Polynomial-degree-robust multilevel and domain decomposition methods with optimal step-sizes for mixed finite element discretizations of elliptic problems

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- Introduction
- The model problem and its mixed finite element approximation
- Solvers for mixed finite elements
- 2 Multigrid for high-order mixed finite elements
  - A hierarchy of meshes and spaces
  - The solver
  - Functional writing
  - Main results
- Oomain decomposition for high-order mixed finite elements
  - Numerical experiments
    - Smooth solution and uniform mesh refinement
    - Rough solution and adaptive mesh refinement
- 5 Conclusions



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### The model problem

### The Dracy porous media flow problem

Find the pressure head  $\gamma : \Omega \to \mathbb{R}$  and the Darcy velocity  $\mathbf{u} : \Omega \to \mathbb{R}^d$  such that

$oldsymbol{u}=-oldsymbol{K} abla\gamma$	in $\Omega$ ,
$ abla \cdot oldsymbol{u} = f$	in $\Omega$ ,
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### Setting

- $\Omega \subset \mathbb{R}^d$ ,  $1 \le d \le 3$ : interval/polygon/polyhedron
- $\mathbf{K} \in [L^{\infty}(\Omega)]^{d \times d}$ : symmetric and positive definite diffusion tensor
- $f \in L^2(\Omega)$  of mean value 0: source term

### Mixed finite element approximation

#### Mixed finite element approximation

Find  $\boldsymbol{u}_J \in \boldsymbol{V}_J$  and  $\gamma_J \in \boldsymbol{W}_J$  such that

$$\begin{aligned} (\boldsymbol{K}^{-1}\boldsymbol{u}_J,\boldsymbol{v}_J) - (\gamma_J,\nabla\cdot\boldsymbol{v}_J) &= 0 \qquad \forall \boldsymbol{v}_J \in \boldsymbol{V}_J, \\ (\nabla\cdot\boldsymbol{u}_J,w_J) &= (f,w_J) \qquad \forall w_J \in \boldsymbol{W}_J. \end{aligned}$$

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### Setting

- $\mathcal{T}_{J}$ : simplicial mesh of  $\Omega$
- *V<sub>J</sub>* := {*v<sub>J</sub>* ∈ *H*<sub>0</sub>(div, Ω), *v<sub>J</sub>*|<sub>K</sub> ∈ **RT**<sub>p</sub>(K) ∀K ∈ *T<sub>J</sub>*}: Raviart–Thomas space (piecewise vector-valued polynomials on *T<sub>J</sub>*) of degree *p*, normal trace continuous and 0 on ∂Ω (*H*<sub>0</sub>(div, Ω)-conforming)
- $W_J$ : piecewise polynomials on  $\mathcal{T}_J$  of degree p and mean value 0 on  $\Omega$



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### MG solvers for mixed finite elements

### Saddle-point solvers

• after a choice of basis: find algebraic vectors U and  $\Gamma$  such that

$$\begin{pmatrix} \mathbb{A} & \mathbb{B}^t \\ \mathbb{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathsf{U} \\ \mathsf{\Gamma} \end{pmatrix} = \begin{pmatrix} \mathsf{0} \\ \mathsf{F} \end{pmatrix}$$

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### SPD reformulations and solvers

• equivalent reformulation via hybridization: find algebraic vector Λ such that

$$\mathbb{S}\Lambda=G$$

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- preconditioned conjugate gradients possible,

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### SPD reformulations and solvers

• equivalent reformulation via hybridization: find algebraic vector Λ such that

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- symmetric and positive definite system matrix
- preconditioned conjugate gradients possible, multigrid not straightforward: Λ (pressure heads on the mesh faces) belong to non-nested spaces (Brenner (1992), Chen (1996), Wheeler, Yotov (2000))

### Flux-only reformulation and corresponding MG solvers

### Equivalent reformulation

Find  $\boldsymbol{u}_{J} \in \boldsymbol{V}_{J}^{f}$  such that

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• 
$$V_J^g := \{ v_J \in V_J : (\nabla \cdot v_J, w_J) = (g, w_J) \ \forall w_J \in W_J \}$$

- only flux unknowns
- multigrid becomes easily possible (Mathew (1993), Ewing, Wang (1994), Hiptmair, Hoppe (1999))

### DD solvers for mixed finite elements

#### **Domain decomposition solvers**

- Glowinski, Wheeler (1988), Cowsar, Mandel, Wheeler (1995), ...
- Ewing, Wang (1992), ...

