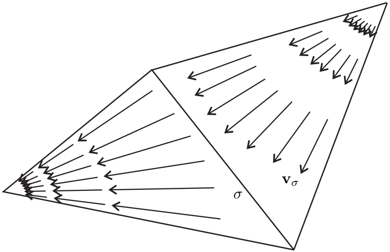
Potential reconstruction s_h

Flux reconstruction in multiple space dimensions

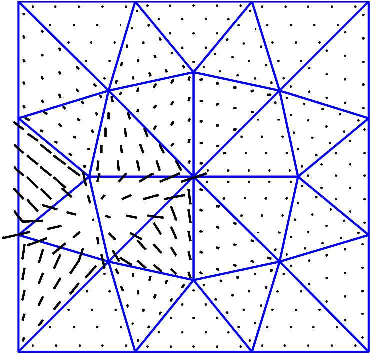
Assumption B (Equilibrated flux reconstruction)

There exists an **equilibrated flux reconstruction** $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *such that*

$$(\nabla\cdot\mathbf{t}_h, 1)_K = (f, 1)_K \qquad \forall K \in \mathcal{T}_h.$$



Raviart–Thomas–Nédélec lowest-order basis function



Flux reconstruction $\mathbf{t}_h$

Outline

A posteriori error estimate for $-\nabla\cdot(\nabla u) = f$ ($\underline{\mathbf{K}} = \mathbb{I}$)

Theorem (A guaranteed a posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h \in V(\mathcal{T}_h) := \{v \in L^2(\Omega),\ v|_K \in H^1(K) \forall K \in \mathcal{T}_h\}$ *be arbitrary,*
- *Assumptions A and B hold.*

Then

$$\|\nabla(u - u_h)\|^2 \le \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{F},K} + \eta_{\mathrm{R},K})^2 + \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2,$$

where $\eta_{\mathrm{F},K}, \eta_{\mathrm{R},K}, \eta_{\mathrm{NC},K}$ *are fully computable from* u_h, $\mathbf{t}_h$, s_h.

A posteriori error estimate for $-\nabla\cdot(\nabla u) = f$ ($\underline{\mathbf{K}} = \mathbb{I}$)

Theorem (A guaranteed a posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h \in V(\mathcal{T}_h) := \{v \in L^2(\Omega),\, v|_K \in H^1(K)\, \forall K \in \mathcal{T}_h\}$ *be arbitrary,*
- *Assumptions A and B hold.*

Then

$$\|\nabla(u-u_h)\|^2 \leq \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{F},K} + \eta_{\mathrm{R},K})^2 + \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2,$$

where $\eta_{\mathrm{F},K}$, $\eta_{\mathrm{R},K}$, $\eta_{\mathrm{NC},K}$ *are fully computable from* u_h, $\mathbf{t}_h$, s_h.

Estimators

- *nonconformity estimator*

$$\eta_{\mathrm{NC},K} := \|\nabla(u_h - s_h)\|_K$$

  - evaluates the departure of u_h from V
  - constraint $u \in V$

- *flux estimator*

$$\eta_{\mathrm{F},K} := \|\nabla u_h + \mathbf{t}_h\|_K$$

  - evaluates the departure of ∇u_h from $\mathbf{H}(\mathrm{div}, \Omega)$
  - constitutive law $\mathbf{t} = -\nabla u$ and constraint $\mathbf{t} \in \mathbf{H}(\mathrm{div}, \Omega)$

- *residual estimator*

$$\eta_{\mathrm{R},K} := \frac{h_K}{\pi}\|f - \nabla\cdot\mathbf{t}_h\|_K$$

  - strong form of the PDE evaluated for the flux $\mathbf{t}_h$
  - equilibrium $\nabla\cdot\mathbf{t} = f$

Estimators

- *nonconformity estimator*

$$\eta_{\mathrm{NC},K} := \|\nabla(u_h - s_h)\|_K$$

  - evaluates the departure of u_h from V
  - constraint $u \in V$

- *flux estimator*

$$\eta_{\mathrm{F},K} := \|\nabla u_h + \mathbf{t}_h\|_K$$

  - evaluates the departure of ∇u_h from $\mathbf{H}(\mathrm{div}, \Omega)$
  - constitutive law $\mathbf{t} = -\nabla u$ and constraint $\mathbf{t} \in \mathbf{H}(\mathrm{div}, \Omega)$

- *residual estimator*

$$\eta_{\mathrm{R},K} := \frac{h_K}{\pi}\|f - \nabla\cdot\mathbf{t}_h\|_K$$

  - strong form of the PDE evaluated for the flux $\mathbf{t}_h$
  - equilibrium $\nabla\cdot\mathbf{t} = f$

Estimators

- *nonconformity estimator*

$$\eta_{\mathrm{NC},K} := \|\nabla(u_h - s_h)\|_K$$

  - evaluates the departure of u_h from V
  - constraint $u \in V$

- *flux estimator*

$$\eta_{\mathrm{F},K} := \|\nabla u_h + \mathbf{t}_h\|_K$$

  - evaluates the departure of ∇u_h from $\mathbf{H}(\mathrm{div}, \Omega)$
  - constitutive law $\mathbf{t} = -\nabla u$ and constraint $\mathbf{t} \in \mathbf{H}(\mathrm{div}, \Omega)$

- *residual estimator*

$$\eta_{\mathrm{R},K} := \frac{h_K}{\pi}\|f - \nabla\cdot\mathbf{t}_h\|_K$$

  - strong form of the PDE evaluated for the flux $\mathbf{t}_h$
  - equilibrium $\nabla\cdot\mathbf{t} = f$

Outline

Assumptions for efficiency

Assumption C (Technical assumption)

Let $\mathcal{T}_h$ be shape-regular and u_h, f, and $\mathbf{t}_h$ pw polynomials.

Assumption D (Potential reconstruction)

Let the potential reconstruction s_h be a piecewise polynomial constructed from u_h by local averaging.

Assumption E (Approximation property – flux reconstruction)

For all $K \in \mathcal{T}_h$, there holds

$$\eta_{\mathrm{F},K} \lesssim \eta_{\sharp,\mathfrak{T}_K},$$

where

$$\eta_{\sharp,\mathfrak{T}_K} := \left\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^2 \|f + \Delta u_h\|_{K'}^2 \right\}^{1/2} + \left\{ \sum_{e \in \mathfrak{E}_K^{\mathrm{int}}} h_e \|[\![\nabla u_h]\!] \cdot \mathbf{n}_e\|_e^2 \right\}^{1/2} + \left\{ \sum_{e \in \mathfrak{E}_K} h_e^{-1} \|[\![u_h]\!]\|_e^2 \right\}^{1/2}.$$

Assumptions for efficiency

Assumption C (Technical assumption)

Let $\mathcal{T}_h$ be shape-regular and u_h, f, and $\mathbf{t}_h$ pw polynomials.

Assumption D (Potential reconstruction)

Let the potential reconstruction s_h be a piecewise polynomial constructed from u_h by local averaging.

Assumption E (Approximation property – flux reconstruction)

For all $K \in \mathcal{T}_h$, there holds

$$\eta_{\mathrm{F},K} \lesssim \eta_{\sharp,\mathfrak{T}_K},$$

where

$$\eta_{\sharp,\mathfrak{T}_K} := \left\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^2 \|f + \Delta u_h\|_{K'}^2 \right\}^{1/2} + \left\{ \sum_{e \in \mathfrak{E}_K^{\mathrm{int}}} h_e \|[\![\nabla u_h]\!]\cdot \mathbf{n}_e\|_e^2 \right\}^{1/2} + \left\{ \sum_{e \in \mathfrak{E}_K} h_e^{-1} \|[\![u_h]\!]\|_e^2 \right\}^{1/2}.$$

Assumptions for efficiency

Assumption C (Technical assumption)

Let $\mathcal{T}_h$ be shape-regular and u_h, f, and $\mathbf{t}_h$ pw polynomials.

Assumption D (Potential reconstruction)

Let the potential reconstruction s_h be a piecewise polynomial constructed from u_h by local averaging.

Assumption E (Approximation property – flux reconstruction)

For all $K \in \mathcal{T}_h$, there holds

$$\eta_{\mathrm{F},K} \lesssim \eta_{\sharp,\mathfrak{T}_K},$$

where

$$\eta_{\sharp,\mathfrak{T}_K} := \left\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^2 \|f + \Delta u_h\|_{K'}^2 \right\}^{1/2} + \left\{ \sum_{e \in \mathfrak{E}_K^{\mathrm{int}}} h_e \|[\![\nabla u_h]\!] \cdot \mathbf{n}_e\|_e^2 \right\}^{1/2} + \left\{ \sum_{e \in \mathfrak{E}_K} h_e^{-1} \|[\![u_h]\!]\|_e^2 \right\}^{1/2}.$$

Local efficiency

Theorem (Local efficiency)

Let $\langle [\![u_h]\!], 1 \rangle_e = 0$ for all faces e of the mesh $\mathcal{T}_h$. Then, under Assumptions C to E,

$$\eta_{\mathrm{NC},K} + \eta_{\mathrm{R},K} + \eta_{\mathrm{F},K} \lesssim \|\nabla(u - u_h)\|_{\mathfrak{T}_K}$$

for all $K \in \mathcal{T}_h$.

Summary for $-\nabla\cdot(\nabla u) = f$

Summary

- **guaranteed upper bound**:

$$\|\nabla(u-u_h)\| \le \left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}$$

- **local efficiency**:

$$\eta_K \lesssim \|\nabla(u-u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- close to **asymptotic exactness**:

$$\frac{\left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}}{\|\nabla(u-u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K)
- **small evaluation cost** of η_K

Summary for $-\nabla\cdot(\nabla u) = f$

Summary

- **guaranteed upper bound**:

$$\|\nabla(u-u_h)\| \leq \left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}$$

- **local efficiency**:

$$\eta_K \lesssim \|\nabla(u-u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- close to **asymptotic exactness**:

$$\frac{\left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}}{\|\nabla(u-u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K)
- **small evaluation cost** of η_K

Summary for $-\nabla\cdot(\nabla u) = f$

Summary

- **guaranteed upper bound**:

$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K\in\mathcal{T}_h} \eta_K^2 \right\}^{1/2}$$

- **local efficiency**:

$$\eta_K \lesssim \|\nabla(u - u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- close to **asymptotic exactness**:

$$\frac{\left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}}{\|\nabla(u - u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K)
- **small evaluation cost** of η_K

Summary for $-\nabla\cdot(\nabla u) = f$

Summary

- **guaranteed upper bound**:

$$\|\nabla(u-u_h)\| \le \left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}$$

- **local efficiency**:

$$\eta_K \lesssim \|\nabla(u-u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- close to **asymptotic exactness**:

$$\frac{\left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}}{\|\nabla(u-u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K)
- **small evaluation cost** of η_K

Summary for $-\nabla\cdot(\nabla u) = f$

Summary

- **guaranteed upper bound**:

$$\|\nabla(u-u_h)\| \leq \left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}$$

- **local efficiency**:

$$\eta_K \lesssim \|\nabla(u-u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- close to **asymptotic exactness**:

$$\frac{\left\{\sum_{K\in\mathcal{T}_h} \eta_K^2\right\}^{1/2}}{\|\nabla(u-u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K)
- **small evaluation cost** of η_K

Outline

Application

Application

- specification of the potential reconstruction s_h and flux reconstruction $\mathbf{t}_h$
- $s_h = u_h$ in conforming methods (FE, VCFV) $\Rightarrow \eta_{\mathrm{NC},K} = 0$
- $\mathbf{t}_h = -\underline{\mathbf{K}}\nabla u_h$ in flux-conforming methods (CCFV, MFE) $\Rightarrow \eta_{\mathrm{F},K} = 0$
- verification of Assumptions A to E

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin method
- various finite volumes
- mixed finite elements

Application

Application

- specification of the potential reconstruction s_h and flux reconstruction $\mathbf{t}_h$
- $s_h = u_h$ in conforming methods (FE, VCFV) $\Rightarrow \eta_{\mathrm{NC},K} = 0$
- $\mathbf{t}_h = -\underline{\mathbf{K}}\nabla u_h$ in flux-conforming methods (CCFV, MFE) $\Rightarrow \eta_{\mathrm{F},K} = 0$
- verification of Assumptions A to E

