A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows

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joint work with C. Cancès, I. S. Pop, and M. F. Wheeler

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Outline

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Multiphase porous media flows

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- highly nonlinear (degenerate) systems of PDEs
- involve phase transitions
- feature evolving sharp fronts
- different time and space scales
- highly contrasted, discontinuous coefficients
- unstructured and nonmatching grids

Goals of this work

- derive fully computable a posteriori **error upper bounds**
- distinguish different **error components**
  - time step choice & mesh adaptivity
  - **stopping criteria** for **linear** and **nonlinear** solvers

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Previous results

Model analysis

- Kröner & Luckhaus (1984)
- Chavent & Jaffré (1986)
- Antontsev, Kazhikhov, & Monakhov (1990)
- Arbogast (1992)
- Chen (2001)
- lately: van Duijn, Mikelić, & Pop; Cancès, Gallouët, & Porretta; Khalil & Saad; Amaziane, Jurak, & Keko ...

Convergence and a priori estimates

- Chen & Ewing (2001)
- Michel (2003)
- Eymard, Herbin, & Michel (2003)
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A posteriori indicators

- Chen & Ewing (2003)
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1. Introduction
2. Setting
   - The two-phase flow model
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   - Extension to multiphase compositional flows
5. Conclusions and future directions

Two-phase flow

The model

$$\partial_t(\phi s_\alpha) + \nabla\cdot\Big(\overbrace{-\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z)}^{\text{Darcy velocity } \mathbf{u}_\alpha}\Big) = q_\alpha, \qquad \alpha \in \{\mathrm{n}, \mathrm{w}\},$$

$$s_\mathrm{n} + s_\mathrm{w} = 1,$$

$$p_\mathrm{n} - p_\mathrm{w} = p_\mathrm{c}(s_\mathrm{w})$$

- two immiscible, incompressible fluids
- space–time domain $\Omega \times (0, T)$
- + initial & boundary conditions
- p_n, p_w: unknown nonwetting and wetting phase pressures
- s_n, s_w: unknown nonwetting and wetting phase saturations
- $p_\mathrm{c}(\cdot)$: the nonlinear capillary pressure
- $k_{\mathrm{r},\alpha}(\cdot)$: the nonlinear relative permeability
- ϕ porosity; $\underline{\mathbf{K}}$ permeability tensor; μ_α, ρ_α, q_α: viscosities, densities, sources; z vertical coordinate; g gravity

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Outline

Global and complementary pressures

Global pressure

$$\mathfrak{p}(s_{\rm w}, p_{\rm w}) := p_{\rm w} + \int_0^{s_{\rm w}} \frac{\lambda_{\rm n}(a)}{\lambda_{\rm w}(a) + \lambda_{\rm n}(a)} p_{\rm c}'(a) {\rm d}a$$

Complementary pressure

$$\mathfrak{q}(s_{\rm w}) := -\int_0^{s_{\rm w}} \frac{\lambda_{\rm w}(a)\lambda_{\rm n}(a)}{\lambda_{\rm w}(a) + \lambda_{\rm n}(a)} p_{\rm c}'(a) {\rm d}a$$

Comments

- phase mobilities $\lambda_\alpha(a) := k_{{\rm r},\alpha}(a)/\mu_\alpha$, $\alpha \in \{{\rm n}, {\rm w}\}$
- necessary for the correct definition of the weak solution
- equivalent Darcy velocities expressions

$$\mathbf{v}_{\rm w}(s_{\rm w}, p_{\rm w}) := -\underline{\mathbf{K}}\big(\lambda_{\rm w}(s_{\rm w})\nabla\mathfrak{p}(s_{\rm w}, p_{\rm w}) + \nabla\mathfrak{q}(s_{\rm w}) + \lambda_{\rm w}(s_{\rm w})\rho_{\rm w} g\nabla z\big),$$
$$\mathbf{v}_{\rm n}(s_{\rm w}, p_{\rm w}) := -\underline{\mathbf{K}}\big(\lambda_{\rm n}(s_{\rm w})\nabla\mathfrak{p}(s_{\rm w}, p_{\rm w}) - \nabla\mathfrak{q}(s_{\rm w}) + \lambda_{\rm n}(s_{\rm w})\rho_{\rm n} g\nabla z\big)$$

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Weak formulation

Energy space

$$X := L^2((0,T); H^1_{\mathrm{D}}(\Omega))$$

Definition (Weak solution (Chen 2001))

