Estimation d'erreur a posteriori pour l'équation de la chaleur basée sur la reconstruction du flux et du potentiel : un cadre unifié

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Outline

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What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|u - u_h\|^2_{\Omega\times(0,T)} \leq \sum_{n=1}^{N} \sum_{T\in\mathcal{T}^n} \eta_T^n(u_h)^2$
- no undetermined constant: error control

Asymptotic exactness

- $\sum_{n=1}^{N} \sum_{T\in\mathcal{T}^n} \eta_T^n(u_h)^2 / \|u - u_h\|^2_{\Omega\times(0,T)} \to 1$
- overestimation factor goes to one with meshes size

Local efficiency (local error lower bound)

- $\eta_T^n(u_h)^2 \leq (C^n_{\mathrm{eff},T})^2 \sum_{T' \text{ close to } T} \|u - u_h\|^2_{T'\times(t^{n-1},t^n)}$
- necessary for optimal space–time mesh refinement

Robustness

- $C^n_{\mathrm{eff},T}$ independent of data, domain, final time, meshes, or solution

Negligible evaluation cost

- estimators can be evaluated locally in space and time

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Previous results

Continuous finite elements

- Bieterman and Babuška (1982), introduction
- Picasso (1998), no derefinement allowed
- Babuška, Feistauer, and Šolín (2001), continuous-in-time discretization
- Strouboulis, Babuška, and Datta (2003), guaranteed estimates
- Verfürth (2003), efficiency, robustness with respect to the final time
- Makridakis and Nochetto (2003), elliptic reconstruction
- Bergam, Bernardi, and Mghazli (2005), efficiency (not optimal)
- Lakkis and Makridakis (2006), elliptic reconstruction

Previous results

Finite volumes

- Ohlberger (2001), non energy norm estimates
- Amara, Nadau, and Trujillo (2004), energy-norm estimates

Discontinuous Galerkin finite elements

- Sun and Wheeler (2005, 2006), non energy norm estimates
- Georgoulis and Lakkis (2009)

Nonconforming finite elements

- Nicaise and Soualem (2005)

Mixed finite elements

- Cascón, Ferragut, and Asensio (2006)

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Outline

The heat equation

The heat equation

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{a.e. in } Q := \Omega \times (0, T), \\ u &= 0 && \text{a.e. on } \partial\Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{a.e. in } \Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a polygonal domain
- $T > 0$ is the final simulation time

Spaces

- $X := L^2(0, T; H_0^1(\Omega))$
- $X' = L^2(0, T, H^{-1}(\Omega))$
- $Y := \{y \in X; \partial_t y \in X'\}$

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The heat equation

Norms

- energy norm $\|y\|_X^2 := \int_0^T \|\nabla y\|^2(t)\,\mathrm{d}t$
- dual norm $\|y\|_Y := \|y\|_X + \|\partial_t y\|_{X'}$

$$\|\partial_t y\|_{X'} = \left\{\int_0^T \|\partial_t y\|_{H^{-1}}^2(t)\,\mathrm{d}t\right\}^{1/2}$$

Weak solution

Find $u \in Y$ such that, for a.e. $t \in (0, T)$ and for all $v \in H_0^1(\Omega)$,

$$\langle \partial_t u, v\rangle(t) + (\nabla u, \nabla v)(t) = (f, v)(t)$$

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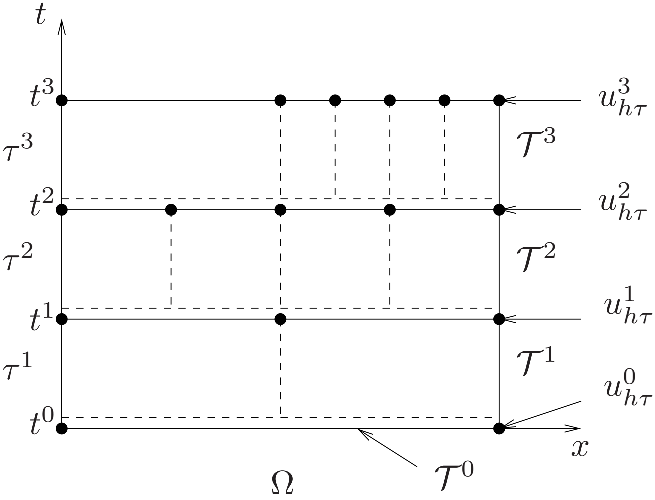
Time-dependent meshes and discrete solutions

Approximate solutions

- discrete times $\{t^n\}_{0\le n\le N}$, $t^0 = 0$ and $t^N = T$
- $I_n := (t^{n-1}, t^n]$, $\tau^n := t^n - t^{n-1}$, $1 \le n \le N$
- a different simplicial mesh $\mathcal{T}^n$ on all $0 \le n \le N$
- $u_{h\tau}^n \in V_h^n := V_h(\mathcal{T}^n)$, $0 \le n \le N$
- $u_{h\tau}^n$ possibly nonconforming, not included in $H_0^1(\Omega)$
- $u_{h\tau} : Q \to \mathbb{R}$ continuous and piecewise affine in time

$$u_{h\tau}(\cdot, t) := (1-\varrho)u_{h\tau}^{n-1} + \varrho u_{h\tau}^n, \qquad \varrho = \frac{1}{\tau^n}(t - t^{n-1})$$

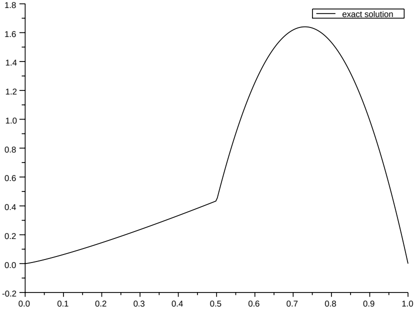
Time-dependent meshes and discrete solutions

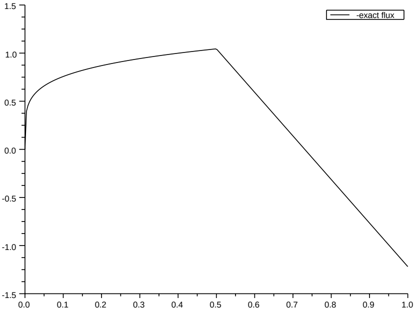


Time-dependent meshes and discrete solutions

Outline

Potential and flux

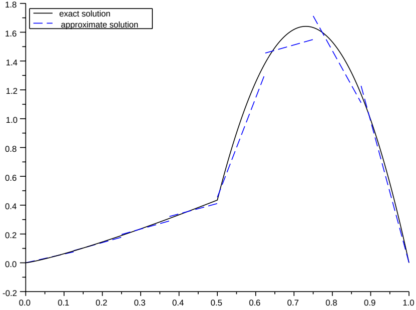


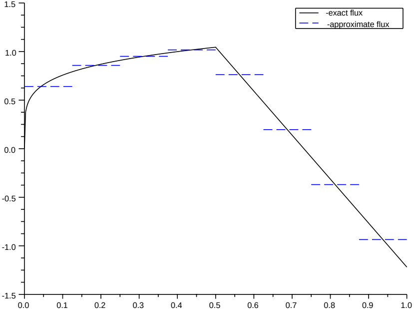


Potential u^n is in $H_0^1(\Omega)$

Flux $-\nabla u^n$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Approximate potential and approximate flux





