Estimations d'erreur a posteriori robustes et solveurs entièrement adaptifs

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en collaboration avec

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Outline

1. Residuals and their dual norms
2. The Laplace equation
   - Guaranteed upper bound
   - Polynomial-degree-robust local efficiency
   - Applications of the unified framework
   - Numerical results
3. The Laplace eigenvalue problem
   - Three equivalence results
   - Guaranteed bounds
   - Numerical results
4. The nonlinear Laplace problem
   - Localization of the dual residual norm
   - Guaranteed upper bound
   - Stopping criteria, efficiency, and robustness
   - Numerical results
5. Conclusions and ongoing work

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Residual and its dual norm for Laplacian, $u_h \in H_0^1(\Omega)$

The Laplace problem

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\begin{aligned}
-\Delta u &= f \qquad \text{in } \Omega, \\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
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- polytope $\Omega \subset \mathbb{R}^d$, $d \geq 1$, $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Residual $\mathcal{R}(u_h) \in H^{-1}(\Omega)$

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\nabla u_h, \nabla v), \qquad v \in H_0^1(\Omega) \qquad \text{weak form. misfit}$$

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Equivalence energy error–dual norm of the residual

Theorem (Equivalence energy error–dual norm of the residual)

Let $u_h \in H_0^1(\Omega)$. Then

$$\|\mathcal{R}(u_h)\|_{-1} = \|\nabla(u - u_h)\|.$$

Proof.

- residual and its dual norm definition

$$\|\mathcal{R}(u_h)\|_{-1} = \sup_{v \in H_0^1(\Omega),\, \|\nabla v\|=1} \{(f, v) - (\nabla u_h, \nabla v)\}$$

- weak solution definition

$$(f, v) = (\nabla u, \nabla v)$$

- conformity ($(u - u_h) \in H_0^1(\Omega)$) and duality:

$$\sup_{v \in H_0^1(\Omega),\, \|\nabla v\|=1} (\nabla(u - u_h), \nabla v) = \|\nabla(u - u_h)\|$$

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$$\|\mathcal{R}(u_h)\|_{-1} = \|\nabla(u - u_h)\| = \overbrace{\left\{ \sum_{K \in \mathcal{T}_h} \|\nabla(u - u_h)\|_K^2 \right\}^{\frac{1}{2}}}^{\textit{localization}}.$$

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The nonconforming case, $u_h \notin H_0^1(\Omega)$

Theorem (Energy error in the nonconforming case)

Let $u_h \notin H_0^1(\Omega)$. Then

$$\|\nabla(u-u_h)\|^2 = \underbrace{\sup_{v\in H_0^1(\Omega);\, \|\nabla v\|=1} \{(f,v)-(\nabla u_h, \nabla v)\}^2}_{\|\mathcal{R}(u_h)\|_{-1},\ \text{dual norm of the residual}} + \underbrace{\min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}.$$

Proof.

- define $s \in H_0^1(\Omega)$ by (projection)
$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$
- develop (Pythagoras)
$$\|\nabla(u-u_h)\|^2 = \|\nabla(u-s)\|^2 + \|\nabla(s-u_h)\|^2$$
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- norm characterization by duality, definition of s:
$$\|\nabla(u-s)\|^2 = \sup_{v\in H_0^1(\Omega);\, \|\nabla v\|=1} (\nabla(u- \ ), \nabla v)^2$$

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The Laplace equation

The game

How to give tight **computable bounds** on

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$$\|\mathcal{R}(u_h)\|_{-1}^2 + \min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|^2 \text{ ?}$$

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Braess, Pillwein, & Schöberl (2009)

Laplace eigenvalues

Energy minimization

Find $u_1 \in H_0^1(\Omega)$ such that $(u_1, 1) > 0$ and

$$u_1 := \arg \min_{v \in H_0^1(\Omega),\, \|v\|=1} \left\{ \frac{1}{2} \|\nabla v\|^2 \right\}.$$

Euler–Lagrange conditions, full problem, weak formulation

Find $(u_k, \lambda_k) \in H_0^1(\Omega) \times \mathbb{R}^+$, $k \geq 1$, with $\|u_k\| = 1$, such that

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- $0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \to \infty$
- u_k, $k \geq 1$, form an orthonormal basis of $L^2(\Omega)$

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The Laplace eigenvalue problem

The game

How to **link**

$$\|\nabla(u_1 - u_h)\| \quad \text{with} \quad \|\mathcal{R}(u_h, \lambda_h)\|_{-1} \text{ ?}$$

How to give tight and robust **computable bounds** on $\|\nabla(u_1 - u_h)\|$ (from above) and $\sqrt{\lambda_h - \lambda_1}$ (from above & below)?

