Numerical simulation of flow and contaminant transport in porous media

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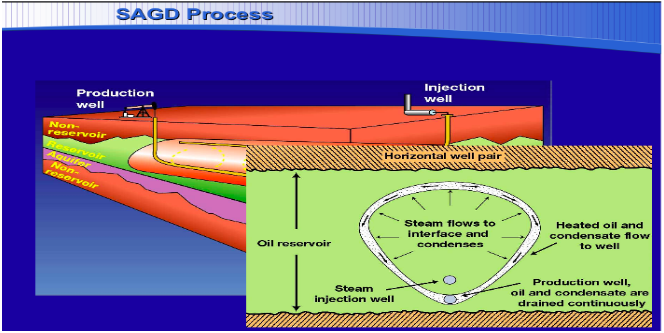
Outline

Outline

Oil production

Oil production

- oil – one of the major energy supply of today's world
- need for efficient production
- high prices – question of rentability



Reservoir

Changing the viscosity by heating

Changing the viscosity by heating

- inject steam into the reservoir
- increase the pressure
- heat the oil
- objective: viscosity reduction of the oil

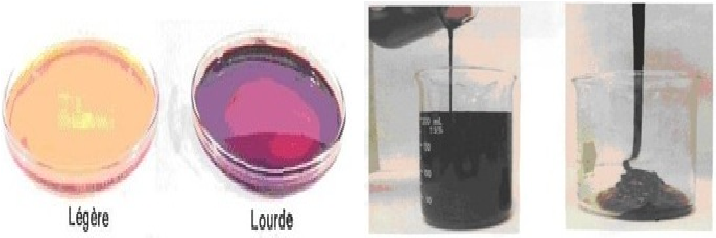


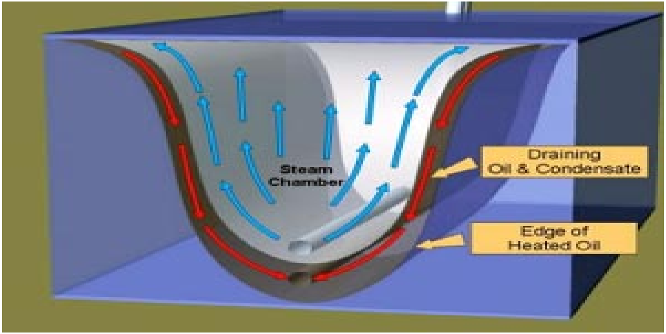
FIG. 1.3 – Huiles lourdes

Heated and cold oil

Efficient oil production

Efficient oil production

- capture the heated oil
- increase the production efficiency



Steam-assisted gravity drainage

Outline

Basic mathematical model: two-phase flow

Mathematical model

$$\partial_t(\phi s_\alpha) + \nabla\cdot\overbrace{\left(-\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z)\right)}^{\text{Darcy velocity } \mathbf{u}_\alpha} = q_\alpha, \qquad \alpha \in \{\mathrm{n}, \mathrm{w}\},$$

$$s_\mathrm{n} + s_\mathrm{w} = 1,$$

$$p_\mathrm{n} - p_\mathrm{w} = p_\mathrm{c}(s_\mathrm{w})$$

- two immiscible, incompressible fluids
- space–time domain $\Omega \times (0, T)$
- + initial & boundary conditions
- p_n, p_w: unknown nonwetting and wetting phase pressures
- s_n, s_w: unknown nonwetting and wetting phase saturations
- $p_\mathrm{c}(\cdot)$: the nonlinear capillary pressure
- $k_{\mathrm{r},\alpha}(\cdot)$: the nonlinear relative permeability
- ϕ porosity; $\underline{\mathbf{K}}$ permeability tensor; μ_α, ρ_α, q_α: viscosities, densities, sources; z vertical coordinate; g gravity