Approximate potential u_h^n is not in $H_0^1(\Omega)$

Approximate flux $-\nabla u_h^n$ is not in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

Potential and flux reconstructions

General form

- potential reconstruction s is continuous and piecewise affine in time with $s^n \in H_0^1(\Omega)$ for all $0 \leq n \leq N$ (s^n are in the correct space)
- flux reconstruction θ is piecewise constant in time with $\theta|_{I_n} \in \mathbf{H}(\mathrm{div}, \Omega)$ for all $1 \leq n \leq N$ ($\theta|_{I_n}$ are in the correct space)

Two additional assumptions

- s^n **preserves the mean values** of $u^n_{h\tau}$ on $\mathcal{T}^{n,n+1}$, a common refinement of $\mathcal{T}^n$ and $\mathcal{T}^{n+1}$

$$(s^n, 1)_{T'} = (u^n_{h\tau}, 1)_{T'} \qquad \forall T' \in \mathcal{T}^{n,n+1}$$

- θ^n satisfies a **local conservation property**

$$(\tilde{f}^n - \partial_t u^n_{h\tau} - \nabla \cdot \theta^n, 1)_T = 0 \qquad \forall T \in \mathcal{T}^n$$

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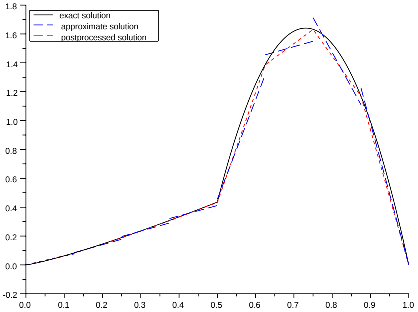
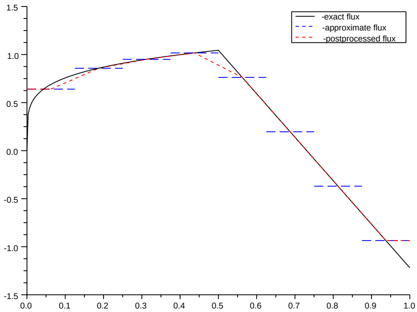


A postprocessed potential s_h^n is in $H_0^1(\Omega)$



A postprocessed flux θ^n is in $\mathbf{H}(\mathrm{div}, \Omega)$

Practical construction of s and θ

Construction of s^n

$$s^n := \mathcal{I}_{\mathrm{av}}^n(u_{h\tau}^n) + \sum_{T' \in \mathcal{T}^{n,n+1}} \alpha_{T'}^n b_{T'},$$

$$\alpha_{T'}^n := \frac{1}{(b_{T'}, 1)_{T'}} (u_{h\tau}^n - \mathcal{I}_{\mathrm{av}}^n(u_{h\tau}^n), 1)_{T'}$$

- $\mathcal{I}_{\mathrm{av}}^n$: the averaging interpolate on the mesh $\mathcal{T}^n$
- $b_{T'}$ standard (time-independent) bubble function supported on T'
- the mean value is preserved on all $T' \in \mathcal{T}^{n,n+1}$
- specificity of the parabolic case
- **independent** of the numerical **scheme**

Construction of θ^n

- inspired from the elliptic case
- **depends** on the numerical **scheme**

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A posteriori error estimate

Theorem (A posteriori error estimate)

Let

- *u be the weak solution*
- *$u_{h\tau}$ be arbitrary*
- *s be the mean values-preserving potential reconstruction and θ locally conservative flux reconstruction.*

Then

$$\|u - u_{h\tau}\|_Y \leq 3 \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{T \in \mathcal{T}^n} (\eta_{\mathrm{R},T}^n + \eta_{\mathrm{DF},T}^n(t))^2 \, \mathrm{d}t \right\}^{1/2} + \left\{ \sum_{n=1}^{N} \int_{I_n} \sum_{T \in \mathcal{T}^n} (\eta_{\mathrm{NC1},T}^n)^2(t) \, \mathrm{d}t \right\}^{1/2} + \left\{ \sum_{n=1}^{N} \tau^n \sum_{T \in \mathcal{T}^n} (\eta_{\mathrm{NC2},T}^n)^2 \right\}^{1/2} + \eta_{\mathrm{IC}} + 3\|f - \tilde{f}\|_{X'}.$$

- **unified setting**: no specification of the numerical scheme

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Estimators

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},T}^n(t) := \|\nabla s(t) + \boldsymbol{\theta}^n\|_T, \quad t \in I_n$
  - penalizes the fact that $-\nabla u_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},T}^n := C_{\mathrm{P}} h_T \|\widetilde{f}^n - \partial_t s^n - \nabla \cdot \boldsymbol{\theta}^n\|_T$
  - residue evaluated for $\boldsymbol{\theta}^n$
  - $C_{\mathrm{P}} = 1/\pi$
- *nonconformity estimators*
  - $\eta_{\mathrm{NC1},T}^n(t) := \|\nabla^{n-1,n}(s - u_{h\tau})(t)\|_T, \quad t \in I_n$
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- *initial condition estimator*
  - $\eta_{\mathrm{IC}} := 2^{1/2} \|s^0 - u^0\|$
- *data oscillation estimator*
  - $\|f - \widetilde{f}\|_{X'}$

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  - $\eta^n_{\mathrm{DF},T}(t) := \|\nabla s(t) + \boldsymbol{\theta}^n\|_T, \quad t \in I_n$
  - penalizes the fact that $-\nabla u^n_{h\tau} \notin \mathbf{H}(\mathrm{div}, \Omega)$
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  - $C_{\mathrm{P}} = 1/\pi$
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- *initial condition estimator*
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- *data oscillation estimator*
  - $\|f - \widetilde{f}\|_{X'}$

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Remarks

Conforming methods

- in conforming methods (FEs, VCFVs) $u_{h\tau}^n \in H_0^1(\Omega)$
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- notice that $\int_{I_n}(\eta^n_{\mathrm{DF},T})^2 \leq (\eta^n_{\mathrm{DF},T,1})^2 + (\eta^n_{\mathrm{DF},T,2})^2$, where

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A space-time adaptive time-marching algorithm

Algorithm for achieving a given relative precision ε

$$\frac{\sum_{n=1}^{N}\left\{(\eta_{\mathrm{sp}}^{n})^2+(\eta_{\mathrm{tm}}^{n})^2\right\}}{\sum_{n=1}^{N}\|u_{h\tau}\|_{Z(I_n)}^2}\leq\varepsilon^2$$