The Laplace eigenvalue problem

The game

How to **link**

$$\|\nabla(u_1 - u_h)\| \text{ with } \|\mathcal{R}(u_h, \lambda_h)\|_{-1} \ ?$$

How to give tight and robust **computable bounds** on $\|\nabla(u_1 - u_h)\|$ (from above) and $\sqrt{\lambda_h - \lambda_1}$ (from above & below)?

The Laplace eigenvalue problem

The game

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Carstensen and Gedicke (2014), Hu, Huang, Lin (2014), Šebestová and Vejchodský (2014), Kuznetsov and Repin (2013), Liu and Oishi (2013)

Nonlinear Laplacian

Quasi-linear elliptic problem

$$
\begin{aligned}
-\nabla\cdot\overline{\sigma}(u,\nabla u) &= f \qquad \text{in } \Omega,\\
u &= 0 \qquad \text{on } \partial\Omega
\end{aligned}
$$

- $p > 1$, $q := \frac{p}{p-1}$, $f \in L^q(\Omega)$
- example: p-Laplacian with $\overline{\sigma}(u,\nabla u) = |\nabla u|^{p-2}\nabla u$

Weak formulation

Find $u \in W_0^{1,p}(\Omega)$ such that

$$(\overline{\sigma}(u,\nabla u),\nabla v) = (f,v) \qquad \forall v \in W_0^{1,p}(\Omega)$$

Residual $\mathcal{R}(u_h^{k,i}) \in W_0^{1,p}(\Omega)'$ and its dual norm

$$\langle \mathcal{R}(u_h^{k,i}), v\rangle := (f,v) - (\overline{\sigma}(u_h^{k,i},\nabla u_h^{k,i}),\nabla v), \qquad v \in W_0^{1,p}(\Omega)$$

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The nonlinear Laplace equation

The game

Is it possible to **localize** the dual norm of the residual

$$\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'} \approx \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}(u_h^{k,i})\|^q_{W_0^{1,p}(\omega_{\mathbf{a}})'} \right\}^{\frac{1}{q}} \; ?$$

- $\mathcal{V}_h$ vertices, $\omega_{\mathbf{a}}$ patches of elements of a partition $\mathcal{T}_h$ of Ω;
- the constant hidden in $\approx$ must not depend on p, Ω, and the regularity of u.

How to give tight and robust **computable bounds** on $\|\mathcal{R}(u_h^{k,i})\|_{W_0^{1,p}(\Omega)'}$ on each Newton step k and algebraic step i?

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Eisenstat and Walker (1994), Deuflhard (1996), Chaillou and Suri (2006, 2007), Kim (2007)

Outline

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A posteriori estimate via flux & potential reconstruction

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

Weak solution properties

- $u \in H_0^1(\Omega)$ (constraint)
- $\sigma := -\nabla u$ (constitutive relation)
- $\nabla \cdot \sigma = f$ (equilibrium)
- $\sigma \in \mathbf{H}(\mathrm{div}, \Omega)$ (constraint)

Theorem (A posteriori estimate, ≈ Prager and Synge (1947), Ladevèze (1975), Dari *et al.* (1996), Repin (1997), Destuynder and Métivet (1999), Ainsworth (2005), Kim (2007)**)**

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
- *$u_h \in H^1(\mathcal{T}_h)$ (piecewise $H^1(K)$) be arbitrary;*
- *$s_h \in H_0^1(\Omega)$, $\sigma_h \in \mathbf{H}(\mathrm{div}, \Omega)$, $(\nabla \cdot \sigma_h, 1)_K = (f, 1)_K$ $\forall K \in \mathcal{T}_h$.*

Then

$$\|\nabla(u - u_h)\|^2 \le \sum_{K \in \mathcal{T}_h} \Big(\underbrace{\|\nabla u_h + \sigma_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \nabla \cdot \sigma_h\|_K}_{\text{equilibrium}} \Big)^2 + \sum_{K \in \mathcal{T}_h} \underbrace{\|\nabla(u_h - s_h)\|_K^2}_{\text{constraint}}.$$

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A posteriori error estimate

Proof.