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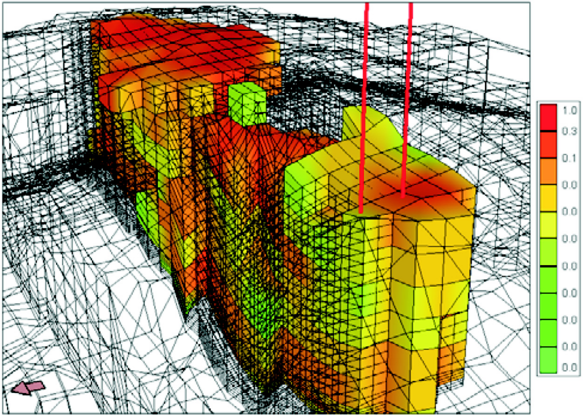
Geometry and meshes



Geometry and meshes example

Outline

Numerical difficulties

Numerical difficulties

- highly nonlinear (degenerate) system of partial differential equations
- coupled with nonlinear algebraic equations
- involves phase transitions
- different time and space scales (orders of magnitude difference)
- highly contrasted, discontinuous coefficients
- complicated 3D geometries
- unstructured and nonmatching grids
- presence of evolving sharp fronts
- combination of diffusive, advective, and reactive effects

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- a posteriori error estimators for determining the parts of the spatio-temporal domain with increased error (material discontinuities, sources and sinks, moving sharp fronts)
- adaptive stopping criteria
- adaptive mesh refinement

Goals

- increase the precision while simultaneously decreasing the cost (**increase efficiency**)
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Estimate distinguishing different error components

Theorem (Distinguishing different error components)

Consider

- *time step n,*
- *linearization step k,*
- *iterative algebraic solver step i,*

and the corresponding approximations $s_{\mathrm{w},h\tau}^{n,k,i}$ and $p_{\mathrm{w},h\tau}^{n,k,i}$. Then

$$|||(s_{\mathrm{w}} - s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w}} - p_{\mathrm{w},h\tau}^{n,k,i})|||_{I_n} \leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Error components

- $\eta_{\mathrm{sp}}^{n,k,i}$: spatial discretization
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Local estimators

- *spatial estimators*

$$\eta_{\mathrm{sp},K}^{n,k,i}(t) := \Bigg\{ \sum_{\alpha\in\{\mathrm{n},\mathrm{w}\}} \Big(\|\mathbf{d}_{\alpha,h}^{n,k,i} - \mathbf{v}_\alpha(p_{\mathrm{w},h}^{n,k,i}, s_{\mathrm{w},h}^{n,k,i})\|_K$$

$$+ h_K/\pi \|q_\alpha^n - \partial_t^n(\phi s_{\alpha,h\tau}^{n,k,i}) - \nabla\cdot\mathbf{u}_{\alpha,h}^{n,k,i}\|_K\Big)^2$$

$$+ \Big(\|\underline{\mathbf{K}}(\lambda_{\mathrm{w}}(s_{\mathrm{w},h\tau}^{n,k,i}) + \lambda_{\mathrm{n}}(s_{\mathrm{w},h\tau}^{n,k,i}))\nabla(\mathfrak{p}(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i}) - \bar{\mathfrak{p}}_{h\tau}^{n,k,i})\|_K(t)\Big)^2$$

$$+ \Big(\|\underline{\mathbf{K}}\nabla(\mathfrak{q}(s_{\mathrm{w},h\tau}^{n,k,i}) - \bar{\mathfrak{q}}_{h\tau}^{n,k,i})\|_K(t)\Big)^2 \Bigg\}^{\frac{1}{2}}$$

- *temporal estimators*

$$\eta_{\mathrm{tm},K,\alpha}^{n,k,i}(t) := \|\mathbf{v}_\alpha(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i})(t) - \mathbf{v}_\alpha(p_{\mathrm{w},h\tau}^{n,k,i}, s_{\mathrm{w},h\tau}^{n,k,i})(t^n)\|_K \quad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

- *linearization estimators*

$$\eta_{\mathrm{lin},K,\alpha}^{n,k,i} := \|\mathbf{l}_{\alpha,h}^{n,k,i}\|_K \qquad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