1. Initialization
   1. choose an initial mesh $\mathcal{T}^0$;
   2. select an initial time step τ^0 and set $n := 1$;
2. Loop in time: while $\sum_i \tau^i < T$,
   1. set $\mathcal{T}^{n*} := \mathcal{T}^{n-1}$ and $\tau^{n*} := \tau^{n-1}$;
   2. solve $u_{h\tau}^{n*} := \mathrm{Sol}(u_{h\tau}^{n-1}, \tau^{n*}, \mathcal{T}^{n*})$;
   3. estimate the space and time errors by η_{sp}^{n} and η_{tm}^{n};
   4. when η_{sp}^{n} or η_{tm}^{n} are too much above or below $\varepsilon\|u_{h\tau}\|_{Z(I_n)}/\sqrt{2}$ or not of similar size, refine or derefine the time step τ^{n*} and the space mesh $\mathcal{T}^{n*}$ and return to step (2-2), otherwise save approximate solution, mesh, and time step as $u_{h\tau}^{n}$, $\mathcal{T}^{n}$, and τ^{n} and set $n := n+1$.

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Outline

Efficiency

Theorem (Efficiency)

Under the approximation property, there holds

$$\eta_{\mathrm{sp}}^n + \eta_{\mathrm{tm}}^n \lesssim \|u - u_{h\tau}\|_{Y(I_n)} + \mathcal{J}^n(u_{h\tau}) + \mathcal{E}_f^n$$

Notation

- $\mathcal{J}^n(u_{h\tau})^2 := \tau^n \sum_{T \in \mathcal{T}^{n-1}} |[\![u_{h\tau}^{n-1}]\!]|^2_{-\frac{1}{2},\mathfrak{F}_T^{n-1}} + \tau^n \sum_{T \in \mathcal{T}^n} |[\![u_{h\tau}^n]\!]|^2_{-\frac{1}{2},\mathfrak{F}_T^n}$
- $(\mathcal{E}_f^n)$ is space-time data oscillation term

Comments on $\mathcal{J}^n$

- $\mathcal{J}^n$ is a typical jump seminorm
- it can bounded by the energy error if the jumps in $u_{h\tau}$ have zero mean values (MFEs, FCFVs, NCFEs); it can also be bounded in DGs, using the scheme
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Assumptions for the lower bound proof

Main assumption: approximation property of the flux reconstruction

$$\|\nabla u_{h\tau}^n + \boldsymbol{\theta}^n\|_T \lesssim \left\{ \sum_{T' \in \mathfrak{T}_T} h_{T'}^2 \|\widetilde{f}^n - \partial_t u_{h\tau}^n + \Delta u_{h\tau}^n\|_{T'}^2 \right\}^{1/2} + |\mathbf{n} \cdot [\![\nabla^n u_{h\tau}^n]\!]|_{+\frac{1}{2}, \mathfrak{F}_T^{\mathrm{i},n}} + |[\![u_{h\tau}^n]\!]|_{-\frac{1}{2}, \mathfrak{F}_T^n}$$

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Other assumptions

- the meshes $\{\mathcal{T}^n\}_{0 \le n \le N}$ are shape regular uniformly in n;
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General concept

Upper bound

- for $0 \leq n \leq N$, we only have to construct $\theta^n \in \mathbf{H}(\mathrm{div}, \Omega)$ which is locally conservative, i.e., such that

$$(\tilde{f}^n - \partial_t u^n_{h\tau} - \nabla \cdot \theta^n, 1)_T = 0, \qquad \forall T \in \mathcal{T}^n$$

- we construct θ^n in some mixed finite element space; example: Raviart–Thomas–Nédélec spaces

$$\mathbf{RTN}_l(\mathcal{T}^n) := \left\{ \mathbf{v}_h \in \mathbf{H}(\mathrm{div}, \Omega)\,; \mathbf{v}_h|_T \in \mathbf{RTN}_l(T) \quad \forall T \in \mathcal{T}^n \right\}$$

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Discontinuous Galerkin method

Definition (DG method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $u_{h\tau}^n \in V_h^n := \mathbb{P}_k(\mathcal{T}^n)$, $k \ge 1$, such that

$$(\partial_t u_{h\tau}^n, v_h) - \sum_{F \in \mathcal{F}^n} \{(\mathbf{n}_F \cdot \{\!\{\nabla^n u_{h\tau}^n\}\!\}, [\![v_h]\!])_F + \theta(\mathbf{n}_F \cdot \{\!\{\nabla^n v_h\}\!\}, [\![u_{h\tau}^n]\!])_F\}$$
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- jump operator $[\![v_h]\!]_F = v_h^- - v_h^+$
- average operator $\{\!\{v_h\}\!\} = \frac{1}{2}(v_h^- + v_h^+)$
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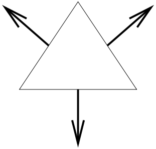
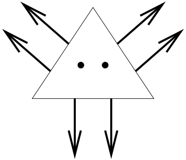
DG flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}^n)$: Raviart–Thomas–Nédélec spaces of degree l



$l = 0$



$l = 1$

Flux reconstruction $\theta^n \in \mathbf{RTN}_l(\mathcal{T}^n)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(F)$,

$$(\theta^n \cdot \mathbf{n}_F, q_h)_F = (-\mathbf{n}_F \cdot \{\!\{\nabla^n u^n_{h\tau}\}\!\} + \alpha_F h_F^{-1} [\![u^n_{h\tau}]\!], q_h)_F$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}^d_{l-1}(T)$, $(\theta^n, \mathbf{r}_h)_T = -(\nabla^n u^n_{h\tau}, \mathbf{r}_h)_T + \theta \sum_{F \in \mathcal{F}^n_T} \omega_F (\mathbf{n}_F \cdot \mathbf{r}_h, [\![u^n_{h\tau}]\!])_F$

Reconstructed flux property

For $l = k$ and when $\mathcal{T}^{n-1} = \mathcal{T}^n$

$$\partial_t u^n_{h\tau} + \nabla \cdot \theta^n = \Pi_{V^n_h} \tilde{f}^n$$

DG flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}^n)$: Raviart–Thomas–Nédélec spaces of degree l

$l = 0$ $\qquad$ $l = 1$

Flux reconstruction $\theta^n \in \mathbf{RTN}_l(\mathcal{T}^n)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(F)$,

$$(\boldsymbol{\theta}^n \cdot \mathbf{n}_F, q_h)_F = (-\mathbf{n}_F \cdot \{\!\{\nabla^n u^n_{h\tau}\}\!\} + \alpha_F h_F^{-1} [\![u^n_{h\tau}]\!], q_h)_F$$

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  $(\boldsymbol{\theta}^n, \mathbf{r}_h)_T = -(\nabla^n u^n_{h\tau}, \mathbf{r}_h)_T + \theta \sum_{F \in \mathcal{F}^n_T} \omega_F (\mathbf{n}_F \cdot \mathbf{r}_h, [\![u^n_{h\tau}]\!])_F$

