A posteriori error estimation based on potential and flux reconstruction for the heat equation: a unified framework

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Outline

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What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|u - u_h\|^2_{\Omega\times(0,T)} \le \sum_{n=1}^{N} \tau^n \sum_{T\in\mathcal{T}^n} \eta_T^n(u_h)^2$
- no undetermined constant: error control

Asymptotic exactness

- $\sum_{n=1}^{N} \tau^n \sum_{T\in\mathcal{T}^n} \eta_T^n(u_h)^2 / \|u - u_h\|^2_{\Omega\times(0,T)} \to 1$
- overestimation factor goes to one with meshes size

Local efficiency (local error lower bound)

- $\tau^n \eta_T^n(u_h)^2 \le (C^n_{\mathrm{eff},T})^2 \sum_{T' \text{ close to } T} \|u - u_h\|^2_{T'\times(t^{n-1},t^n)}$
- necessary for optimal space–time mesh refinement

Robustness

- $C^n_{\mathrm{eff},T}$ independent of data, domain, final time, meshes, or solution

Negligible evaluation cost

- estimators can be evaluated locally in space and time

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Previous results

Continuous finite elements

- Bieterman and Babuška (1982), introduction
- Picasso (1998), no derefinement allowed
- Babuška, Feistauer, and Šolín (2001), continuous-in-time discretization
- Strouboulis, Babuška, and Datta (2003), guaranteed estimates
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction
- Bergam, Bernardi, and Mghazli (2005), efficiency (not optimal)
- Lakkis and Makridakis (2006), elliptic reconstruction

Previous results

Finite volumes

- Ohlberger (2001), non energy norm estimates
- Amara, Nadau, and Trujillo (2004), energy-norm estimates

Discontinuous Galerkin finite elements

- Sun and Wheeler (2005, 2006), non energy norm estimates
- Georgoulis and Lakkis (2009)

Nonconforming finite elements

- Nicaise and Soualem (2005)

Mixed finite elements

- Cascón, Ferragut, and Asensio (2006)

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The heat equation

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\begin{aligned}
\partial_t u - \Delta u &= f && \text{a.e. in } Q := \Omega \times (0, T), \\
u &= 0 && \text{a.e. on } \partial\Omega \times (0, T), \\
u(\cdot, 0) &= u_0 && \text{a.e. in } \Omega
\end{aligned}
$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polygonal domain
- $T > 0$ is the final simulation time

Spaces

- $X := L^2(0, T; H_0^1(\Omega))$
- $X' = L^2(0, T, H^{-1}(\Omega))$
- $Y := \{y \in X; \partial_t y \in X'\}$

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The heat equation

Norms

- energy norm $\|y\|_X^2 := \int_0^T \|\nabla y\|^2(t)\,\mathrm{d}t$
- dual norm $\|y\|_Y := \|y\|_X + \|\partial_t y\|_{X'}$

$$\|\partial_t y\|_{X'} = \left\{ \int_0^T \|\partial_t y\|_{H^{-1}}^2(t)\,\mathrm{d}t \right\}^{1/2}$$

Weak solution

Find $u \in Y$ such that, for a.e. $t \in (0, T)$ and for all $v \in H_0^1(\Omega)$,

$$\langle \partial_t u, v\rangle(t) + (\nabla u, \nabla v)(t) = (f, v)(t)$$

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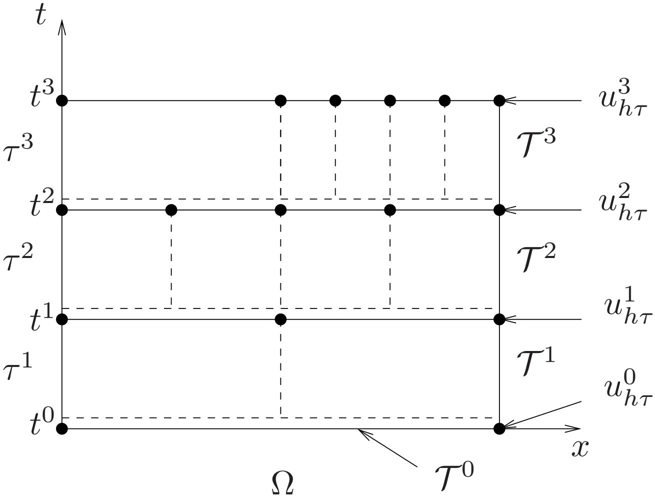
Time-dependent meshes and discrete solutions