- *algebraic estimators*

$$\eta_{\mathrm{alg},K,\alpha}^{n,k,i} := \|\mathbf{a}_{\alpha,h}^{n,k,i}\|_K \qquad \alpha \in \{\mathrm{n},\mathrm{w}\}$$

Model problem

Horizontal flow

$$\partial_t(\phi s_\alpha) - \nabla\cdot\left(\frac{k_{\mathrm{r},\alpha}(s_{\mathrm{w}})}{\mu_\alpha}\underline{\mathbf{K}}\nabla p_\alpha\right) = 0,$$
$$s_{\mathrm{n}} + s_{\mathrm{w}} = 1,$$
$$p_{\mathrm{n}} - p_{\mathrm{w}} = p_{\mathrm{c}}(s_{\mathrm{w}})$$

Brooks–Corey model

- relative permeabilities

$$k_{\mathrm{r,w}}(s_{\mathrm{w}}) = s_{\mathrm{e}}^4, \quad k_{\mathrm{r,n}}(s_{\mathrm{w}}) = (1-s_{\mathrm{e}})^2(1-s_{\mathrm{e}}^2)$$

- capillary pressure

$$p_{\mathrm{c}}(s_{\mathrm{w}}) = p_{\mathrm{d}} s_{\mathrm{e}}^{-\frac{1}{2}}$$

- $$s_{\mathrm{e}} := \frac{s_{\mathrm{w}} - s_{\mathrm{rw}}}{1 - s_{\mathrm{rw}} - s_{\mathrm{rn}}}$$

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Fully implicit cell-centered finite volume scheme

Fully implicit cell-centered two-point finite volumes

For all $1 \le n \le N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},e_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{e_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},e_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the normal fluxes are given by

$$F_{\mathrm{w},e_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\lambda_{\mathrm{r,w}}(s^n_{\mathrm{w},K}) + \lambda_{\mathrm{r,w}}(s^n_{\mathrm{w},L})}{2} |\underline{\mathbf{K}}| \frac{\bar{p}^n_{\mathrm{w},L} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_L|} |e_{KL}|,$$

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Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{e_{KL}\in\mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},e_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

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Velocities reconstructions

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$$
\begin{aligned}
(\mathbf{d}_{\alpha,h}^{n,k,i} \cdot \mathbf{n}_K, 1)_{e_{KL}} &:= F_{\alpha,e_{KL}}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
((\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i}) \cdot \mathbf{n}_K, 1)_{e_{KL}} &:= F_{\alpha,e_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),\\
\mathbf{a}_{\alpha,h}^{n,k,i} &:= \mathbf{d}_{\alpha,h}^{n,k,i+\nu} + \mathbf{l}_{\alpha,h}^{n,k,i+\nu} - (\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})
\end{aligned}
$$

Comments

- phase velocities reconstructions:
  $\mathbf{u}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i} + \mathbf{a}_{\alpha,h}^{n,k,i}$
- $\mathbf{d}_{\alpha,h}^{n,k,i}, \mathbf{l}_{\alpha,h}^{n,k,i}, \mathbf{a}_{\alpha,h}^{n,k,i}$ used to identify error components

Velocities reconstructions

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$$(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K, 1)_{e_{KL}} := F_{\alpha,e_{KL}}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),$$

$$((\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K, 1)_{e_{KL}} := F_{\alpha,e_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),$$

$$\mathbf{a}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i+\nu} + \mathbf{l}_{\alpha,h}^{n,k,i+\nu} - (\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})$$

Comments

- phase velocities reconstructions: $\mathbf{u}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i} + \mathbf{a}_{\alpha,h}^{n,k,i}$
- $\mathbf{d}_{\alpha,h}^{n,k,i}, \mathbf{l}_{\alpha,h}^{n,k,i}, \mathbf{a}_{\alpha,h}^{n,k,i}$ used to identify error components

