Estimations d'erreur a posteriori pour les volumes finis et les éléments finis mixtes

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Motivation

Problems

- $-\nabla \cdot \mathbf{S}\nabla p = f$
  - flow in porous media
  - $\mathbf{S}$ inhomogeneous and anisotropic
- $-\nabla \cdot \mathbf{S}\nabla p + \nabla \cdot (p\mathbf{w}) + rp = f$
  - transport in porous media
  - convection-dominated

Mixed finite elements / finite volumes

- accurate for inhomogeneous and anisotropic pbs
- upwinding for convection-dominated pbs

need for good a posteriori error control

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What is an a posteriori error estimate

A posteriori error estimate

- Let p be a weak solution of a PDE.
- Let p_h be its approximate numerical solution.
- A priori error estimate: $\|p - p_h\|_\Omega \leq f(p) h^q$. Dependent on p, not computable. Useful in theory.
- A posteriori error estimate: $\|p - p_h\|_\Omega \lesssim f(p_h)$. Only uses p_h, computable. Great in practice.

Usual form

- $f(p_h)^2 = \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$, where $\eta_K(p_h)$ is an element indicator.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

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(Mathematical) reliability (global upper bound)

- $\|p - p_h\|_{\Omega}^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- Problems:
  - What is C?
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- Mathematicians often do not care about constants...

(Real-life) reliability (global upper bound)

- $\|p - p_h\|_{\Omega}^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- Real life depends on constants very much!

Global efficiency (global lower bound)

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 \leq C_{\mathrm{eff},\Omega}^2 \|p - p_h\|_{\Omega}^2$

Local efficiency (local lower bound)

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$

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Semi-robustness

- $C_{\mathrm{eff},K}$ only depends on inhomogeneities and anisotropies around K (Bernardi and Verfürth '00, Galerkin FEMs)
- $C_{\mathrm{eff},K}$ affine dependence on the local Péclet number around K (energy norm), $\mathrm{Pe}_K := h_K \frac{|\mathbf{w}_K|}{|\mathbf{S}_K|}$, (Verfürth '98, Galerkin FEMs)

Previous works on a posteriori error estimates for FVMs and MFEMs

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- Ohlberger '01
- Lazarov and Tomov '02
- Achdou, Bernardi, and Coquel '03
- Nicaise '05

Previous works on MFEMs . . .

- Alonso '96
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- evaluation of the constants (with the exception of Nicaise)
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A convection–diffusion–reaction problem

Problem

$$\begin{aligned} -\nabla\cdot\mathbf{S}\nabla p+\nabla\cdot(p\mathbf{w})+rp &= f \quad \text{in } \Omega\,,\\ p &= g \quad \text{on } \Gamma_{\mathrm{D}}\,,\\ -\mathbf{S}\nabla p\cdot\mathbf{n} &= u \quad \text{on } \Gamma_{\mathrm{N}} \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- data related to a basic triangulation $\widetilde{\mathcal{T}}_h$ of Ω
- $\mathbf{S}|_K$ is a constant SPD matrix, $c_{\mathbf{S},K}$ its smallest, and $C_{\mathbf{S},K}$ its largest eigenvalue on each $K \in \widetilde{\mathcal{T}}_h$
- $\left(\frac{1}{2}\nabla\cdot\mathbf{w}+r\right)|_K \geq c_{\mathbf{w},r,K} \geq 0$ (from pure diffusion to convection–diffusion–reaction cases)

Difficulties

- $\mathbf{S}$ inhomogeneous and anisotropic
- $\mathbf{w}$ dominates

Bilinear form, weak solution, energy (semi-)norm

Definition (Bilinear form $\mathcal{B}$)

We define a bilinear form $\mathcal{B}$ for $p, \varphi \in H^1(\mathcal{T}_h)$ by

$$\mathcal{B}(p,\varphi) := \sum_{K\in\mathcal{T}_h} \left\{ (\mathbf{S}\nabla p, \nabla\varphi)_K + (\nabla\cdot(p\mathbf{w}), \varphi)_K + (rp, \varphi)_K \right\}.$$

Definition (Weak solution)

Weak solution: $p \in H^1(\Omega)$ with $p|_{\Gamma_{\mathrm{D}}} = g$ such that

$$\mathcal{B}(p,\varphi) = (f,\varphi)_\Omega - \langle u, \varphi\rangle_{\Gamma_{\mathrm{N}}} \qquad \forall \varphi \in H^1_{\mathrm{D}}(\Omega).$$

Definition (Energy (semi-)norm)

We define the energy (semi-)norm for $\varphi \in H^1(\mathcal{T}_h)$ by

$$|||\varphi|||^2_\Omega := \sum_{K\in\mathcal{T}_h} |||\varphi|||^2_K, \; |||\varphi|||^2_K := \left\|\mathbf{S}^{\frac{1}{2}}\nabla\varphi\right\|^2_K + \left\|\left(\frac{1}{2}\nabla\cdot\mathbf{w}+r\right)^{\frac{1}{2}}\varphi\right\|^2_K.$$

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General finite volume scheme

Definition (FV scheme for $-\nabla \cdot \mathbf{S}\nabla p + \nabla \cdot (p\mathbf{w}) + rp = f$)

Find p_K, $K \in \mathcal{T}_h$, such that

$$\sum_{\sigma \in \mathcal{E}_K} S_{K,\sigma} + \sum_{\sigma \in \mathcal{E}_K} W_{K,\sigma} + r_K p_K |K| = f_K |K| \qquad \forall K \in \mathcal{T}_h \,.$$