Find $(s_{\mathrm{w}}, p_{\mathrm{w}})$ such that, with $s_{\mathrm{n}} := 1 - s_{\mathrm{w}}$,

$$s_{\mathrm{w}} \in C([0,T]; L^2(\Omega)), \ s_{\mathrm{w}}(\cdot, 0) = s^0_{\mathrm{w}},$$
$$\partial_t s_{\mathrm{w}} \in L^2((0,T); (H^1_{\mathrm{D}}(\Omega))'),$$
$$\mathfrak{p}(s_{\mathrm{w}}, p_{\mathrm{w}}) \in X,$$
$$\mathfrak{q}(s_{\mathrm{w}}) \in X,$$
$$\int_0^T \{\langle \partial_t(\phi s_\alpha), \varphi \rangle - (\mathbf{v}_\alpha(s_{\mathrm{w}}, p_{\mathrm{w}}), \nabla\varphi) - (q_\alpha, \varphi)\} \mathrm{d}t = 0$$
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Error measure

Error measure $|||(s_{\rm w} - s_{{\rm w},h\tau}, p_{\rm w} - p_{{\rm w},h\tau})|||$

$$\Bigg\{ \sum_{\alpha\in\{{\rm n},{\rm w}\}} \Bigg\{ \sup_{\varphi\in X, \|\varphi\|_X=1} \int_0^T \{\langle \partial_t(\phi s_\alpha) - \partial_t(\phi s_{\alpha,h\tau}), \varphi\rangle$$

$$-(\mathbf{v}_\alpha(s_{\rm w}, p_{\rm w}) - \mathbf{v}_\alpha(s_{{\rm w},h\tau}, p_{{\rm w},h\tau}), \nabla\varphi)\}{\rm d}t \Bigg\}^2 \Bigg\}^{\frac{1}{2}}$$

$$+ \left\{ \inf_{\hat{\mathfrak{p}}\in X} \int_0^T \|\underline{\mathbf{K}}(\lambda_{\rm w}(s_{{\rm w},h\tau}) + \lambda_{\rm n}(s_{{\rm w},h\tau}))\nabla(\mathfrak{p}(s_{{\rm w},h\tau}, p_{{\rm w},h\tau}) - \hat{\mathfrak{p}})\|^2 {\rm d}t \right\}^{\frac{1}{2}}$$

$$+ \left\{ \inf_{\hat{\mathfrak{q}}\in X} \int_0^T \|\underline{\mathbf{K}}\nabla(\mathfrak{q}(s_{{\rm w},h\tau}) - \hat{\mathfrak{q}})\|^2 {\rm d}t \right\}^{\frac{1}{2}}$$

Comments

- dual norm of the residual
- $\mathfrak{p}(s_{{\rm w},h\tau}, p_{{\rm w},h\tau}) \notin X \Rightarrow$ global pressure nonconformity
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Link to functional spaces error measure

Theorem (Link to energy-type error (Cancès, Pop, V. 2013))

Let $(s_{\rm w}, p_{\rm w})$ be the weak solution. Let $(s_{{\rm w},h\tau}, p_{{\rm w},h\tau})$ be arbitrary but such that $\mathfrak{p}(s_{{\rm w},h\tau}, p_{{\rm w},h\tau}) \in X$ and $\mathfrak{q}(s_{{\rm w},h\tau}) \in X + IC$. Then

$$
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&\|s_{\rm w} - s_{{\rm w},h\tau}\|_{L^2((0,T);(H^1_{\rm D}(\Omega))')} + \|\mathfrak{q}(s_{\rm w}) - \mathfrak{q}(s_{{\rm w},h\tau})\|_{L^2(\Omega\times(0,T))} \\
&+ \|\mathfrak{p}(s_{\rm w}, p_{\rm w}) - \mathfrak{p}(s_{{\rm w},h\tau}, p_{{\rm w},h\tau})\|_{L^2((0,T);H^1_{\rm D}(\Omega))} \\
&\le C\left\{ \sum_{\alpha\in\{\rm n,w\}} \left\{ \sup_{\varphi\in X,\, \|\varphi\|_X=1} \int_0^T \{\langle \partial_t(\phi s_\alpha) - \partial_t(\phi s_{\alpha,h\tau}), \varphi\rangle \right.\right. \\
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&+ \|\mathfrak{p}(s_{\rm w}, p_{\rm w}) - \mathfrak{p}(s_{{\rm w},h\tau}, p_{{\rm w},h\tau})\|_{L^2((0,T);H^1_{\rm D}(\Omega))} \\
&\leq C \left\{ \sum_{\alpha\in\{{\rm n},{\rm w}\}} \left\{ \sup_{\varphi\in X,\, \|\varphi\|_X=1} \int_0^T \{ \langle \partial_t(\phi s_\alpha) - \partial_t(\phi s_{\alpha,h\tau}), \varphi\rangle \right.\right. \\
&\left.\left. -(\mathbf{v}_\alpha(s_{\rm w}, p_{\rm w}) - \mathbf{v}_\alpha(s_{{\rm w},h\tau}, p_{{\rm w},h\tau}), \nabla\varphi)\}{\rm d}t \right\}^2 \right\}^{\frac{1}{2}}
\end{aligned}$$

