Un cadre unifié pour les estimations a posteriori pour le problème de Stokes

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Outline

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Previous results

A posteriori error estimation

- Prager and Synge (1947), energy error equality
- Babuška and Rheinboldt (1978), mathematical analysis
- Ladevèze and Leguillon (1983), equilibrated fluxes estimates
- Verfürth (1989), local efficiency
- Ainsworth and Oden (1993), equilibration
- Repin (1997), functional a posteriori error estimates
- Luce and Wohlmuth (2004), dual meshes estimates
- Dörfler and Ainsworth (2005), guaranteed upper bound in the Stokes setting
- Ainsworth (2005), unified framework
- Carstensen (2005–2009), unified framework

Outline

The Stokes problem

Stokes problem

Find $\mathbf{u}$ and p such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \partial\Omega \end{aligned}$$

Weak solution

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= 0 \quad \forall q \in Q \end{aligned}$$

- $\mathbf{V} := [H_0^1(\Omega)]^d$, $Q := L_0^2(\Omega)$
- $a(\mathbf{u}, \mathbf{v}) := (\nabla \mathbf{u}, \nabla \mathbf{v})$, $b(\mathbf{v}, q) := -(q, \nabla \cdot \mathbf{v})$

inf–sup **condition**

$$\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\nabla \mathbf{v}\| \, \|q\|} = \beta$$

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Babuška–Brezzi splitting

Energy norm

$$|||(\mathbf{v}, q)|||^2 := \|\nabla \mathbf{v}\|^2 + \beta^2 \|q\|^2 \qquad (\mathbf{v}, q) \in \mathbf{V} \times Q$$

Babuška–Brezzi splitting

- $\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r)) := a(\mathbf{v}, \mathbf{z}) + b(\mathbf{z}, q) + b(\mathbf{v}, r)$
- equivalent formulation: find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\mathcal{B}((\mathbf{u}, p), (\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}) \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q$$

- inf–sup condition on $\mathbf{V} \times Q$

$$\inf_{(\mathbf{v},q)\in\mathbf{V}\times Q} \sup_{(\mathbf{z},r)\in\mathbf{V}\times Q} \frac{\mathcal{B}((\mathbf{v}, q), (\mathbf{z}, r))}{|||(\mathbf{z}, r)|||\,|||(\mathbf{v}, q)|||} = \frac{\sqrt{5}-1}{2} =: C_{\mathrm{S}}$$

(β disappears thanks to the definition of the energy norm)
(value of C_{S} communicated to us by J.-F. Maître)

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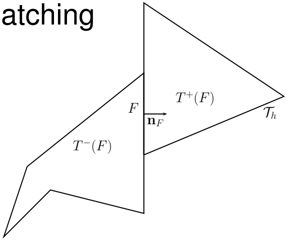
Discrete setting

Mesh $\mathcal{T}_h$

- a polygonal (polyhedral) partition of Ω
- elements T of $\mathcal{T}_h$ can be nonconvex or non star-shaped
- $\mathcal{T}_h$ can be nonmatching



Nonmatching polygonal mesh $\mathcal{T}_h$

Broken Sobolev space

$$\mathbf{V}(\mathcal{T}_h) := \{\mathbf{v}_h \in \mathbf{L}^2(\Omega);\ \mathbf{v}_h|_T \in [H^1(T)]^d \quad \forall T \in \mathcal{T}_h\}$$

- jump of $\mathbf{v}_h$ over a side F: $[\![\mathbf{v}_h]\!]_F := \mathbf{v}_h|_{T^-(F)} - \mathbf{v}_h|_{T^+(F)}$
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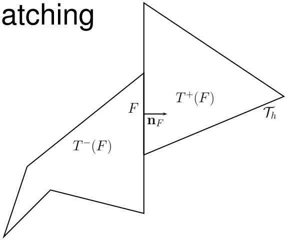

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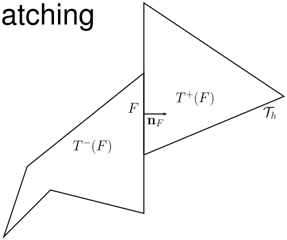

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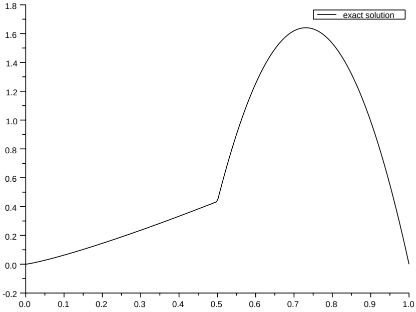
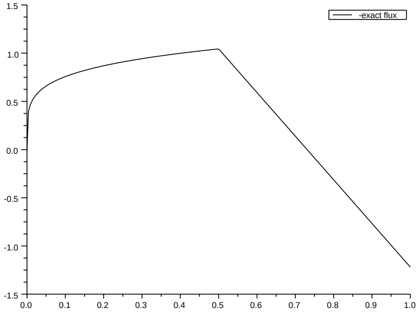
Outline

Potential and flux ($-\Delta u = f$)

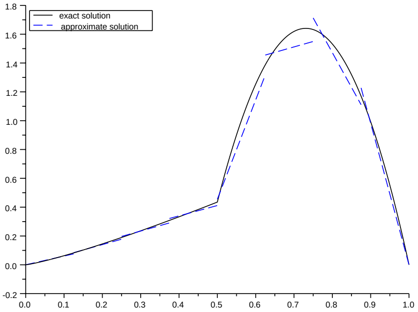


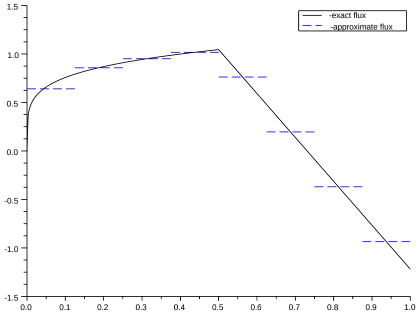


Potential u is in $H_0^1(\Omega)$

Flux $-\nabla u$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Approximate potential and approximate flux ($-\Delta u = f$)