- we know from the Introduction

$$\|\nabla(u-u_h)\|^2 = \sup_{v\in H_0^1(\Omega);\,\|\nabla v\|=1} \{(f,v)-(\nabla u_h,\nabla v)\}^2 + \min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\|^2$$

- nonconformity upper bound:

$$\min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\| \le \|\nabla(u_h-s_h)\|$$

- equilibrated flux, Green theorem:

$$(f,v)-(\nabla u_h,\nabla v) = (f-\nabla\cdot\sigma_h, v) - (\nabla u_h+\sigma_h,\nabla v)$$

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Global potential and flux reconstructions

Ideally

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h} f} \|\nabla u_h + \mathbf{v}_h\|$$

$$s_h := \arg \min_{v_h \in V_h} \|\nabla(u_h - v_h)\|$$

- $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, $Q_h \subset L^2(\Omega)$, $V_h \subset H_0^1(\Omega)$
- too expensive, **global** minimization problems (the hypercircle method)

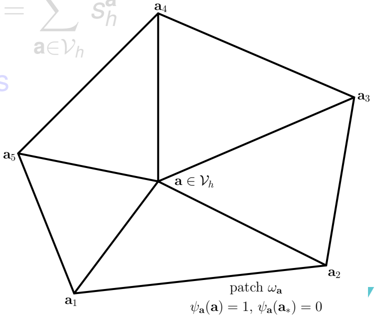
Local potential and flux reconstructions

Partition of unity localization

$$\sigma_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

- cut-off by hat basis functions $\psi_{\mathbf{a}} \in \mathbb{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$
- $\sigma_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \sigma_h^{\mathbf{a}}, \quad s_h := \sum_{\mathbf{a} \in \mathcal{V}_h} s_h^{\mathbf{a}}$
- local minimizations



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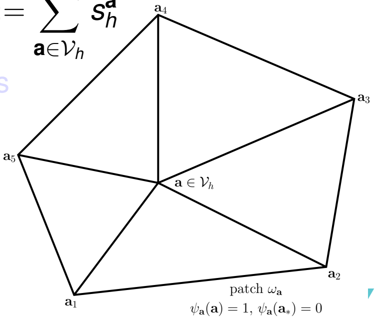


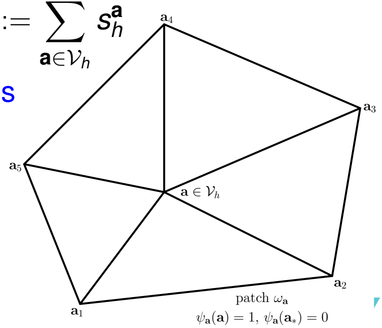
Local potential and flux reconstructions

Partition of unity localization

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

- cut-off by hat basis functions $\psi_{\mathbf{a}} \in \mathbb{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$
- $\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}, \quad s_h := \sum_{\mathbf{a} \in \mathcal{V}_h} s_h^{\mathbf{a}}$
- **local** minimizations



Local equilibrated flux reconstruction

Assumption A (Galerkin orthogonality wrt hat functions)

There holds $u_h \in H^1(\mathcal{T}_h)$ *and*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

$\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$: MFE spaces (hom. Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$)

Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving the **local mixed FE problem**

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$\Updownarrow$$

$$\begin{aligned}
(\boldsymbol{\sigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}}, \\
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Local potential reconstruction

$V_h^{\mathbf{a}}$: FE space (hom. Dirichlet BC on $\partial\omega_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{V}_h$)

Definition (Construction of s_h, ≈ Carstensen and Merdon (2013))

Let $u_h \in H^1(\mathcal{T}_h)$. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ by solving the **local conforming finite element problem**

$$s_h^{\mathbf{a}} := \arg\min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

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Outline

Assumptions

Assumption B (Weak continuity)

There holds $\langle [\![u_h]\!], 1 \rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$

Assumption C (Piecewise polynomials, data, and meshes)

*The approximation u_h and the datum f are piecewise polynomial. The **degrees** of the MFE reconstructions σ_h and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.*

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Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency)

Let u be the weak solution. Under Assumptions A, B, and C,

$$\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \leq C_{\rm st} C_{\rm cont,PF} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}},$$

$$\|\nabla(u_h - s_h)\|_K \leq C_{\rm st} C_{\rm cont,bPF} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Remarks

- $C_{\rm cont,PF}$: $\approx 1 + 2/\pi$ on convex patches $\omega_{\mathbf{a}}$
- $C_{\rm st}$ can be bounded by solving the local Neumann problems by conforming FEs: find $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}} \subset H_*^1(\omega_{\mathbf{a}})$ s.t.