Reconstructed flux property

For $l = k$ and when $\mathcal{T}^{n-1} = \mathcal{T}^n$

$$\partial_t u^n_{h\tau} + \nabla \cdot \boldsymbol{\theta}^n = \Pi_{V^n_h} \tilde{f}^n$$

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Reconstructed flux property

For $l = k$ and when $\mathcal{T}^{n-1} = \mathcal{T}^n$

$$\partial_t u^n_{h\tau} + \nabla \cdot \boldsymbol{\theta}^n = \Pi_{V^n_h} \widetilde{f}^n$$

Outline

Cell-centered finite volume method

Definition (CCFV method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $\bar{u}_{h\tau}^n \in \bar{V}_h^n := \mathbb{P}_0(\mathcal{T}^n)$ such that

$$\frac{1}{\tau^n}(\bar{u}_{h\tau}^n - u_{h\tau}^{n-1}, 1)_T + \sum_{F \in \mathcal{F}_T^n} S_{T,F}^n = (\tilde{f}^n, 1)_T \qquad \forall T \in \mathcal{T}^n.$$

Flux $\theta^n \in$ RTN$_0(\mathcal{T}^n)$

$$(\theta^n \cdot \mathbf{n}, 1)_F := S_{T,F}^n$$

Postprocessing of the potential

- $\bar{u}_{h\tau}^n \in \bar{V}_h^n$ not suitable for energy error estimates ($\nabla \bar{u}_{h\tau}^n = 0$)
- $u_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_1(\mathcal{T}^n)$ enriched elementwise by parabolas
- $$-\nabla u_{h\tau}^n = \theta^n,$$ $$(u_{h\tau}^n, 1)_T = (\bar{u}_{h\tau}^n, 1)_T$$
- diffusive flux estimator $\eta_{\mathrm{DF},T,1}^n$ vanishes (flux-conf. method)

Cell-centered finite volume method

Definition (CCFV method)

On I_n, $\mathcal{T}^n$, $1 \leq n \leq N$, find $\bar{u}_{h\tau}^n \in \bar{V}_h^n := \mathbb{P}_0(\mathcal{T}^n)$ such that

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Flux $\theta^n \in$ RTN$_0(\mathcal{T}^n)$

$$(\theta^n \cdot \mathbf{n}, 1)_F := S_{T,F}^n$$

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Cell-centered finite volume method

Definition (CCFV method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $\bar{u}^n_{h\tau} \in \bar{V}^n_h := \mathbb{P}_0(\mathcal{T}^n)$ such that

$$\frac{1}{\tau^n}(\bar{u}^n_{h\tau} - u^{n-1}_{h\tau}, 1)_T + \sum_{F \in \mathcal{F}^n_T} S^n_{T,F} = (\tilde{f}^n, 1)_T \qquad \forall T \in \mathcal{T}^n.$$

Flux $\boldsymbol{\theta}^n \in \mathbf{RTN}_0(\mathcal{T}^n)$

$$(\boldsymbol{\theta}^n \cdot \mathbf{n}, 1)_F := S^n_{T,F}$$

Postprocessing of the potential

- $\bar{u}^n_{h\tau} \in \bar{V}^n_h$ not suitable for energy error estimates ($\nabla \bar{u}^n_{h\tau} = 0$)
- $u^n_{h\tau} \in V^n_h$, V^n_h is $\mathbb{P}_1(\mathcal{T}^n)$ enriched elementwise by parabolas
- $$-\nabla u^n_{h\tau} = \boldsymbol{\theta}^n,$$ $$(u^n_{h\tau}, 1)_T = (\bar{u}^n_{h\tau}, 1)_T$$
- diffusive flux estimator $\eta^n_{\mathrm{DF},T,1}$ vanishes (flux-conf. method)

Outline

Mixed finite element method

Definition (MFE method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $\boldsymbol{\sigma}_{h\tau}^n \in \mathbf{W}_h^n$ and $\bar{u}_{h\tau}^n \in \bar{V}_h^n$ such that

$$(\boldsymbol{\sigma}_{h\tau}^n, \mathbf{w}_h) - (\bar{u}_{h\tau}^n, \nabla\cdot\mathbf{w}_h) = 0 \qquad \forall\, \mathbf{w}_h \in \mathbf{W}_h^n,$$

$$(\nabla\cdot\boldsymbol{\sigma}_{h\tau}^n, v_h) + \frac{1}{\tau^n}(\bar{u}_{h\tau}^n - u_{h\tau}^{n-1}, v_h) = (\widetilde{f}^n, v_h) \qquad \forall v_h \in \bar{V}_h^n.$$

Flux $\theta^n \in \mathbf{W}_h^n$

$\theta^n := \boldsymbol{\sigma}_{h\tau}^n$ directly

Postprocessing of the potential

- $u_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_{l+1}(\mathcal{T}^n)$ enriched by bubbles (Arbogast and Chen, 1995)
- $$\Pi_{\mathbf{W}_h^n}(-\nabla^n u_{h\tau}^n) = \boldsymbol{\sigma}_{h\tau}^n,$$ $$\Pi_{\bar{V}_h^n}(u_{h\tau}^n) = \bar{u}_{h\tau}^n$$
- diffusive flux estimator $\eta_{\mathrm{DF},T,1}^n$ vanishes (flux-conf. method)

Mixed finite element method

Definition (MFE method)

On I_n, $\mathcal{T}^n$, $1 \leq n \leq N$, find $\boldsymbol{\sigma}_{h\tau}^n \in \mathbf{W}_h^n$ and $\bar{u}_{h\tau}^n \in \bar{V}_h^n$ such that

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$$(\nabla\cdot\boldsymbol{\sigma}_{h\tau}^n, v_h) + \frac{1}{\tau^n}(\bar{u}_{h\tau}^n - u_{h\tau}^{n-1}, v_h) = (\widetilde{f}^n, v_h) \qquad \forall v_h \in \bar{V}_h^n.$$

Flux $\theta^n \in \mathbf{W}_h^n$

$\theta^n := \boldsymbol{\sigma}_{h\tau}^n$ directly

Postprocessing of the potential

- $u_{h\tau}^n \in V_h^n$, V_h^n is $\mathbb{P}_{l+1}(\mathcal{T}^n)$ enriched by bubbles (Arbogast and Chen, 1995)
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Mixed finite element method

Definition (MFE method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $\boldsymbol{\sigma}^n_{h\tau} \in \mathbf{W}^n_h$ and $\bar{u}^n_{h\tau} \in \bar{V}^n_h$ such that

$$(\boldsymbol{\sigma}^n_{h\tau}, \mathbf{w}_h) - (\bar{u}^n_{h\tau}, \nabla\cdot\mathbf{w}_h) = 0 \qquad \forall\, \mathbf{w}_h \in \mathbf{W}^n_h,$$

$$(\nabla\cdot\boldsymbol{\sigma}^n_{h\tau}, v_h) + \frac{1}{\tau^n}(\bar{u}^n_{h\tau} - u^{n-1}_{h\tau}, v_h) = (\widetilde{f}^n, v_h) \qquad \forall v_h \in \bar{V}^n_h.$$