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin method
- various finite volumes
- mixed finite elements

Application

Application

- specification of the potential reconstruction s_h and flux reconstruction $\mathbf{t}_h$
- $s_h = u_h$ in conforming methods (FE, VCFV) $\Rightarrow \eta_{\mathrm{NC},K} = 0$
- $\mathbf{t}_h = -\underline{\mathbf{K}}\nabla u_h$ in flux-conforming methods (CCFV, MFE) $\Rightarrow \eta_{\mathrm{F},K} = 0$
- verification of Assumptions A to E

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin method
- various finite volumes
- mixed finite elements

Application

Application

- specification of the potential reconstruction s_h and flux reconstruction $\mathbf{t}_h$
- $s_h = u_h$ in conforming methods (FE, VCFV) $\Rightarrow \eta_{\mathrm{NC},K} = 0$
- $\mathbf{t}_h = -\underline{\mathbf{K}}\nabla u_h$ in flux-conforming methods (CCFV, MFE) $\Rightarrow \eta_{\mathrm{F},K} = 0$
- verification of Assumptions A to E

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin method
- various finite volumes
- mixed finite elements

Application

Application

- specification of the potential reconstruction s_h and flux reconstruction $\mathbf{t}_h$
- $s_h = u_h$ in conforming methods (FE, VCFV) $\Rightarrow \eta_{\mathrm{NC},K} = 0$
- $\mathbf{t}_h = -\underline{\mathbf{K}}\nabla u_h$ in flux-conforming methods (CCFV, MFE) $\Rightarrow \eta_{\mathrm{F},K} = 0$
- verification of Assumptions A to E

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin method
- various finite volumes
- mixed finite elements

Numerics: finite elements in 1D

Model problem

$$\begin{aligned} -u'' &= \pi^2 sin(\pi x) \quad \text{in } (0,1), \\ u &= 0 \quad \text{in } 0,1 \end{aligned}$$

Exact solution

$$u(x) = sin(\pi x)$$

Discretization

conforming finite elements

Numerics: finite elements in 1D

Model problem

$$\begin{aligned} -u'' &= \pi^2 sin(\pi x) \quad \text{in } (0,1), \\ u &= 0 \quad \text{in } 0, 1 \end{aligned}$$

Exact solution

$$u(x) = sin(\pi x)$$

Discretization

conforming finite elements

Numerics: finite elements in 1D

Model problem

$$\begin{aligned} -u'' &= \pi^2 sin(\pi x) \quad \text{in } (0,1), \\ u &= 0 \quad \text{in } 0,1 \end{aligned}$$

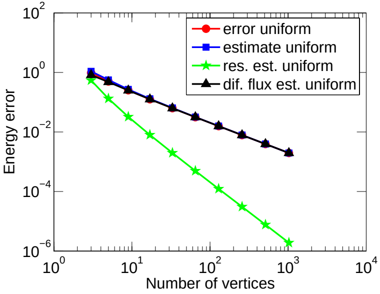
Exact solution

$$u(x) = sin(\pi x)$$

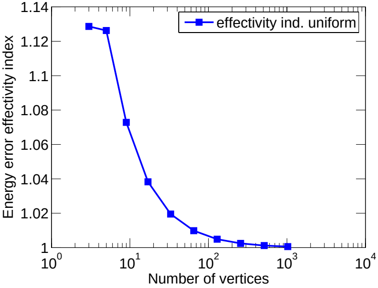
Discretization

conforming finite elements

Estimated and actual errors, effectivity index



Actual error and estimator and its components



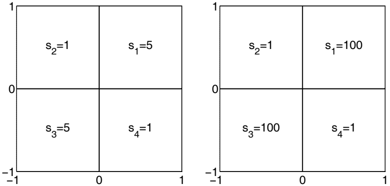
Effectivity index

Numerics: cell-centered finite volumes

- diffusion equation

$$-\nabla\cdot(\underline{\mathbf{K}}\nabla u) = 0 \quad \text{in} \quad \Omega = (-1,1)\times(-1,1)$$

- discontinuous and inhomogeneous $\underline{\mathbf{K}}$, two cases:



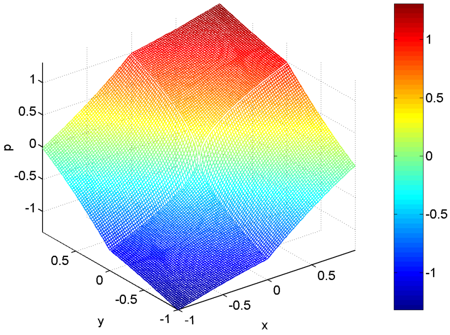
- analytical solution: singularity at the origin

$$u(r,\theta)|_{\Omega_i} = r^\alpha(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

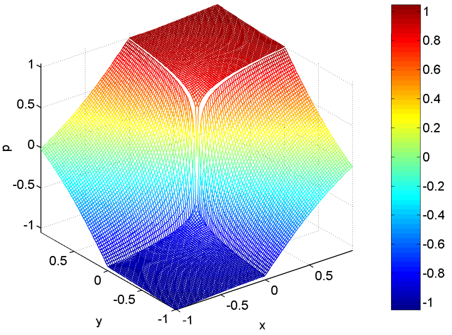
  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

- discretization by cell-centered finite volumes

Analytical solutions

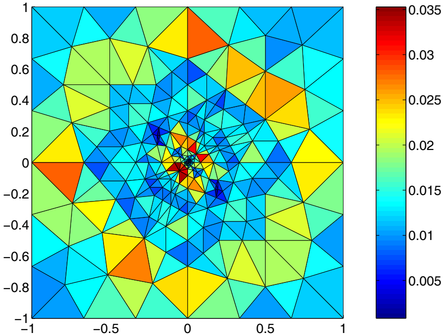


Case 1

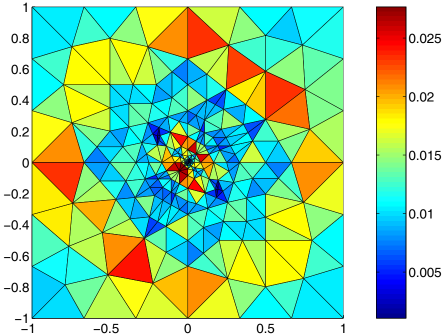


Case 2

Error distribution on an adaptively refined mesh, case 1

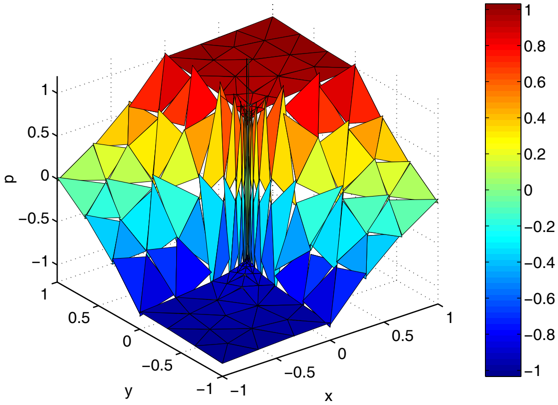


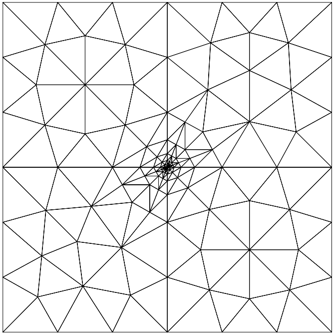
Estimated error distribution



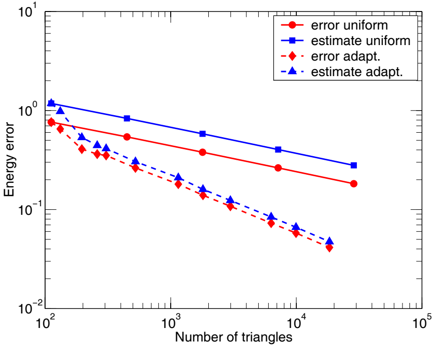
Exact error distribution

Approximate solution and the corresponding adaptively refined mesh, case 2

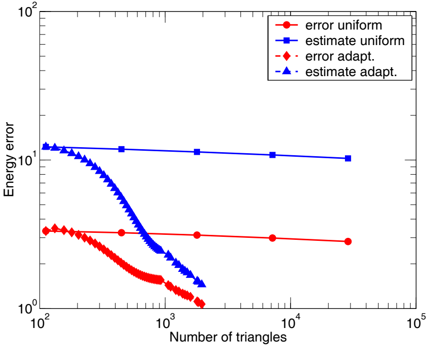




Estimated and actual errors in uniformly/adaptively refined meshes

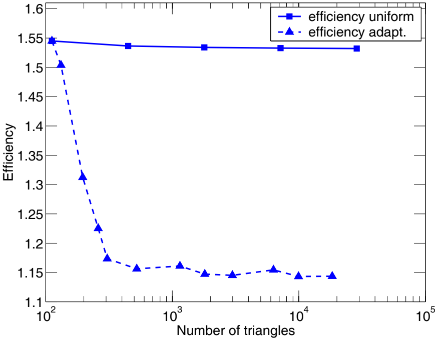


Case 1

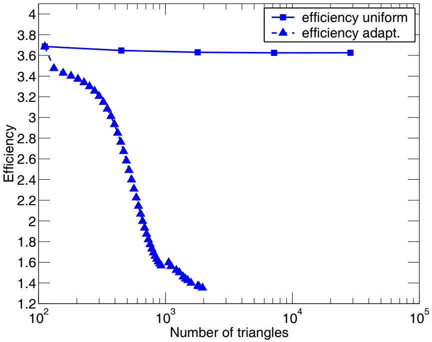


Case 2

Effectivity indices in uniformly/adaptively refined meshes



Case 1



Case 2

Outline

Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A}:\mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*

$$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

3.
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do 1 algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*

   $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i+1$, back to 3.2.*
4. *Convergence? OK $\Rightarrow$ finish. KO $\Rightarrow k := k+1$, back to 2.*

Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A}: \mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*
   $$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$
3. 1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do 1 algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*
      $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$
   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i + 1$, back to 3.2.*
4. *Convergence? OK $\Rightarrow$ finish. KO $\Rightarrow k := k + 1$, back to 2.*

Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector* U^0. *Set* $k := 1$.
2. $U^{k-1} \Rightarrow$ *matrix* $\mathbb{A}^{k-1}$ *and vector* F^{k-1}: *find* U^k *s.t.*

$$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

3.
   1. *Set* $U^{k,0} := U^{k-1}$ *and* $i := 1$.
   2. *Do 1 algebraic solver step* $\Rightarrow U^{k,i}$ *s.t. ($R^{k,i}$ algebraic res.)*

$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK* $\Rightarrow U^k := U^{k,i}$. *KO* $\Rightarrow i := i + 1$, *back to 3.2.*
4. *Convergence? OK* $\Rightarrow$ *finish. KO* $\Rightarrow k := k + 1$, *back to 2.*

Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*

$$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

3.
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do 1 algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*

$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i + 1$, back to 3.2.*
4. *Convergence? OK $\Rightarrow$ finish. KO $\Rightarrow k := k + 1$, back to 2.*

Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. $U^{k-1} \Rightarrow$ *matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*
   $$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$
3.
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do 1 algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*
      $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$
   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i + 1$, back to 3.2.*
4. *Convergence? OK $\Rightarrow$ finish. KO $\Rightarrow k := k + 1$, back to 2.*

Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A}:\mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector U^0. Set $k := 1$.*
2. *$U^{k-1} \Rightarrow$ matrix $\mathbb{A}^{k-1}$ and vector F^{k-1}: find U^k s.t.*

$$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$

3.
   1. *Set $U^{k,0} := U^{k-1}$ and $i := 1$.*
   2. *Do 1 algebraic solver step $\Rightarrow U^{k,i}$ s.t. ($R^{k,i}$ algebraic res.)*

   $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

   3. *Convergence? OK $\Rightarrow U^k := U^{k,i}$. KO $\Rightarrow i := i+1$, back to 3.2.*
4. *Convergence? OK $\Rightarrow$ finish. KO $\Rightarrow k := k+1$, back to 2.*