Comments

- conforming approximations: energy-type error controlled by the dual norm of the residual

Link to functional spaces error measure

Theorem (Link to energy-type error (Cancès, Pop, V. 2013))

Let $(s_{\rm w}, p_{\rm w})$ be the weak solution. Let $(s_{{\rm w},h\tau}, p_{{\rm w},h\tau})$ be arbitrary but such that $\mathfrak{p}(s_{{\rm w},h\tau}, p_{{\rm w},h\tau}) \in X$ and $\mathfrak{q}(s_{{\rm w},h\tau}) \in X + IC$. Then

$$
\begin{aligned}
&\|s_{\rm w} - s_{{\rm w},h\tau}\|_{L^2((0,T);(H^1_{\rm D}(\Omega))')} + \|\mathfrak{q}(s_{\rm w}) - \mathfrak{q}(s_{{\rm w},h\tau})\|_{L^2(\Omega\times(0,T))}\\
&+ \|\mathfrak{p}(s_{\rm w}, p_{\rm w}) - \mathfrak{p}(s_{{\rm w},h\tau}, p_{{\rm w},h\tau})\|_{L^2((0,T);H^1_{\rm D}(\Omega))}\\
&\le C\left\{\sum_{\alpha\in\{{\rm n},{\rm w}\}}\left\{\sup_{\varphi\in X,\, \|\varphi\|_X=1}\int_0^T\{\langle\partial_t(\phi s_\alpha) - \partial_t(\phi s_{\alpha,h\tau}), \varphi\rangle\right.\right.\\
&\left.\left.-(\mathbf{v}_\alpha(s_{\rm w}, p_{\rm w}) - \mathbf{v}_\alpha(s_{{\rm w},h\tau}, p_{{\rm w},h\tau}), \nabla\varphi)\}{\rm d}t\right\}^2\right\}^{\frac{1}{2}}
\end{aligned}
$$

Comments

- conforming approximations: energy-type error controlled by the dual norm of the residual

Outline

Outline

Pressure reconstructions

Definition (Global & complementary pressure reconstructions)

Piecewise affine-in-time scalar fields such that

$$\hat{\mathfrak{p}}_{h\tau} \in X,$$
$$\hat{\mathfrak{q}}_{h\tau} \in X.$$

Comments

- $\hat{\mathfrak{p}}_{h\tau}$: global pressure reconstruction if $\mathfrak{p}(s_{\mathrm{w},h\tau}, p_{\mathrm{w},h\tau}) \notin X$
- $\hat{\mathfrak{q}}_{h\tau}$: complementary pressure reconstruction if $\mathfrak{q}(s_{\mathrm{w},h\tau}) \notin X$
- continuity of traces as for the exact solution
- practice: continuous, piecewise polynomials by averaging

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Phase velocities reconstructions

Definition (Equilibrated phase velocities reconstructions)

Piecewise constant-in-time vector fields such that, for all $1 \leq n \leq N$ and $\alpha \in \{\mathrm{n}, \mathrm{w}\}$,

$$\mathbf{u}_{\alpha,h\tau}|_{I_n} \in \mathbf{H}(\mathrm{div}, \Omega)$$

and such that

$$(q_\alpha^n - \partial_t(\phi s_{\alpha,h\tau}|_{I_n}) - \nabla\cdot\mathbf{u}_{\alpha,h\tau}|_{I_n}, 1)_K = 0 \qquad \forall K \in \mathcal{T}_h^n.$$

Comments

- $\mathbf{u}_{\mathrm{n},h\tau}$: nonwetting phase velocity reconstruction
- $\mathbf{u}_{\mathrm{w},h\tau}$: wetting phase velocity reconstruction
- continuity of normal traces and local conservation as for the exact Darcy velocities $\mathbf{u}_\alpha$
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Outline

Basic a posteriori error estimate

Theorem (Basic a posteriori error estimate)