Flux $\theta^n \in \mathbf{W}^n_h$

$\theta^n := \boldsymbol{\sigma}^n_{h\tau}$ directly

Postprocessing of the potential

- $u^n_{h\tau} \in V^n_h$, V^n_h is $\mathbb{P}_{l+1}(\mathcal{T}^n)$ enriched by bubbles (Arbogast and Chen, 1995)
- $$\Pi_{\mathbf{W}^n_h}(-\nabla^n u^n_{h\tau}) = \boldsymbol{\sigma}^n_{h\tau},$$
  $$\Pi_{\bar{V}^n_h}(u^n_{h\tau}) = \bar{u}^n_{h\tau}$$
- diffusive flux estimator $\eta^n_{\mathrm{DF},T,1}$ vanishes (flux-conf. method)

Outline

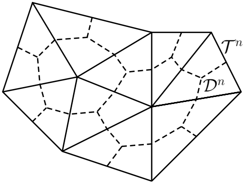
Vertex-centered finite volume method

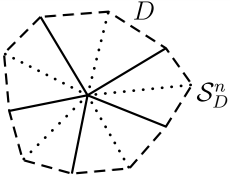
Definition (VCFV method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $u_{h\tau}^n \in V_h^n := \mathbb{P}_1(\mathcal{T}^n) \cap H_0^1(\Omega)$ s.t.

$$(\partial_t u_{h\tau}^n, 1)_D - (\nabla u_{h\tau}^n \cdot \mathbf{n}_D, 1)_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}^{\mathrm{i},n}.$$

Setting





- triangulation $\mathcal{T}^n$, dual mesh $\mathcal{D}^n$, simplicial submesh $\mathcal{S}^n$
- $s = u_{h\tau}$, nonconformity estimators $\eta_{\mathrm{NC1},T}^n$ and $\eta_{\mathrm{NC2},T}^n$ vanish (conforming method)

Flux $\theta^n \in \mathbf{RTN}_0(\mathcal{S}^n)$

- by prescription: $\theta^n \cdot \mathbf{n}_F|_F := -\{\!\{\nabla u_{h\tau}^n \cdot \mathbf{n}_F\}\!\}$ on faces F of $\mathcal{S}^n$
- by MFE solution of local Neumann problems on patches $\mathcal{S}_D^n$

Vertex-centered finite volume method

Definition (VCFV method)

On I_n, $\mathcal{T}^n$, $1 \leq n \leq N$, find $u_{h\tau}^n \in V_h^n := \mathbb{P}_1(\mathcal{T}^n) \cap H_0^1(\Omega)$ s.t.

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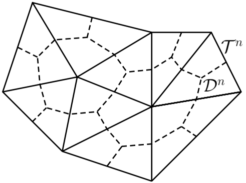

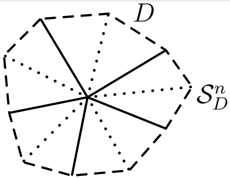

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Vertex-centered finite volume method

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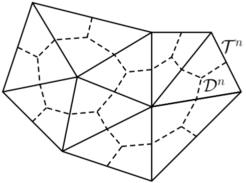

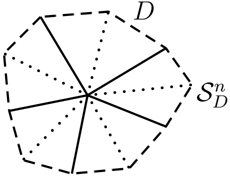

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Outline

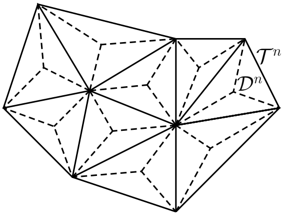
Face-centered finite volume method

Definition (FCFV method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $u_{h\tau}^n \in V_h^n$ (Crouzeix–Raviart sp.) s.t.

$$(\partial_t u_{h\tau}^n, 1)_D - (\nabla^n u_{h\tau}^n \cdot \mathbf{n}_D, 1)_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}^{\mathrm{i},n}.$$

Setting



- triangulation $\mathcal{T}^n$, dual mesh $\mathcal{D}^n$, simplicial submesh $\mathcal{S}^n$
- $u_{h\tau}^n \notin H_0^1(\Omega)$, $-\nabla u_{h\tau}^n \notin \mathbf{H}(\mathrm{div}, \Omega)$

Flux $\theta^n \in \mathbf{RTN}_0(\mathcal{S}^n)$

- by prescription: $\theta^n \cdot \mathbf{n}_F|_F := -\{\!\{\nabla u_{h\tau}^n \cdot \mathbf{n}_F\}\!\}$ on faces F of $\mathcal{S}^n$
- by solution of local Neumann problems on patches $\mathcal{S}_D^n$

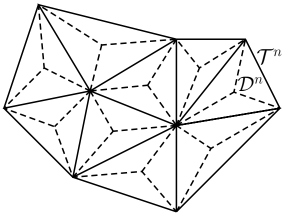
Face-centered finite volume method

Definition (FCFV method)

On I_n, $\mathcal{T}^n$, $1 \le n \le N$, find $u^n_{h\tau} \in V^n_h$ (Crouzeix–Raviart sp.) s.t.

$$(\partial_t u^n_{h\tau}, 1)_D - (\nabla^n u^n_{h\tau} \cdot \mathbf{n}_D, 1)_{\partial D} = (\tilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}^{\mathrm{i},n}.$$

Setting



- triangulation $\mathcal{T}^n$, dual mesh $\mathcal{D}^n$, simplicial submesh $\mathcal{S}^n$
- $u^n_{h\tau} \notin H^1_0(\Omega)$, $-\nabla u^n_{h\tau} \notin \mathbf{H}(\mathrm{div}, \Omega)$

Flux $\theta^n \in \mathbf{RTN}_0(\mathcal{S}^n)$

- by prescription: $\theta^n \cdot \mathbf{n}_F|_F := -\{\!\{\nabla u^n_{h\tau} \cdot \mathbf{n}_F\}\!\}$ on faces F of $\mathcal{S}^n$
- by solution of local Neumann problems on patches $\mathcal{S}^n_D$

Face-centered finite volume method

Definition (FCFV method)

On I_n, $\mathcal{T}^n$, $1 \leq n \leq N$, find $u^n_{h\tau} \in V^n_h$ (Crouzeix–Raviart sp.) s.t.