Data from Klieber & Rivière (2006)

Data

$$\Omega = (0,300)\text{m} \times (0,300)\text{m}, \quad T = 4{\cdot}10^6\text{s},$$
$$\phi = 0.2, \quad \underline{\mathbf{K}} = 10^{-11}\underline{\mathbf{I}}\,\text{m}^2,$$
$$\mu_{\rm w} = 5{\cdot}10^{-4}\text{kg}\,\text{m}^{-1}\text{s}^{-1}, \quad \mu_{\rm n} = 2{\cdot}10^{-3}\text{kg}\,\text{m}^{-1}\text{s}^{-1},$$
$$s_{\rm rw} = s_{\rm rn} = 0, \quad p_{\rm d} = 5{\cdot}10^3\text{kg}\,\text{m}^{-1}\text{s}^{-2}$$

Initial condition ($\widetilde{K}$ 18m × 18m lower left corner block)

$$s_{\rm w}^0 = 0.2 \text{ on } K \in \mathcal{T}_h,\ K \notin \widetilde{K},$$
$$s_{\rm w}^0 = 0.95 \text{ on } K \in \mathcal{T}_h,\ K \in \widetilde{K}$$

Boundary conditions ($\widehat{K}$ 18m × 18m upper right corner block)

- no flow Neumann boundary conditions everywhere except of $\partial\widetilde{K} \cap \partial\Omega$ and $\partial\widehat{K} \cap \partial\Omega$
- $\widetilde{K}$ – injection well: $s_{\rm w} = 0.95$, $p_{\rm w} = 3.45{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$
- $\widehat{K}$ – production well: $s_{\rm w} = 0.2$, $p_{\rm w} = 2.41{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$

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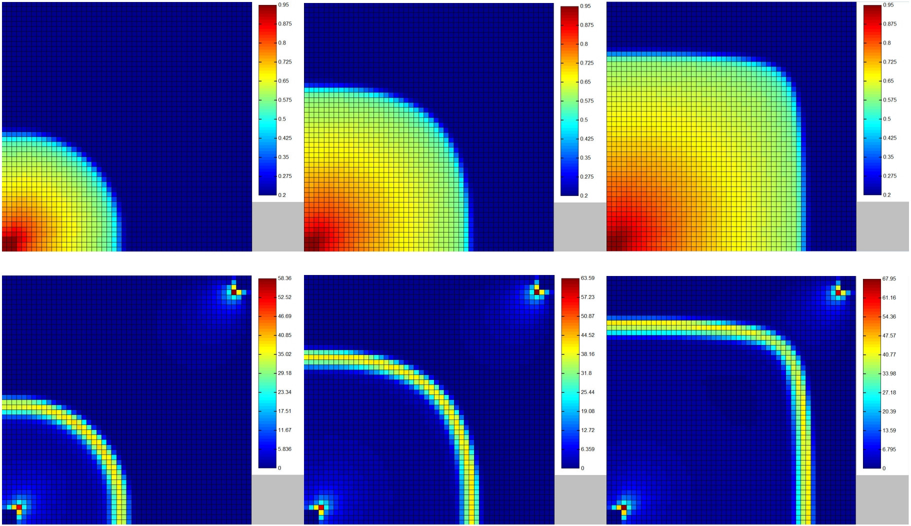
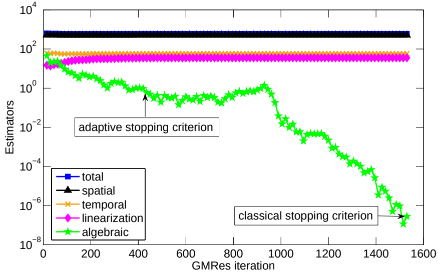
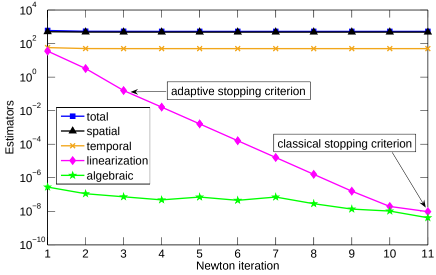
Water saturation/estimators evolution