Approximate solutions

- discrete times $\{t^n\}_{0\le n\le N}$, $t^0 = 0$ and $t^N = T$
- $I_n := (t^{n-1}, t^n]$, $\tau^n := t^n - t^{n-1}$, $1 \le n \le N$
- a different simplicial mesh $\mathcal{T}^n$ on all $0 \le n \le N$
- $u^n_{h\tau} \in V^n_h := V_h(\mathcal{T}^n)$, $0 \le n \le N$
- $u^n_{h\tau}$ possibly nonconforming, not included in $H^1_0(\Omega)$
- $u_{h\tau} : Q \to \mathbb{R}$ continuous and piecewise affine in time

$$u_{h\tau}(\cdot, t) := (1-\varrho)u^{n-1}_{h\tau} + \varrho u^n_{h\tau}, \qquad \varrho = \frac{1}{\tau^n}(t - t^{n-1})$$

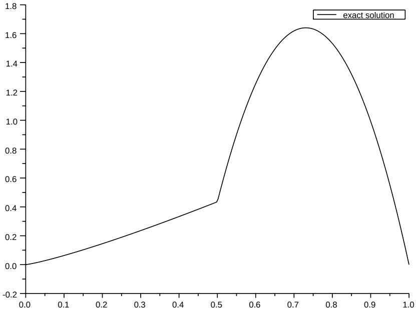
Time-dependent meshes and discrete solutions



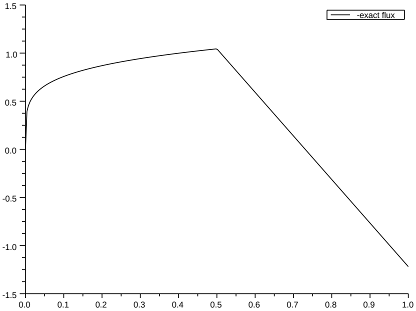
Time-dependent meshes and discrete solutions

Outline

Properties of the weak solution

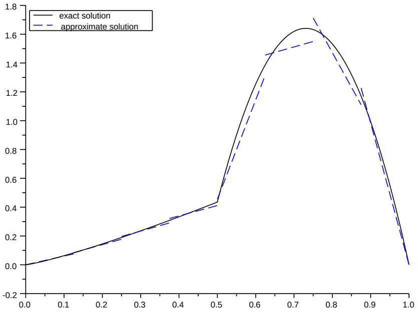


Potential u^n is in $H_0^1(\Omega)$

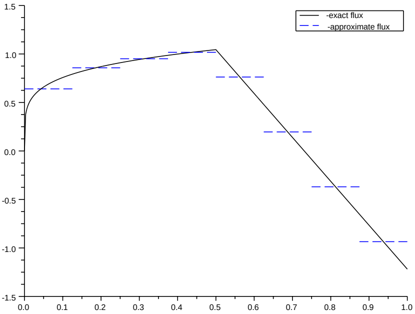


Flux $-\nabla u^n$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Approximate potential and approximate flux



Approximate potential u_h^n is not in $H_0^1(\Omega)$



Approximate flux $-\nabla u_h^n$ is not in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

Potential and flux reconstructions

General form

- potential reconstruction s is continuous and piecewise affine in time with $s^n \in H_0^1(\Omega)$ for all $0 \leq n \leq N$ (s^n are in the correct space)
- flux reconstruction θ is piecewise constant in time with $\theta|_{I_n} \in \mathbf{H}(\mathrm{div}, \Omega)$ for all $1 \leq n \leq N$ ($\theta|_{I_n}$ are in the correct space)

Two additional assumptions

- s^n preserves the mean values of $u^n_{h\tau}$ on $\mathcal{T}^{n,n+1}$, a common refinement of $\mathcal{T}^n$ and $\mathcal{T}^{n+1}$

$$(s^n, 1)_{T'} = (u^n_{h\tau}, 1)_{T'} \qquad \forall T' \in \mathcal{T}^{n,n+1}$$