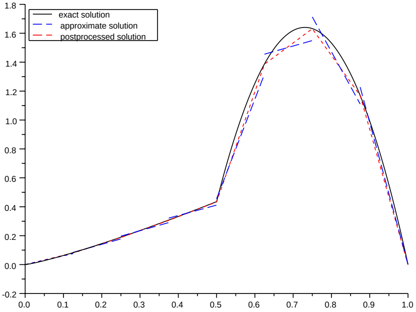


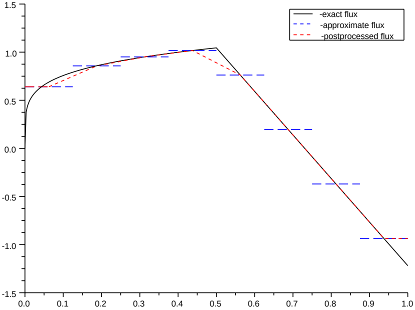


Approximate potential u_h is not in $H_0^1(\Omega)$

Approximate flux $-\nabla u_h$ is not in $\mathbf{H}(\mathrm{div}, \Omega)$

Potential and flux reconstructions ($-\Delta u = f$)





A postprocessed potential s_h is in $H_0^1(\Omega)$

A postprocessed flux $\boldsymbol{\sigma}_h$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

Assumption 1: velocity and stress reconstructions

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$

Stress reconstruction

- stress reconstruction $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{H}}(\mathrm{div}, \Omega)$
- elementwise **local conservation** holds:

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{e}_i)_T = 0, \quad i = 1, \ldots, d, \quad \forall T \in \mathcal{T}_h$$

or

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Assumption 1: velocity and stress reconstructions

$-\Delta \mathbf{u} + \nabla p = \mathbf{f}$

$\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$ **constitutive law**

$\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}$ **equilibrium**

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Outline

Guaranteed upper bound

Theorem (A posteriori error estimate)

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the weak solution. Let $(\mathbf{u}_h, p_h) \in \mathbf{V}(\mathcal{T}_h) \times Q$ be arbitrary. Let the velocity reconstruction $\mathbf{s}_h$ and the stress reconstruction $\underline{\sigma}_h$ satisfy **Assumption 1***. Then,*

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \le \left\{ \sum_{T \in \mathcal{T}_h} \eta_{\mathrm{NC},T}^2 \right\}^{1/2} + \frac{1}{C_{\mathrm{S}}} \left\{ \sum_{T \in \mathcal{T}_h} \left\{ (\eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T})^2 + \eta_{\mathrm{D},T}^2 \right\} \right\}^{1/2}.$$

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Estimators, $\underline{\sigma} = \nabla \mathbf{u} - p\underline{\mathbf{I}}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

$$\eta_{\mathrm{DF},T} := \|\nabla \mathbf{s}_h - p_h \underline{\mathbf{I}} - \underline{\sigma}_h\|_T$$

- *residual estimator*

$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla \cdot \underline{\sigma}_h + \mathbf{f}\|_T$$

  - $C_{\mathrm{P},T}$: Poincaré cnst ($1/\pi$ when T convex); h_T: cell diameter

- *nonconformity estimator*

$$\eta_{\mathrm{NC},T} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_T$$

- *divergence estimator*

$$\eta_{\mathrm{D},T} := \frac{\|\nabla \cdot \mathbf{s}_h\|_T}{\beta}$$

Continuous level

- **constitutive law**: $\nabla \mathbf{u} - p\underline{\mathbf{I}} - \underline{\sigma} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla \cdot \underline{\sigma} + \mathbf{f} = \mathbf{0}$
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Continuous level

- **constitutive law**: $\nabla \mathbf{u} - p\underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}} = \underline{\mathbf{0}}$
- **equilibrium**: $\nabla \cdot \underline{\boldsymbol{\sigma}} + \mathbf{f} = \mathbf{0}$
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Estimators, $\underline{\boldsymbol{\sigma}} = \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$

Estimators for $T \in \mathcal{T}_h$

- *diffusive flux estimator*

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$$\eta_{\mathrm{R},T} := C_{\mathrm{P},T} h_T \|\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_T$$

  - $C_{\mathrm{P},T}$: Poincaré cnst ($1/\pi$ when T convex); h_T: cell diameter

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Main steps of the proof, $\underline{\sigma} = \nabla \mathbf{u} - p\underline{I}$

Main steps of the proof, cf. Prager–Synge (1947), Repin (2002).

- triangle inequality:
$$|||(\mathbf{u}-\mathbf{u}_h, p-p_h)||| \le \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\| + |||(\mathbf{u}-\mathbf{s}_h, p-p_h)|||$$
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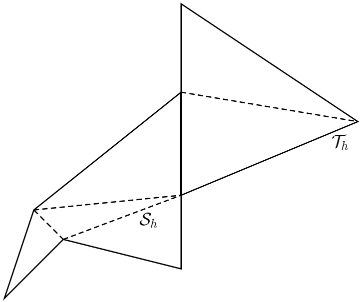
Outline

Assumption 2

Technical aspects

- there exists a shape-regular matching simplicial submesh $\mathcal{S}_h$ of $\mathcal{T}_h$ such that, for each $T \in \mathcal{T}_h$, the number of subelements $T' \subset T$, $T' \in \mathcal{S}_h$, is uniformly bounded
- $\mathbf{u}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^d$, $p_h \in \mathbb{P}_k(\mathcal{T}_h)$, $\mathbf{f} \in [\mathbb{P}_k(\mathcal{T}_h)]^d$, $\underline{\boldsymbol{\sigma}}_h \in [\mathbb{P}_k(\mathcal{S}_h)]^{d\times d}$ for some fixed $k \geq 1$



Nonmatching polygonal mesh $\mathcal{T}_h$ and its simplicial submesh $\mathcal{S}_h$

Assumption 3

Approximation property

There holds

$$\|\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \eta_{\mathrm{res},T} \qquad \forall T \in \mathcal{T}_h,$$

or

$$\|\nabla \mathbf{u}_h - \underline{\boldsymbol{\sigma}}_h\|_T \lesssim \eta_{\mathrm{res},T} \qquad \forall T \in \mathcal{T}_h,$$

where

$$\eta_{\mathrm{res},T}^2 := \sum_{T \in \mathfrak{T}_T} \left\{ h_T^2 \|\mathbf{f} + \Delta \mathbf{u}_h - \nabla p_h\|_T^2 + \|\nabla \cdot \mathbf{u}_h\|_T^2 \right\} + \sum_{F \in \mathfrak{F}_T^{\mathrm{int}}} h_F \|[\![(\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}}) \mathbf{n}_F]\!]\|_F^2 + \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2$$

is the classical local residual error indicator.