Let $(s_{\mathrm{w}}, p_{\mathrm{w}})$ be the weak solution. Let $(s_{\mathrm{w},h\tau}, p_{\mathrm{w},h\tau})$ be arbitrary (satisfying BC+IC). Let $\hat{\mathfrak{p}}_{h\tau}$, $\hat{\mathfrak{q}}_{h\tau}$, and $\mathbf{u}_{\alpha,h\tau}$, $\alpha \in \{\mathrm{n}, \mathrm{w}\}$, be the pressure and phase velocities reconstructions. Then

$$|||(s_{\mathrm{w}} - s_{\mathrm{w},h\tau}, p_{\mathrm{w}} - p_{\mathrm{w},h\tau})|||$$
$$\leq \left\{ \sum_{n=1}^{N} \sum_{K \in \mathcal{T}_h^n} \left(\eta_K^n (s_{\mathrm{w},h\tau}, p_{\mathrm{w},h\tau}, \hat{\mathfrak{p}}_{h\tau}, \hat{\mathfrak{q}}_{h\tau}, \mathbf{u}_{\mathrm{n},h\tau}, \mathbf{u}_{\mathrm{w},h\tau}) \right)^2 \right\}^{\frac{1}{2}}.$$

Comments

- overall error control
- a posteriori error estimators η_K^n fully computable

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$$|||(s_{\mathrm{w}} - s_{\mathrm{w},h\tau}, p_{\mathrm{w}} - p_{\mathrm{w},h\tau})||| \leq \left\{ \sum_{n=1}^{N} \sum_{K \in \mathcal{T}_h^n} \left(\eta_K^n(s_{\mathrm{w},h\tau}, p_{\mathrm{w},h\tau}, \hat{\mathfrak{p}}_{h\tau}, \hat{\mathfrak{q}}_{h\tau}, \mathbf{u}_{\mathrm{n},h\tau}, \mathbf{u}_{\mathrm{w},h\tau}) \right)^2 \right\}^{\frac{1}{2}}.$$

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Outline

Estimate distinguishing different error components

Theorem (Distinguishing different error components)

Consider

- *time step n,*
- *linearization step k,*
- *iterative algebraic solver step i,*

and the corresponding approximations $s_{\mathrm{w},h\tau}^{n,k,i}$ and $p_{\mathrm{w},h\tau}^{n,k,i}$. Then

$$|||(s_{\mathrm{w}} - s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w}} - p_{\mathrm{w},h\tau}^{n,k,i})|||_{I_n} \leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Error components

- $\eta_{\mathrm{sp}}^{n,k,i}$: spatial discretization
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- $\eta_{\mathrm{lin}}^{n,k,i}$: linearization
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Outline

Model problem

Horizontal flow

$$\partial_t(\phi s_\alpha) - \nabla\cdot\left(\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\underline{\mathbf{K}}\nabla p_\alpha\right) = 0,$$

$$s_\mathrm{n} + s_\mathrm{w} = 1,$$

$$p_\mathrm{n} - p_\mathrm{w} = p_\mathrm{c}(s_\mathrm{w})$$

Brooks–Corey model

- relative permeabilities

$$k_{\mathrm{r,w}}(s_\mathrm{w}) = s_\mathrm{e}^4, \quad k_{\mathrm{r,n}}(s_\mathrm{w}) = (1-s_\mathrm{e})^2(1-s_\mathrm{e}^2)$$

- capillary pressure

$$p_\mathrm{c}(s_\mathrm{w}) = p_\mathrm{d} s_\mathrm{e}^{-\frac{1}{2}}$$

-

$$s_\mathrm{e} := \frac{s_\mathrm{w} - s_\mathrm{rw}}{1 - s_\mathrm{rw} - s_\mathrm{rn}}$$

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Outline

Fully implicit cell-centered finite volume scheme

Fully implicit cell-centered two-point finite volumes

For all $1 \le n \le N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},e_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},e_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the normal fluxes are given by

$$F_{\mathrm{w},e_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\lambda_{\mathrm{r,w}}(s^n_{\mathrm{w},K}) + \lambda_{\mathrm{r,w}}(s^n_{\mathrm{w},L})}{2} |\underline{\mathbf{K}}| \frac{\bar{p}^n_{\mathrm{w},L} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_L|} |e_{KL}|,$$

$$F_{\mathrm{n},e_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\lambda_{\mathrm{r,n}}(s^n_{\mathrm{w},K}) + \lambda_{\mathrm{r,n}}(s^n_{\mathrm{w},L})}{2} |\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},L} + p_{\mathrm{c}}(s^n_{\mathrm{w},L}) - (\bar{p}^n_{\mathrm{w},K} + p_{\mathrm{c}}(s^n_{\mathrm{w},K}))}{|\mathbf{x}_K - \mathbf{x}_L|} |e_{KL}|$$

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Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KL} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KL} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},e_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized normal fluxes are given by

$$F_{\alpha,e_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) := F_{\alpha,e_{KL}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) + \sum_{M \in \{K,L\}} \frac{\partial F_{\alpha,e_{KL}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) + \sum_{M \in \{K,L\}} \frac{\partial F_{\alpha,e_{KL}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}).$$