- θ^n satisfies a local conservation property

$$(\tilde{f}^n - \partial_t u^n_{h\tau} - \nabla \cdot \theta^n, 1)_T = 0 \qquad \forall T \in \mathcal{T}^n$$

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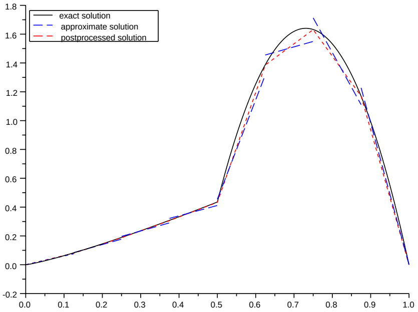

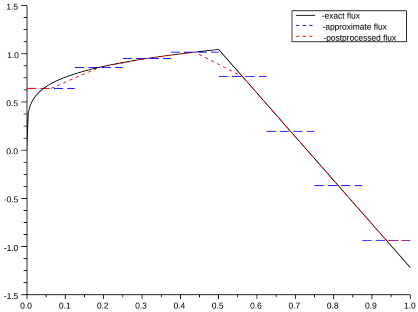

A postprocessed potential s_h^n is in $H_0^1(\Omega)$

A postprocessed flux θ^n is in $\mathbf{H}(\mathrm{div}, \Omega)$

Practical construction of s and θ

Construction of s^n

$$s^n := \mathcal{I}_{\mathrm{av}}^n(u_{h\tau}^n) + \sum_{T' \in \mathcal{T}^{n,n+1}} \alpha_{T'}^n b_{T'},$$

$$\alpha_{T'}^n := \frac{1}{(b_{T'}, 1)_{T'}} (u_{h\tau}^n - \mathcal{I}_{\mathrm{av}}^n(u_{h\tau}^n), 1)_{T'}$$

- $\mathcal{I}_{\mathrm{av}}^n$: the averaging interpolate on the mesh $\mathcal{T}^n$
- $b_{T'}$ standard (time-independent) bubble function supported on T'
- the mean value is preserved on all $T' \in \mathcal{T}^{n,n+1}$
- specificity of the parabolic case
- independent of the numerical scheme

Construction of θ^n

- inspired from the elliptic case
- depends on the numerical scheme

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A posteriori error estimate

Theorem (A posteriori error estimate)

There holds

$$\|u - u_{h\tau}\|_Y \leq 3 \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{T\in\mathcal{T}^n} (\eta^n_{\mathrm{R},T} + \eta^n_{\mathrm{DF},T}(t))^2 \,\mathrm{d}t \right\}^{1/2}$$
$$+ \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{T\in\mathcal{T}^n} (\eta^n_{\mathrm{NC1},T})^2(t) \,\mathrm{d}t \right\}^{1/2}$$
$$+ \left\{ \sum_{n=1}^{N} \tau^n \sum_{T\in\mathcal{T}^n} (\eta^n_{\mathrm{NC2},T})^2 \right\}^{1/2} + \eta_{\mathrm{IC}} + 3\|f - \widetilde{f}\|_{X'}.$$

- **unified setting**: no specification of the numerical scheme
- only mean values-preserving potential reconstruction s and locally conservative flux reconstruction θ needed

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Estimators

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},T}^{n}(t) := \|\nabla s(t) + \boldsymbol{\theta}^{n}\|_{T}, \quad t \in I_n$
  - penalizes the fact that $-\nabla u_{h\tau}^{n} \notin \mathbf{H}(\mathrm{div}, \Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},T}^{n} := C_{\mathrm{P}} h_T \|\widetilde{f}^{n} - \partial_t s^{n} - \nabla \cdot \boldsymbol{\theta}^{n}\|_{T}$
  - residue evaluated for $\boldsymbol{\theta}^{n}$
  - $C_{\mathrm{P}} = 1/\pi$
- *nonconformity estimators*
  - $\eta_{\mathrm{NC1},T}^{n}(t) := \|\nabla^{n-1,n}(s - u_{h\tau})(t)\|_{T}, \quad t \in I_n$
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- *initial condition estimator*
  - $\eta_{\mathrm{IC}} := 2^{1/2} \|s^0 - u^0\|$
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  - $\eta_{\mathrm{DF},T}^n(t) := \|\nabla s(t) + \boldsymbol{\theta}^n\|_T, \quad t \in I_n$
  - penalizes the fact that $-\nabla u_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},T}^n := C_{\mathrm{P}} h_T \|\widetilde{f}^n - \partial_t s^n - \nabla\cdot\boldsymbol{\theta}^n\|_T$
  - residue evaluated for $\boldsymbol{\theta}^n$
  - $C_{\mathrm{P}} = 1/\pi$
- *nonconformity estimators*
  - $\eta_{\mathrm{NC1},T}^n(t) := \|\nabla^{n-1,n}(s - u_{h\tau})(t)\|_T, \quad t \in I_n$
  - $\eta_{\mathrm{NC2},T}^n := C_{\mathrm{P}} h_T \|\partial_t(s - u_{h\tau})^n\|_T$
  - penalize the fact that $u_{h\tau}^n \notin H_0^1(\Omega)$
- *initial condition estimator*
  - $\eta_{\mathrm{IC}} := 2^{1/2}\|s^0 - u^0\|$
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Remarks