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \nabla v_h)_{\omega_{\mathbf{a}}} + (\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, v_h)_{\omega_{\mathbf{a}}} \ \forall v_h \in V_h^{\mathbf{a}};$$

  then $C_{\rm st} \leq \|\psi_{\mathbf{a}} \nabla u_h + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} / \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$
- $\Rightarrow$ maximal overestimation factor guaranteed

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Conforming finite elements

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Find $u_h \in V_h$ such that

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- $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$
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- NIPG, IIPG: broken Poincaré–Friedrichs inequality & jumps
- symmetric version:

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Outline

Numerics: smooth test case

Model problem

$$-\Delta u = f \quad \text{in } \Omega :=]0,1[^2$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured nested triangular grids
- uniform refinement
- simulations by V. Dolejší (Charles University Prague)

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Exact solution

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$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured nested triangular grids
- uniform refinement
- simulations by V. Dolejší (Charles University Prague)

Numerics: smooth test case

Model problem

$$-\Delta u = f \quad \text{in } \Omega :=]0,1[^2$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
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Discretization

- incomplete interior penalty discontinuous Galerkin method
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Estimates, errors, and effectivity indices

h	p	$\|\nabla(u-u_h)\|$	$\|u-u_h\|_{DG}$	$\|\nabla u_h+\sigma_h\|$	$\|\nabla(u_h-s_h)\|$	η_{osc}	η	η_{DG}	I^{eff}	I^{eff}_{DG}
$h_0/1$	1	1.21E+00	1.22E+00	1.24E+00	1.07E-01	5.56E-02	1.30E+00	1.31E+00	1.07	1.07
$h_0/2$		6.18E-01	6.22E-01	6.38E-01	5.09E-02	7.02E-03	6.47E-01	6.50E-01	1.05	1.05
		(0.97)	(0.97)	(0.96)	(1.07)	(2.99)	(1.01)	(1.01)		
$h_0/4$		3.12E-01	3.13E-01	3.22E-01	2.43E-02	8.80E-04	3.24E-01	3.25E-01	1.04	1.04
		(0.99)	(0.99)	(0.99)	(1.07)	(3.00)	(1.00)	(1.00)		
$h_0/8$		1.56E-01	1.57E-01	1.61E-01	1.18E-02	1.10E-04	1.62E-01	1.63E-01	1.04	1.04
		(1.00)	(1.00)	(1.00)	(1.05)	(3.00)	(1.00)	(1.00)		
$h_0/1$	2	1.50E-01	1.53E-01	1.49E-01	2.76E-02	5.10E-03	1.56E-01	1.59E-01	1.04	1.04
$h_0/2$		3.85E-02	3.92E-02	3.83E-02	7.99E-03	3.22E-04	3.94E-02	4.01E-02	1.03	1.02
		(1.96)	(1.96)	(1.96)	(1.79)	(3.98)	(1.98)	(1.98)		
$h_0/4$		9.70E-03	9.88E-03	9.68E-03	2.12E-03	2.02E-05	9.93E-03	1.01E-02	1.02	1.02
		(1.99)	(1.99)	(1.98)	(1.92)	(4.00)	(1.99)	(1.99)		
$h_0/8$		2.43E-03	2.48E-03	2.43E-03	5.42E-04	1.26E-06	2.49E-03	2.54E-03	1.02	1.02
		(1.99)	(1.99)	(1.99)	(1.96)	(4.00)	(1.99)	(1.99)		
$h_0/1$	3	1.32E-02	1.34E-02	1.29E-02	2.52E-03	3.58E-04	1.35E-02	1.37E-02	1.03	1.03
$h_0/2$		1.67E-03	1.69E-03	1.65E-03	3.13E-04	1.13E-05	1.70E-03	1.71E-03	1.01	1.01
		(2.98)	(2.98)	(2.97)	(3.01)	(4.99)	(3.00)	(3.00)		
$h_0/4$		2.11E-04	2.13E-04	2.09E-04	3.83E-05	3.53E-07	2.12E-04	2.15E-04	1.01	1.01
		(2.99)	(2.99)	(2.99)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/8$		2.64E-05	2.67E-05	2.61E-05	4.69E-06	1.10E-08	2.66E-05	2.69E-05	1.01	1.01
		(3.00)	(3.00)	(3.00)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/1$	4	9.36E-04	9.54E-04	9.05E-04	2.41E-04	2.12E-05	9.57E-04	9.74E-04	1.02	1.02
$h_0/2$		5.93E-05	6.05E-05	5.77E-05	1.68E-05	3.36E-07	6.04E-05	6.16E-05	1.02	1.02
		(3.98)	(3.98)	(3.97)	(3.84)	(5.98)	(3.99)	(3.98)		
$h_0/4$		3.72E-06	3.80E-06	3.63E-06	1.10E-06	5.31E-09	3.80E-06	3.87E-06	1.02	1.02
		(3.99)	(3.99)	(3.99)	(3.94)	(5.98)	(3.99)	(3.99)		
$h_0/8$		2.33E-07	2.38E-07	2.27E-07	7.02E-08	8.30E-11	2.38E-07	2.43E-07	1.02	1.02
		(4.00)	(4.00)	(4.00)	(3.97)	(6.00)	(4.00)	(3.99)		
$h_0/1$	5	5.41E-05	5.50E-05	5.22E-05	1.38E-05	1.06E-06	5.50E-05	5.58E-05	1.02	1.02
$h_0/2$		1.70E-06	1.72E-06	1.65E-06	4.39E-07	9.35E-09	1.72E-06	1.74E-06	1.01	1.01
		(4.99)	(5.00)	(4.98)	(4.98)	(6.82)	(5.00)	(5.00)		
$h_0/4$		5.32E-08	5.39E-08	5.19E-08	1.40E-08	7.67E-11	5.38E-08	5.45E-08	1.01	1.01
		(5.00)	(5.00)	(4.99)	(4.97)	(6.93)	(5.00)	(5.00)		
$h_0/8$		1.66E-09	1.69E-09	1.62E-09	4.41E-10	5.99E-13	1.68E-09	1.70E-09	1.01	1.01
		(5.00)	(5.00)	(5.00)	(4.99)	(7.00)	(5.00)	(5.00)		