Estimators and stopping criteria

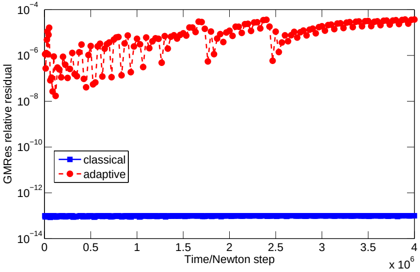


Estimators in function of GMRes iterations

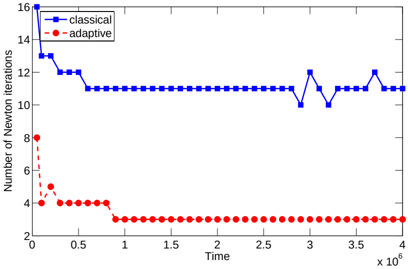


Estimators in function of Newton iterations

GMRes relative residual/Newton iterations

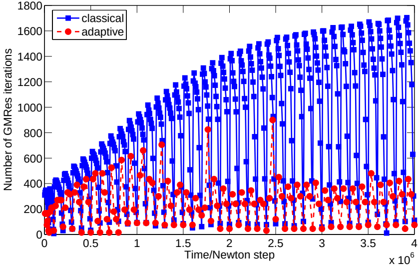


GMRes relative residual

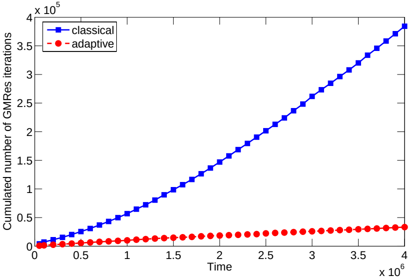


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Outline

Extension to multiphase compositional flows

Multiphase compositional flows

- $N_{\mathcal{P}}$ phases, $N_{\mathcal{C}}$ components
- miscible, compressible
- isothermal/thermal
- Ph.D. theses of Carole Heinry and Soleiman Yousef (Paris 6/IFPEN)

Multiphase compositional flows

Unknowns

- phase saturations S_p
- component molar fractions $C_{p,c}$
- reference pressure P

Constitutive laws

- phase pressures – reference pressure – capillary pressure

$$P_p := P + P_{c_p}(\boldsymbol{S})$$

- Darcy's law

$$\boldsymbol{v}_p(P_p, \boldsymbol{C}_p) := -\Lambda\left(\nabla P_p - \rho_p(P_p, \boldsymbol{C}_p)\boldsymbol{g}\right)$$

- component fluxes

$$\Phi_c := \sum_{p \in \mathcal{P}_c} \Phi_{p,c}, \quad \Phi_{p,c} := \nu_p(P_p, \boldsymbol{S}, \boldsymbol{C}_p) C_{p,c} \boldsymbol{v}_p(P_p, \boldsymbol{C}_p)$$

- amount of moles of component c per unit volume

$$l_c := \phi \sum_{p \in \mathcal{P}_c} \zeta_p(P_p, \boldsymbol{C}_p) S_p C_{p,c}$$

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Multiphase compositional flows

Governing partial differential equations

- conservation of mass

$$\partial_t l_c + \nabla \cdot \mathbf{\Phi}_c = q_c, \qquad \forall c \in \mathcal{C}$$

- + boundary & initial conditions

Closure algebraic equations

- conservation of pore volume

$$\sum_{p \in \mathcal{P}} S_p = 1$$

- conservation of the quantity of the matter

$$\sum_{c \in \mathcal{C}_p} C_{p,c} = 1 \qquad \forall p \in \mathcal{P}$$

- thermodynamic equilibrium

$$\sum_{c \in \mathcal{C}} (N_{\mathcal{P}_c} - 1) \text{ equations}$$

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