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Efficiency

Theorem (Local efficiency; bubble functions, Verfürth (1989))

Let **Assumptions 2** *and* **3** *hold. Let* $\mathbf{s}_h = \mathcal{I}_{\rm av}(\mathbf{u}_h)$ *and let* $(\mathbf{u}, p) \in \mathbf{V} \times Q$ *be the weak solution. Then,*

$$\eta_{\mathrm{NC},T} + \eta_{\mathrm{R},T} + \eta_{\mathrm{DF},T} + \eta_{\mathrm{D},T} \lesssim |||(\mathbf{u} - \mathbf{u}_h, p - p_h)|||_{\mathfrak{T}_T} + \left\{ \sum_{F \in \mathfrak{F}_T} h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F^2 \right\}^{1/2}.$$

Remark

- $h_F^{-1} \|[\![\mathbf{u}_h]\!]\|_F = 0$ for conforming methods
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Outline

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Discontinuous Galerkin method

Discontinuous approximation spaces

$$\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d, \; Q_h := \mathbb{P}_{k-1}(\mathcal{T}_h) \cap Q \qquad k \geq 1$$

Bilinear and linear forms

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{T \in \mathcal{T}_h} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_T + \sum_{F \in \partial \mathcal{T}_h} \gamma_F h_F^{-1} \langle [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!] \rangle_F$$
$$- \sum_{F \in \partial \mathcal{T}_h} \{ \langle \{\!\{\nabla \mathbf{u}_h\}\!\} \mathbf{n}_F, [\![\mathbf{v}_h]\!] \rangle_F + \theta \langle \{\!\{\nabla \mathbf{v}_h\}\!\} \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F \},$$

$$b_h(\mathbf{v}_h, q_h) := - \sum_{T \in \mathcal{T}_h} (q_h, \nabla \cdot \mathbf{v}_h)_T + \sum_{F \in \partial \mathcal{T}_h} \langle \{\!\{q_h\}\!\}, [\![\mathbf{v}_h]\!] \cdot \mathbf{n}_F \rangle_F$$

Discontinuous Galerkin method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

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$$\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d, \; Q_h := \mathbb{P}_{k-1}(\mathcal{T}_h) \cap Q \qquad k \geq 1$$

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$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{T \in \mathcal{T}_h} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_T + \sum_{F \in \partial \mathcal{T}_h} \gamma_F h_F^{-1} \langle [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!] \rangle_F$$
$$- \sum_{F \in \partial \mathcal{T}_h} \left\{ \langle \{\!\{ \nabla \mathbf{u}_h \}\!\} \mathbf{n}_F, [\![\mathbf{v}_h]\!] \rangle_F + \theta \langle \{\!\{ \nabla \mathbf{v}_h \}\!\} \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F \right\},$$

$$b_h(\mathbf{v}_h, q_h) := - \sum_{T \in \mathcal{T}_h} (q_h, \nabla \cdot \mathbf{v}_h)_T + \sum_{F \in \partial \mathcal{T}_h} \langle \{\!\{ q_h \}\!\}, [\![\mathbf{v}_h]\!] \cdot \mathbf{n}_F \rangle_F$$

Discontinuous Galerkin method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

Discontinuous Galerkin method

Discontinuous approximation spaces

$$\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d,\ Q_h := \mathbb{P}_{k-1}(\mathcal{T}_h) \cap Q \qquad k \geq 1$$

Bilinear and linear forms

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_{T\in\mathcal{T}_h} (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_T + \sum_{F\in\partial\mathcal{T}_h} \gamma_F h_F^{-1} \langle [\![\mathbf{u}_h]\!], [\![\mathbf{v}_h]\!]\rangle_F$$
$$- \sum_{F\in\partial\mathcal{T}_h} \{\langle \{\!\{\nabla \mathbf{u}_h\}\!\}\mathbf{n}_F, [\![\mathbf{v}_h]\!]\rangle_F + \theta \langle \{\!\{\nabla \mathbf{v}_h\}\!\}\mathbf{n}_F, [\![\mathbf{u}_h]\!]\rangle_F\},$$

$$b_h(\mathbf{v}_h, q_h) := -\sum_{T\in\mathcal{T}_h} (q_h, \nabla\cdot\mathbf{v}_h)_T + \sum_{F\in\partial\mathcal{T}_h} \langle \{\!\{q_h\}\!\}, [\![\mathbf{v}_h]\!]\cdot\mathbf{n}_F\rangle_F$$

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Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

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Velocity and stress reconstructions in DG

Reconstructed velocity $\mathbf{s}_h$

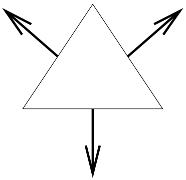
$$\mathbf{s}_h = \mathcal{I}_{\mathrm{av}}(\mathbf{u}_h)$$

Reconstructed stress $\underline{\sigma}_h$

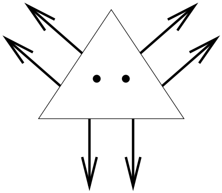
$$\underline{\sigma}_h \in \underline{\Sigma}^l(\mathcal{T}_h) := \{\underline{v}_h \in \underline{\boldsymbol{H}}(\mathrm{div},\Omega);\ \underline{v}_h|_T \in \underline{\Sigma}^l(T) \quad \forall T \in \mathcal{T}_h\},$$

$$\underline{\Sigma}^l(T) := [\mathbb{P}_l(T)]^{d\times d} + [\mathbb{P}_l(T)]^d \otimes \mathbf{x},$$

Raviart–Thomas–Nédélec space of tensor functions of order l, $l = k-1$ or k (simplicial meshes)



$l = 0$



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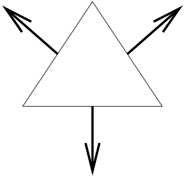
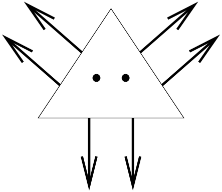


$l = 0$



$l = 1$

Velocity and stress reconstructions in DG

Specification of degrees of freedom of $\underline{\sigma}_h$

- for all $F \in \mathcal{F}_T$ and all $\mathbf{q}_h \in [\mathbb{P}_l(F)]^d$

$$\langle \underline{\sigma}_h \mathbf{n}_F, \mathbf{q}_h \rangle_F = \langle \{\!\{ \nabla \mathbf{u}_h - p_h \underline{\mathbf{I}} \}\!\} \mathbf{n}_F - \gamma_F h_F^{-1} [\![\mathbf{u}_h]\!], \mathbf{q}_h \rangle_F$$

- for all $\underline{\tau}_h \in [\mathbb{P}_{l-1}(T)]^{d \times d}$

$$(\underline{\sigma}_h, \underline{\tau}_h)_T = (\nabla \mathbf{u}_h - p_h \underline{\mathbf{I}}, \underline{\tau}_h)_T - \theta \sum_{F \in \mathcal{F}_T} \langle \omega_F \underline{\tau}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!] \rangle_F$$

Velocity and stress reconstructions in DG

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Application of the framework to DG

Lemma (Reconstructed stress in the DG method)

For all $T \in \mathcal{T}_h$, there holds

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h + \mathbf{f}, \mathbf{v}_h)_T = 0 \quad \forall \mathbf{v}_h \in [\mathbb{P}_l(T)]^d.$$

In particular, **Assumption 1** *holds true.*

Proof.