Conforming methods

- in conforming methods (FEs, VCFVs) $u_{h\tau}^n \in H_0^1(\Omega)$
- set $s^n := u_{h\tau}^n$
- $\Rightarrow$ nonconformity estimators $\eta_{\mathrm{NC1},T}^n$ and $\eta_{\mathrm{NC2},T}^n$ vanish

Flux-conforming methods

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- notice that $\int_{I_n}(\eta^n_{\mathrm{DF},T})^2 \le (\eta^n_{\mathrm{DF},T,1})^2 + (\eta^n_{\mathrm{DF},T,2})^2$, where

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- this yields

$$\|u - u_{h\tau}\|_Y \le \left\{\sum_{n=1}^{N} (\eta^n_{\mathrm{sp}})^2\right\}^{1/2} + \left\{\sum_{n=1}^{N} (\eta^n_{\mathrm{tm}})^2\right\}^{1/2} + \eta_{\mathrm{IC}} + 3\|f - \tilde{f}\|_{X'}$$

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A space-time adaptive time-marching algorithm

Algorithm for achieving a given relative precision ε

$$\frac{\sum_{n=1}^{N}\left\{(\eta_{\mathrm{sp}}^{n})^2+(\eta_{\mathrm{tm}}^{n})^2\right\}}{\sum_{n=1}^{N}\|u_{h\tau}\|_{Z(I_n)}^2}\leq\varepsilon^2$$

1. Initialization
   1. choose an initial mesh $\mathcal{T}^0$;
   2. select an initial time step τ^0 and set $n := 1$;
2. Loop in time: while $\sum_i \tau^i < T$,
   1. set $\mathcal{T}^{n*} := \mathcal{T}^{n-1}$ and $\tau^{n*} := \tau^{n-1}$;
   2. solve $u_{h\tau}^{n*} := \mathrm{Sol}(u_{h\tau}^{n-1}, \tau^{n*}, \mathcal{T}^{n*})$;
   3. estimate the space and time errors by η_{sp}^{n} and η_{tm}^{n};
   4. when η_{sp}^{n} or η_{tm}^{n} are too much above or below $\varepsilon\|u_{h\tau}\|_{Z(I_n)}/\sqrt{2}$ or not of similar size, refine or derefine the time step τ^{n*} and the space mesh $\mathcal{T}^{n*}$ and return to step (2-2), otherwise save approximate solution, mesh, and time step as $u_{h\tau}^{n}$, $\mathcal{T}^{n}$, and τ^{n} and set $n := n+1$.

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Outline

Efficiency

Theorem (Efficiency)

There holds

$$\eta_{\mathrm{sp}}^n + \eta_{\mathrm{tm}}^n \lesssim \|u - u_{h\tau}\|_{Y(I_n)} + \mathcal{J}^n(u_{h\tau}) + \mathcal{E}_f^n$$

Notation

- $\mathcal{J}^n(u_{h\tau})^2 := \tau^n \sum_{T \in \mathcal{T}^{n-1}} |[\![u_{h\tau}^{n-1}]\!]|^2_{-\frac{1}{2},\mathfrak{F}_T^{n-1}} + \tau^n \sum_{T \in \mathcal{T}^n} |[\![u_{h\tau}^n]\!]|^2_{-\frac{1}{2},\mathfrak{F}_T^n}$
- $(\mathcal{E}_f^n)$ is space-time data oscillation term