Inexact Newton method

System of nonlinear algebraic equations

Nonlinear operator $\mathcal{A}:\mathbb{R}^N \to \mathbb{R}^N$, vector $F \in \mathbb{R}^N$: find $U \in \mathbb{R}^N$ s.t.

$$\mathcal{A}(U) = F$$

Algorithm (Inexact iterative linearization)

1. *Choose initial vector* U^0*. Set* $k := 1$.
2. $U^{k-1} \Rightarrow$ *matrix* $\mathbb{A}^{k-1}$ *and vector* F^{k-1}*: find* U^k *s.t.*
$$\mathbb{A}^{k-1} U^k \approx F^{k-1}.$$
3.
    1. *Set* $U^{k,0} := U^{k-1}$ *and* $i := 1$.
    2. *Do 1 algebraic solver step* $\Rightarrow U^{k,i}$ *s.t. (*$R^{k,i}$ *algebraic res.)*
    $$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$
    3. *Convergence? OK* $\Rightarrow U^k := U^{k,i}$*. KO* $\Rightarrow i := i+1$*, back to 3.2.*
4. *Convergence? OK* $\Rightarrow$ *finish. KO* $\Rightarrow k := k+1$*, back to 2.*

Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error)

- *How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?*

Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error)

- *How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?*

Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error)

- *How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?*

ematics

Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error)

- *How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?*

Context and questions

Approximate solution

- approximate solution $U^{k,i}$ does not solve $\mathcal{A}(U^{k,i}) = F$

Numerical method

- underlying numerical method: the vector $U^{k,i}$ is associated with a (piecewise polynomial) approximation $u_h^{k,i}$

Partial differential equation

- underlying PDE, u its weak solution: $A(u) = f$

Question (Stopping criteria)

- *What is a good stopping criterion for the linear solver?*
- *What is a good stopping criterion for the nonlinear solver?*

Question (Error)

- *How big is the error $\|u - u_h^{k,i}\|$ on Newton step k and algebraic solver step i, how is it distributed?*

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\xi) = \underline{\mathbf{A}}(v)\xi \qquad \forall (v,\xi) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\xi) = \underline{\mathbf{A}}(\xi)\xi \qquad \forall \xi \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\xi) = |\xi|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall(v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall\boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- quasi-linear diffusion problem
$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$
- Leray–Lions problem
$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$
- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that
$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$-\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$\begin{aligned} -\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\xi) = \underline{\mathbf{A}}(v)\xi \qquad \forall (v,\xi) \in \mathbb{R} \times \mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\xi) = \underline{\mathbf{A}}(\xi)\xi \qquad \forall \xi \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\xi) = |\xi|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Quasi-linear elliptic problem

Quasi-linear elliptic problem

$$\begin{aligned} -\nabla\cdot\boldsymbol{\sigma}(u,\nabla u) &= f \qquad \text{in } \Omega,\\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

- quasi-linear diffusion problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(v)\boldsymbol{\xi} \qquad \forall (v,\boldsymbol{\xi}) \in \mathbb{R}\times\mathbb{R}^d$$

- Leray–Lions problem

$$\boldsymbol{\sigma}(v,\boldsymbol{\xi}) = \underline{\mathbf{A}}(\boldsymbol{\xi})\boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathbb{R}^d$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$

Example

p-Laplacian: Leray–Lions setting with $\underline{\mathbf{A}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\underline{\mathbf{I}}$

Nonlinear operator $A : V := W_0^{1,p}(\Omega) \to V'$

$$\langle A(u), v\rangle_{V',V} := (\boldsymbol{\sigma}(u,\nabla u), \nabla v)$$

Weak formulation

Find $u \in V$ such that

$$A(u) = f \text{ in } V'$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega), v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\sigma(u,\nabla u) - \sigma(u_h^{k,i}, \nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u - u_h^{k,i}]\!]\|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\sigma(u, \nabla u) - \sigma(u_h^{k,i}, \nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\sigma(u,\nabla u) - \sigma(u_h^{k,i}, \nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u - u_h^{k,i}]\!]\|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\sigma(u,\nabla u) - \sigma(u_h^{k,i}, \nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u - u_h^{k,i}]\!]\|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \leq \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla \varphi\|_p = 1} (\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}), \nabla \varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K \in \mathcal{T}_h} \sum_{e \in \mathcal{E}_K} h_e^{1-q} \| [\![u - u_h^{k,i}]\!] \|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u, \nabla u) - \boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\ \|\nabla\varphi\|_p = 1} (\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K\in\mathcal{T}_h} \sum_{e\in\mathcal{E}_K} h_e^{1-q} \|[\![u - u_h^{k,i}]\!]\|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \leq \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Approximate solution and error measure

Approximate solution

- $u_h^{k,i} \in V(\mathcal{T}_h) \not\subset V$, $u_h^{k,i}$ not necessarily in V
- $V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\}$

Error measure

$$\mathcal{J}_u(u_h^{k,i}) := \sup_{\varphi \in V;\, \|\nabla\varphi\|_p = 1} (\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i}), \nabla\varphi) + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

$$\mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i}) := \left\{ \sum_{K\in\mathcal{T}_h} \sum_{e\in\mathcal{E}_K} h_e^{1-q} \| [\![u - u_h^{k,i}]\!] \|_{q,e}^q \right\}^{1/q}$$

- weak difference of the fluxes (dual norm of the residual) + nonconformity (computable jump term)
- there holds $\mathcal{J}_u(u_h^{k,i}) = 0$ if and only if $u = u_h^{k,i}$
- physical relevance: **strong** difference of the **fluxes** + **nonconformity**

$$\mathcal{J}_u(u_h^{k,i}) \le \mathcal{J}_u^{\mathrm{up}}(u_h^{k,i}) := \|\boldsymbol{\sigma}(u,\nabla u) - \boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i})\|_q + \mathcal{J}_{u,\mathrm{NC}}(u_h^{k,i})$$

Outline

A posteriori error estimate

Assumption A (Total flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and an algebraic remainder* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla \cdot \mathbf{t}_h^{k,i} = f_h - \rho_h^{k,i},$$

with the data approximation f_h *s.t.* $(f_h, 1)_K = (f, 1)_K \ \forall K \in \mathcal{T}_h$.

Theorem (A guaranteed a posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

A posteriori error estimate

Assumption A (Total flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and an algebraic remainder* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla \cdot \mathbf{t}_h^{k,i} = f_h - \rho_h^{k,i},$$

with the data approximation f_h *s.t.* $(f_h, 1)_K = (f, 1)_K \ \ \forall K \in \mathcal{T}_h$.

Theorem (A guaranteed a posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

A posteriori error estimate

Assumption A (Total flux reconstruction)

There exists a flux reconstruction $\mathbf{t}_h^{k,i} \in \mathbf{H}^q(\mathrm{div}, \Omega)$ *and an algebraic remainder* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla\cdot\mathbf{t}_h^{k,i} = f_h - \rho_h^{k,i},$$

with the data approximation f_h *s.t.* $(f_h, 1)_K = (f, 1)_K \;\; \forall K \in \mathcal{T}_h$.

Theorem (A guaranteed a posteriori error estimate)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumption A hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \overline{\eta}^{k,i},$$

where $\overline{\eta}^{k,i}$ *is fully computable from* $u_h^{k,i}$, $\mathbf{t}_h^{k,i}$, *and* $\rho_h^{k,i}$.

Distinguishing error components

Assumption B (Discretization, linearization, and algebraic errors)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i} = \mathbf{t}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

(iii) *as the nonlinear solver converges,* $\|\mathbf{l}_h^{k,i}\|_q \to 0$.

Comments

- $\mathbf{d}_h^{k,i}$: *discretization flux reconstruction*
- $\mathbf{l}_h^{k,i}$: *linearization* ***error*** *flux reconstruction*
- $\mathbf{a}_h^{k,i}$: *algebraic* ***error*** *flux reconstruction*

Distinguishing error components

Assumption B (Discretization, linearization, and algebraic errors)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i} = \mathbf{t}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

(iii) *as the nonlinear solver converges,* $\|\mathbf{l}_h^{k,i}\|_q \to 0$.

Comments

- $\mathbf{d}_h^{k,i}$: *discretization flux reconstruction*
- $\mathbf{l}_h^{k,i}$: *linearization* ***error*** *flux reconstruction*
- $\mathbf{a}_h^{k,i}$: *algebraic* ***error*** *flux reconstruction*

Estimate distinguishing error components

Theorem (Estimate distinguishing different error components)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumptions A and B hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i} := \eta_{\text{disc}}^{k,i} + \eta_{\text{lin}}^{k,i} + \eta_{\text{alg}}^{k,i} + \eta_{\text{rem}}^{k,i} + \eta_{\text{quad}}^{k,i} + \eta_{\text{osc}}^{k,i}.$$

Estimate distinguishing error components

Theorem (Estimate distinguishing different error components)

Let

- *$u \in V$ be the weak solution,*
- *$u_h^{k,i} \in V(\mathcal{T}_h)$ be arbitrary,*
- *Assumptions A and B hold.*

Then there holds

$$\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i} := \eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i}.$$

Estimators

- *discretization* estimator

$$\eta_{\mathrm{disc},K}^{k,i} := 2^{1/p}\left(\|\overline{\boldsymbol{\sigma}}_h^{k,i} + \mathbf{d}_h^{k,i}\|_{q,K} + \left\{\sum_{e\in\mathcal{E}_K} h_e^{1-q}\|[\![u_h^{k,i}]\!]\|_{q,e}^q\right\}^{\frac{1}{q}}\right)$$

- *linearization* estimator

$$\eta_{\mathrm{lin},K}^{k,i} := \|\mathbf{l}_h^{k,i}\|_{q,K}$$

- *algebraic* estimator

$$\eta_{\mathrm{alg},K}^{k,i} := \|\mathbf{a}_h^{k,i}\|_{q,K}$$

- *algebraic remainder estimator*

$$\eta_{\mathrm{rem},K}^{k,i} := h_\Omega\|\rho_h^{k,i}\|_{q,K}$$

- *quadrature estimator*

$$\eta_{\mathrm{quad},K}^{k,i} := \|\boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i}) - \overline{\boldsymbol{\sigma}}_h^{k,i}\|_{q,K}$$

- *data oscillation estimator*

$$\eta_{\mathrm{osc},K}^{k,i} := C_{\mathrm{P},p} h_K\|f - f_h\|_{q,K}$$

- $\eta_{\cdot}^{k,i} := \left\{\sum_{K\in\mathcal{T}_h}\left(\eta_{\cdot,K}^{k,i}\right)^q\right\}^{1/q}$

Outline

Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\rm rem}^{k,i} \le \gamma_{\rm rem} \max\{\eta_{\rm disc}^{k,i}, \eta_{\rm lin}^{k,i}, \eta_{\rm alg}^{k,i}\},$$
$$\eta_{\rm alg}^{k,i} \le \gamma_{\rm alg} \max\{\eta_{\rm disc}^{k,i}, \eta_{\rm lin}^{k,i}\},$$
$$\eta_{\rm lin}^{k,i} \le \gamma_{\rm lin} \eta_{\rm disc}^{k,i}$$

- $\gamma_{\rm rem}, \gamma_{\rm alg}, \gamma_{\rm lin} \approx 0.1$

Local stopping criteria

- stop whenever:

$$\eta_{{\rm rem},K}^{k,i} \le \gamma_{{\rm rem},K} \max\{\eta_{{\rm disc},K}^{k,i}, \eta_{{\rm lin},K}^{k,i}, \eta_{{\rm alg},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{{\rm alg},K}^{k,i} \le \gamma_{{\rm alg},K} \max\{\eta_{{\rm disc},K}^{k,i}, \eta_{{\rm lin},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{{\rm lin},K}^{k,i} \le \gamma_{{\rm lin},K} \eta_{{\rm disc},K}^{k,i} \qquad \forall K \in \mathcal{T}_h$$