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Velocities reconstructions

Velocities reconstructions

$$
\begin{aligned}
(\mathbf{d}_{\alpha,h}^{n,k,i} \cdot \mathbf{n}_K, 1)_{e_{KL}} &:= F_{\alpha,e_{KL}}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
((\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i}) \cdot \mathbf{n}_K, 1)_{e_{KL}} &:= F_{\alpha,e_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
\mathbf{a}_{\alpha,h}^{n,k,i} &:= \mathbf{d}_{\alpha,h}^{n,k,i+\nu} + \mathbf{l}_{\alpha,h}^{n,k,i+\nu} - (\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})
\end{aligned}
$$

Comments

- phase velocities reconstructions: $\mathbf{u}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i} + \mathbf{a}_{\alpha,h}^{n,k,i}$
- $\mathbf{d}_{\alpha,h}^{n,k,i}, \mathbf{l}_{\alpha,h}^{n,k,i}, \mathbf{a}_{\alpha,h}^{n,k,i}$ used to identify error components

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$$

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- $\mathbf{d}_{\alpha,h}^{n,k,i}, \mathbf{l}_{\alpha,h}^{n,k,i}, \mathbf{a}_{\alpha,h}^{n,k,i}$ used to identify error components

Global and complementary pressure reconstructions

Global and complementary pressures reconstructions

- piecewise quadratic global and complementary pressures:

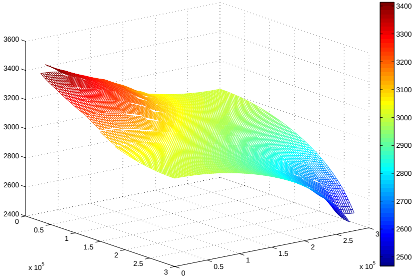
$$-(\lambda_{\rm w}(s_{{\rm w},K}^{n,k,i}) + \lambda_{\rm n}(s_{{\rm w},K}^{n,k,i}))\underline{\mathbf{K}}\nabla(\mathfrak{p}_h^{n,k,i}|_K) = (\mathbf{d}_{{\rm w},h}^{n,k,i} + \mathbf{d}_{{\rm n},h}^{n,k,i})|_K,$$
$$\mathfrak{p}_h^{n,k,i}(\mathbf{x}_K) = \mathfrak{p}(\bar{p}_{{\rm w},K}^{n,k,i}, s_{{\rm w},K}^{n,k,i}),$$

$$\underline{\mathbf{K}}\nabla(\mathfrak{q}_h^{n,k,i}|_K) = \lambda_{\rm n}(s_{{\rm w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(\mathfrak{p}_h^{n,k,i}|_K) + \mathbf{d}_{{\rm n},h}^{n,k,i}|_K,$$
$$\mathfrak{q}_h^{n,k,i}(\mathbf{x}_K) = \mathfrak{q}(s_{{\rm w},K}^{n,k,i})$$

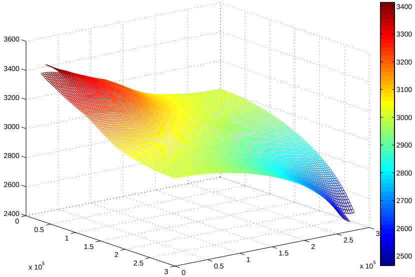
- reconstructions:

$$\hat{\mathfrak{p}}_h^{n,k,i} := \mathcal{I}_{\rm av}(\mathfrak{p}_h^{n,k,i}),$$
$$\hat{\mathfrak{q}}_h^{n,k,i} := \mathcal{I}_{\rm av}(\mathfrak{q}_h^{n,k,i})$$

Global pressure reconstructions



Approximate global pressure $\mathfrak{p}(s_{\mathrm{w},h_\tau}, p_{\mathrm{w},h_\tau}) \notin X$



Averaged approximate global pressure $\hat{\mathfrak{p}}_{h_\tau} \in X$

Data from Klieber & Rivière (2006)

Data

$$\Omega = (0,300)\mathrm{m} \times (0,300)\mathrm{m}, \quad T = 4{\cdot}10^6\mathrm{s},$$

$$\phi = 0.2, \quad \underline{\mathbf{K}} = 10^{-11}\underline{\mathbf{I}}\,\mathrm{m}^2,$$

$$\mu_{\mathrm{w}} = 5{\cdot}10^{-4}\mathrm{kg\,m^{-1}s^{-1}}, \quad \mu_{\mathrm{n}} = 2{\cdot}10^{-3}\mathrm{kg\,m^{-1}s^{-1}},$$

$$s_{\mathrm{rw}} = s_{\mathrm{rn}} = 0, \quad p_{\mathrm{d}} = 5{\cdot}10^3\mathrm{kg\,m^{-1}s^{-2}}$$