Green theorem, definition of $\underline{\boldsymbol{\sigma}}_h$, definition of the DG method:

$$\begin{aligned}
-(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \mathbf{v}_h)_T = {} & (\underline{\boldsymbol{\sigma}}_h, \nabla\mathbf{v}_h)_T - \sum_{F\in\mathcal{F}_T} \langle \underline{\boldsymbol{\sigma}}_h \mathbf{n}_T, \mathbf{v}_h\rangle_F \\
& + (\nabla\mathbf{u}_h - p_h\underline{\mathbf{I}}, \nabla\mathbf{v}_h)_T - \theta \sum_{F\in\mathcal{F}_T} \langle \omega_F \nabla\mathbf{v}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!]\rangle_F \\
& - \sum_{F\in\mathcal{F}_T} \langle \{\!\{\nabla\mathbf{u}_h - p_h\underline{\mathbf{I}}\}\!\}\mathbf{n}_F + \gamma_F h_F^{-1}[\![\mathbf{u}_h]\!], \mathbf{n}_T\cdot\mathbf{n}_F \mathbf{v}_h\rangle_F \\
= {} & a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h)_T \quad \forall \mathbf{v}_h \in [\mathbb{P}_l(T)]^d.
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Green theorem, definition of $\underline{\boldsymbol{\sigma}}_h$, definition of the DG method:

$$\begin{aligned}-(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \mathbf{v}_h)_T &= (\underline{\boldsymbol{\sigma}}_h, \nabla\mathbf{v}_h)_T - \sum_{F\in\mathcal{F}_T} \langle \underline{\boldsymbol{\sigma}}_h \mathbf{n}_T, \mathbf{v}_h\rangle_F \\ &\quad + (\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}}, \nabla\mathbf{v}_h)_T - \theta \sum_{F\in\mathcal{F}_T} \langle \omega_F \nabla\mathbf{v}_h \mathbf{n}_F, [\![\mathbf{u}_h]\!]\rangle_F \\ &\quad - \sum_{F\in\mathcal{F}_T} \langle \{\!\{\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}}\}\!\}\mathbf{n}_F + \gamma_F h_F^{-1}[\![\mathbf{u}_h]\!], \mathbf{n}_T\cdot\mathbf{n}_F \mathbf{v}_h\rangle_F \\ &= a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h)_T \quad \forall \mathbf{v}_h \in [\mathbb{P}_l(T)]^d.\end{aligned}$$

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Green theorem, definition of $\underline{\boldsymbol{\sigma}}_h$, definition of the DG method:

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$$+ (\nabla\mathbf{u}_h - p_h\underline{\boldsymbol{I}}, \nabla\mathbf{v}_h)_T - \theta\sum_{F\in\mathcal{F}_T}\langle\omega_F\nabla\mathbf{v}_h\mathbf{n}_F, [\![\mathbf{u}_h]\!]\rangle_F$$

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Application of the framework to DG

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Application of the framework to DG

Remark (Orthogonal projection)
We thus have $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}^l(\mathcal{T}_h)$ and $(\nabla\cdot\underline{\boldsymbol{\sigma}}_h)|_T = -(\boldsymbol{\Pi}_l\mathbf{f})|_T$ for simplicial meshes ($\boldsymbol{\Pi}_l$ is the $\mathbf{L}^2$-orthogonal projection onto $[\mathbb{P}_l(\mathcal{T}_h)]^d$), i.e., the same quality result as for mixed methods, by **local** postprocessing.

Lemma (Approximation property)

Assumption 3 *holds true.*

Remark (General meshes)
For general polygonal nonmatching meshes, we reconstruct the stress on a simplicial submesh $\mathcal{S}_h$ of $\mathcal{T}_h$.

Application of the framework to DG

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Outline

Conforming and conforming stabilized methods

Conforming methods

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h \subset \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \quad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$

- Taylor–Hood family
- mini element
- cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element
- $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element

Conforming stabilized methods

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$$

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Conforming and conforming stabilized methods

$\mathbf{s}_h := \mathbf{u}_h$

Conforming methods

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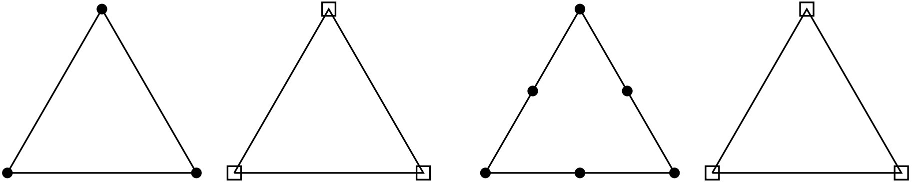
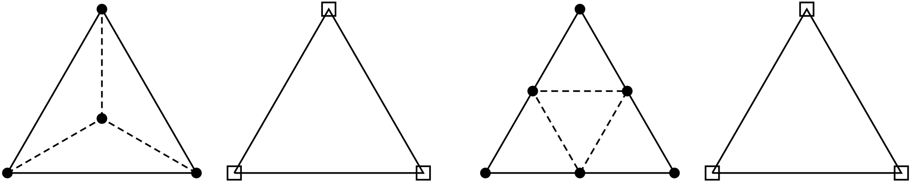
Conforming and conforming stabilized methods



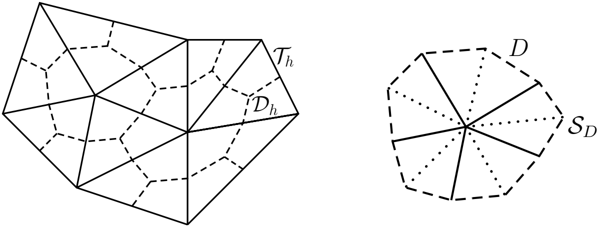
$\mathbb{P}_1$–$\mathbb{P}_1$ element (left) and $\mathbb{P}_2$–$\mathbb{P}_1$ element (right)



Cross-grid $\mathbb{P}_1$–$\mathbb{P}_1$ element (left) and $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ element (right)

Local conservativity, lowest-order conforming methods

Dual mesh



Dual mesh $\mathcal{D}_h$ (left) and simplicial submesh $\mathcal{S}_D$ of $D \in \mathcal{D}_h$ (right)