Comments on $\mathcal{J}^n$

- $\mathcal{J}^n$ is a typical jump seminorm
- it can bounded by the energy error if the jumps in $u_{h\tau}$ have zero mean values (MFEs, FCFVs, NCFEs); it can also be bounded in DGs, using the scheme
- it can alternatively be added to the error measure (note that $\mathcal{J}^n(u_{h\tau}) = \mathcal{J}^n(u - u_{h\tau})$)

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Assumptions for the lower bound proof

Main assumption: approximation property of the flux reconstruction

$$\|\nabla u_{h\tau}^n + \boldsymbol{\theta}^n\|_T \lesssim \left\{ \sum_{T' \in \mathfrak{T}_T} h_{T'}^2 \|\widetilde{f}^n - \partial_t u_{h\tau}^n + \Delta u_{h\tau}^n\|_{T'}^2 \right\}^{1/2} + |\mathbf{n}\cdot[\![\nabla^n u_{h\tau}^n]\!]|_{+\frac{1}{2},\mathfrak{F}_T^{\mathrm{i},n}} + |[\![u_{h\tau}^n]\!]|_{-\frac{1}{2},\mathfrak{F}_T^n}$$

- $\|\nabla u_{h\tau}^n + \boldsymbol{\theta}^n\|_T$ is a local lower bound for the classical residual estimators

Other assumptions

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Outline

General concept

Upper bound

- for $0 \le n \le N$, we only have to construct $\boldsymbol{\theta}^n \in \mathbf{H}(\mathrm{div}, \Omega)$ which is locally conservative, i.e., such that

$$(\tilde{f}^n - \partial_t u_{h\tau}^n - \nabla\cdot\boldsymbol{\theta}^n, 1)_T = 0, \qquad \forall T \in \mathcal{T}^n$$

- we construct $\boldsymbol{\theta}^n$ in some mixed finite element space; example: Raviart–Thomas–Nédélec spaces

$$\mathbf{RTN}_l(\mathcal{T}^n) := \{\mathbf{v}_h \in \mathbf{H}(\mathrm{div}, \Omega)\,; \mathbf{v}_h|_T \in \mathbf{RTN}_l(T) \quad \forall T \in \mathcal{T}^n\}$$

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We achieve this by a straightforward generalization of the elliptic case (previous works)

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Discontinuous Galerkin method

Definition (DG method)

On I_n, $\mathcal{T}^n$, $1 \leq n \leq N$, find $u^n_{h\tau} \in V^n_h := \mathbb{P}_k(\mathcal{T}^n)$, $k \geq 1$, such that

$$(\partial_t u^n_{h\tau}, v_h) - \sum_{F \in \mathcal{F}^n} \{(\mathbf{n}_F \cdot \{\!\{\nabla^n u^n_{h\tau}\}\!\}, [\![v_h]\!])_F + \theta(\mathbf{n}_F \cdot \{\!\{\nabla^n v_h\}\!\}, [\![u^n_{h\tau}]\!])_F\}$$
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Flux $\theta^n \in \mathbf{RTN}_l(\mathcal{T}^n)$, $l \in \{k-1, k\}$

For all $T \in \mathcal{T}^n$, all $F \in \mathcal{F}^n_T$, all $q_h \in \mathbb{P}_l(F)$ (face normal comp.),

$$(\theta^n \cdot \mathbf{n}_F, q_h)_F = (-\mathbf{n}_F \cdot \{\!\{\nabla^n u^n_{h\tau}\}\!\} + \alpha_F h_F^{-1} [\![u^n_{h\tau}]\!], q_h)_F,$$

and for all $\mathbf{r}_h \in \mathbb{P}^d_{l-1}(T)$ (element components),

$(\theta^n, \mathbf{r}_h)_T = -(\nabla^n u^n_{h\tau}, \mathbf{r}_h)_T + \theta \sum_{F \in \mathcal{F}^n_T} \omega_F (\mathbf{n}_F \cdot \mathbf{r}_h, [\![u^n_{h\tau}]\!])_F.$

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Outline

Cell-centered finite volume method

Definition (CCFV method)