- $\gamma_{{\rm rem},K}, \gamma_{{\rm alg},K}, \gamma_{{\rm lin},K} \approx 0.1$

Stopping criteria

Global stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem}}^{k,i} \le \gamma_{\mathrm{rem}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}, \eta_{\mathrm{alg}}^{k,i}\},$$
$$\eta_{\mathrm{alg}}^{k,i} \le \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\},$$
$$\eta_{\mathrm{lin}}^{k,i} \le \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

- $\gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$

Local stopping criteria

- stop whenever:

$$\eta_{\mathrm{rem},K}^{k,i} \le \gamma_{\mathrm{rem},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}, \eta_{\mathrm{alg},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{alg},K}^{k,i} \le \gamma_{\mathrm{alg},K} \max\{\eta_{\mathrm{disc},K}^{k,i}, \eta_{\mathrm{lin},K}^{k,i}\} \qquad \forall K \in \mathcal{T}_h,$$
$$\eta_{\mathrm{lin},K}^{k,i} \le \gamma_{\mathrm{lin},K} \eta_{\mathrm{disc},K}^{k,i} \qquad \forall K \in \mathcal{T}_h$$

- $\gamma_{\mathrm{rem},K}, \gamma_{\mathrm{alg},K}, \gamma_{\mathrm{lin},K} \approx 0.1$

Assumption for efficiency

Assumption C (Approximation property)

For all $K \in \mathcal{T}_h$, there holds

$$\|\overline{\boldsymbol{\sigma}}_h^{k,i} + \mathbf{d}_h^{k,i}\|_{q,K} \lesssim \eta_{\sharp,\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i},$$

where

$$\eta_{\sharp,\mathfrak{T}_K}^{k,i} := \left\{ \sum_{K' \in \mathfrak{T}_K} h_{K'}^q \|f_h + \nabla\cdot\overline{\boldsymbol{\sigma}}_h^{k,i}\|_{q,K'}^q + \sum_{e \in \mathfrak{E}_K^{\mathrm{int}}} h_e \|[\![\overline{\boldsymbol{\sigma}}_h^{k,i}\cdot\mathbf{n}_e]\!]\|_{q,e}^q + \sum_{e \in \mathcal{E}_K} h_e^{1-q} \|[\![u_h^{k,i}]\!]\|_{q,e}^q \right\}^{\frac{1}{q}}.$$

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\text{quad}}^{k,i} + \eta_{\text{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Global efficiency

Theorem (Global efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the global stopping criteria hold. Recall that $\mathcal{J}_u(u_h^{k,i}) \leq \eta^{k,i}$. Then, under Assumption C,

$$\eta^{k,i} \lesssim \mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}^{k,i},$$

where $\lesssim$ means up to a constant independent of σ and q.

- **robustness** with respect to the **nonlinearity** thanks to the choice of the dual norm as error measure

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

$$\eta_{\mathrm{disc},K}^{k,i} + \eta_{\mathrm{lin},K}^{k,i} + \eta_{\mathrm{alg},K}^{k,i} + \eta_{\mathrm{rem},K}^{k,i} \lesssim \mathcal{J}_{u,\mathfrak{T}_K}^{\mathrm{up}}(u_h^{k,i}) + \eta_{\mathrm{quad},\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i}$$

for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

$$\eta_{\mathrm{disc},K}^{k,i} + \eta_{\mathrm{lin},K}^{k,i} + \eta_{\mathrm{alg},K}^{k,i} + \eta_{\mathrm{rem},K}^{k,i} \lesssim \mathcal{J}_{u,\mathfrak{T}_K}^{\mathrm{up}}(u_h^{k,i}) + \eta_{\mathrm{quad},\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i}$$

for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

Local efficiency

Theorem (Local efficiency)

Let the mesh $\mathcal{T}_h$ be shape-regular and let the local stopping criteria hold. Then, under Assumption C,

$$\eta_{\mathrm{disc},K}^{k,i} + \eta_{\mathrm{lin},K}^{k,i} + \eta_{\mathrm{alg},K}^{k,i} + \eta_{\mathrm{rem},K}^{k,i} \lesssim \mathcal{J}_{u,\mathfrak{T}_K}^{\mathrm{up}}(u_h^{k,i}) + \eta_{\mathrm{quad},\mathfrak{T}_K}^{k,i} + \eta_{\mathrm{osc},\mathfrak{T}_K}^{k,i}$$

for all $K \in \mathcal{T}_h$.

- robustness and **local efficiency** for an upper bound on the dual norm

Outline

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^{k-1}U^{k,i} = F^{k-1} - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^{k-1}U^{k,i+\nu} = F^{k-1} - R^{k,i+\nu}.$$

- ν chosen adaptively so that $\eta_{\mathrm{rem},K}^{k,i}$ or $\eta_{\mathrm{rem}}^{k,i}$ are small enough.
- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^{k-1} U^{k,i+\nu} = F^{k-1} - R^{k,i+\nu}.$$

- ν chosen adaptively so that $\eta_{\mathrm{rem},K}^{k,i}$ or $\eta_{\mathrm{rem}}^{k,i}$ are small enough.
- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^{k-1} U^{k,i+\nu} = F^{k-1} - R^{k,i+\nu}.$$

- ν chosen adaptively so that $\eta_{\mathrm{rem},K}^{k,i}$ or $\eta_{\mathrm{rem}}^{k,i}$ are small enough.
- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^{k-1} U^{k,i+\nu} = F^{k-1} - R^{k,i+\nu}.$$

- ν chosen adaptively so that $\eta_{\mathrm{rem},K}^{k,i}$ or $\eta_{\mathrm{rem}}^{k,i}$ are small enough.
- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have
$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$
- Do ν additional steps of the algebraic solver, yielding
$$\mathbb{A}^{k-1} U^{k,i+\nu} = F^{k-1} - R^{k,i+\nu}.$$
- ν chosen adaptively so that $\eta_{\mathrm{rem},K}^{k,i}$ or $\eta_{\mathrm{rem}}^{k,i}$ are small enough.
- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set
$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$
- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have
$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$
- Do ν additional steps of the algebraic solver, yielding
$$\mathbb{A}^{k-1} U^{k,i+\nu} = F^{k-1} - R^{k,i+\nu}.$$
- ν chosen adaptively so that $\eta_{\mathrm{rem},K}^{k,i}$ or $\eta_{\mathrm{rem}}^{k,i}$ are small enough.
- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set
$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$
- Independent of the algebraic solver.

Algebraic error flux reconstruction and algebraic remainder

Construction of $\mathbf{a}_h^{k,i}$ and $\rho_h^{k,i}$

- On linearization step k and algebraic step i, we have

$$\mathbb{A}^{k-1} U^{k,i} = F^{k-1} - R^{k,i}.$$

- Do ν additional steps of the algebraic solver, yielding

$$\mathbb{A}^{k-1} U^{k,i+\nu} = F^{k-1} - R^{k,i+\nu}.$$

- ν chosen adaptively so that $\eta_{\mathrm{rem},K}^{k,i}$ or $\eta_{\mathrm{rem}}^{k,i}$ are small enough.
- Construct the function $\rho_h^{k,i}$ from the algebraic residual vector $R^{k,i+\nu}$ (lifting into appropriate discrete space).
- Suppose we can obtain discretization and linearization flux reconstructions $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}$ on each algebraic step. Then set

$$\mathbf{a}_h^{k,i} := (\mathbf{d}_h^{k,i+\nu} + \mathbf{l}_h^{k,i+\nu}) - (\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}).$$

- Independent of the algebraic solver.

Nonconforming finite elements for the p-Laplacian

Discretization

Find $u_h \in V_h$ such that

$$(\boldsymbol{\sigma}(\nabla u_h), \nabla v_h) = (f_h, v_h) \qquad \forall v_h \in V_h.$$

- $\boldsymbol{\sigma}(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- V_h the Crouzeix–Raviart space
- $f_h := \Pi_0 f$
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

Nonconforming finite elements for the p-Laplacian

Discretization

Find $u_h \in V_h$ such that

$$(\boldsymbol{\sigma}(\nabla u_h), \nabla v_h) = (f_h, v_h) \qquad \forall v_h \in V_h.$$

- $\boldsymbol{\sigma}(\nabla u_h) = |\nabla u_h|^{p-2} \nabla u_h$
- V_h the Crouzeix–Raviart space
- $f_h := \Pi_0 f$
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

Linearization

Linearization

Find $u_h^k \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f_h, \psi_e) \qquad \forall e \in \mathcal{E}_h^{\text{int}}.$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization

$$\sigma^{k-1}(\xi) := |\nabla u_h^{k-1}|^{p-2}\xi$$

- Newton linearization

$$\sigma^{k-1}(\xi) := |\nabla u_h^{k-1}|^{p-2}\xi + (p-2)|\nabla u_h^{k-1}|^{p-4} (\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\xi - \nabla u_h^{k-1})$$

- leads to the system of linear algebraic equations

$$\mathbb{A}^{k-1} U^k = F^{k-1}$$

Linearization

Linearization

Find $u_h^k \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^k), \nabla \psi_e) = (f_h, \psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}.$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2}\boldsymbol{\xi}$$

- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(\boldsymbol{\xi}) := |\nabla u_h^{k-1}|^{p-2}\boldsymbol{\xi} + (p-2)|\nabla u_h^{k-1}|^{p-4} (\nabla u_h^{k-1} \otimes \nabla u_h^{k-1})(\boldsymbol{\xi} - \nabla u_h^{k-1})$$

- leads to the system of linear algebraic equations

$$\mathbb{A}^{k-1} U^k = F^{k-1}$$

Algebraic solution

Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e) = (f_h, \psi_e) - R_e^{k,i} \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}.$$

- algebraic residual vector $R^{k,i} = \{R_e^{k,i}\}_{e \in \mathcal{E}_h^{\mathrm{int}}}$
- discrete system

$$\mathbb{A}^{k-1} U^k = F^{k-1} - R^{k,i}$$

Algebraic solution

Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla \psi_e) = (f_h, \psi_e) - R_e^{k,i} \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}.$$

- algebraic residual vector $R^{k,i} = \{R_e^{k,i}\}_{e \in \mathcal{E}_h^{\mathrm{int}}}$
- discrete system

$$\mathbb{A}^{k-1} U^k = F^{k-1} - R^{k,i}$$

Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,

$$(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x}-\mathbf{x}_K) - \sum_{e\in\mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x}-\mathbf{x}_K)|_{K_e},$$

where, $R_e^{k,i} = (f_h, \psi_e) - (\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla\psi_e) \qquad \forall e \in \mathcal{E}_h^{\rm int}$.

Definition (Construction of $\mathbf{d}_h^{k,i}$)

For all $K \in \mathcal{T}_h$,

$$\mathbf{d}_h^{k,i}|_K := -\boldsymbol{\sigma}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x}-\mathbf{x}_K) - \sum_{e\in\mathcal{E}_K} \frac{\bar{R}_e^{k,i}}{d|D_e|}(\mathbf{x}-\mathbf{x}_K)|_{K_e},$$

where $\bar{R}_e^{k,i} := (f_h, \psi_e) - (\boldsymbol{\sigma}(\nabla u_h^{k,i}), \nabla\psi_e) \qquad \forall e \in \mathcal{E}_h^{\rm int}$.

Definition (Construction of $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $\overline{\boldsymbol{\sigma}}_h^{k,i} := \boldsymbol{\sigma}(\nabla u_h^{k,i})$. Consequently, $\eta_{{\rm quad},K}^{k,i} = 0$ for all $K \in \mathcal{T}_h$.

Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,

$$(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x}-\mathbf{x}_K) - \sum_{e\in\mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x}-\mathbf{x}_K)|_{K_e},$$

where, $R_e^{k,i} = (f_h, \psi_e) - (\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla\psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}$.

Definition (Construction of $\mathbf{d}_h^{k,i}$)

For all $K \in \mathcal{T}_h$,

$$\mathbf{d}_h^{k,i}|_K := -\boldsymbol{\sigma}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x}-\mathbf{x}_K) - \sum_{e\in\mathcal{E}_K} \frac{\bar{R}_e^{k,i}}{d|D_e|}(\mathbf{x}-\mathbf{x}_K)|_{K_e},$$

where $\bar{R}_e^{k,i} := (f_h, \psi_e) - (\boldsymbol{\sigma}(\nabla u_h^{k,i}), \nabla\psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}$.