Normal flux functions

$\Upsilon_F(\mathbf{u}_h) := (\nabla \mathbf{u}_h \mathbf{n}_F)|_F \qquad F \in \partial\mathcal{S}_h^{\text{int}}$ such that $F \subset \partial D$, $D \in \mathcal{D}_h$

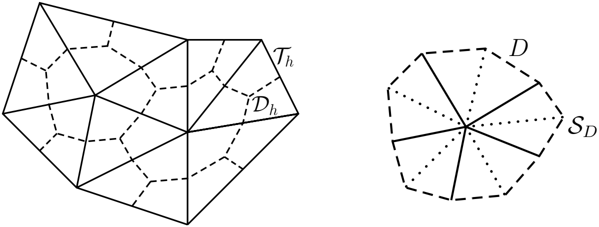
Lemma (Conservativity on $\mathcal{D}_h^{\text{int}}$; ~ Luce and Wohlmuth (2004))

For $\mathbf{f}$ *piecewise constant on* $\mathcal{T}_h$, *there holds*

$$\sum_{F \in \mathcal{F}_D} \langle \Upsilon_F(\mathbf{u}_h)\mathbf{n}_D \cdot \mathbf{n}_F, \mathbf{e}_i \rangle_F - (\nabla p_h, \mathbf{e}_i)_D + (\mathbf{f}, \mathbf{e}_i)_D = 0, \qquad i = 1, \ldots, d \quad \forall D \in \mathcal{D}_h^{\text{int}}.$$

Local conservativity, lowest-order conforming methods

Dual mesh



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Local conservativity, lowest-order conforming methods

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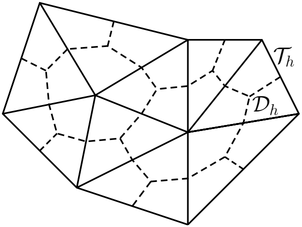

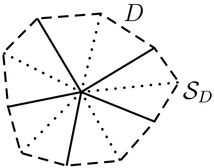

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Stress reconstruction, lowest-order conf. methods

Local Raviart–Thomas–Nédélec spaces on each $D \in \mathcal{D}_h$

$$\underline{\mathbf{\Sigma}}^0_{\mathrm{N}}(\mathcal{S}_D) := \{\underline{\boldsymbol{v}}_h \in \underline{\mathbf{\Sigma}}^0(\mathcal{S}_D);\ \underline{\boldsymbol{v}}_h \mathbf{n}_F = \boldsymbol{\Upsilon}_F(\mathbf{u}_h) \quad \forall F \in \partial\mathcal{S}_h^{\mathrm{int}},\ F \subset \partial D\}$$

- normal trace fixed on $\partial D \setminus \partial\Omega$ by the flux functions $\boldsymbol{\Upsilon}_F(\mathbf{u}_h)$: Neumann data in equilibrium with the load $\nabla p_h - \mathbf{f}$ on $D \in \mathcal{D}_h^{\mathrm{int}}$

Stress reconstruction $\underline{\boldsymbol{\sigma}}_h$

$$\underline{\boldsymbol{\sigma}}_h|_D := \arg \inf_{\underline{\boldsymbol{v}}_h \in \underline{\mathbf{\Sigma}}^0_{\mathrm{N}}(\mathcal{S}_D),\ \nabla\cdot\underline{\boldsymbol{v}}_h = \nabla p_h - \mathbf{f}} \|\nabla \mathbf{u}_h - \underline{\boldsymbol{v}}_h\|_D$$

Equivalently

Find $\underline{\boldsymbol{\sigma}}_h \in \underline{\mathbf{\Sigma}}^0_{\mathrm{N}}(\mathcal{S}_D)$ and $\mathbf{r}_h \in [\mathbb{P}_0^*(\mathcal{S}_D)]^d$ such that

$$(\underline{\boldsymbol{\sigma}}_h - \nabla \mathbf{u}_h, \underline{\boldsymbol{v}}_h)_D + (\mathbf{r}_h, \nabla\cdot\underline{\boldsymbol{v}}_h)_D = 0 \qquad \forall \underline{\boldsymbol{v}}_h \in \underline{\mathbf{\Sigma}}^0_{\mathrm{N},0}(\mathcal{S}_D),$$

$$-(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \boldsymbol{\phi}_h)_D - (\mathbf{f} - \nabla p_h, \boldsymbol{\phi}_h)_D = 0 \qquad \forall \boldsymbol{\phi}_h \in [\mathbb{P}_0^*(\mathcal{S}_D)]^d$$

Stress reconstruction, lowest-order conf. methods

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Stress reconstruction, lowest-order conf. methods

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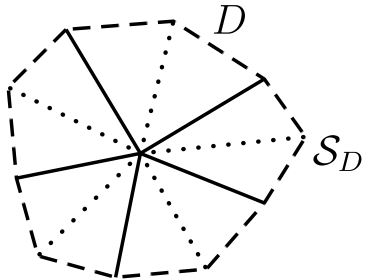
Stress reconstruction, lowest-order conf. methods



- **local** Raviart–Thomas–Nédélec **MFE problem** on $\mathcal{S}_D$ (Neumann BC given by $\Upsilon_F(\mathbf{u}_h)$ on $\partial D \setminus \partial\Omega$, homogeneous Dirichlet BC given on $\partial D \cap \partial\Omega$)
- complementary energy minimization with constraints
- both minimizer and constraint form our estimators
- Poisson-type and not a local Stokes problem

Application of the framework to CG

Lemma (Reconstructed stress in the CG method)

For $\mathbf{f}$ *piecewise constant, there holds,*

$$(\nabla\cdot\underline{\boldsymbol{\sigma}}_h)|_T = (\nabla p_h - \mathbf{f})|_T \qquad \forall T \in \mathcal{S}_h.$$

In particular, **Assumption 1** *holds true.*

Application of the framework to CG

Lemma (Approximation property)

Assumption 3 *holds true*.

Main elements of the proof

- construction of $\underline{\sigma}_h$ from $\mathbf{u}_h$ and p_h
- local postprocessing of potentials in MFEs
- duality
- properties of Raviart–Thomas–Nédélec spaces, scaling arguments, equivalence of norms on finite-dimensions spaces
- inverse inequality
- Cauchy–Schwarz inequality, discrete Poincaré and Friedrichs inequalities

Application of the framework to CG

Lemma (Approximation property)

Assumption 3 *holds true*.