On I_n, $\mathcal{T}^n$, $1 \leq n \leq N$, find $\bar{u}^n_{h\tau} \in \bar{V}^n_h := \mathbb{P}_0(\mathcal{T}^n)$ such that

$$\frac{1}{\tau^n}(\bar{u}^n_{h\tau} - u^{n-1}_{h\tau}, 1)_T + \sum_{F \in \mathcal{F}^n_T} S^n_{T,F} = (\tilde{f}^n, 1)_T \qquad \forall T \in \mathcal{T}^n.$$

Flux $\theta^n \in \mathbf{RTN}_0(\mathcal{T}^n)$

$$(\theta^n \cdot \mathbf{n}, 1)_F := S^n_{T,F}$$

Postprocessing of the potential

- $\bar{u}^n_{h\tau} \in \bar{V}^n_h$ not suitable for energy error estimates ($\nabla \bar{u}^n_{h\tau} = 0$)
- $u^n_{h\tau} \in V^n_h$, V^n_h is $\mathbb{P}_1(\mathcal{T}^n)$ enriched elementwise by the parabolas $\sum_{i=1}^d x_i^2$
- $$-\nabla u^n_{h\tau} = \theta^n,$$ $$(u^n_{h\tau}, 1)_T = (\bar{u}^n_{h\tau}, 1)_T$$

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$$(\theta^n \cdot \mathbf{n}, 1)_F := S_{T,F}^n$$

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- $\bar{u}_{h\tau}^n \in \bar{V}_h^n$ not suitable for energy error estimates ($\nabla \bar{u}_{h\tau}^n = 0$)
- $u_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_1(\mathcal{T}^n)$ enriched elementwise by the parabolas $\sum_{i=1}^d x_i^2$
- $$-\nabla u_{h\tau}^n = \theta^n,$$
  $$(u_{h\tau}^n, 1)_T = (\bar{u}_{h\tau}^n, 1)_T$$

Cell-centered finite volume method

Definition (CCFV method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $\bar{u}^n_{h\tau} \in \bar{V}^n_h := \mathbb{P}_0(\mathcal{T}^n)$ such that

$$\frac{1}{\tau^n}(\bar{u}^n_{h\tau} - u^{n-1}_{h\tau}, 1)_T + \sum_{F \in \mathcal{F}^n_T} S^n_{T,F} = (\tilde{f}^n, 1)_T \qquad \forall T \in \mathcal{T}^n.$$

Flux $\boldsymbol{\theta}^n \in \mathbf{RTN}_0(\mathcal{T}^n)$

$$(\boldsymbol{\theta}^n \cdot \mathbf{n}, 1)_F := S^n_{T,F}$$

Postprocessing of the potential

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- $$-\nabla u^n_{h\tau} = \boldsymbol{\theta}^n,$$ $$(u^n_{h\tau}, 1)_T = (\bar{u}^n_{h\tau}, 1)_T$$

Outline

Mixed finite element method

Definition (MFE method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $\boldsymbol{\sigma}_{h\tau}^n \in \mathbf{W}_h^n$ and $\bar{u}_{h\tau}^n \in \bar{V}_h^n$ such that

$$(\boldsymbol{\sigma}_{h\tau}^n, \mathbf{w}_h) - (\bar{u}_{h\tau}^n, \nabla\cdot\mathbf{w}_h) = 0 \qquad \forall\, \mathbf{w}_h \in \mathbf{W}_h^n,$$

$$(\nabla\cdot\boldsymbol{\sigma}_{h\tau}^n, v_h) + \frac{1}{\tau^n}(\bar{u}_{h\tau}^n - u_{h\tau}^{n-1}, v_h) = (\tilde{f}^n, v_h) \qquad \forall v_h \in \bar{V}_h^n.$$

Flux $\theta^n \in \mathbf{W}_h^n$

$\theta^n := \boldsymbol{\sigma}_{h\tau}^n$ directly

Postprocessing of the potential

- $u_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_{l+1}(\mathcal{T}^n)$ enriched by bubbles (Arbogast and Chen, 1995)
- $$\Pi_{\mathbf{W}_h^n}(-\nabla^n u_{h\tau}^n) = \boldsymbol{\sigma}_{h\tau}^n,$$ $$\Pi_{\bar{V}_h^n}(u_{h\tau}^n) = \bar{u}_{h\tau}^n$$