Numerics: singular test case & *hp*-adaptivity

Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := \Omega :=]-1,1[^2 \backslash [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

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$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := \Omega :=]-1,1[^2 \backslash [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

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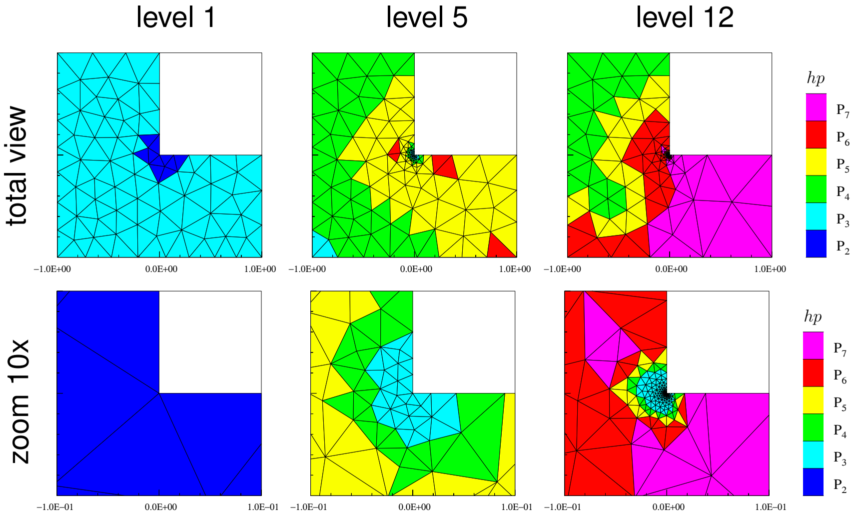
- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

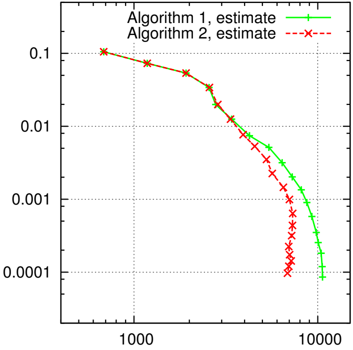
Estimates, errors, and effectivity indices