- $S_{K,\sigma}$: diffusive flux; $W_{K,\sigma}$: convective flux } no specific form, just conservativity needed
- $r_K := (r, 1)/|K|$
- $f_K := (f, 1)/|K|$

Example

- $S_{K,\sigma} = -s_{K,L} \frac{|\sigma_{K,L}|}{d_{K,L}} (p_L - p_K)$
- $W_{K,\sigma} = p_\sigma \langle \mathbf{w} \cdot \mathbf{n}, 1 \rangle_\sigma$: weighted-upwind

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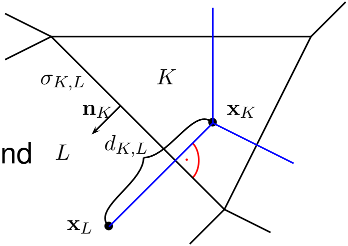


Outline

First-order system reformulation

Second-order PDE

$$-\nabla \cdot \mathbf{S}\nabla p + \nabla \cdot (p\mathbf{w}) + rp = f$$

$$\Downarrow$$

First-order system

$$\mathbf{u} = -\mathbf{S}\nabla p\,,$$
$$\nabla \cdot \mathbf{u} + \nabla \cdot (p\mathbf{w}) + rp = f$$

$\Gamma_{\mathrm{D}} = \partial\Omega$ and $g = 0$ for the sake of simplicity

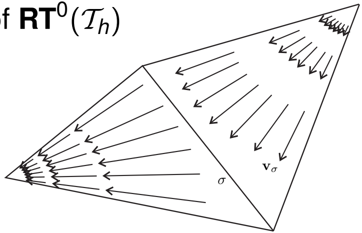
A centered scheme

Definition (A centered mixed finite element scheme)

Find $\mathbf{u}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ and $p_h \in \Phi(\mathcal{T}_h)$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla \cdot \mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)\,,$$
$$(\nabla \cdot \mathbf{u}_h, \phi_h)_\Omega - (\mathbf{S}^{-1}\mathbf{u}_h\mathbf{w}, \phi_h)_\Omega + ((r + \nabla \cdot \mathbf{w})p_h, \phi_h)_\Omega$$
$$= (f, \phi_h)_\Omega \qquad \forall \phi_h \in \Phi(\mathcal{T}_h)\,.$$

Basis of $\mathbf{RT}^0(\mathcal{T}_h)$



Basis of $\Phi(\mathcal{T}_h)$

$$\phi_K = 1|_K$$

An upwind-weighted scheme

Definition (An upwind-weighted mixed finite element scheme)

Find $\mathbf{u}_h \in \mathbf{RT}^0(\mathcal{T}_h)$ and $p_h \in \Phi(\mathcal{T}_h)$ such that

$$(\mathbf{S}^{-1}\mathbf{u}_h, \mathbf{v}_h)_\Omega - (p_h, \nabla\cdot\mathbf{v}_h)_\Omega = 0 \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}^0(\mathcal{T}_h)\,,$$

$$(\nabla\cdot\mathbf{u}_h, \phi_h)_\Omega + \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_K} \hat{p}_\sigma w_{K,\sigma}\phi_K + (rp_h, \phi_h)_\Omega$$

$$= (f, \phi_h)_\Omega \qquad \forall \phi_h \in \Phi_h\,.$$

- $w_{K,\sigma} := \langle \mathbf{w}\cdot\mathbf{n}, 1\rangle_\sigma$: flux of $\mathbf{w}$ through a side σ
- $\hat{p}_\sigma$: weighted upwind value,

$$\hat{p}_\sigma := \begin{cases} (1-\nu_\sigma)p_K + \nu_\sigma p_L & \text{if } w_{K,\sigma} \geq 0 \\ (1-\nu_\sigma)p_L + \nu_\sigma p_K & \text{if } w_{K,\sigma} < 0 \end{cases}$$

- ν_σ: coefficient of the amount of upstream weighting, e.g.

$$\nu_\sigma := \min\{c_{\mathbf{S},\sigma}|\sigma|/(h_\sigma|w_{K,\sigma}|), 1/2\}$$

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A locally postprocessed scalar variable $\tilde{p}_h$

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K / |K| = p_K$ (mean of $\tilde{p}_h$ on K is p_K).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
- means of traces of $\tilde{p}_h$ on the sides continuous, $\tilde{p}_h \in W_0(\mathcal{T}_h)$
- the means are equal to the Lagrange multipliers from the hybridized forms of the schemes

Remarks

- $\tilde{p}_h$ is a connection of p_h and $\mathbf{u}_h$
- basis of our a posteriori error estimates

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- $\tilde{p}_h$ is a connection of p_h and $\mathbf{u}_h$
- basis of our a posteriori error estimates

A locally postprocessed scalar variable $\tilde{p}_h$

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

- $-\mathbf{S}_K \nabla \tilde{p}_h|_K = \mathbf{u}_h|_K$ (flux of $\tilde{p}_h$ is $\mathbf{u}_h$),
- $(\tilde{p}_h, 1)_K / |K| = p_K$ (mean of $\tilde{p}_h$ on K is p_K).

Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique (it is a pw second-order polynomial)
- $\tilde{p}_h$ is nonconforming, $\notin H_0^1(\Omega)$, only $\in H^1(\mathcal{T}_h)$ in general
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Proof:
$$\begin{aligned} 0 &= -(\nabla \tilde{p}_h, \mathbf{v}_{\sigma_{K,L}})_{K\cup L} - (\tilde{p}_h, \nabla\cdot\mathbf{v}_{\sigma_{K,L}})_{K\cup L} \\ &= -\langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}, \tilde{p}_h\rangle_{\partial K} - \langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}, \tilde{p}_h\rangle_{\partial L} \\ &= \langle \mathbf{v}_{\sigma_{K,L}}\cdot\mathbf{n}_K, \tilde{p}_h|_L - \tilde{p}_h|_K\rangle_{\sigma_{K,L}} \end{aligned}$$

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Outline

A posteriori error estimate for the centered scheme

Theorem (A posteriori error estimate for the centered scheme)

There holds

$$|||p - \tilde{p}_h|||_\Omega \le \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{C},K})^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_K$
  - $\mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)$: modified Oswald int. (preserves means of traces)
- **residual estimator**
  - $\eta_{\mathrm{R},K} := m_K \|f + \nabla \cdot \mathbf{S}_K \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r\tilde{p}_h\|_K$
  - $m_K^2 := \min \left\{ C_{\mathrm{P},d} \frac{h_K^2}{c_{\mathbf{S},K}}, \frac{1}{c_{\mathbf{w},r,K}} \right\}$
- **convection estimator**
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A posteriori error estimate for the centered scheme

Sketch of proof.

- $s \in H_0^1(\Omega)$ arbitrary: $|||p - \tilde{p}_h|||_\Omega \leq |||p - s|||_\Omega + |||s - \tilde{p}_h|||_\Omega$
- $(p - s) \in H_0^1(\Omega)$: coercivity of $\mathcal{B}$ implies

$$|||p - s|||_\Omega \leq \frac{\mathcal{B}(p-s, p-s)}{|||p-s|||_\Omega} \leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||_\Omega = 1} \mathcal{B}(p - s, \varphi)$$

$$= \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||_\Omega = 1} \{\mathcal{B}(p - \tilde{p}_h, \varphi) + \mathcal{B}(\tilde{p}_h - s, \varphi)\}$$

- Green theorem and weak solution definition: $\mathcal{B}(p - \tilde{p}_h, \varphi)$

$$= (f, \varphi)_\Omega - \sum_{K \in \mathcal{T}_h} \left\{(\mathbf{S}\nabla\tilde{p}_h, \nabla\varphi)_K + \left(\nabla\cdot(\tilde{p}_h\mathbf{w}), \varphi\right)_K + (r\tilde{p}_h, \varphi)_K\right\}$$

$$= \sum_{K \in \mathcal{T}_h} \left(f + \nabla\cdot\mathbf{S}_K\nabla\tilde{p}_h - \nabla\cdot(\tilde{p}_h\mathbf{w}) - r\tilde{p}_h, \varphi\right)_K + \sum_{K \in \mathcal{T}_h} \langle \mathbf{u}_h\cdot\mathbf{n}, \varphi\rangle_{\partial K}$$

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A posteriori error estimate for the centered scheme

Sketch of proof (cont.)

- definition of the centered scheme:

$$\big(f + \nabla \cdot \mathbf{S}_K \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r\tilde{p}_h, \varphi_K\big)_K = 0 \qquad \forall K \in \mathcal{T}_h$$

- hence:

$$\mathcal{B}(p - \tilde{p}_h, \varphi) = \sum_{K \in \mathcal{T}_h} \big(f + \nabla \cdot \mathbf{S}_K \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r\tilde{p}_h, \varphi - \varphi_K\big)_K$$

- remain the nonconformity and convection terms

Remarks

- completely based on the primal abstract framework
- generalization of the techniques due to Verfürth '98 for Galerkin FEMs
- nonconformity (techniques due to El Alaoui and Ern '04, Karakashian and Pascal '03)

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A posteriori error estimate for the upwind-weighted scheme

Theorem (A posteriori error estimate for the upwind-weighted scheme)

There holds

$$|||p - \tilde{p}_h|||_\Omega \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{C},K} + \eta_{\mathrm{U},K})^2 \right\}^{\frac{1}{2}}.$$

- **upwinding estimator**
  - $\eta_{\mathrm{U},K} := \sum_{\sigma \in \mathcal{E}_K} m_\sigma \|(\hat{p}_\sigma - \tilde{p}_\sigma)\mathbf{w} \cdot \mathbf{n}\|_\sigma$
  - $\hat{p}_\sigma$: the weighted upwind value
  - $\tilde{p}_\sigma$: the mean of $\tilde{p}_h$ over the side σ
  - m_σ: function of $c_{\mathbf{S},K}$, $c_{\mathbf{w},r,K} = \left(\frac{1}{2}\nabla \cdot \mathbf{w} + r\right)|_K$, d, h_K, $|\sigma|$, $|K|$
  - all dependencies evaluated explicitly

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Outline

Local efficiency of the estimates

Theorem (Local efficiency of the residual estimator)

There holds

$$\eta_{\mathrm{R},K} \leq |||p - \tilde{p}_h|||_K C \left\{ \max\left\{1, \frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}}\right\} + \min\{\mathrm{Pe}_K, \varrho_K\} \right\}.$$