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Outline

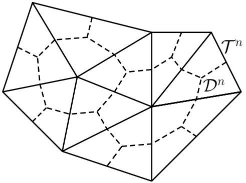
Vertex-centered finite volume method

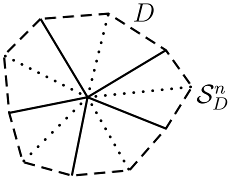
Definition (VCFV method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $u^n_{h\tau} \in V^n_h := \mathbb{P}_1(\mathcal{T}^n) \cap H^1_0(\Omega)$ s. t.

$$(\partial_t u^n_{h\tau}, 1)_D - (\nabla u^n_{h\tau} \cdot \mathbf{n}_D, 1)_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}^{\mathrm{i},n}.$$

Setting





- triangulation $\mathcal{T}^n$, dual mesh $\mathcal{D}^n$, simplicial submesh $\mathcal{S}^n$

Flux $\theta^n \in \mathbf{RTN}_0(\mathcal{S}^n)$

- by prescription: $\theta^n \cdot \mathbf{n}_F|_F := -\{\!\{\nabla u^n_{h\tau} \cdot \mathbf{n}_F\}\!\}$ on faces F of $\mathcal{S}^n$
- by solution of local Neumann problems on patches $\mathcal{S}^n_D$

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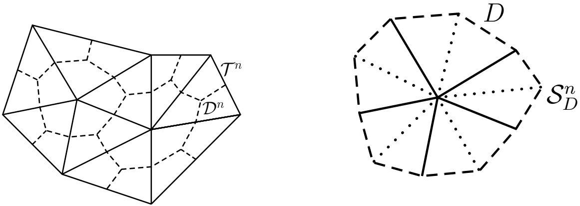

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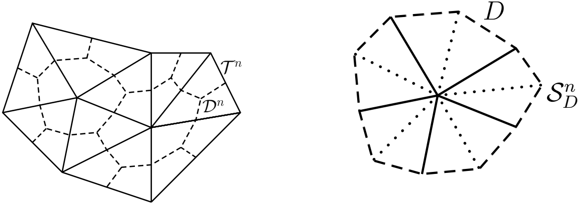

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Outline

Face-centered finite volume method

Definition (FCFV method)

On I_n, $\mathcal{T}^n$, $1 \leq n \leq N$, find $u_{h\tau}^n \in V_h^n$ (Crouzeix–Raviart sp.) s. t.

$$(\partial_t u_{h\tau}^n, 1)_D - (\nabla^n u_{h\tau}^n \cdot \mathbf{n}_D, 1)_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}^{\mathrm{i},n}.$$

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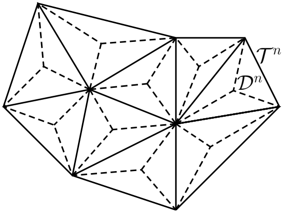

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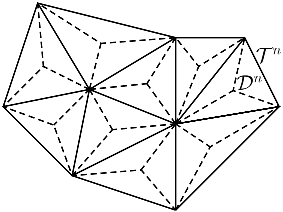

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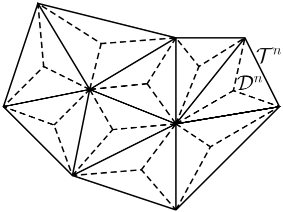

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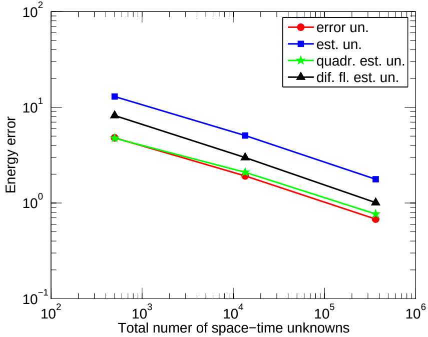
Outline