lev	$\|\mathcal{T}_h\|$	DoF	$\|\nabla(u-u_h)\|$	$\|\nabla u_h + \boldsymbol{\sigma}_h\|$	η_{osc}	$\|\nabla(u_h - s_h)\|$	η_{BC}	η	I^{eff}
0	114	684	6.22E-02	6.63E-02	1.89E-15	4.48E-02	3.81E-02	1.05E-01	1.69
1	122	1180	4.28E-02	4.27E-02	1.18E-14	3.08E-02	2.92E-02	7.29E-02	1.70
2	139	1919	3.28E-02	3.37E-02	8.21E-14	2.09E-02	2.12E-02	5.36E-02	1.64
3	165	2573	2.32E-02	2.30E-02	3.88E-13	1.50E-02	1.03E-02	3.41E-02	1.47
4	174	2858	1.02E-02	1.01E-02	4.48E-13	8.22E-03	9.19E-03	1.99E-02	1.96
5	199	3351	6.27E-03	6.21E-03	1.12E-12	4.81E-03	6.18E-03	1.25E-02	2.00
6	237	3926	4.21E-03	4.23E-03	1.98E-12	3.15E-03	3.29E-03	7.66E-03	1.82
7	285	4537	2.84E-03	2.91E-03	7.47E-12	2.13E-03	2.42E-03	5.33E-03	1.88
8	338	5257	2.04E-03	2.19E-03	4.63E-11	1.45E-03	1.32E-03	3.51E-03	1.72
9	372	5658	1.21E-03	1.23E-03	1.11E-11	9.07E-04	9.99E-04	2.26E-03	1.87
10	426	6500	7.70E-04	7.69E-04	5.69E-11	5.55E-04	6.95E-04	1.46E-03	1.89
11	453	7010	4.95E-04	5.04E-04	9.77E-11	3.97E-04	4.74E-04	9.91E-04	2.00
12	469	7308	3.41E-04	3.47E-04	1.13E-10	2.55E-04	2.88E-04	6.40E-04	1.88
13	463	7286	2.42E-04	2.42E-04	1.39E-10	1.73E-04	1.94E-04	4.37E-04	1.81
14	458	7215	1.69E-04	1.69E-04	1.23E-10	1.19E-04	1.53E-04	3.17E-04	1.88
15	440	6955	1.29E-04	1.31E-04	1.45E-10	9.21E-05	9.10E-05	2.24E-04	1.73
16	435	7035	9.71E-05	9.91E-05	1.39E-10	6.89E-05	7.63E-05	1.74E-04	1.79
17	434	7167	8.52E-05	8.97E-05	1.41E-10	5.76E-05	5.47E-05	1.42E-04	1.67
18	419	6960	7.51E-05	7.97E-05	1.44E-10	5.00E-05	4.15E-05	1.21E-04	1.60
19	410	6838	6.06E-05	6.35E-05	1.47E-10	3.87E-05	3.65E-05	9.69E-05	1.60

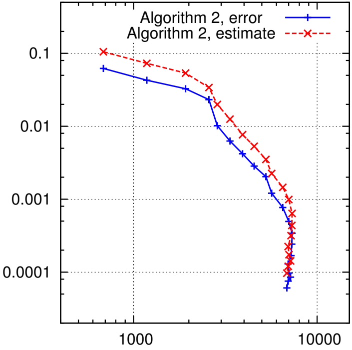
hp-refinement grids



hp-adaptive refinement algorithms



Algorithm 1 (only refinement) and Algorithm 2 (refinement & derefinement) wrt DoF



Exponential convergence of Algorithm 2 wrt DoF

Outline

Setting

Weak formulation

Find $(u_k, \lambda_k) \in V := H_0^1(\Omega) \times \mathbb{R}^+$ with $\|u_k\| = 1$, $k \geq 1$, s. t.

$$(\nabla u_k, \nabla v) = \lambda_k (u_k, v) \qquad \forall v \in V$$

Assumption A (Conforming variational approximation of the first eigenvalue – eigenvector pair (u_1, λ_1))

There holds

- $(u_h, \lambda_h) \in V \times \mathbb{R}^+$
- $\|u_h\| = 1$
- $(u_h, 1) > 0$
- $\|\nabla u_h\|^2 = \lambda_h$

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$L^2(\Omega)$ bound

Riesz representation of the residual $r_{(h)} \in V$

$$(\nabla r_{(h)}, \nabla v) = \langle \mathcal{R}(u_h, \lambda_h), v\rangle \qquad \forall v \in V$$

(note that $\|\nabla r_{(h)}\| = \|\mathcal{R}(u_h, \lambda_h)\|_{-1}$)

Lemma ($L^2(\Omega)$ bound via a quadratic residual inequality)

Let Assumption A hold and let

$$\lambda_h < \lambda_2$$

and

$$\beta_h := \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1} \|r_{(h)}\| < 1,$$

$$\alpha_h^2 := 2\left(1 - \sqrt{1 - \beta_h^2}\right) \leq |\Omega|^{-1}(u_h, 1)^2.$$

Then

$$\|u_1 - u_h\| \leq \alpha_h.$$

$L^2(\Omega)$ bound

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$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof I.