- residual estimator is **locally efficient** (lower bound for error on K), **robust** with respect to **S**, and **semi-robust** with respect to **w**
- $C_{\mathrm{eff},K}$:
  - C independent of h_K, **S**, **w**, and r
  - no dependency on inhomogeneities and anisotropies
  - $\frac{C_{\mathbf{w},r,K}}{c_{\mathbf{w},r,K}} \leq 2$ for r nonnegative
  - $C_{\mathrm{eff},K}$ depends affinely on Pe_K
  - $\varrho_K := \frac{|\mathbf{w}|_K|}{\sqrt{c_{\mathbf{w},r,K}}\sqrt{c_{\mathbf{S},K}}}$ prevents $C_{\mathrm{eff},K}$ from exploding in convection-dominated cases on rough grids

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Local efficiency of the estimates

Theorem (Local efficiency of the nonconformity and convection estimators)

There holds

$$\eta_{\mathrm{NC},K}^2+\eta_{\mathrm{C},K}^2 \le \alpha \sum_{L;L\cap K\neq\emptyset} |||p-\tilde{p}_h|||_L^2+\beta \inf_{s_h\in\mathbb{P}_2(\mathcal{T}_h)\cap H_0^1(\Omega)} \sum_{L;L\cap K\neq\emptyset} \|p-s_h\|_L^2 .$$

- nonconformity and convection estimators are **locally efficient** (up to higher-order terms if $c_{\mathbf{w},r,K} \neq 0$) and **semi-robust**
- $C_{\mathrm{eff},K}$:
  - depends on maximal ratio of inhomogeneities around K
  - depends on anisotropy in each L around K by $\frac{C_{\mathbf{S},L}}{c_{\mathbf{S},L}}$
  - again $\min\{\mathrm{Pe}_L, \varrho_L\}$ in each L around K prevents $C_{\mathrm{eff},K}$ from exploding in convection-dominated cases on rough grids

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- $C_{\mathrm{eff},K}$:
  - depends on maximal ratio of inhomogeneities around K
  - depends on anisotropy in each L around K by $\frac{C_{\mathbf{S},L}}{c_{\mathbf{S},L}}$
  - again $\min\{\mathrm{Pe}_L, \varrho_L\}$ in each L around K prevents $C_{\mathrm{eff},K}$ from exploding in convection-dominated cases on rough grids

Outline

A locally postprocessed scalar variable $\tilde{p}_h$

Definition (Postprocessed scalar variable $\tilde{p}_h$)

We define $\tilde{p}_h$ such that, separately on each $K \in \mathcal{T}_h$,

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\begin{aligned}
-\nabla \cdot \mathbf{S}\nabla \tilde{p}_h &= \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}_K} S_{K,\sigma}\,, \\
(1-\mu_K)(\tilde{p}_h, 1)_K/|K| + \mu_K \tilde{p}_h(\mathbf{x}_K) &= p_K\,, \\
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Properties of $\tilde{p}_h$

- $\tilde{p}_h$ exists and is unique
- flux of $\tilde{p}_h$ is given by $S_{K,\sigma}$, point or mean value by p_K
- $\tilde{p}_h$ is nonconforming, $\notin H^1(\Omega)$, $\notin W_0(\mathcal{T}_h)$, only $\in H^1(\mathcal{T}_h)$
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A posteriori error estimate

Theorem (A posteriori error estimate)

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- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)|||_K$
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A posteriori error estimate

- **upwinding estimator**
  - $\eta_{\mathrm{U},K} := \sum_{\sigma\in\mathcal{E}_K\setminus\mathcal{E}_h^{\mathrm{N}}} m_\sigma \|(p_\sigma - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)_\sigma)\mathbf{w}\cdot\mathbf{n}\|_\sigma$
  - p_σ: the weighted upwind value
  - $\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)_\sigma$: the mean of $\mathcal{I}_{\mathrm{Os}}(\tilde{p}_h)$ over the side σ
  - m_σ: function of $c_{\mathbf{S},K}$, $c_{\mathbf{w},r,K} = \left(\frac{1}{2}\nabla\cdot\mathbf{w} + r\right)|_K$, d, h_K, $|\sigma|$, $|K|$
  - all dependencies evaluated explicitly
- **reaction quadrature estimator**
  - $\eta_{\mathrm{RQ},K} := \frac{1}{\sqrt{c_{\mathbf{w},r,K}}}\|r_K p_K - (r\tilde{p}_h, 1)_K|K|^{-1}\|_K$
  - disappears when r pw constant and $\tilde{p}_h$ fixed by mean
- **Neumann boundary estimator**
  - $\eta_{\Gamma_{\mathrm{N}},K} := 0 + \frac{1}{\sqrt{c_{\mathbf{S},K}}}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\mathrm{N}}} \sqrt{C_{\mathrm{t},K,\sigma}}\sqrt{h_K}\|u_\sigma - u\|_\sigma$

Outline

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Comments on the estimates and their efficiency

- no saturation assumption
- $p \in H^1(\Omega)$, no additional regularity
- no convexity of Ω needed
- no "monotonicity" hypothesis on inhomogeneities distribution as El Alaoui and Ern '04 or Bernardi and Verfürth '00
- the only important tool: optimal discrete Poincaré–Friedrichs inequalities (Vohralík, NFAO 2005)
- holds from diffusion to convection–diffusion–reaction cases
- no efficiency for the upwinding estimator (since the upwind-weighted scheme does not change to the centered one; improvement: smooth transition upwind-weighted $\to$ centered scheme)
- no efficiency for the reaction quadrature and Neumann boundary estimators