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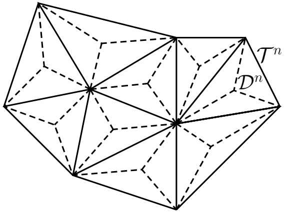
Setting



- triangulation $\mathcal{T}^n$, dual mesh $\mathcal{D}^n$, simplicial submesh $\mathcal{S}^n$
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- by solution of local Neumann problems on patches $\mathcal{S}^n_D$

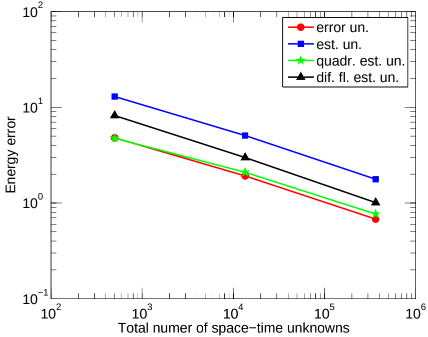
Outline

Numerical experiment

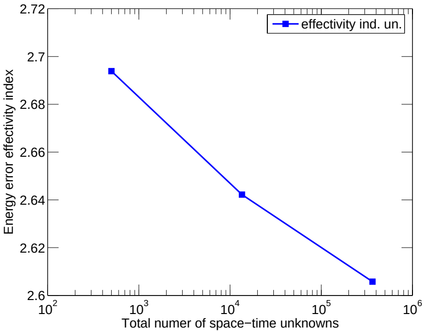
Numerical experiment

- exact solution $u = e^{x+y+t-3}$ on square domain $\Omega = (0,3) \times (0,3)$, $T = 1.5$ or $T = 3$
- square meshes: 10×10, 30×30, 90×90
- time steps: 0.3, 0.1, 0.3333
- vertex-centered finite volumes
- additional quadrature/mass lumping estimator

Energy norm results, $T = 1.5$

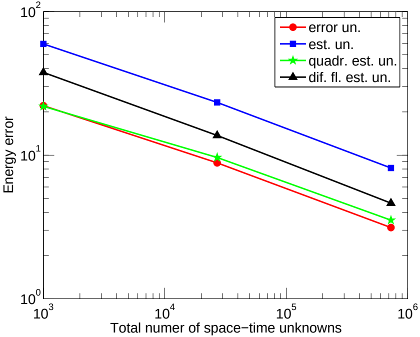


Energy error and estimators

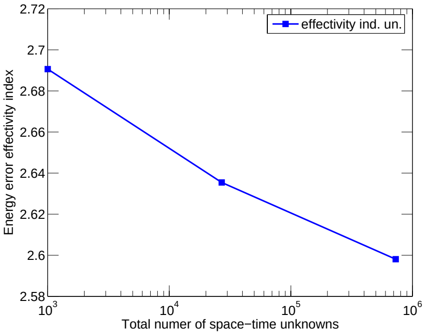


Effectivity index

Energy norm results, $T = 3$

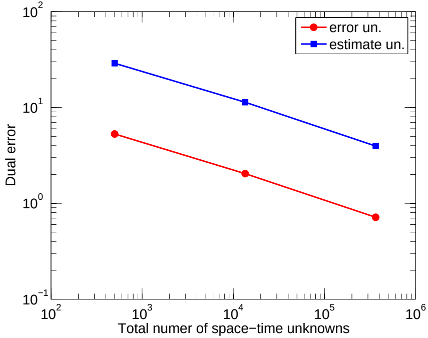


Energy error and estimators

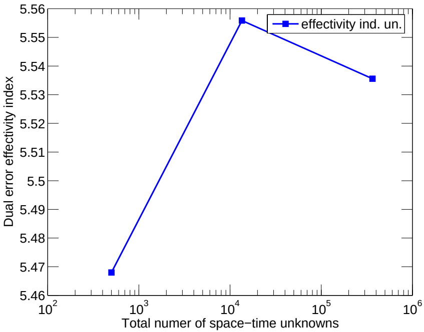


Effectivity index

Dual norm results, $T = 1.5$

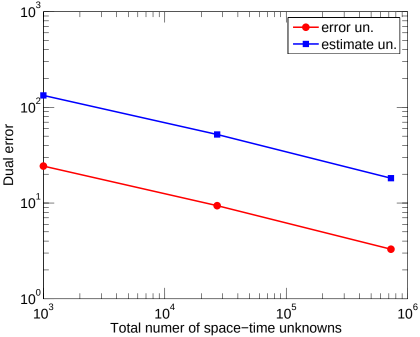


Dual error and estimators

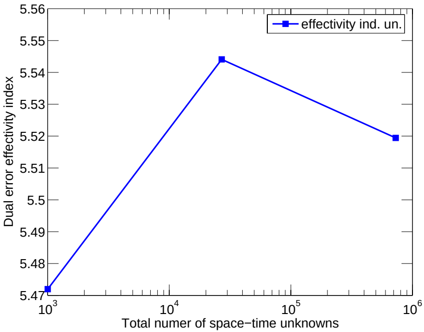


Effectivity index

Dual norm results, $T = 3$



Dual error and estimators



Effectivity index

Outline

A parabolic convection–diffusion–reaction problem

Model problem

$$
\begin{aligned}
u_t - \nabla\cdot(\mathbf{S}\nabla u) + \nabla\cdot(u\mathbf{v}) + ru &= f && \text{in } \Omega\times(0,T),\\
u(\cdot,0) &= u_0 && \text{in } \Omega,\\
u &= 0 && \text{on } \partial\Omega\times(0,T)
\end{aligned}
$$

Energy norm

$$X := L^2(0,T;H_0^1(\Omega))$$

$$\|v\|_X^2 := \int_0^T |||v(\cdot,t)|||^2\,\mathrm{d}t,$$

$$|||v|||^2 := \left\|\mathbf{S}^{\frac{1}{2}}\nabla v\right\|^2 + \left\|\left(\tfrac{1}{2}\nabla\cdot\mathbf{v} + r\right)^{\frac{1}{2}} v\right\|^2$$

A parabolic convection–diffusion–reaction problem

Model problem

$$\begin{aligned} u_t - \nabla\cdot(\mathbf{S}\nabla u) + \nabla\cdot(u\mathbf{v}) + ru = f & \quad \text{in } \Omega\times(0,T), \\ u(\cdot,0) = u_0 & \quad \text{in } \Omega, \\ u = 0 & \quad \text{on } \partial\Omega\times(0,T) \end{aligned}$$

Energy norm

$$X := L^2(0,T;H^1_0(\Omega))$$

$$\begin{aligned} \|v\|_X^2 &:= \int_0^T |||v(\cdot,t)|||^2 \,\mathrm{d}t, \\ |||v|||^2 &:= \left\|\mathbf{S}^{\frac12}\nabla v\right\|^2 + \left\|\left(\tfrac12\nabla\cdot\mathbf{v} + r\right)^{\frac12} v\right\|^2 \end{aligned}$$

Estimate for $u_t - \nabla\cdot(\mathbf{S}\nabla u) + \nabla\cdot(u\mathbf{v}) + ru = f$

Theorem (A posteriori error estimate)