Definition (Construction of $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $\overline{\boldsymbol{\sigma}}_h^{k,i} := \boldsymbol{\sigma}(\nabla u_h^{k,i})$. Consequently, $\eta_{\mathrm{quad},K}^{k,i} = 0$ for all $K \in \mathcal{T}_h$.

Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$)

For all $K \in \mathcal{T}_h$,

$$(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i})|_K := -\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x}-\mathbf{x}_K) - \sum_{e\in\mathcal{E}_K} \frac{R_e^{k,i}}{d|D_e|}(\mathbf{x}-\mathbf{x}_K)|_{K_e},$$

where, $R_e^{k,i} = (f_h, \psi_e) - (\boldsymbol{\sigma}^{k-1}(\nabla u_h^{k,i}), \nabla\psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}$.

Definition (Construction of $\mathbf{d}_h^{k,i}$)

For all $K \in \mathcal{T}_h$,

$$\mathbf{d}_h^{k,i}|_K := -\boldsymbol{\sigma}(\nabla u_h^{k,i})|_K + \frac{f_h|_K}{d}(\mathbf{x}-\mathbf{x}_K) - \sum_{e\in\mathcal{E}_K} \frac{\bar{R}_e^{k,i}}{d|D_e|}(\mathbf{x}-\mathbf{x}_K)|_{K_e},$$

where $\bar{R}_e^{k,i} := (f_h, \psi_e) - (\boldsymbol{\sigma}(\nabla u_h^{k,i}), \nabla\psi_e) \qquad \forall e \in \mathcal{E}_h^{\mathrm{int}}$.

Definition (Construction of $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $\overline{\boldsymbol{\sigma}}_h^{k,i} := \boldsymbol{\sigma}(\nabla u_h^{k,i})$. Consequently, $\eta_{\mathrm{quad},K}^{k,i} = 0$ for all $K \in \mathcal{T}_h$.

Verification of the assumptions – upper bound

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_0(\mathcal{S}_h) \Rightarrow$ $\mathbf{a}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$.

Verification of the assumptions – upper bound

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_0(\mathcal{S}_h) \Rightarrow$ $\mathbf{a}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_0(\mathcal{S}_h)$.

Verification of the assumptions – efficiency

Lemma (Assumption C)

Assumption C holds.

Comments

- $\mathbf{d}_h^{k,i}$ close to $\sigma(\nabla u_h^{k,i})$
- approximation properties of Raviart–Thomas–Nédélec spaces

Verification of the assumptions – efficiency

Lemma (Assumption C)

Assumption C holds.

Comments

- $\mathbf{d}_h^{k,i}$ close to $\boldsymbol{\sigma}(\nabla u_h^{k,i})$
- approximation properties of Raviart–Thomas–Nédélec spaces

Discontinuous Galerkin for the quasi-linear diffusion

Discretization

Find $u_h \in V_h := \mathbb{P}_m(\mathcal{T}_h)$, $m \geq 1$, such that, for all $v_h \in V_h$,

$$(\boldsymbol{\sigma}(u_h, \nabla u_h), \nabla v_h) - \sum_{e \in \mathcal{E}_h} \{\langle \{\!\{\boldsymbol{\sigma}(u_h, \nabla u_h)\}\!\} \cdot \mathbf{n}_e, [\![v_h]\!]\rangle_e$$

$$+ \theta \langle \{\!\{\underline{\mathbf{A}}(u_h) \nabla v_h\}\!\} \cdot \mathbf{n}_e, [\![u_h]\!]\rangle_e\} + \sum_{e \in \mathcal{E}_h} \langle \bar{\alpha}_e h_e^{-1} [\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h).$$

- $\theta \in \{-1, 0, 1\}$
- $\bar{\alpha}_e := \|\underline{\mathbf{A}}\|_{L^\infty(\mathbb{R})} \chi_e$, χ_e large enough
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

Discontinuous Galerkin for the quasi-linear diffusion

Discretization

Find $u_h \in V_h := \mathbb{P}_m(\mathcal{T}_h)$, $m \geq 1$, such that, for all $v_h \in V_h$,

$$(\boldsymbol{\sigma}(u_h, \nabla u_h), \nabla v_h) - \sum_{e \in \mathcal{E}_h} \{ \langle \{\!\{ \boldsymbol{\sigma}(u_h, \nabla u_h) \}\!\} \cdot \mathbf{n}_e, [\![v_h]\!] \rangle_e$$

$$+ \theta \langle \{\!\{ \underline{\mathbf{A}}(u_h) \nabla v_h \}\!\} \cdot \mathbf{n}_e, [\![u_h]\!] \rangle_e \} + \sum_{e \in \mathcal{E}_h} \langle \bar{\alpha}_e h_e^{-1} [\![u_h]\!], [\![v_h]\!] \rangle_e = (f, v_h).$$

- $\theta \in \{-1, 0, 1\}$
- $\bar{\alpha}_e := \|\underline{\mathbf{A}}\|_{L^\infty(\mathbb{R})} \chi_e$, χ_e large enough
- leads to the system of nonlinear algebraic equations

$$\mathcal{A}(U) = F$$

Linearization

Linearization

Find $u_h^k \in V_h$ such that, for all $K \in \mathcal{T}_h$ and all $j \in \mathcal{C}_K := \{1, \ldots, \dim(\mathbb{P}_m(K))\}$,

$$(\boldsymbol{\sigma}^{k-1}(u_h^k, \nabla u_h^k), \nabla \psi_{K,j}) - \sum_{e \in \mathcal{E}_h} \{ \langle \{\!\{ \boldsymbol{\sigma}^{k-1}(u_h^k, \nabla u_h^k) \}\!\} \cdot \mathbf{n}_e, [\![\psi_{K,j}]\!] \rangle_e$$

$$+ \theta \langle \{\!\{ \underline{\mathbf{A}}^{k-1}(u_h^k) \nabla \psi_{K,j} \}\!\} \cdot \mathbf{n}_e, [\![u_h^k]\!] \rangle_e \} + \sum_{e \in \mathcal{E}_h} \langle \bar{\alpha}_e h_e^{-1} [\![u_h^k]\!], [\![\psi_{K,j}]\!] \rangle_e = (f, \psi_{K,j}).$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization $\boldsymbol{\sigma}^{k-1}(v, \xi) := \underline{\mathbf{A}}(u_h^{k-1})\xi$
- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(v, \xi) := \underline{\mathbf{A}}(u_h^{k-1})\xi + (v - u_h^{k-1}) \partial_v \underline{\mathbf{A}}(u_h^{k-1}) \nabla u_h^{k-1},$$

$$\underline{\mathbf{A}}^{k-1}(v) := \underline{\mathbf{A}}(u_h^{k-1}) + \partial_v \underline{\mathbf{A}}(u_h^{k-1})(v - u_h^{k-1})$$

- leads to the system of linear algebraic equations

$$\mathbb{A}^{k-1} U^k = F^{k-1}$$

Linearization

Linearization

Find $u_h^k \in V_h$ such that, for all $K \in \mathcal{T}_h$ and all $j \in \mathcal{C}_K := \{1, \ldots, \dim(\mathbb{P}_m(K))\}$,

$$(\boldsymbol{\sigma}^{k-1}(u_h^k, \nabla u_h^k), \nabla\psi_{K,j}) - \sum_{e \in \mathcal{E}_h} \{ \langle \{\!\{ \boldsymbol{\sigma}^{k-1}(u_h^k, \nabla u_h^k) \}\!\} \cdot \mathbf{n}_e, [\![\psi_{K,j}]\!] \rangle_e$$
$$+ \theta \langle \{\!\{ \underline{\mathbf{A}}^{k-1}(u_h^k) \nabla\psi_{K,j} \}\!\} \cdot \mathbf{n}_e, [\![u_h^k]\!] \rangle_e \} + \sum_{e \in \mathcal{E}_h} \langle \bar{\alpha}_e h_e^{-1} [\![u_h^k]\!], [\![\psi_{K,j}]\!] \rangle_e = (f, \psi_{K,j}).$$

- $u_h^0 \in V_h$ yields the initial vector U^0
- fixed-point linearization $\boldsymbol{\sigma}^{k-1}(v, \xi) := \underline{\mathbf{A}}(u_h^{k-1})\xi$
- Newton linearization

$$\boldsymbol{\sigma}^{k-1}(v, \xi) := \underline{\mathbf{A}}(u_h^{k-1})\xi + (v - u_h^{k-1})\partial_v \underline{\mathbf{A}}(u_h^{k-1}) \nabla u_h^{k-1},$$
$$\underline{\mathbf{A}}^{k-1}(v) := \underline{\mathbf{A}}(u_h^{k-1}) + \partial_v \underline{\mathbf{A}}(u_h^{k-1})(v - u_h^{k-1})$$

- leads to the system of linear algebraic equations

$$\mathbb{A}^{k-1} U^k = F^{k-1}$$

Algebraic solution

Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(u_h^{k,i}, \nabla u_h^{k,i}), \nabla \psi_{K,j}) - \sum_{e \in \mathcal{E}_h} \{ \langle \{\!\{ \boldsymbol{\sigma}^{k-1}(u_h^{k,i}, \nabla u_h^{k,i}) \}\!\} \cdot \mathbf{n}_e, [\![\psi_{K,j}]\!] \rangle_e$$

$$+ \theta \langle \{\!\{ \underline{\mathbf{A}}^{k-1}(u_h^{k,i}) \nabla \psi_{K,j} \}\!\} \cdot \mathbf{n}_e, [\![u_h^{k,i}]\!] \rangle_e \} + \sum_{e \in \mathcal{E}_h} \langle \bar{\alpha}_e h_e^{-1} [\![u_h^{k,i}]\!], [\![\psi_{K,j}]\!] \rangle_e$$

$$= (f, \psi_{K,j}) - R_{K,j}^{k,i}.$$

- algebraic residual vector $R^{k,i} = \{R_{K,j}^{k,i}\}_{K \in \mathcal{T}_h, j \in \mathcal{C}_K}$
- discrete system

$$\mathbb{A}^{k-1} U^k = F^{k-1} - R^{k,i}$$

Algebraic solution

Algebraic solution

Find $u_h^{k,i} \in V_h$ such that

$$(\boldsymbol{\sigma}^{k-1}(u_h^{k,i}, \nabla u_h^{k,i}), \nabla \psi_{K,j}) - \sum_{e \in \mathcal{E}_h} \{ \langle \{\!\{ \boldsymbol{\sigma}^{k-1}(u_h^{k,i}, \nabla u_h^{k,i}) \}\!\} \cdot \mathbf{n}_e, [\![\psi_{K,j}]\!] \rangle_e$$

$$+ \theta \langle \{\!\{ \underline{\mathbf{A}}^{k-1}(u_h^{k,i}) \nabla \psi_{K,j} \}\!\} \cdot \mathbf{n}_e, [\![u_h^{k,i}]\!] \rangle_e \} + \sum_{e \in \mathcal{E}_h} \langle \bar{\alpha}_e h_e^{-1} [\![u_h^{k,i}]\!], [\![\psi_{K,j}]\!] \rangle_e$$

$$= (f, \psi_{K,j}) - R_{K,j}^{k,i}.$$

- algebraic residual vector $R^{k,i} = \{R_{K,j}^{k,i}\}_{K \in \mathcal{T}_h, j \in \mathcal{C}_K}$
- discrete system

$$\mathbb{A}^{k-1} U^k = F^{k-1} - R^{k,i}$$

Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}) \in \mathbf{RTN}_l(\mathcal{T}_h)$, $l := m-1/m$)

For all $K \in \mathcal{T}_h$ and all $e \in \mathcal{E}_K$,

$$\langle(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i})\cdot\mathbf{n}_e, q_h\rangle_e := \langle -\{\!\!\{\boldsymbol{\sigma}^{k-1}(u_h^{k,i}, \nabla u_h^{k,i})\}\!\!\}\cdot\mathbf{n}_e + \bar{\alpha}_e h_e^{-1}[\![u_h^{k,i}]\!], q_h\rangle_e,$$

$$(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i}, \mathbf{r}_h)_K := -(\boldsymbol{\sigma}^{k-1}(u_h^{k,i}, \nabla u_h^{k,i}), \mathbf{r}_h)_K + \theta \sum_{e\in\mathcal{E}_K} w_e \langle \underline{\mathbf{A}}^{k-1}(u_h^{k,i})\mathbf{r}_h\cdot\mathbf{n}_e, [\![u_h^{k,i}]\!]\rangle_e,$$

for all $q_h \in \mathbb{P}_l(e)$ and all $\mathbf{r}_h \in [\mathbb{P}_{l-1}(K)]^d$.