Comparison with Galerkin FEMs

- **residual estimator** ($\|f + \nabla \cdot \mathbf{S}_K \nabla \tilde{p}_h - \nabla \cdot (\tilde{p}_h \mathbf{w}) - r\tilde{p}_h\|_K$)
  - very good sense in MFEM/FVM ($\tilde{p}_h$ is piecewise quadratic)
  - $\nabla \cdot \mathbf{S}_K \nabla p_h|_K = 0$ in piecewise linear Galerkin FEM
- edge **mass balance estimator** ($\|[\mathbf{S}\nabla \tilde{p}_h \cdot \mathbf{n}]\|_\sigma$)
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- **nonconformity estimator** ($|||\tilde{p}_h - \mathcal{I}(\tilde{p}_h)|||_K$)
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  - SUPG (Verfürth '98): different form, efficient

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Pure diffusion problem $-\nabla \cdot \mathbf{S}\nabla p = f$, $\Gamma_{\mathrm{D}} = \partial\Omega$, $g = 0$

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- asymptotically efficient with constant 1 for however large inhomogeneities and anisotropies

$$\inf_{s\in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega \le 1|||p - \tilde{p}_h|||_\Omega$$

- MFEM: is in fact also an a priori estimate:

$$\inf_{s\in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega \le |||\tilde{p}_h - \mathcal{I}_{\mathrm{MO}}(\tilde{p}_h)|||_\Omega \le Ch$$

using that $\tilde{p}_h \in W_0(\mathcal{T}_h)$ (cont. means of traces)

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- asymptotically efficient with constant 1 for however large inhomogeneities and anisotropies

$$\inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega \leq 1 |||p - \tilde{p}_h|||_\Omega$$

- MFEM: is in fact also an a priori estimate:

$$\inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega \leq |||\tilde{p}_h - \mathcal{I}_{\rm MO}(\tilde{p}_h)|||_\Omega \leq Ch$$

using that $\tilde{p}_h \in W_0(\mathcal{T}_h)$ (cont. means of traces)

Pure diffusion problem $-\nabla \cdot \mathbf{S}\nabla p = f$, $\Gamma_{\mathrm{D}} = \partial\Omega$, $g = 0$

Theorem (Mixed FEM for the diffusion problem)

There holds

$$|||p - \tilde{p}_h|||_\Omega \leq \inf_{s \in H_0^1(\Omega)} |||\tilde{p}_h - s|||_\Omega + \left\{ \sum_{K \in \mathcal{T}_h} C_{\mathrm{P},d} \frac{h_K^2}{c_{\mathbf{S},K}} \|f - f_K\|_K^2 \right\}^{\frac{1}{2}}.$$

Theorem (Galerkin FEM for the diffusion problem)

There holds

$$|||p - p_h|||_\Omega \leq \inf_{s_h \in V_h} |||p - s_h|||_\Omega .$$

Mixed FEM 1D:

- no nonconformity, $\tilde{p}_h \in H_0^1(\Omega)$
- $|||p - \tilde{p}_h|||_\Omega \leq Ch^2$ when $f \in H^1(\mathcal{T}_h)$
- $\tilde{p}_h = p$, the exact solution, for pw constant $\mathbf{S}$ (arbitrary inhomogeneities) and pw constant f

Galerkin FEM 1D:

- $|||p - \tilde{p}_h|||_\Omega \leq Ch$

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Definition (Generalized weak solution)

Generalized weak solution: $\tilde{p} \in W_0(\mathcal{T}_h)$ such that

$$\mathcal{B}(\tilde{p}, \varphi) = (f, \varphi)_\Omega \qquad \forall \varphi \in W_0(\mathcal{T}_h).$$

Theorem (A posteriori error estimate for mixed FEM and the generalized weak solution of the pure diffusion problem)

There holds

$$|||\tilde{p} - \tilde{p}_h|||_\Omega \le \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}}.$$

- **residual estimator**
  - $\eta_{\mathrm{R},K}^2 := C_{\mathrm{P},d} \frac{h_K^2}{c_{\mathbf{S},K}} \|f - f_K\|_K^2$
  - only dependent on sources f
  - $f \in H^1(\mathcal{T}_h) \Rightarrow |||\tilde{p} - \tilde{p}_h|||_\Omega \le Ch^2$
  - if f piecewise constant $\Rightarrow$ $\tilde{p}_h$ coincides with $\tilde{p}$ for arbitrary inhomogeneities and anisotropies

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Outline

Relations of mixed finite elements to finite volumes

- MFE matrix problem: $\begin{pmatrix} \mathbb{A} & \mathbb{C} \\ \mathbb{B} & \mathbb{D} \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$
- local flux expression formula Vohralík (M2AN, 2006): local linear problem $\mathbb{M}_V U_V^{\mathrm{int}} = F_V - \mathbb{A}_V P_V$ for each vertex V
- $\Rightarrow$ the matrix problem can be exactly, without any numerical integration, rewritten as $\mathbb{S}P = H$
- lowest-order RT MFEM is thus equivalent to a particular multi-point FV scheme
- $\mathbb{S}$ is sparse, in general positive definite but nonsymmetric
- works from pure diffusion to convection–diffusion–reaction cases, for upwind schemes, in 2D and 3D, for nonlinear cases

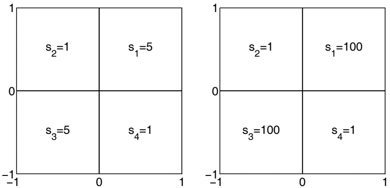
Outline

Problem with discontinuous and inhomogeneous diffusion tensor: finite volumes

- consider the pure diffusion equation

$$-\nabla \cdot \mathbf{S} \nabla p = 0 \quad \text{in} \quad \Omega = (-1,1) \times (-1,1)$$