Initial condition ($\widetilde{K}$ 18m × 18m lower left corner block)

$$s_{\mathrm{w}}^0 = 0.2 \text{ on } K \in \mathcal{T}_h,\ K \notin \widetilde{K},$$

$$s_{\mathrm{w}}^0 = 0.95 \text{ on } K \in \mathcal{T}_h,\ K \in \widetilde{K}$$

Boundary conditions ($\widehat{K}$ 18m × 18m upper right corner block)

- no flow Neumann boundary conditions everywhere except of $\partial\widetilde{K} \cap \partial\Omega$ and $\partial\widehat{K} \cap \partial\Omega$
- $\widetilde{K}$ – injection well: $s_{\mathrm{w}} = 0.95$, $p_{\mathrm{w}} = 3.45{\cdot}10^6\mathrm{kg\,m^{-1}s^{-2}}$
- $\widehat{K}$ – production well: $s_{\mathrm{w}} = 0.2$, $p_{\mathrm{w}} = 2.41{\cdot}10^6\mathrm{kg\,m^{-1}s^{-2}}$

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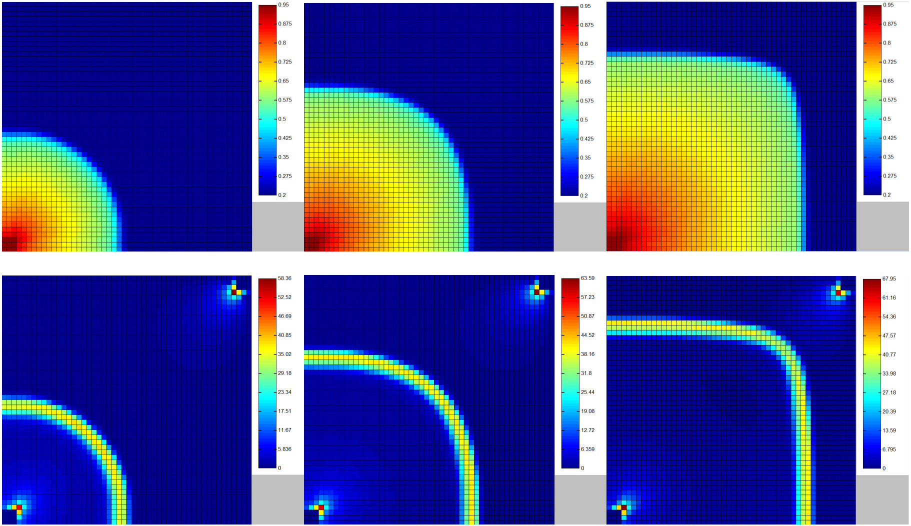
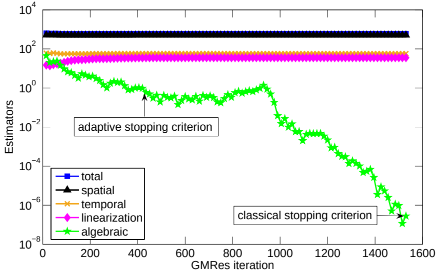
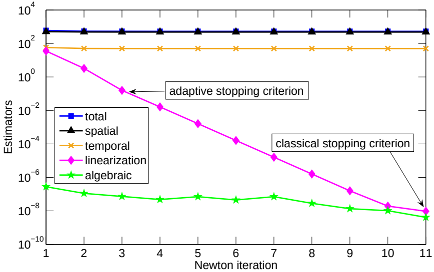
Water saturation/estimators evolution