Definition (Construction of $\mathbf{d}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$, $l := m-1$ or $l := m$)

For all $K \in \mathcal{T}_h$ and all $e \in \mathcal{E}_K$,

$$\langle \mathbf{d}_h^{k,i}\cdot\mathbf{n}_e, q_h\rangle_e := \langle -\{\!\!\{\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i})\}\!\!\}\cdot\mathbf{n}_e + \bar{\alpha}_e h_e^{-1}[\![u_h^{k,i}]\!], q_h\rangle_e,$$

$$(\mathbf{d}_h^{k,i}, \mathbf{r}_h)_K := -(\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}), \mathbf{r}_h)_K + \theta \sum_{e\in\mathcal{E}_K} w_e \langle \underline{\mathbf{A}}(u_h^{k,i})\mathbf{r}_h\cdot\mathbf{n}_e, [\![u_h^{k,i}]\!]\rangle_e,$$

for all $q_h \in \mathbb{P}_l(e)$ and all $\mathbf{r}_h \in [\mathbb{P}_{l-1}(K)]^d$.

Flux reconstructions

Definition (Construction of $(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i}) \in \mathbf{RTN}_l(\mathcal{T}_h)$, $l := m-1/m$)

For all $K \in \mathcal{T}_h$ and all $e \in \mathcal{E}_K$,

$$\langle(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i})\cdot\mathbf{n}_e, q_h\rangle_e := \langle -\{\!\{\boldsymbol{\sigma}^{k-1}(u_h^{k,i},\nabla u_h^{k,i})\}\!\}\cdot\mathbf{n}_e + \bar{\alpha}_e h_e^{-1}[\![u_h^{k,i}]\!], q_h\rangle_e,$$

$$(\mathbf{d}_h^{k,i}+\mathbf{l}_h^{k,i},\mathbf{r}_h)_K := -(\boldsymbol{\sigma}^{k-1}(u_h^{k,i},\nabla u_h^{k,i}),\mathbf{r}_h)_K + \theta\sum_{e\in\mathcal{E}_K} w_e\langle \underline{\mathbf{A}}^{k-1}(u_h^{k,i})\mathbf{r}_h\cdot\mathbf{n}_e, [\![u_h^{k,i}]\!]\rangle_e,$$

for all $q_h \in \mathbb{P}_l(e)$ and all $\mathbf{r}_h \in [\mathbb{P}_{l-1}(K)]^d$.

Definition (Construction of $\mathbf{d}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$, $l := m-1$ or $l := m$)

For all $K \in \mathcal{T}_h$ and all $e \in \mathcal{E}_K$,

$$\langle\mathbf{d}_h^{k,i}\cdot\mathbf{n}_e, q_h\rangle_e := \langle -\{\!\{\boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i})\}\!\}\cdot\mathbf{n}_e + \bar{\alpha}_e h_e^{-1}[\![u_h^{k,i}]\!], q_h\rangle_e,$$

$$(\mathbf{d}_h^{k,i},\mathbf{r}_h)_K := -(\boldsymbol{\sigma}(u_h^{k,i},\nabla u_h^{k,i}),\mathbf{r}_h)_K + \theta\sum_{e\in\mathcal{E}_K} w_e\langle \underline{\mathbf{A}}(u_h^{k,i})\mathbf{r}_h\cdot\mathbf{n}_e, [\![u_h^{k,i}]\!]\rangle_e,$$

for all $q_h \in \mathbb{P}_l(e)$ and all $\mathbf{r}_h \in [\mathbb{P}_{l-1}(K)]^d$.

Verification of the assumptions – upper bound

Definition (Construction of f_h, $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $f_h := \Pi_l f$ and $\overline{\boldsymbol{\sigma}}_h^{k,i} := \mathbf{I}_l^{\mathrm{RTN}}(\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}))$.

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_l(\mathcal{T}_h) \Rightarrow$ $\mathbf{a}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$.

Verification of the assumptions – upper bound

Definition (Construction of f_h, $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $f_h := \Pi_l f$ and $\overline{\boldsymbol{\sigma}}_h^{k,i} := \mathbf{I}_l^{\mathrm{RTN}}(\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}))$.

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_l(\mathcal{T}_h) \Rightarrow$ $\mathbf{a}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$.

Verification of the assumptions – upper bound

Definition (Construction of f_h, $\overline{\boldsymbol{\sigma}}_h^{k,i}$)

Set $f_h := \Pi_l f$ and $\overline{\boldsymbol{\sigma}}_h^{k,i} := \mathbf{I}_l^{\mathrm{RTN}}(\boldsymbol{\sigma}(u_h^{k,i}, \nabla u_h^{k,i}))$.

Lemma (Assumptions A and B)

Assumptions A and B hold.

Comments

- $\|\mathbf{a}_h^{k,i}\|_{q,K} \to 0$ as the linear solver converges by definition.
- $\|\mathbf{l}_h^{k,i}\|_{q,K} \to 0$ as the nonlinear solver converges by the construction of $\mathbf{l}_h^{k,i}$.
- Both $(\mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i})$ and $\mathbf{d}_h^{k,i}$ belong to $\mathbf{RTN}_l(\mathcal{T}_h) \Rightarrow$ $\mathbf{a}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$ and $\mathbf{t}_h^{k,i} \in \mathbf{RTN}_l(\mathcal{T}_h)$.

Verification of the assumptions – efficiency

Lemma (Assumption C)

Assumption C holds.

Comments

- $\mathbf{d}_h^{k,i}$ close to $\overline{\sigma}_h^{k,i}$
- approximation properties of Raviart–Thomas–Nédélec spaces

Verification of the assumptions – efficiency

Lemma (Assumption C)

Assumption C holds.

Comments

- $\mathbf{d}_h^{k,i}$ close to $\overline{\boldsymbol{\sigma}}_h^{k,i}$
- approximation properties of Raviart–Thomas–Nédélec spaces

Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

Summary

Discretization methods

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

Linearizations

- fixed point
- Newton

Linear solvers

- independent of the linear solver

. . . all Assumptions A to C verified

Numerical experiment I

Model problem

- p-Laplacian

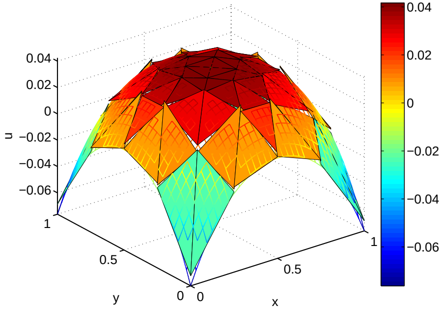
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = u_0 \qquad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

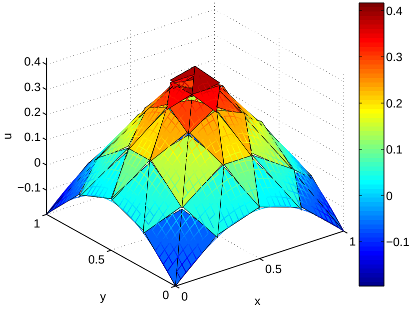
$$u(x,y) = -\frac{p-1}{p}\left((x-\tfrac{1}{2})^2 + (y-\tfrac{1}{2})^2\right)^{\frac{p}{2(p-1)}} + \frac{p-1}{p}\left(\frac{1}{2}\right)^{\frac{p}{p-1}}$$

- tested values $p = 1.5$ and 10
- nonconforming finite elements

Analytical and approximate solutions

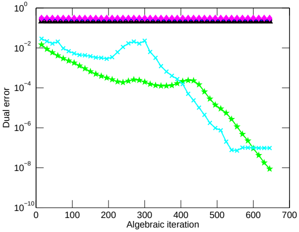


Case $p = 1.5$

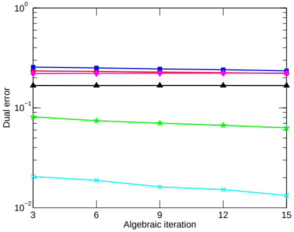


Case $p = 10$

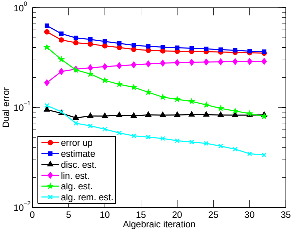
Error and estimators as a function of CG iterations, $p = 10$, 6th level mesh, 6th Newton step.



Newton

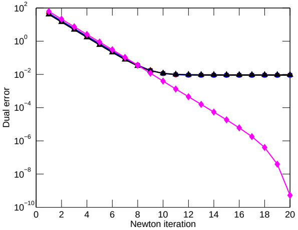


inexact Newton

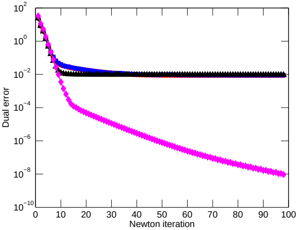


ad. inexact Newton

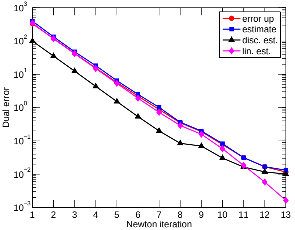
Error and estimators as a function of Newton iterations, $p = 10$, 6th level mesh



Newton

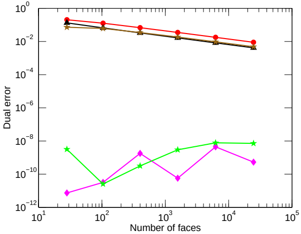


inexact Newton

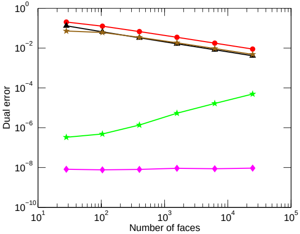


ad. inexact Newton

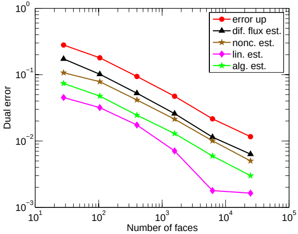
Error and estimators, $p = 10$



Newton

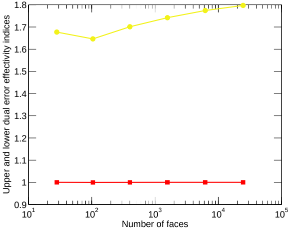


inexact Newton

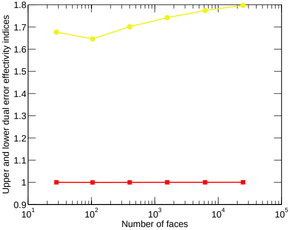


ad. inexact Newton

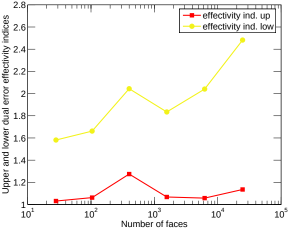
Effectivity indices, $p = 10$



Newton

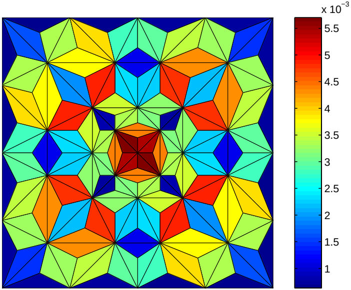


inexact Newton

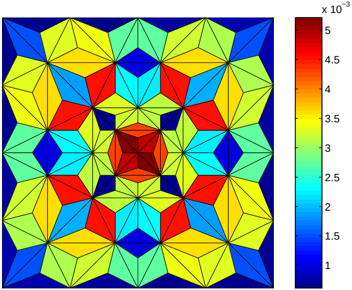


ad. inexact Newton

Error distribution, $p = 10$

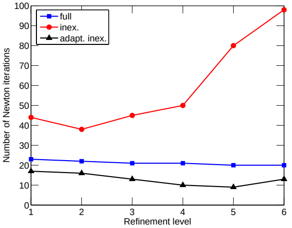


Estimated error distribution

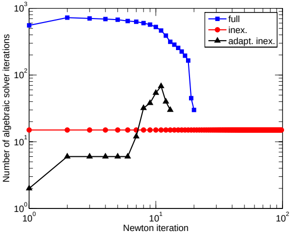


Exact error distribution

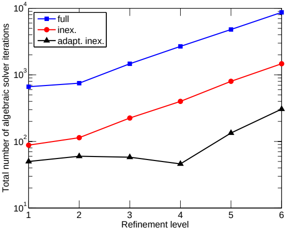
Newton and algebraic iterations, $p = 10$