- discontinuous and inhomogeneous **S**, two cases:

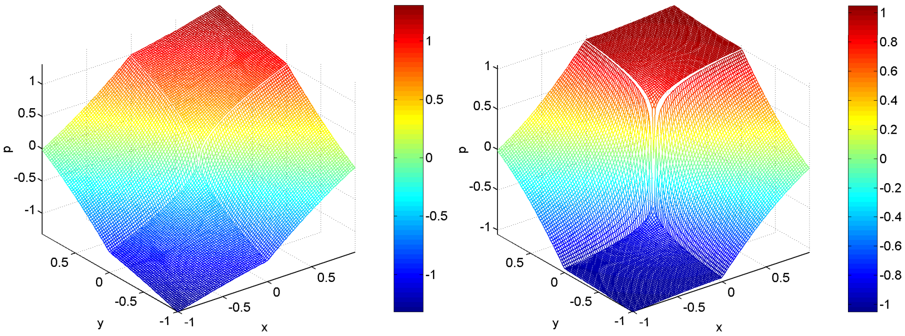


- analytical solution: singularity at the origin

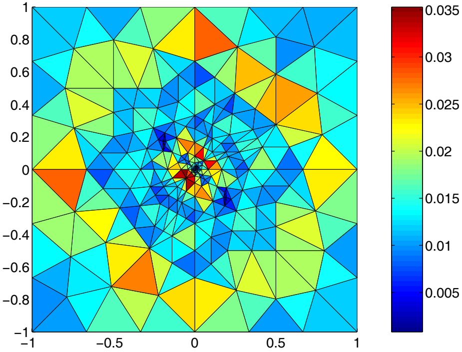
$$p(r,\theta) = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

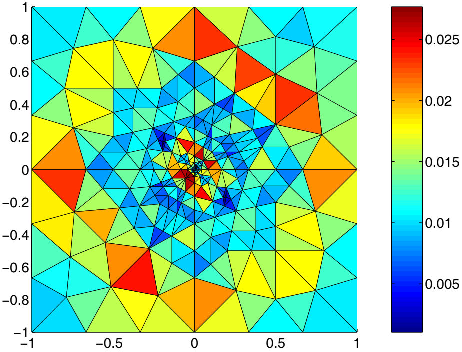
  - (r, θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

Analytical solutions

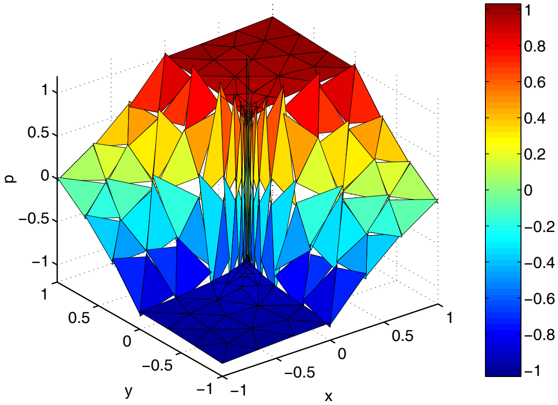


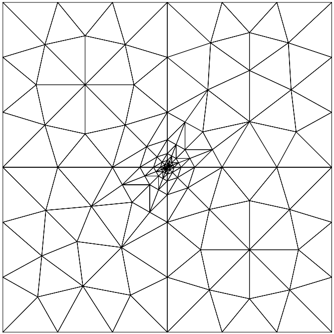
Estimated and actual error distribution on an adaptively refined mesh, case 1



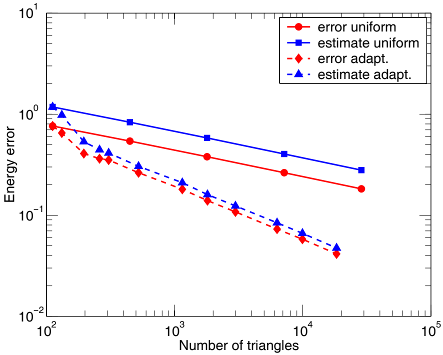


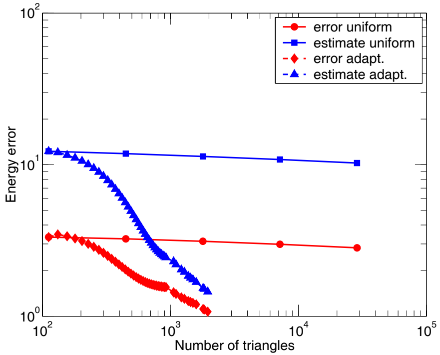
Approximate solution and the corresponding adaptively refined mesh, case 2



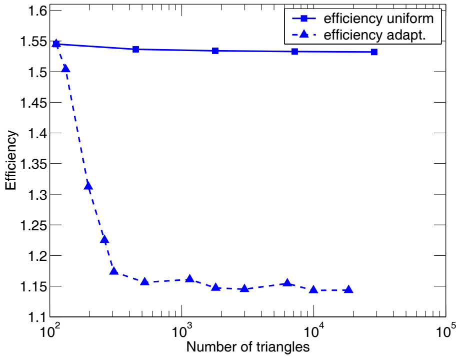


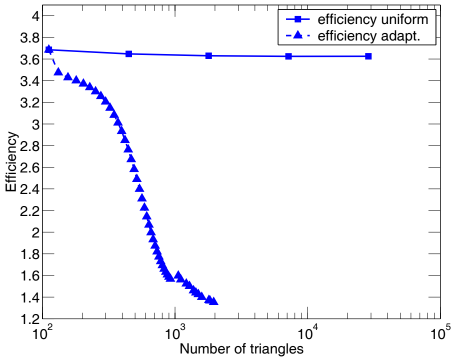
Estimated and actual error against the number of elements in uniformly/adaptively refined meshes