Let

- *u be the weak solution,*
- *$u_h \in X$ be arbitrary,*
- *$\mathcal{D}^n$, $n \in \{1,\ldots,N\}$, be a partition of Ω on $(t_{n-1}, t_n]$,*
- *$\mathbf{t}_h^n$ (diffusive flux reconstruction), $\mathbf{w}_h^n$ (convective flux rec.), $n \in \{1,\ldots,N\}$, be arbitrary in $\mathbf{H}(\mathrm{div},\Omega)$, such that*

$$\frac{u_D^n - u_D^{n-1}}{\tau_n}|D| + \langle \mathbf{t}_h^n\cdot\mathbf{n}, 1\rangle_{\partial D} + \langle \mathbf{w}_h^n\cdot\mathbf{n}, 1\rangle_{\partial D} + r_D^n u_D^n|D| = f_D^n|D|$$

$$\forall n \in \{1,\ldots,N\},\ \forall D \in \mathcal{D}^n.$$

Then

$$\|(u-u_{h\tau})(\cdot,T)\|^2 + \|u-u_{h\tau}\|_X^2 \le \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \left(\left\{ \sum_{D\in\mathcal{D}^n} \left(\eta_{\mathrm{R},D}^n + \eta_{\mathrm{DCF},D}^n(t)\right)^2 \right\}^{\frac12} + \left\{ \sum_{D\in\mathcal{D}^n} \left(\eta_{\mathrm{DOQ},D}^n(t)\right)^2 \right\}^{\frac12} \right)^2 \mathrm{d}t + \|u_0 - u_{h\tau}(\cdot,0)\|^2.$$

Estimate for $u_t - \nabla \cdot (\mathbf{S}\nabla u) + \nabla \cdot (u\mathbf{v}) + ru = f$

Theorem (A posteriori error estimate)

Let

- *u be the weak solution,*
- *$u_h \in X$ be arbitrary,*
- *$\mathcal{D}^n$, $n \in \{1, \ldots, N\}$, be a partition of Ω on $(t_{n-1}, t_n]$,*
- *$\mathbf{t}_h^n$ (diffusive flux reconstruction), $\mathbf{w}_h^n$ (convective flux rec.), $n \in \{1, \ldots, N\}$, be arbitrary in $\mathbf{H}(\mathrm{div}, \Omega)$, such that*

$$\frac{u_D^n - u_D^{n-1}}{\tau_n}|D| + \langle \mathbf{t}_h^n \cdot \mathbf{n}, 1\rangle_{\partial D} + \langle \mathbf{w}_h^n \cdot \mathbf{n}, 1\rangle_{\partial D} + r_D^n u_D^n |D| = f_D^n |D|$$

$$\forall n \in \{1, \ldots, N\}, \ \forall D \in \mathcal{D}^n.$$

Then

$$\|(u - u_{h\tau})(\cdot, T)\|^2 + \|u - u_{h\tau}\|_X^2 \leq \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \left(\left\{ \sum_{D \in \mathcal{D}^n} (\eta_{\mathrm{R},D}^n + \eta_{\mathrm{DCF},D}^n(t))^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{D \in \mathcal{D}^n} (\eta_{\mathrm{DOQ},D}^n(t))^2 \right\}^{\frac{1}{2}} \right)^2 \mathrm{d}t + \|u_0 - u_{h\tau}(\cdot, 0)\|^2.$$

A posteriori error estimates

Estimators

For each $n \in \{1, \ldots, N\}$ and $D \in \mathcal{D}^n$, define

- *residual estimator*

$$\eta_{\mathrm{R},D}^n := m_D^n \left\| f_D^n - \frac{u_D^n - u_D^{n-1}}{\tau_n} - \nabla \cdot \mathbf{t}_h^n - \nabla \cdot \mathbf{w}_h^n - r_D^n u_D^n \right\|_D$$

where

$$m_D^n := \min\left\{ C_{\mathrm{P},D}^{\frac{1}{2}} h_D (c_{\mathbf{S},D}^n)^{-\frac{1}{2}}, (c_{\mathbf{v},r,D}^n)^{-\frac{1}{2}} \right\}$$

- *diffusive and convective flux estimator*

$$\eta_{\mathrm{DCF},D}^n(t) := \left\| \mathbf{S}^{\frac{1}{2}} \nabla u_{h\tau} + \mathbf{S}^{-\frac{1}{2}} \mathbf{t}_h^n - \mathbf{S}^{-\frac{1}{2}} u_{h\tau} \mathbf{v} + \mathbf{S}^{-\frac{1}{2}} \mathbf{w}_h^n \right\|_D (t)$$

- *data oscillation–quadrature estimator*

$$\eta_{\mathrm{DOQ},D}^n(t) := \overline{m}^n \left\| f - f_D^n - (u_{h\tau})_t + \frac{u_D^n - u_D^{n-1}}{\tau_n} - r u_{h\tau} + r_D^n u_D^n \right\|_D (t)$$

A posteriori error estimates

Estimators

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Distinguishing space and time errors

Distinguishing space and time errors

$$\eta^n \leq \eta_{\mathrm{sp}}^n + \eta_{\mathrm{tm}}^n$$

- η_{sp}^n: spatial estimator
- η_{tm}^n: temporal estimator
- adaptive algorithm which automatically balances η_{sp}^n and η_{tm}^n and **achieves a user-given relative precision**

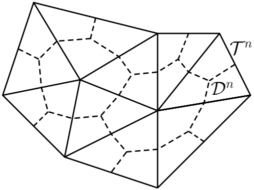
Vertex-centered finite volume scheme

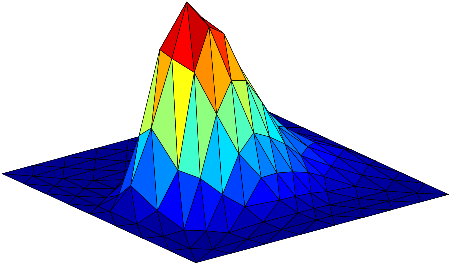
Vertex-centered finite volume scheme

Find the values u_D^n, $n \in \{1, \dots, N\}$, $D \in \mathcal{D}^n$, such that

$$\frac{u_D^n + u_D^{n-1}}{\tau_n}|D| - \langle \mathbf{S}_h^n \nabla u_h^n \cdot \mathbf{n}, 1\rangle_{\partial D} + \langle \mathbf{v}^n \cdot \mathbf{n}, 1\rangle_{\partial D}\, \overline{u_D^n} + r_D^n u_D^n |D| = f_D^n |D|$$

$$\forall n \in \{1, \dots, N\},\ \forall D \in \mathcal{D}^n$$





Numerical experiments

Model problem

- $\mathbf{S} = \nu Id$, ν is a parameter
- $\mathbf{v} = (0.8, 0.4)$
- $r = 0$, $f = 0$

Exact solution

$$u(x,y,t) = \frac{1}{200\nu(t+t_0)+1} e^{-50\frac{(x-x_0-v_1(t+t_0))^2+(y-y_0-v_2(t+t_0))^2}{200\nu(t+t_0)+1}}$$