Estimators and stopping criteria

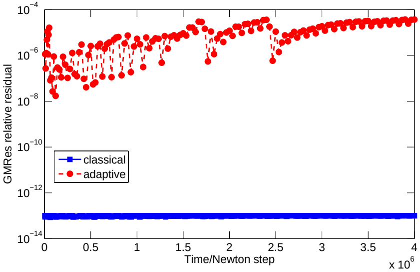


Estimators in function of GMRes iterations

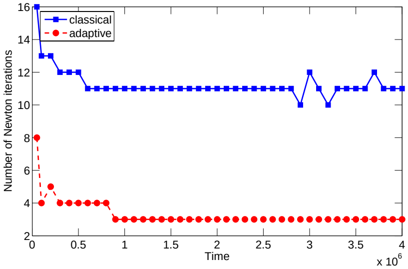


Estimators in function of Newton iterations

GMRes relative residual/Newton iterations

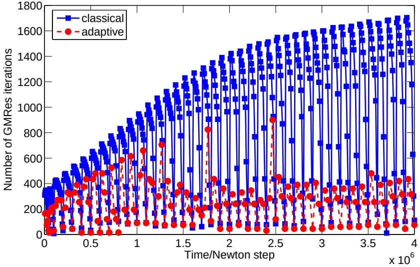


GMRes relative residual

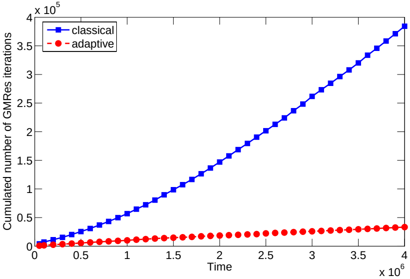


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Outline

Iteratively coupled vertex-centered finite volumes

Implicit pressure equation on step k

$$-\left(\left(\lambda_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \lambda_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\right)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D + \lambda_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{p}_{\mathrm{c}}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\right)_{\partial D\setminus\partial\Omega} = 0 \quad \forall D\in\mathcal{D}_h^{\mathrm{int},n}$$

Explicit saturation equation on step k

$$s_{\mathrm{w},D}^{n,k} := \frac{\tau^n}{\phi|D|}\left(\lambda_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D, 1\right)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1} \quad \forall D\in\mathcal{D}_h^{\mathrm{int},n}$$

Iteratively coupled vertex-centered finite volumes

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Linearization and algebraic solution

Iterative coupling step k and algebraic step i

$$-\left(\left(\lambda_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \lambda_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\right)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \lambda_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{p}_{\mathrm{c}}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\right)_{\partial D\setminus\partial\Omega} = -R_{\mathrm{t},D}^{n,k,i} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

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Velocities reconstructions

Total velocities reconstructions

$$(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D, 1)_e := -\big((\lambda_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i}) + \lambda_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i}))\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \lambda_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\overline{p}_{\mathrm{c}}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D, 1\big)_e,$$

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$$\mathbf{a}_{\mathrm{t},h}^{n,k,i} := \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu} + \mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu} - (\mathbf{d}_{\mathrm{t},h}^{n,k,i} + \mathbf{l}_{\mathrm{t},h}^{n,k,i})$$

Wetting phase velocities reconstructions

$$(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1)_e := -\big(\lambda_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_e,$$

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Pressure reconstructions

- conforming setting ⇒ no pressure reconstructions

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Pressure reconstructions

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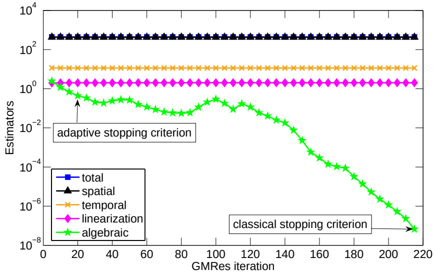
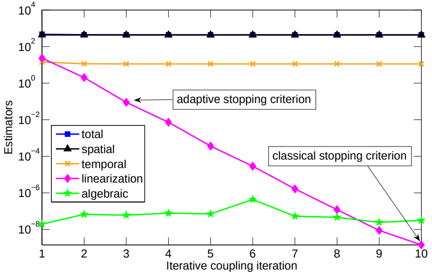
Estimators and stopping criteria