Global efficiency of the estimates





Convection-dominated problem: mixed finite elements

- consider the convection–diffusion–reaction equation

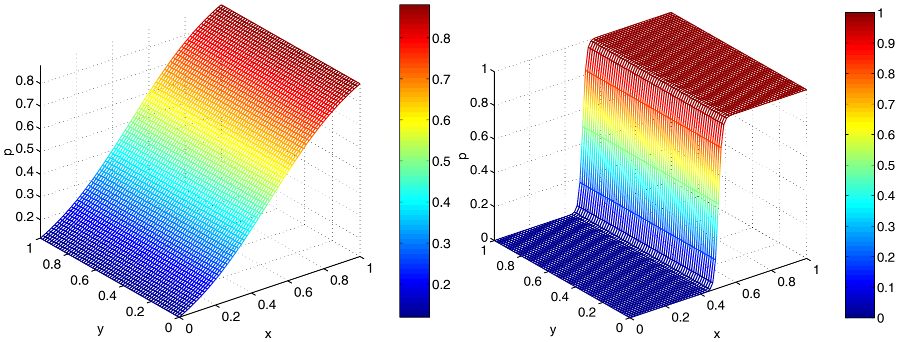
$$-\varepsilon \triangle p + \nabla \cdot (p(0,1)) + p = f \quad \text{in} \quad \Omega = (0,1) \times (0,1)$$

- analytical solution: layer of width a

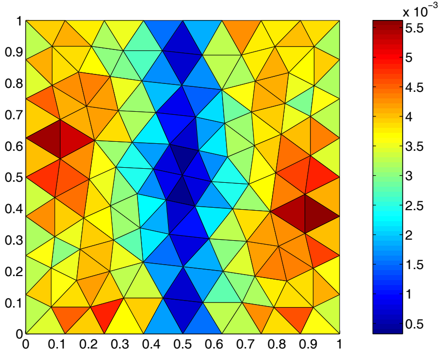
$$p(x,y) = 0.5 \left(1 - \tanh\left(\frac{0.5 - x}{a} \right) \right)$$

- consider
  - $\varepsilon = 1$, $a = 0.5$
  - $\varepsilon = 10^{-2}$, $a = 0.05$
  - $\varepsilon = 10^{-4}$, $a = 0.02$
- unstructured grid of 46 elements given, uniformly/adaptively refined

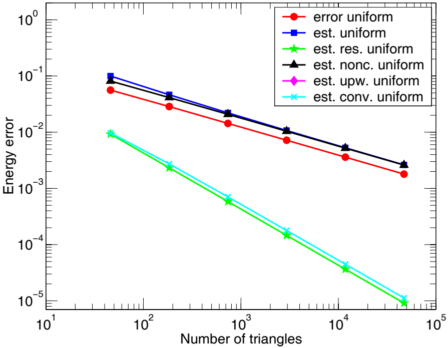
Analytical solutions, $\varepsilon = 1$, $a = 0.5$ and $\varepsilon = 10^{-4}$, $a = 0.02$

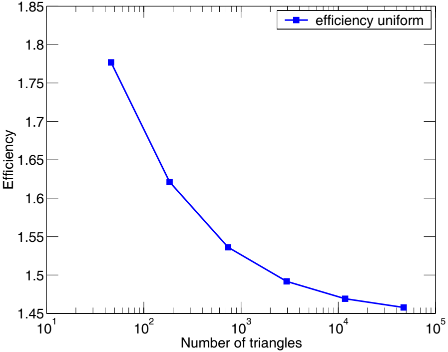


Estimated and actual error distribution, $\varepsilon = 1$, $a = 0.5$

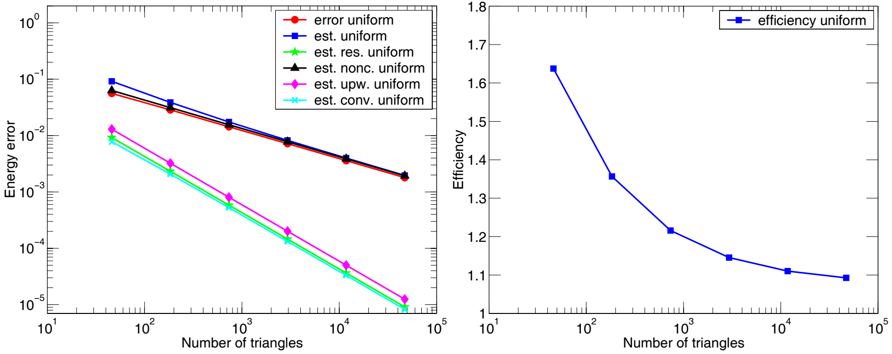


Modified Oswald interpolate: estimated and actual error against the number of elements and global efficiency of the estimates, $\varepsilon = 1$, $a = 0.5$