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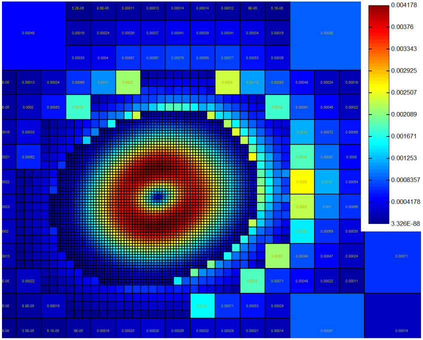
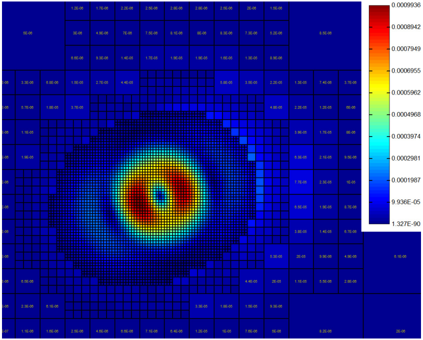
Error distributions

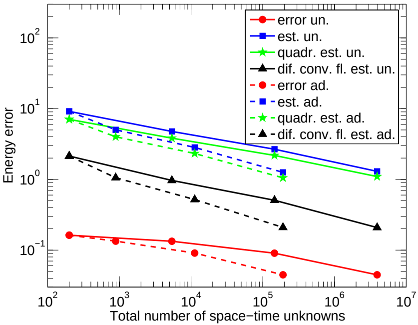


Estimated error distribution, $\nu = 0.001$, $T = 0.6$



Exact error distribution, $\nu = 0.001$, $T = 0.6$

Estimated and actual errors



Estimated and actual errors,
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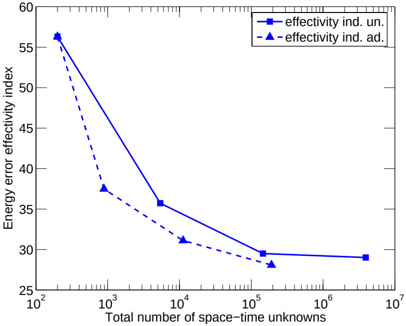
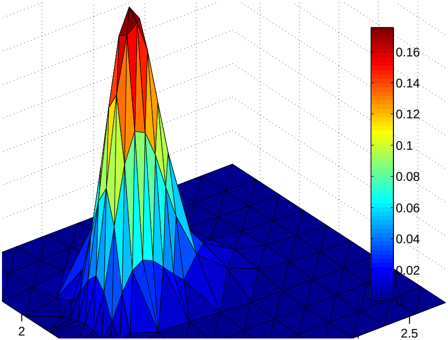
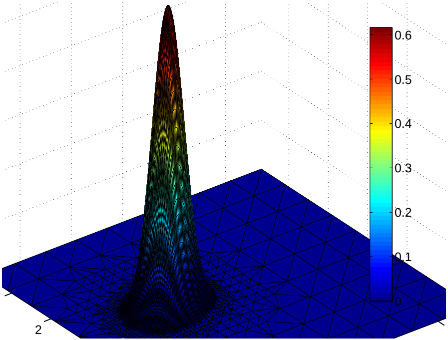


Effectivity indices,
$\nu = 0.001$, $T = 0.6$

Adaptive refinement approximate solutions

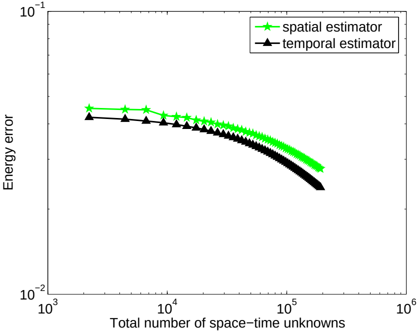


Approximate solutions, $\nu = 0.001$, $T = 0.6$, two levels refinement



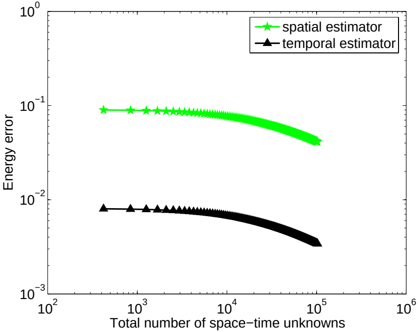
Approximate solutions, $\nu = 0.001$, $T = 0.6$, four levels refinement

Spatial and temporal estimators equilibrated

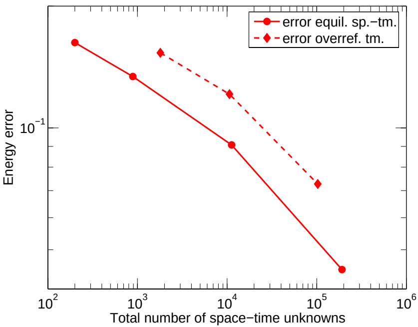


Spatial estimators η^n_{sp} and temporal estimators η^n_{tm} equilibrated, $\nu = 0.001$, $T = 0.6$

Overrefinement in time

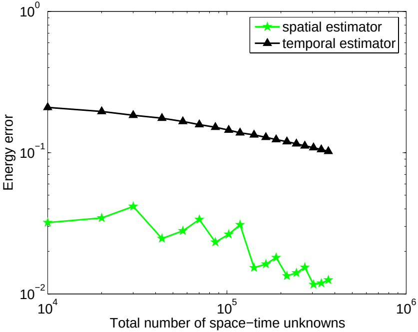


Spatial estimators η_{sp}^{n} and temporal estimators η_{tm}^{n}

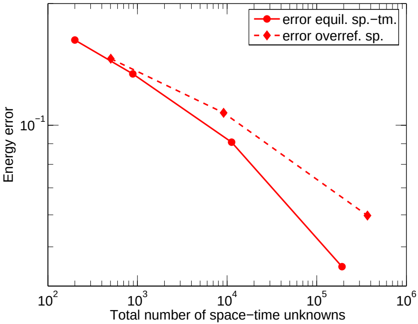


Comparison with the equilibrated case

Overrefinement in space



Spatial estimators η_{sp}^n and temporal estimators η_{tm}^n



Comparison with the equilibrated case

Outline

Conclusions and future work

Conclusions

- **unified framework** for the heat equation (works for all major numerical schemes)
- directly and locally computable estimates
- global-in-space and local-in-time efficiency and robustness with respect to the final time as in Verfürth (2003)

Future work

- nonlinear problems
- extensions to other types of problems

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Bibliography

Papers

- ERN A., VOHRALÍK M., A posteriori error estimation based on potential and flux reconstruction for the heat equation, *SIAM J. Numer. Anal.* **48** (2010), 198–223.
- HILHORST D., VOHRALÍK M., A posteriori error estimates for combined finite volume–finite element discretizations of reactive transport equations on nonmatching grids, *Comput. Methods Appl. Mech. Engrg.*, DOI 10.1016/j.cma.2010.08.017.

Merci de votre attention !