weak solution, residual, and Riesz representation definitions:

$$(\mathfrak{r}_{(h)}, u_k) = \frac{1}{\lambda_k}(\nabla u_k, \nabla \mathfrak{r}_{(h)}) = \frac{1}{\lambda_k}\left(\lambda_h(u_h, u_k) - (\nabla u_h, \nabla u_k)\right)$$
$$= \left(\frac{\lambda_h}{\lambda_k} - 1\right)(u_h, u_k)$$

Parseval equality for $\mathfrak{r}_{(h)}$

$$\|\mathfrak{r}_{(h)}\|^2 =$$

assumption $\lambda_h < \lambda_2$:

$$\min_{k \geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h$$

$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof I.

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$$(r_{(h)}, u_k) = \frac{1}{\lambda_k}(\nabla u_k, \nabla r_{(h)}) = \frac{1}{\lambda_k}\left(\lambda_h(u_h, u_k) - (\nabla u_h, \nabla u_k)\right)$$
$$= \left(\frac{\lambda_h}{\lambda_k} - 1\right)(u_h, u_k)$$

Parseval equality for $r_{(h)}$:

$$\|r_{(h)}\|^2 = \sum_{k \geq 1} (r_{(h)}, u_k)^2$$

assumption $\lambda_h < \lambda_2$:

$$\min_{k \geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h$$

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$$= \left(\frac{\lambda_h}{\lambda_k} - 1\right)(u_h, u_k)$$

Parseval equality for $r_{(h)}$:

$$\|r_{(h)}\|^2 = \sum_{k\geq 1}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h, u_k)^2$$

assumption $\lambda_h < \lambda_2$:

$$\min_{k\geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h$$

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$$= \left(\frac{\lambda_h}{\lambda_k} - 1\right)(u_h, u_k)$$

Parseval equality for $r_{(h)}$:

$$\|r_{(h)}\|^2 = \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + \sum_{k\geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h, u_k)^2$$

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$L^2(\Omega)$ bound via a quadratic residual inequality

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Parseval equality for $r_{(h)}$, u_k orthonormal basis:

$$\|r_{(h)}\|^2 = \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + \sum_{k\geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 (u_h - u_1, u_k)^2$$

assumption $\lambda_h < \lambda_2$:

$$\min_{k\geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h$$

$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof I.

weak solution, residual, and Riesz representation definitions:

$$(\mathfrak{r}_{(h)}, u_k) = \frac{1}{\lambda_k}(\nabla u_k, \nabla \mathfrak{r}_{(h)}) = \frac{1}{\lambda_k}\left(\lambda_h(u_h, u_k) - (\nabla u_h, \nabla u_k)\right)$$
$$= \left(\frac{\lambda_h}{\lambda_k} - 1\right)(u_h, u_k)$$

Parseval equality for $\mathfrak{r}_{(h)}$, u_k orthonormal basis:

$$\|\mathfrak{r}_{(h)}\|^2 = \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + \sum_{k\geq 2} \underbrace{\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2}_{\geq C_h} (u_h - u_1, u_k)^2$$

assumption $\lambda_h < \lambda_2$:

$$\min_{k\geq 2}\left(1 - \frac{\lambda_h}{\lambda_k}\right)^2 = \left(1 - \frac{\lambda_h}{\lambda_2}\right)^2 =: C_h$$

$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof II.

Parseval equality for $(u_h - u_1)$, $(u_h - u_1, u_1) = -\frac{1}{2}\|u_1 - u_h\|^2$:

$$\|\mathfrak{r}(h)\|^2 \geq \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h\|u_1 - u_h\|^2 - \frac{C_h}{4}\|u_1 - u_h\|^4$$

dropping the first term above, $e_h := \|u_1 - u_h\|^2$:

$$\frac{C_h}{4} e_h^2 - C_h e_h + \|\mathfrak{r}(h)\|^2 \geq 0$$

quadratic residual inequality in e_h, under assumption on β_h:

$$e_h \leq 2\left(1 - \sqrt{1 - \beta_h^2}\right) \quad \text{or} \quad e_h \geq 2(1 + \sqrt{1 - \beta_h^2})$$

sign condition $(u_h, 1) > 0$, assumption on α_h:

$$e_h \leq 2\left(1 - \sqrt{1 - \beta_h^2}\right)$$

$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof II.