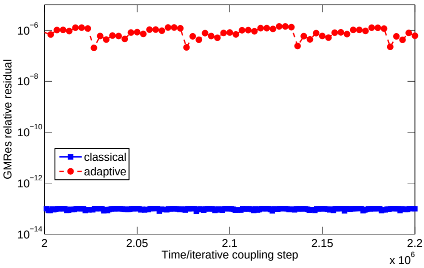


Estimators in function of GMRes iterations

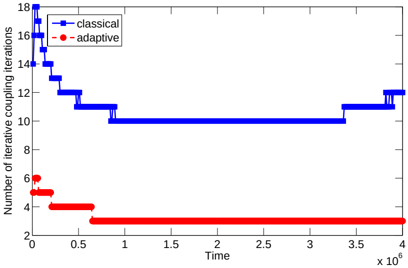


Estimators in function of iterative coupling iterations

GMRes relative residual/iterative coupling iterations

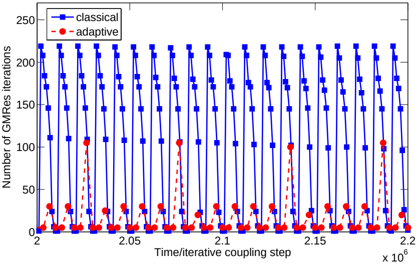


GMRes relative residual

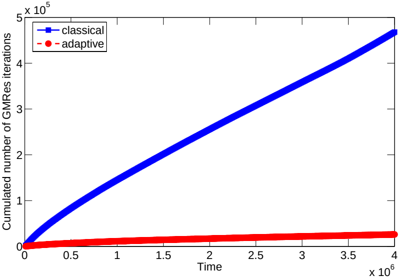


Iterative coupling iterations

GMRes iterations

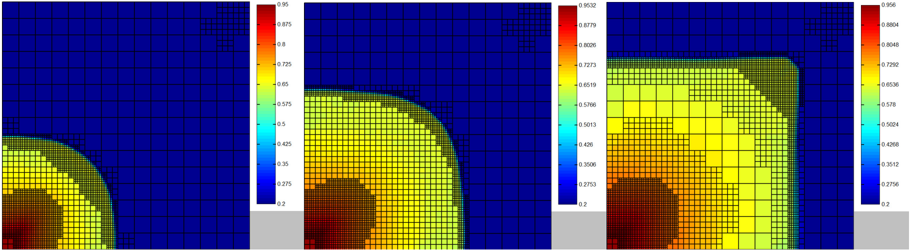


Per time and iterative coupling step



Cumulated

Space/time/nonlinear solver/linear solver adaptivity



Outline

Extension to multiphase compositional flows

Multiphase compositional flows

- $N_{\mathcal{P}}$ phases, $N_{\mathcal{C}}$ components
- miscible, compressible
- isothermal/thermal
- Ph.D. theses of Carole Heinry and Soleiman Yousef (Paris 6/IFPEN)

Discretization and resolution

- fully implicit cell-centered finite volumes
- Newton linearization
- GMRes with ILU0 preconditioning

Test case

- two phases and three components
- heterogeneous permeability distribution

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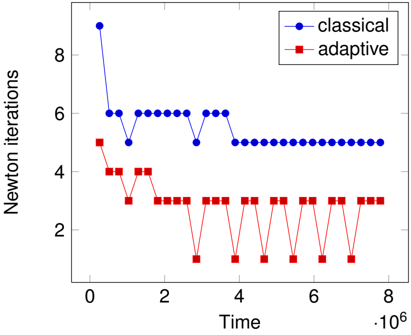
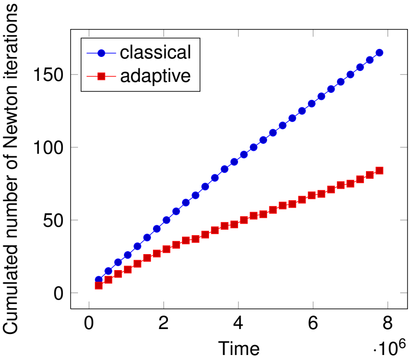
Newton iterations

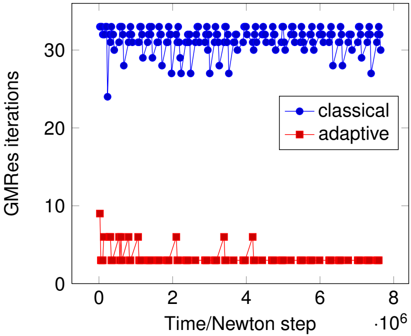


Per time step

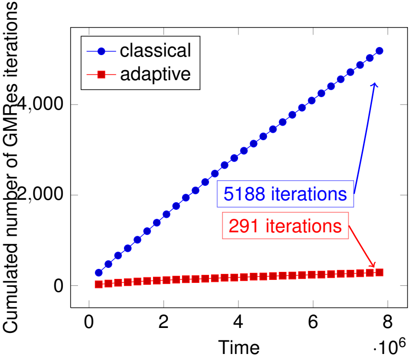


Cumulated

GMRes iterations



Per time and Newton step



Cumulated

Outline

Conclusions

Complete adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- space-time mesh adaptivity
- smart online decisions: algebraic step / linearization step / time step refinement / space mesh refinement
- important **computational savings**
- error upper bound via **a posteriori error estimates**

Future directions

- practical implementations
- other complex problems

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Bibliography

Bibliography

- VOHRALÍK M., WHEELER M. F., A posteriori error estimates, stopping criteria, and adaptivity for two-phase flows, *Comput. Geosci.* (2013), DOI 10.1007/s10596-013-9356-0, in press.
- CANCÈS C., POP I. S., VOHRALÍK M., An a posteriori error estimate for vertex-centered finite volume discretizations of immiscible incompressible two-phase flow, *Math. Comp.* (2013), in press.

Thank you for your attention!