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- p-robustness
- unified treatment of multigrid and DD

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# A hierarchy of meshes

### **Example:** Two mesh hierarchies with J = 3 refinements.

**Assumption:** The meshes  $\{\mathcal{T}_j\}_{0 \le j \le J}$  can be quasi-uniform or graded, satisfying:

- quasi-uniform  $\mathcal{T}_0$ ,
- shape-regularity,
- maximum strength of refinement.

For given polynomial degree p and J, choose *increasing* level-wise polynomial degrees  $p_j$ ,  $j \in \{0, ..., J\}$ ,

and define the spaces

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# -cycle multigrid



# V(0,1)-cycle multigrid



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### V(0,1)-cycle multigrid with block-Jacobi smoothing



- zero pre- and a single post-smoothing step
- cheapest RT<sub>0</sub> coarse solve
- additive Schwarz/block-Jacobi smoothing  $\rho_i^j$

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p-robust multilevel and domain decomposition methods for mixed finite elements

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- level-wise step-sizes  $\lambda_i^i$  in correction stage: optimally chosen by line searchia

# V(0,1)-cycle multigrid with block-Jacobi smoothing and line search



- V-cycle geometric multigrid as in Ewing, Wang (1994)
- zero pre- and a single post-smoothing step
- cheapest RT<sub>0</sub> coarse solve
- additive Schwarz/block-Jacobi smoothing  $\rho_i^i$ : fully parallel on each level
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#### • Functional writing

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#### Pythagorean error formula and bound on the algebraic error

#### Theorem (Pythagorean error representation)

There holds

$$\underbrace{\left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_{J}-\boldsymbol{u}_{J}^{i+1})\right\|^{2}}_{new \ error} = \underbrace{\left\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_{J}-\boldsymbol{u}_{J}^{i})\right\|^{2}}_{old \ error} - \underbrace{\sum_{j=0}^{J} \left(\lambda_{j}^{i} \left\|\boldsymbol{K}^{-1/2} \boldsymbol{\rho}_{j}^{i}\right\|\right)^{2}}_{\left(\eta_{alg}^{i}\right)^{2}}.$$

#### Corollary (Guaranteed lower bound on the algebraic error)

There holds:

$$\eta_{\mathsf{alg}}^i \leq \left\| \boldsymbol{K}^{-1/2} (\boldsymbol{u}_J - \boldsymbol{u}_J^i) \right\|.$$

 similar situation to the conjugate gradients method, see Meurant (1997) and Strakoš and Tichý (2002)

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### *p*-robust error contraction and algebraic estimator efficiency

Theorem (p-robust error contraction of the multilevel solver)

There holds

$$\|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i+1})\| \leq \alpha \|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_J - \boldsymbol{u}_J^{i})\|, \qquad 0 < \alpha(\kappa_{\mathcal{T}}, \boldsymbol{d}, \boldsymbol{K}, J) < 1.$$

Theorem (p-robust reliable and efficient bound on the algebraic error)

There holds 
$$\eta_{\text{alg}}^{i} \leq \|\boldsymbol{K}^{-1/2}(\boldsymbol{u}_{J} - \boldsymbol{u}_{J}^{i})\|$$
 and, with  $\beta = \sqrt{1 - \alpha^{2}}$ ,  
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Corollary (Equivalence of the two main results)

The solver <code>contraction</code> is <code>equivalent</code> to the <code>efficiency</code> of the estimator  $\eta^i_{\mathsf{alc}}$ 

•  $\alpha$  is independent of the polynomial degree p

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#### *p*-robust error contraction and algebraic estimator efficiency

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#### Corollary (Equivalence of the two main results)

The solver contraction is equivalent to the efficiency of the estimator  $\eta_{alg}^i$ .

- $\alpha$  is independent of the polynomial degree p
  - the dependence on J is at most *linear* under minimal H<sup>1</sup>-regularity
  - complete *independence* of *J* is obtained under *H*<sup>2</sup>-regularity
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# Domain decomposition for high-order mixed finite elements



Coarse grid  $\mathcal{T}_H$  (solid line), fine grid  $\mathcal{T}_J$  (dashed line), patch domain  $\omega^a$  (nria

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## Smooth solution and uniform mesh refinement

## Setting

- $\Omega$ : unit square
- **K** = Id
- $\gamma(\mathbf{x}, \mathbf{y}) = \cos(\pi \mathbf{x}) \cos(\pi \mathbf{y})$
- $T_0$  with mesh size  $h_0 = 0.3$ , uniform mesh refinement