Numerical experiment

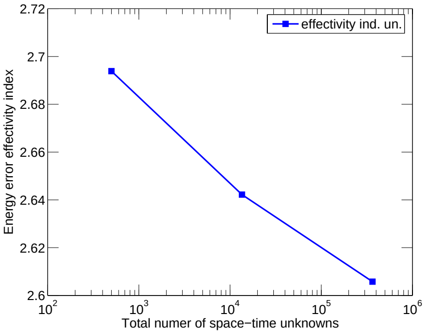
Numerical experiment

- exact solution $u = e^{x+y+t-3}$ on square domain $\Omega = (0,3) \times (0,3)$, $T = 1.5$ or $T = 3$
- square meshes: 10×10, 30×30, 90×90
- time steps: 0.3, 0.1, 0.3333
- vertex-centered finite volumes
- additional quadrature/mass lumping estimator

Energy norm results, $T = 1.5$

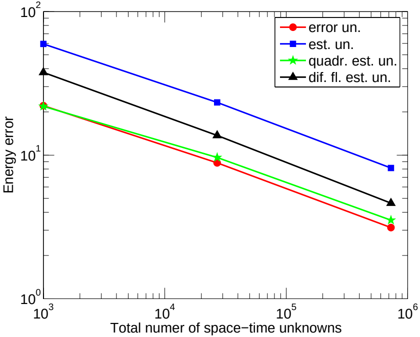


Energy error and estimators



Effectivity index

Energy norm results, $T = 3$



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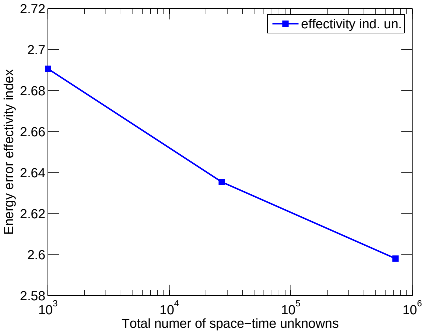
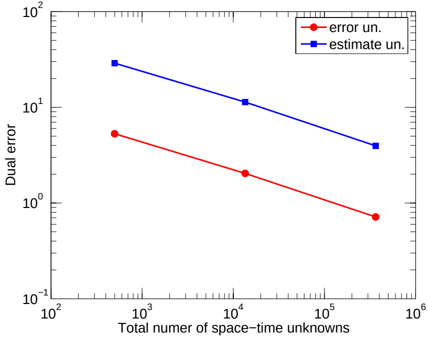
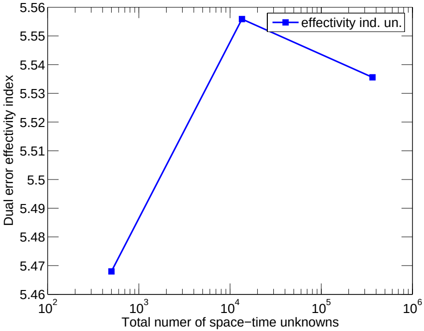


Effectivity index

Dual norm results, $T = 1.5$

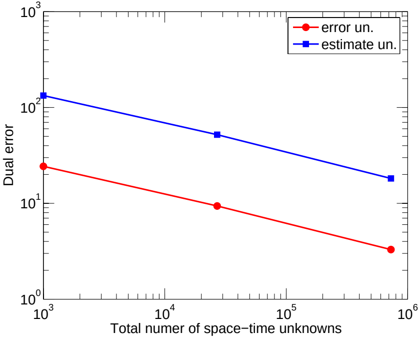


Dual error and estimators

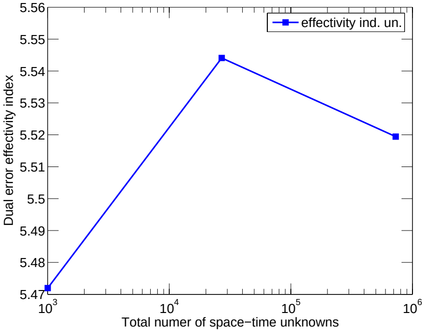


Effectivity index

Dual norm results, $T = 3$



Dual error and estimators



Effectivity index

Outline

Conclusions and future work

Conclusions

- **unified framework** for the heat equation (works for all major numerical schemes)
- directly and locally computable estimates
- global-in-space and local-in-time efficiency and robustness with respect to the final time as in Verfürth (2003)

Future work

- nonlinear problems
- extensions to other types of problems

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Thanks for your attention!