Newton it. / refinement

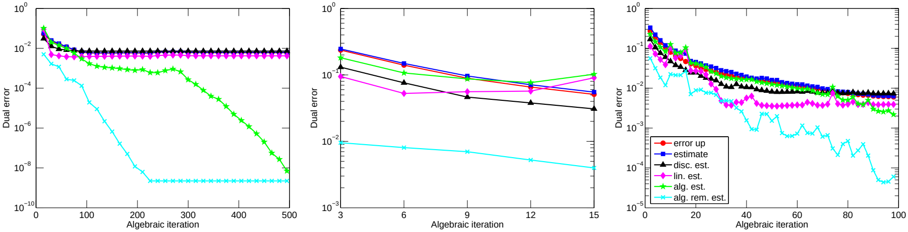


alg. it. / Newton step



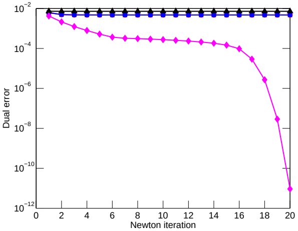
alg. it. / refinement

Error and estimators as a function of CG iterations, $p = 1.5$, 6th level mesh, 1st Newton step.

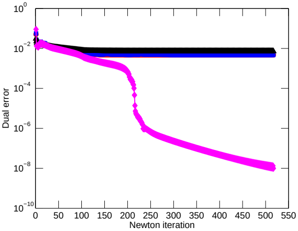


Newton inexact Newton ad. inexact Newton

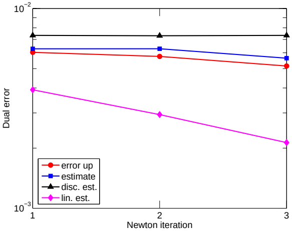
Error and estimators as a function of Newton iterations, $p = 1.5$, 6th level mesh



Newton

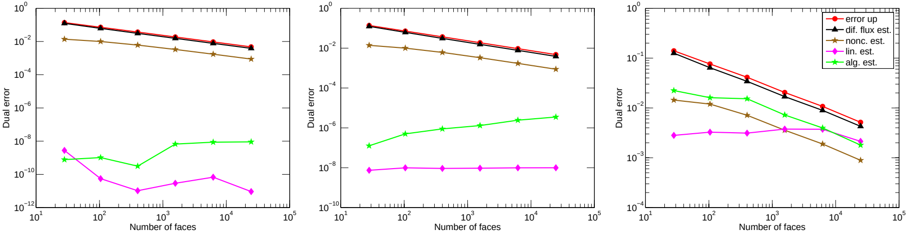


inexact Newton



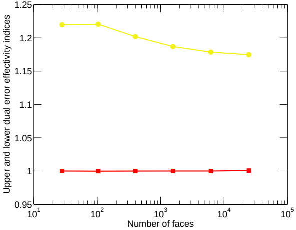
ad. inexact Newton

Error and estimators, $p = 1.5$

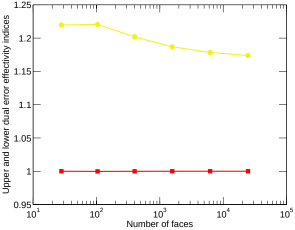


Newton inexact Newton ad. inexact Newton

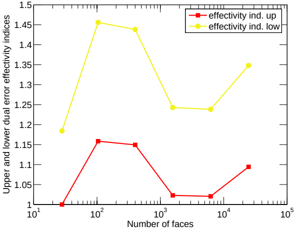
Effectivity indices, $p = 1.5$



Newton

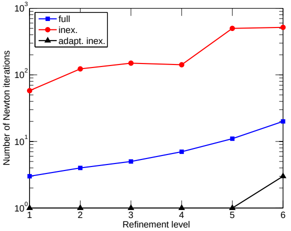


inexact Newton

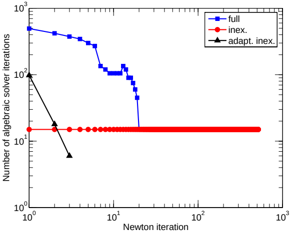


ad. inexact Newton

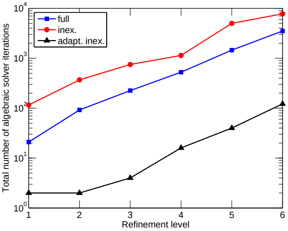
Newton and algebraic iterations, $p = 1.5$



Newton it. / refinement



alg. it. / Newton step



alg. it. / refinement

Numerical experiment II

Model problem

- p-Laplacian

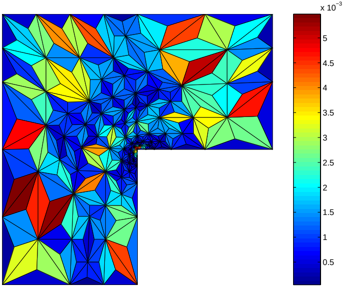
$$\nabla\cdot(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } \Omega,$$
$$u = u_0 \quad \text{on } \partial\Omega$$

- weak solution (used to impose the Dirichlet BC)

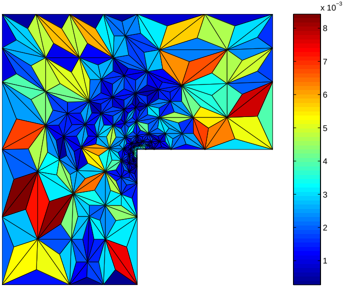
$$u(r,\theta) = r^{\frac{7}{8}} \sin(\theta \tfrac{7}{8})$$

- $p = 4$, L-shape domain, singularity in the origin (Carstensen and Klose (2003))
- nonconforming finite elements

Error distribution on an adaptively refined mesh

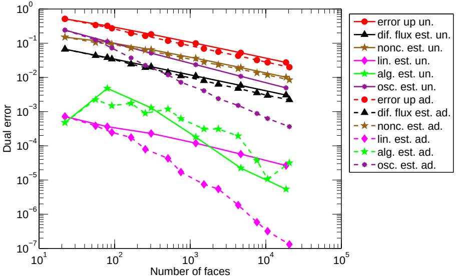


Estimated error distribution

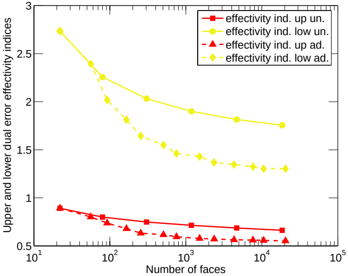


Exact error distribution

Estimated and actual errors and the effectivity index

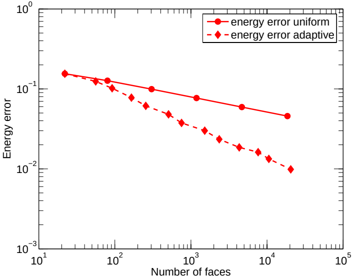


Estimated and actual errors

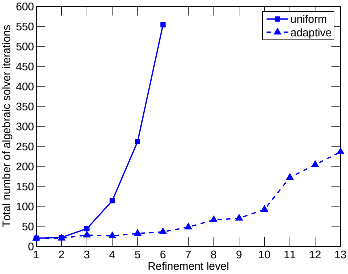


Effectivity index

Energy error and overall performance



Energy error



Overall performance

Outline

Two-phase flow in porous media

Two-phase flow in porous media

$$
\begin{aligned}
\partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha, && \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
-\lambda_\alpha(s_\mathrm{w})\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z) &= \mathbf{u}_\alpha, && \alpha \in \{\mathrm{n}, \mathrm{w}\},\\
s_\mathrm{n} + s_\mathrm{w} &= 1,\\
p_\mathrm{n} - p_\mathrm{w} &= p_c(s_\mathrm{w})
\end{aligned}
$$

Mathematical issues

- coupled system
- unsteady, nonlinear
- elliptic–parabolic degenerate type
- dominant advection

Two-phase flow in porous media

Two-phase flow in porous media

$$\begin{aligned} \partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha, & \alpha \in \{\mathrm{n}, \mathrm{w}\}, \\ -\lambda_\alpha(s_\mathrm{w})\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z) &= \mathbf{u}_\alpha, & \alpha \in \{\mathrm{n}, \mathrm{w}\}, \\ s_\mathrm{n} + s_\mathrm{w} &= 1, \\ p_\mathrm{n} - p_\mathrm{w} &= p_c(s_\mathrm{w}) \end{aligned}$$

Mathematical issues

- coupled system
- unsteady, nonlinear
- elliptic–parabolic degenerate type
- dominant advection

Outline

Two-phase flow in porous media

Theorem (A posteriori error estimate distinguishing the error components)

Let

- *n be the time step,*
- *k be the linearization step,*
- *i be the algebraic solver step,*

with the approximations $(s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w},h\tau}^{n,k,i})$. Then

$$|||(s_{\mathrm{w}} - s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w}} - p_{\mathrm{w},h\tau}^{n,k,i})|||_{I_n} \le \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Error components

- $\eta_{\mathrm{sp}}^{n,k,i}$: spatial discretization
- $\eta_{\mathrm{tm}}^{n,k,i}$: temporal discretization
- $\eta_{\mathrm{lin}}^{n,k,i}$: linearization
- $\eta_{\mathrm{alg}}^{n,k,i}$: algebraic solver

Two-phase flow in porous media

Theorem (A posteriori error estimate distinguishing the error components)

Let

- *n be the time step,*
- *k be the linearization step,*
- *i be the algebraic solver step,*

with the approximations $(s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w},h\tau}^{n,k,i})$. Then

$$|||(s_{\mathrm{w}} - s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w}} - p_{\mathrm{w},h\tau}^{n,k,i})|||_{I_n} \le \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Error components

- $\eta_{\mathrm{sp}}^{n,k,i}$: spatial discretization
- $\eta_{\mathrm{tm}}^{n,k,i}$: temporal discretization
- $\eta_{\mathrm{lin}}^{n,k,i}$: linearization
- $\eta_{\mathrm{alg}}^{n,k,i}$: algebraic solver

Local estimators

- *spatial estimators*

$$\eta_{\mathrm{sp},K}^{n,k,i}(t) := \Bigg\{ \sum_{\alpha\in\{\mathrm{n},\mathrm{w}\}} \Big(\|\mathbf{d}_{\alpha,h}^{n,k,i} - \mathbf{v}_\alpha(p_{\mathrm{w},h}^{n,k,i}, s_{\mathrm{w},h}^{n,k,i})\|_K$$
$$+ h_K/\pi \|q_\alpha^n - \partial_t^n(\phi s_{\alpha,h\tau}^{n,k,i}) - \nabla\cdot\mathbf{u}_{\alpha,h}^{n,k,i}\|_K\Big)^2$$
$$+ \Big(\|\underline{\mathbf{K}}(\lambda_{\mathrm{w}}(s_{\mathrm{w},h\tau}^{n,k,i}) + \lambda_{\mathrm{n}}(s_{\mathrm{w},h\tau}^{n,k,i}))\nabla(\mathfrak{p}(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i}) - \bar{\mathfrak{p}}_{h\tau}^{n,k,i})\|_K(t)\Big)^2$$
$$+ \Big(\|\underline{\mathbf{K}}\nabla(\mathfrak{q}(s_{\mathrm{w},h\tau}^{n,k,i}) - \bar{\mathfrak{q}}_{h\tau}^{n,k,i})\|_K(t)\Big)^2 \Bigg\}^{\frac{1}{2}}$$