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- construction of $\underline{\boldsymbol{\sigma}}_h$ from $\mathbf{u}_h$ and p_h
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Outline

Crouzeix–Raviart nonconforming method

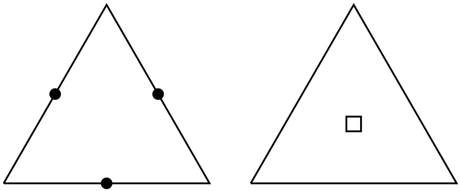
Discontinuous approximation space

$$\mathbf{V}_h := \{\mathbf{v}_h \in [\mathbb{P}_1(\mathcal{T}_h)]^d;\ \langle [\![\mathbf{v}_h]\!], \mathbf{e}_i \rangle_F = 0, \quad i = 1, \dots, d, \quad \forall F \in \partial\mathcal{T}_h\},$$
$$Q_h := \mathbb{P}_0(\mathcal{T}_h) \cap Q$$

Crouzeix–Raviart nonconforming method

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \mathbf{v}_h \in \mathbf{V}_h,$$
$$b(\mathbf{u}_h, q_h) = 0 \qquad \forall q_h \in Q_h$$



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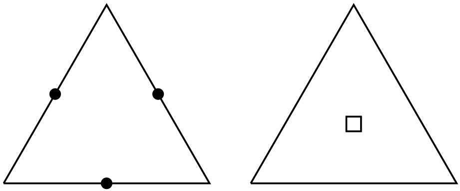
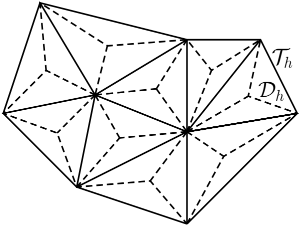
Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h \end{aligned}$$



Local conservativity, Crouzeix–Raviart method

Dual mesh



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Normal flux functions

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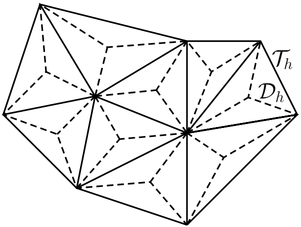

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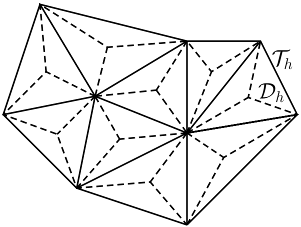

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Stress reconstruction, Crouzeix–Raviart method

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Application of the framework to the Crouzeix–Raviart method

Lemma (Reconstructed stress in the CR method)

For $\mathbf{f}$ *piecewise constant, there holds*

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In particular, **Assumption 1** *holds true*.

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Outline

A general locally conservative method

A general locally conservative method

$$\sum_{F\in\mathcal{F}_T} \Upsilon_F^i(\mathbf{n}_T\cdot\mathbf{n}_F) + (\mathbf{f},\mathbf{e}_i)_T = 0, \qquad i=1,\ldots,d, \quad \forall T\in\mathcal{T}_h$$

- Υ_F: side normal fluxes
- velocities $\mathbf{u}_h \in [\mathbb{P}_0(\mathcal{T}_h)]^d$ and pressures $p_h \in \mathbb{P}_0(\mathcal{T}_h)$ are typically also obtained

Stress reconstruction

$\underline{\sigma}_h \in \underline{\Sigma}^0(\mathcal{T}_h)$ such that

$$\underline{\sigma}_h\mathbf{n}_F|_F := \frac{\Upsilon_F}{|F|} \qquad \forall F\in\mathcal{F}_T, \quad \forall T\in\mathcal{T}_h$$

Elementwise postprocessing of the velocity

$\nabla\mathbf{u}_h = 0 \Rightarrow \tilde{\mathbf{u}}_h \in [\mathbb{P}_2(\mathcal{T}_h)]^d$ such that

$$\nabla\tilde{\mathbf{u}}_h|_T - p_h\underline{I}|_T = \underline{\sigma}_h|_T \qquad \forall T\in\mathcal{T}_h,$$

$$\frac{(\tilde{\mathbf{u}}_h,\mathbf{e}_i)_T}{|T|} = \mathbf{u}_h^i|_T, \qquad i=1,\ldots,d, \quad \forall T\in\mathcal{T}_h$$

A general locally conservative method

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Application of the framework to locally conservative methods

Application of the framework

- **Assumption 1** immediate from the definition of $\underline{\boldsymbol{\sigma}}_h$ and local conservation of the scheme
- **Assumption 3** straightforward: $\|\nabla \tilde{\mathbf{u}}_h - p_h \underline{\mathbf{I}} - \underline{\boldsymbol{\sigma}}_h\|_T = 0$ by the definition of $\tilde{\mathbf{u}}_h$

Remark (General meshes)
For general polygonal meshes, we reconstruct the stress and the postprocessed velocity $\tilde{\mathbf{u}}_h$ on a simplicial submesh $\mathcal{S}_h$ of $\mathcal{T}_h$.

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Outline

Mixed finite element methods

Mixed finite element method

Find $(\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_h, p_h) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h \times Q_h$, the approximation to the stress tensor $\underline{\boldsymbol{\sigma}}$, the velocity $\mathbf{u}$, and the pressure p, respectively, such that

$$
\begin{aligned}
(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + (\mathbf{u}_h, \nabla\cdot\underline{\boldsymbol{\tau}}_h) &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h,\\
-(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h,\\
(\mathbf{u}_h, \nabla q_h) &= 0 && \forall q_h \in Q_h
\end{aligned}
$$

Approximation spaces

$\underline{\boldsymbol{\Sigma}}_h := \underline{\boldsymbol{\Sigma}}^k(\mathcal{T}_h)$, $\mathbf{V}_h := [\mathbb{P}_k(\mathcal{T}_h)]^d$, and $Q_h := \mathbb{P}_{k+1}(\mathcal{T}_h) \cap C(\Omega) \cap Q$, $k \geq 0$

Application of the framework

- $\underline{\boldsymbol{\sigma}}_h$ directly constructed by the MFE $\Rightarrow$ **Assumption 1**
- local postprocessing of $\mathbf{u}_h$ into $\tilde{\mathbf{u}}_h \Rightarrow$ **Assumption 3**

Mixed finite element methods

Mixed finite element method

Find $(\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_h, p_h) \in \underline{\boldsymbol{\Sigma}}_h \times \mathbf{V}_h \times Q_h$, the approximation to the stress tensor $\underline{\boldsymbol{\sigma}}$, the velocity $\mathbf{u}$, and the pressure p, respectively, such that

$$
\begin{aligned}
(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + (\mathbf{u}_h, \nabla\cdot\underline{\boldsymbol{\tau}}_h) &= 0 && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h,\\
-(\nabla\cdot\underline{\boldsymbol{\sigma}}_h, \mathbf{v}_h) + (\nabla p_h, \mathbf{v}_h) &= (\mathbf{f}, \mathbf{v}_h) && \forall \mathbf{v}_h \in \mathbf{V}_h,\\
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\end{aligned}
$$