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## **Contraction factors**



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# Effectivity indices of the guaranteed lower bound $\eta_{alg}^{i}$



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Number of iterations to decrease  $\eta_{\rm alg}^i$  by  $10^5$ 

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p	J = 2	J = 3	J = 4
2	9	9	8
3	9	8	7
6	6	5	4

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## Rough solution and adaptive mesh refinement

## Setting

- Ω: unit square
- *K* = Id

• 
$$\gamma(x, y) = \tan^{-1}(\alpha(r - r_0))$$
, where  $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$ 

- $\alpha = 1000, x_c = 0.5, y_c = 0.5, r_0 = 0.01$
- $T_0$  with mesh size  $h_0 = 0.3$ , adaptive mesh refinement

## Adaptive mesh: J = 12, p = 6, & Dörfler marking parameter $\theta = 0.8$





## **Contraction factors**



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р	J = 3	J = 6	<i>J</i> = 12
2	13	12	9
3	12	11	8
6	13	10	6

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## p-robust MG and DD methods for mixed finite elements

✓ *p*-robust **algebraic error** contraction

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- A. MIRAÇI, J. PAPEŽ, M. VOHRALÍK, A-posteriori-steered *p*-robust multigrid with optimal step-sizes and adaptive number of smoothing steps, *SIAM J. Sci. Comput.* 43 (2021), S117–S145.

# Thank you for your attention!

## Outline







## Localized algebraic error estimate

## Theorem (Localized algebraic error estimate)

There holds

$$(\eta_{alg}^{i})^{2} = \|\boldsymbol{K}^{-1/2}\boldsymbol{\rho}_{0}^{i}\|^{2} + \sum_{j=1}^{J} \lambda_{j}^{i} \sum_{\boldsymbol{a} \in \mathcal{V}_{j}} \|\boldsymbol{K}^{-1/2}\boldsymbol{\rho}_{j,\boldsymbol{a}}^{i}\|_{\omega_{j}^{\boldsymbol{a}}}^{2}.$$









# Solvers for high-order finite elements

# Algebraic problem Find $U_J \in \mathbb{R}^{|V_J^{\rho}|}$ such that

$$\mathbb{A}_J \mathsf{U}_J = \mathsf{F}_J$$

- $A_J$  less and less sparse for big p
- A<sub>J</sub> worse and worse conditioned for big µ
- $A_J$  looses structure on graded meshes  $\mathcal{T}_J$
- $\mathbb{A}_J$  is dependent on the basis of  $V_J^p$

Black-box iterative solvers do not work well for high  $\rho$  & on highly graded meshes  $T_J$ 

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p-robust solver/preconditoner

J. Schöberl, M. Melenk, C. Pechstein, S. Zaglmayr: Additive Schwarz preconditioning for p-version triangular and tetrahedral finite elements (2008): globally coupled p = 1 sub-system; p > 1 treated locally on vertex patches

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# Solvers for high-order finite elements

## Algebraic problem

Find  $U_{I} \in \mathbb{R}^{|V_{J}^{P}|}$  such that

$$\mathbb{A}_J \mathsf{U}_J = \mathsf{F}_J$$

- A<sub>1</sub> less and less sparse for big p
- $A_{I}$  worse and worse conditioned for big p
- $\mathbb{A}_{\mathcal{I}}$  looses structure on graded meshes  $\mathcal{T}_{\mathcal{I}}$
- $\mathbb{A}_{I}$  is dependent on the basis of  $V_{I}^{p}$

## **Solvers** for high-order finite elements

## Algebraic problem

Find U  $\iota \in \mathbb{R}^{|V_J^{\rho}|}$  such that

$$\mathbb{A}_J \mathsf{U}_J = \mathsf{F}_J$$

- A less and less sparse for big p
- $A_{i}$  worse and worse conditioned for big p
- $\mathbb{A}_{\mathcal{I}}$  looses structure on graded meshes  $\mathcal{T}_{\mathcal{I}}$
- A<sub>1</sub> is dependent on the basis of  $V_1^p$

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# Solvers for high-order finite elements

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J. Schöberl, M. Melenk, C. Pechstein, S. Zaglmayr: Additive Schwarz preconditioning for p-version triangular and tetrahedral finite elements (2008): globally coupled p = 1 sub-system; p > 1 treated locally on vertex patches *J*-robust solver on graded meshes J. Xu, L. Chen, R. Nochetto: Optimal multilevel methods for H(grad), H(curl), a H(div) systems on graded and unstructured grids (2009):

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Localized algebraic error estimate High-order finite element solvers

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