- *temporal estimators*

$$\eta_{\mathrm{tm},K,\alpha}^{n,k,i}(t) := \|\mathbf{v}_\alpha(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i})(t) - \mathbf{v}_\alpha(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i})(t^n)\|_K \qquad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

- *linearization estimators*

$$\eta_{\mathrm{lin},K,\alpha}^{n,k,i} := \|\mathbf{l}_{\alpha,h}^{n,k,i}\|_K \qquad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

- *algebraic estimators*

$$\eta_{\mathrm{alg},K,\alpha}^{n,k,i} := \|\mathbf{a}_{\alpha,h}^{n,k,i}\|_K \qquad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

Global estimators

Global estimators

$$\eta_{\mathrm{sp}}^{n,k,i} := \left\{ 3 \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{sp},K}^{n,k,i}(t))^2 \,\mathrm{d}t \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{tm}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \int_{I_n} \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{tm},K,\alpha}^{n,k,i}(t))^2 \,\mathrm{d}t \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{lin}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \tau^n \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{lin},K,\alpha}^{n,k,i})^2 \right\}^{\frac{1}{2}},$$

$$\eta_{\mathrm{alg}}^{n,k,i} := \left\{ \sum_{\alpha \in \{\mathrm{n},\mathrm{w}\}} \tau^n \sum_{K \in \mathcal{T}_h^n} (\eta_{\mathrm{alg},K,\alpha}^{n,k,i})^2 \right\}^{\frac{1}{2}}$$

Outline

Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \leq n \leq N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the fluxes are given by

$$F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\lambda_{\mathrm{w}}(s^n_{\mathrm{w},K}) + \lambda_{\mathrm{w}}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}|\frac{\bar{p}^n_{\mathrm{w},K'} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|,$$

$$F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\lambda_{\mathrm{n}}(s^n_{\mathrm{w},K}) + \lambda_{\mathrm{n}}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},K'} + \pi(s^n_{\mathrm{w},K'}) - (\bar{p}^n_{\mathrm{w},K} + \pi(s^n_{\mathrm{w},K}))}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|$$

Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \leq n \leq N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the fluxes are given by

$$F_{\mathrm{w},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\lambda_\mathrm{w}(s^n_{\mathrm{w},K}) + \lambda_\mathrm{w}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}|\frac{\bar{p}^n_{\mathrm{w},K'} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|,$$

$$F_{\mathrm{n},e_{KK'}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\lambda_\mathrm{n}(s^n_{\mathrm{w},K}) + \lambda_\mathrm{n}(s^n_{\mathrm{w},K'})}{2}|\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},K'} + \pi(s^n_{\mathrm{w},K'}) - (\bar{p}^n_{\mathrm{w},K} + \pi(s^n_{\mathrm{w},K}))}{|\mathbf{x}_K - \mathbf{x}_{K'}|}|e_{KK'}|$$

Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n} |K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$\begin{aligned} F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) :=& F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}). \end{aligned}$$

Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KK'} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$\begin{aligned} F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) :=& F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) \\ &+ \sum_{M \in \{K,K'\}} \frac{\partial F_{\alpha,e_{KK'}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}). \end{aligned}$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K,1)_{e_{KK'}} := F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),$$
$$((\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K,1)_{e_{KK'}} := F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),$$
$$\mathbf{a}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i+\nu}+\mathbf{l}_{\alpha,h}^{n,k,i+\nu}-(\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})$$

Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

$$-\lambda_{\mathrm{w}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{w},h}^{n,k,i}|_K) = \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,$$
$$p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) = \bar{p}_{\mathrm{w},K}^{n,k,i},$$

$$-\lambda_{\mathrm{n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{n},h}^{n,k,i}|_K) = \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,$$
$$p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) = \pi(s_{\mathrm{w},K}^{n,k,i})+\bar{p}_{\mathrm{w},K}^{n,k,i}$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$
\begin{aligned}
(\mathbf{d}_{\alpha,h}^{n,k,i} \cdot \mathbf{n}_K, 1)_{e_{KK'}} &:= F_{\alpha,e_{KK'}}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
((\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i}) \cdot \mathbf{n}_K, 1)_{e_{KK'}} &:= F_{\alpha,e_{KK'}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
\mathbf{a}_{\alpha,h}^{n,k,i} &:= \mathbf{d}_{\alpha,h}^{n,k,i+\nu} + \mathbf{l}_{\alpha,h}^{n,k,i+\nu} - (\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})
\end{aligned}
$$

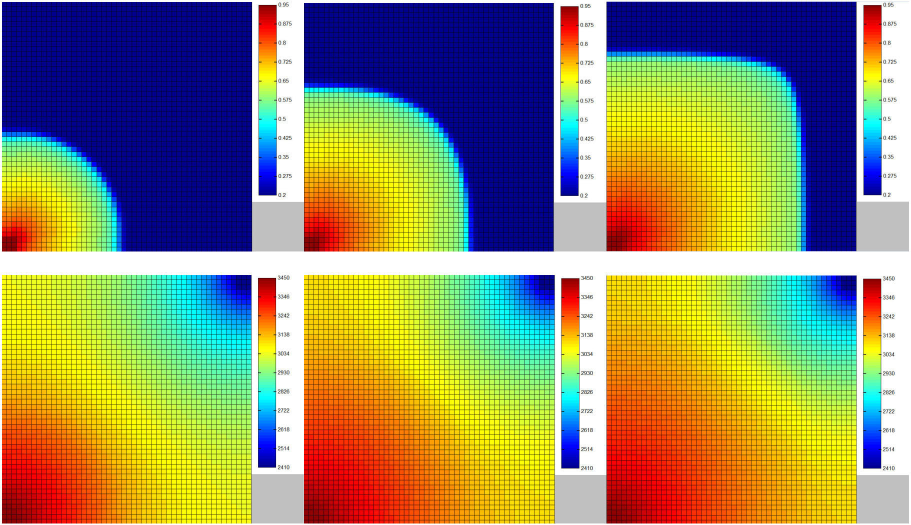
Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

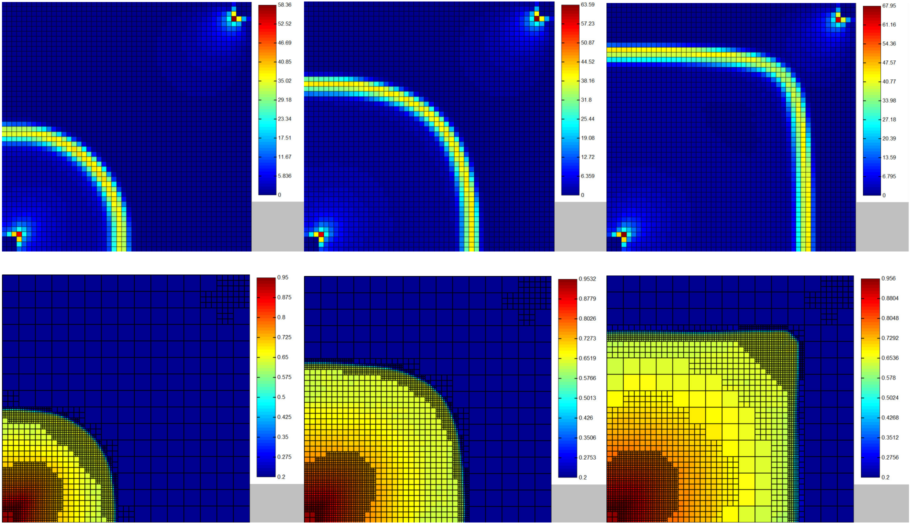
$$
\begin{aligned}
-\lambda_{\mathrm{w}}(s_{\mathrm{w},K}^{n,k,i}) \underline{\mathbf{K}} \nabla (p_{\mathrm{w},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,\\
p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) &= \bar{p}_{\mathrm{w},K}^{n,k,i},
\end{aligned}
$$

$$
\begin{aligned}
-\lambda_{\mathrm{n}}(s_{\mathrm{w},K}^{n,k,i}) \underline{\mathbf{K}} \nabla (p_{\mathrm{n},h}^{n,k,i}|_K) &= \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,\\
p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) &= \pi(s_{\mathrm{w},K}^{n,k,i}) + \bar{p}_{\mathrm{w},K}^{n,k,i}
\end{aligned}
$$

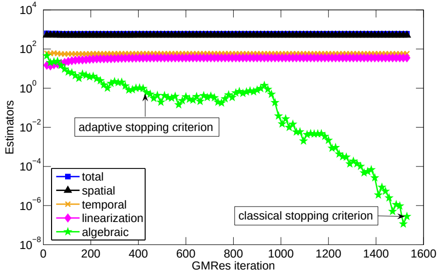
Water saturation/water pressure evolution



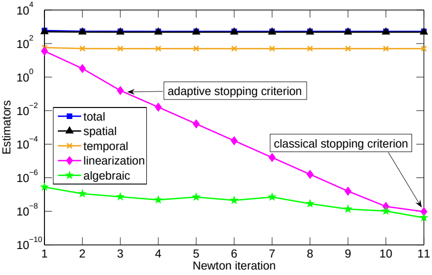
Estimators/meshes evolution



Estimators and stopping criteria

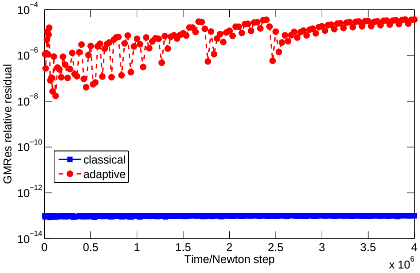


Estimators in function of GMRes iterations

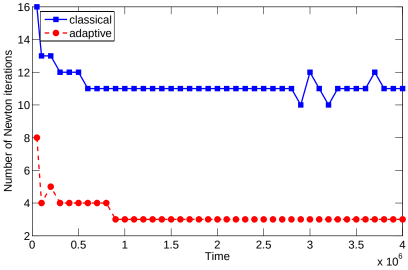


Estimators in function of Newton iterations

GMRes relative residual/Newton iterations

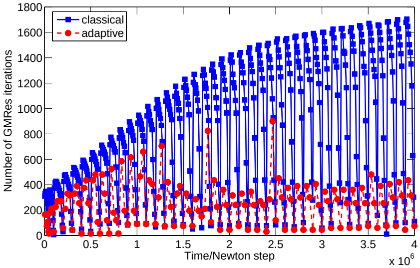


GMRes relative residual

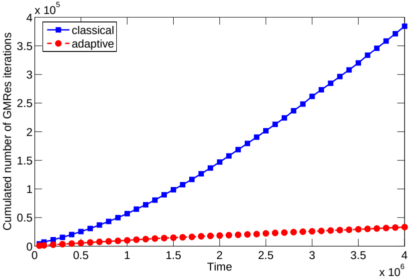


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Outline

Conclusions

Entire adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- **"smart online decisions"**: algebraic step / linearization step / space mesh refinement / time step modification
- important **computational savings**
- guaranteed and robust error upper bound via **a posteriori estimates**

Future directions

- other coupled nonlinear systems
- convergence and optimality

Conclusions

Entire adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- **"smart online decisions"**: algebraic step / linearization step / space mesh refinement / time step modification
- important **computational savings**
- guaranteed and robust error upper bound via **a posteriori estimates**

Future directions

- other coupled nonlinear systems
- convergence and optimality

Bibliography

Bibliography

- VOHRALÍK M., *A posteriori error estimates for efficiency and error control in numerical simulations*, Lecture notes, Course NM497, Université Pierre et Marie Curie, Paris, 2012.
- JIRÁNEK P., STRAKOŠ Z., VOHRALÍK M., A posteriori error estimates including algebraic error and stopping criteria for iterative solvers. *SIAM J.Sci.Comput.* **32** (2010),1567–1590.
- ERN A., VOHRALÍK M., Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs, HAL Preprint 00681422.
- VOHRALÍK M., WHEELER M. F., A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows, HAL Preprint 00633594.

Díky za Vaši pozornost!