Approximation spaces

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Outline

Equilibration

Locally conservative methods

- locally conservative side fluxes readily at disposal

Nonconservative schemes

- locally conservative side fluxes not at disposal at a first sight
- ready on a dual grid for lowest-order schemes
- can be obtained by equilibration of the originally nonconservative side fluxes ($\sim$ Ainsworth and Oden (1993))
  - hat basis functions on an element form a partition of unity
  - solution of a small $(d+1) \times (d+1)$ local system on each $T \in \mathcal{T}_h$ in order to redistribute the mass locally

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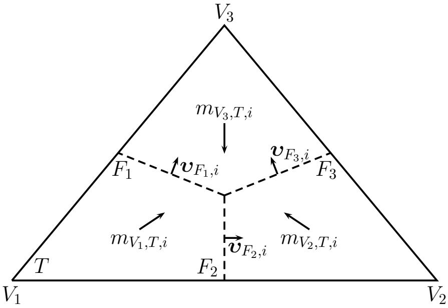
Equilibration in $2D$



Equilibration on a triangle

Linear system to solve

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \upsilon_{F_1,i} \\ \upsilon_{F_2,i} \\ \upsilon_{F_3,i} \end{pmatrix} = \begin{pmatrix} m_{V_1,T,i} \\ m_{V_2,T,i} \\ m_{V_3,T,i} \end{pmatrix}$$

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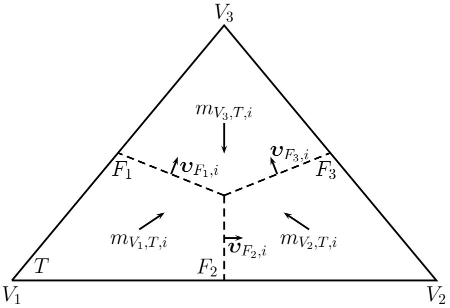

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Outline

Setting

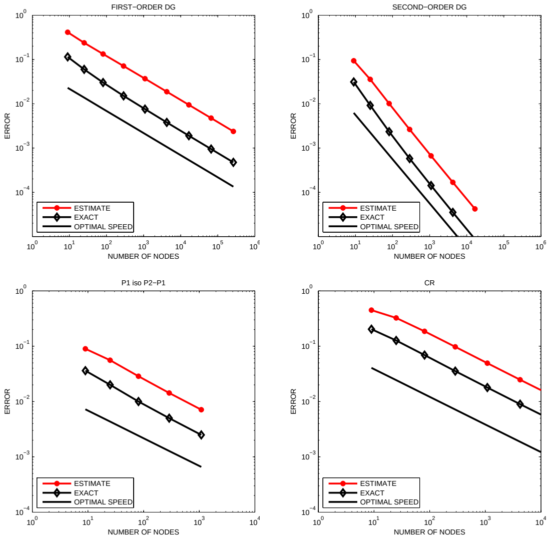
Model problem

- $\Omega = (0,1) \times (0,1)$
- $\mathbf{f}$ chosen according to the solution

$$\mathbf{u} = \nabla \times (x-1)^2 x^{1+\alpha} (y-1)^2 y^2 \mathbf{e}_3, \quad p = x + y - 1$$

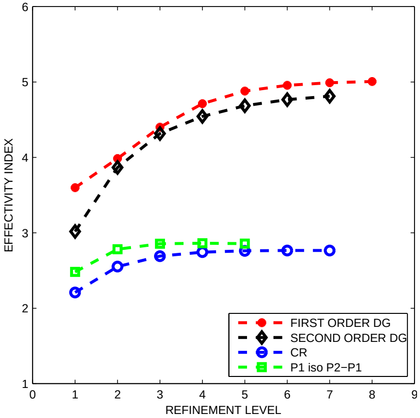
- regularity: $[H^{\frac{1}{2}+\alpha}(\Omega)]^d$ for $\alpha \notin \mathbb{N}$ and $[C^\infty(\Omega)]^d$ for $\alpha \in \mathbb{N}$

Errors and estimates



Estimated and exact errors, smooth case

Effectivity indices



Effectivity indices, smooth case

Estimated and exact error distributions

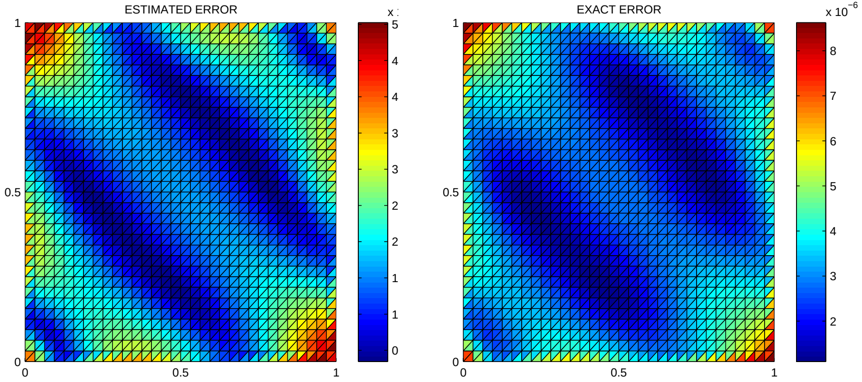


Figure: Estimated (left) and exact (right) error distributions, 2nd order DG method, smooth test case

Estimated and exact error distributions

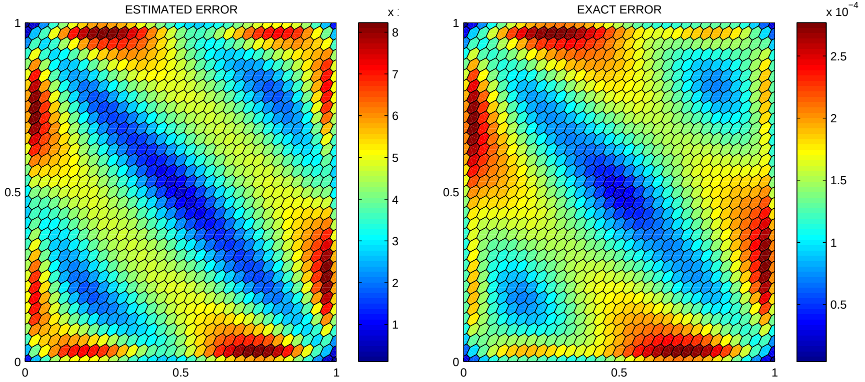


Figure: Estimated (left) and exact (right) error distributions, $\mathbb{P}_1$ iso $\mathbb{P}_2$–$\mathbb{P}_1$ method, smooth test case