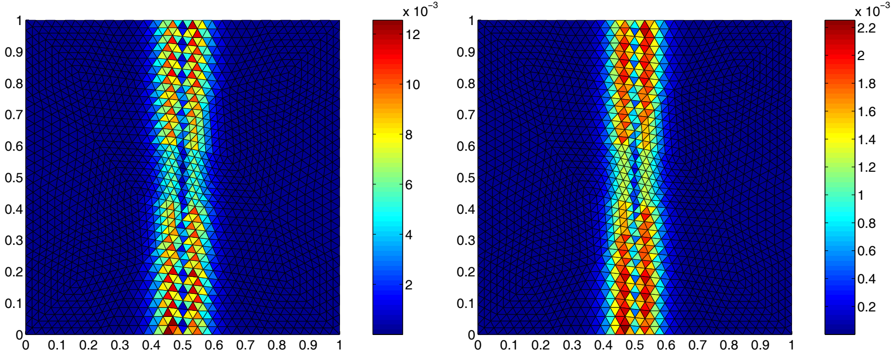




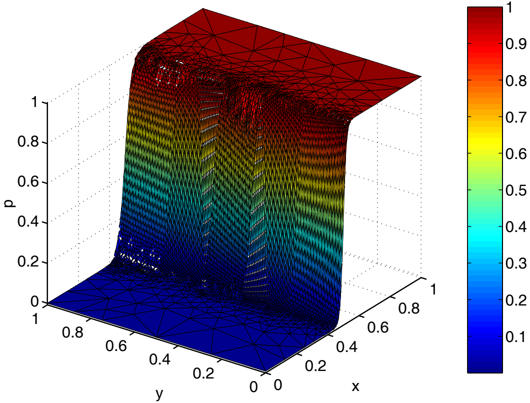
Oswald interpolate: estimated and actual error against the number of elements and global efficiency of the estimates, $\varepsilon = 1$, $a = 0.5$

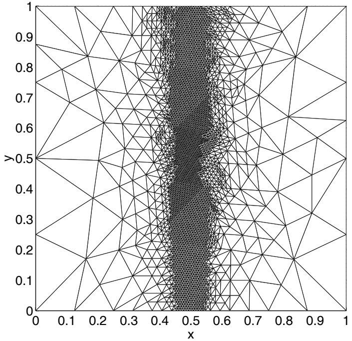


Estimated and actual error distribution, $\varepsilon = 10^{-2}$, $a = 0.05$

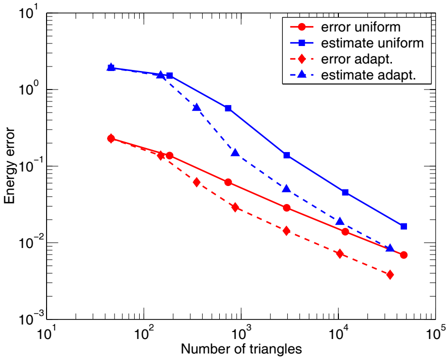


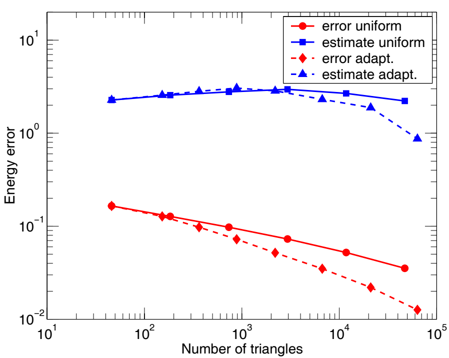
Approximate solution and the corresponding adaptively refined mesh, $\varepsilon = 10^{-4}$, $a = 0.02$



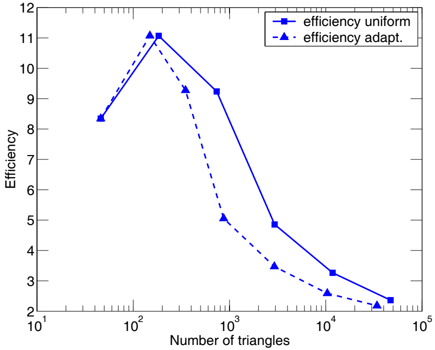


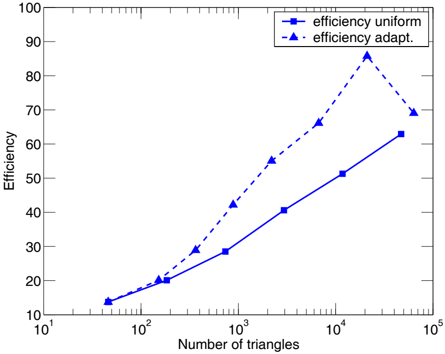
Estimated and actual error against the number of elements in uniformly/adaptively refined meshes, $\varepsilon = 10^{-2}$, $a = 0.05$ and $\varepsilon = 10^{-4}$, $a = 0.02$





Global efficiency of the estimates, $\varepsilon = 10^{-2}$, $a = 0.05$ and $\varepsilon = 10^{-4}$, $a = 0.02$





Outline

Conclusions and future work

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- asymptotically exact, fully reliable, and locally efficient estimates for inhomogeneous and anisotropic and convection-dominated problems
- directly computable—all constants evaluated
- works for mixed finite elements and finite volumes
- based on conformity of the flux variable
- connections mixed finite elements – finite volumes

Future work

- a unified a priori and a posteriori error analysis of mixed finite element / finite volume methods
- nonlinear case

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Bibliography

Papers

- VOHRALÍK M., Asymptotically exact a posteriori error estimates for mixed finite element discretizations of convection–diffusion–reaction equations, to appear in *SIAM J. Numer. Anal.*
- VOHRALÍK M., Residual flux-based a posteriori error estimates for finite volume discretizations of inhomogeneous, anisotropic, and convection-dominated problems, submitted to *Numer. Math.*

Merci de votre attention !