Estimated and exact error distributions

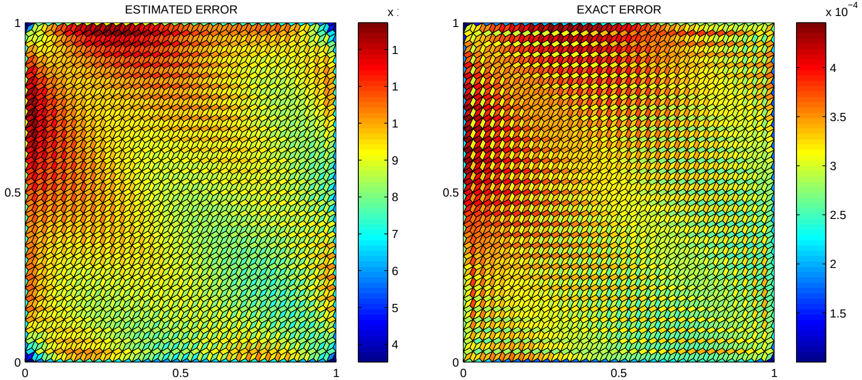
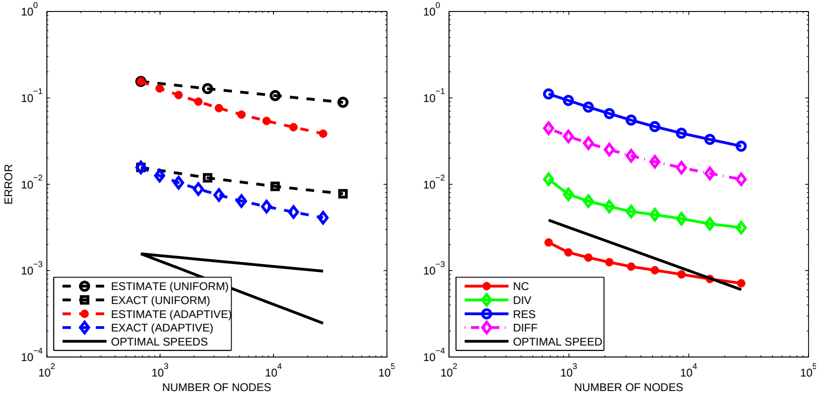


Figure: Estimated (left) and exact (right) error distributions, Crouzeix–Raviart, smooth test case

Singular cases and adaptivity



Estimated and exact errors in uniform/adaptive refinement (left) and components of the estimator in adaptive refinement (right), first-order DG method, singular test case

Adaptive mesh refinement

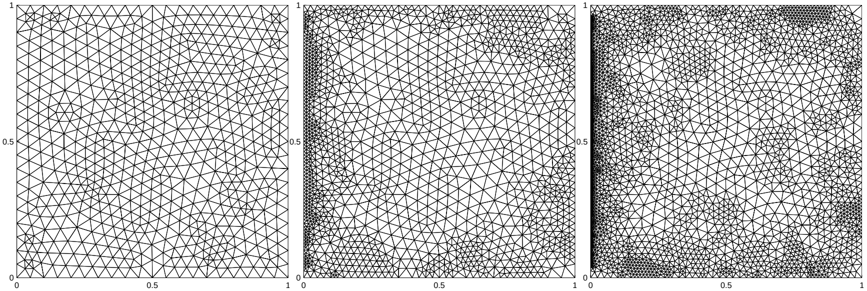


Figure: Adaptively refined meshes, 1st order DG method, singular test case

Outline

Conclusions and future work

Conclusions

- **unified framework** for major numerical methods
- **no discrete** inf–sup **condition** needed
- easily and **fully computable** estimates, **locally efficient**
- estimates **physically relevant**
- based on **local conservation**, built-in in any(?) scheme (directly or after equilibration)

Future work

- instationary Stokes problem
- Navier–Stokes problem

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Bibliography

Papers

- HANNUKAINEN A., STENBERG R., AND VOHRALÍK M., A unified framework for a posteriori error estimation for the Stokes problem, in review for *Numer. Math.*
- ERN A., VOHRALÍK M., A posteriori error estimation based on potential and flux reconstruction for the heat equation, *SIAM J. Numer. Anal.* **48** (2010), 198–223.

Merci de votre attention !

Prague
Czech Republic
June 6-10, 2011

FVCA VI

6th International Symposium on Finite Volumes for Complex Applications

- Problems and Perspectives -

Topics of Main Interest

- New schemes and methods
- New fields of application
- Non homogeneous systems
- Convergence and stability analysis
- Global error analysis
- Purely multidimensional difficulties
- Limits of methods
- Complex geometries and adaptivity
- Complexity, efficiency and large-scale computations
- Distributive computation
- Multiphase problems and fitting
- Combustion problems
- Climate and Ocean modelling, Atmospheric pollution
- Kinetic equations
- Water Waves
- Chaotic problems (turbulence, ignition, mixing)
- Comparisons with experimental results

Confirmed Invited Speakers

- **Daniele DI PIETRO**, IFP Energies nouvelles, France
- **Jerome DRONIOU**, University Montpellier, France
- **Alexandre ERN**, Paris-Est University, France
- **Bernard GEURTS**, University of Twente, Netherlands
- **Jean-Claude LATCHÉ**, IRSN, France
- **Jinghai LI**, Chinese Academy of Science, China
- **Richard LISKA**, CTU Prague, Czech Republic
- **Mohammed SEAID**, University of Durham, UK

• *including the Session on 3D Benchmark for Anisotropic Problems (coordinated by R. Herbin and F. Hubert)*

Important Dates

February 1, 2011 - Submission of full version of papers (including benchmark)
March 1, 2011 - Notice of (conditional) paper acceptance
March 5, 2011 - Early registration for participating delegates (low rate)
April 1, 2011 -Submission of the final version of the paper
April 15, 2011 - Notice of acceptance of the final version of the paper
April 15, 2011 - Deadline registration for authors
May 5, 2011 - Deadline registration for participating delegates

Conference Fee (including lunches and conference dinner)

Industry early registration (prior to March 5, 2011)	360 EUR
Industry regular registration (after March 5, 2011)	400 EUR
University early registration (prior to March 5, 2011)	270 EUR
University regular registration (after March 5, 2011)	300 EUR
Student* early registration (prior to March 5, 2011)	160 EUR
Student* regular registration (after March 5, 2011)	180 EUR

* a copy of student ID card is required

http://fvca6.fs.cvut.cz