Parseval equality for $(u_h - u_1)$, $(u_h - u_1, u_1) = -\frac{1}{2}\|u_1 - u_h\|^2$:

$$\|\mathfrak{r}_{(h)}\|^2 \geq \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h\|u_1 - u_h\|^2 - \frac{C_h}{4}\|u_1 - u_h\|^4$$

dropping the first term above, $e_h := \|u_1 - u_h\|^2$:

$$\frac{C_h}{4} e_h^2 - C_h e_h + \|\mathfrak{r}_{(h)}\|^2 \geq 0$$

quadratic residual inequality in e_h, under assumption on β_h:

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$L^2(\Omega)$ bound via a quadratic residual inequality

Sketch of the proof II.

Parseval equality for $(u_h - u_1)$, $(u_h - u_1, u_1) = -\frac{1}{2}\|u_1 - u_h\|^2$:

$$\|r_{(h)}\|^2 \geq \left(\frac{\lambda_h}{\lambda_1} - 1\right)^2 (u_h, u_1)^2 + C_h\|u_1 - u_h\|^2 - \frac{C_h}{4}\|u_1 - u_h\|^4$$

dropping the first term above, $e_h := \|u_1 - u_h\|^2$:

$$\frac{C_h}{4} e_h^2 - C_h e_h + \|r_{(h)}\|^2 \geq 0$$

quadratic residual inequality in e_h, under assumption on β_h:

$$e_h \leq 2\left(1 - \sqrt{1 - \beta_h^2}\right) \quad \text{or} \quad e_h \geq 2(1 + \sqrt{1 - \beta_h^2})$$

sign condition $(u_h, 1) > 0$, assumption on α_h:

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Outline

Eigenvalue and eigenvector error equivalences

Theorem (Eigenvector error – dual norm of the residual equivalence)

Under the above assumptions, there holds

$$\left(\frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} + 1\right)^{-1} \|\mathcal{R}(u_h, \lambda_h)\|_{-1}^2$$

$$\leq \|\nabla(u_1 - u_h)\|^2 \leq \left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-2} \left(1 - \frac{\alpha_h^2}{4}\right)^{-1} \|\mathcal{R}(u_h, \lambda_h)\|_{-1}^2.$$

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$$\frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2}\right)\left(1 - \frac{\alpha_h^2}{4}\right) \|\nabla(u_1 - u_h)\|^2 \leq \lambda_h - \lambda_1 \leq \|\nabla(u_1 - u_h)\|^2.$$

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Dual norm of the residual equivalences

Theorem (Dual norm of the residual equivalences)

Let $(u_h, \lambda_h) \in V \times \mathbb{R}$ verifying Assumption B be arbitrary. Then

$$\|\mathcal{R}(u_h, \lambda_h)\|_{-1} \leq \|\nabla u_h + \boldsymbol{\sigma}_h\|.$$

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$$\|\nabla u_h + \boldsymbol{\sigma}_h\| \leq (d+1) C_{\rm st} C_{\rm cont,PF} \|\mathcal{R}(u_h, \lambda_h)\|_{-1}.$$

- $C_{\rm st}$, $C_{\rm cont,PF}$: independent of p, computable upper bounds

Assumption B (Galerkin orthogonality of the residual to $\psi_{\mathbf{a}}$)

There holds, for all $\mathbf{a} \in \mathcal{V}_h^{\rm int}$,

$$\lambda_h (u_h, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}(u_h, \lambda_h), \psi_{\mathbf{a}} \rangle = 0.$$

Assumption C (Piecewise polynomial form)

$u_h \in \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$, $\boldsymbol{\sigma}_h$ is chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.

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Outline

Guaranteed bounds for the first eigenvalue

Theorem (Eigenvalue bounds)

Let $0 < \underline{\lambda_2} \leq \lambda_2$ *and* $0 < \underline{\lambda_1} \leq \lambda_1$. *Let Assumptions A and B hold and let* $\lambda_h < \underline{\lambda_2}$. *Let* $\boldsymbol{\sigma}_h$ *be an equilibrated flux and let*

$$\underbrace{\beta_h}_{\searrow 0} := \frac{1}{\sqrt{\underline{\lambda_1}}}\left(1 - \frac{\lambda_h}{\underline{\lambda_2}}\right)^{-1} \|\nabla u_h + \boldsymbol{\sigma}_h\| < 1,$$

$$\underbrace{\alpha_h^2}_{\searrow 0} := 2\left(1 - \sqrt{1 - \beta_h^2}\right) \leq |\Omega|^{-1}(u_h, 1)^2.$$

Then

$$\lambda_1 \geq \lambda_h - \overbrace{\underbrace{\left(1 - \frac{\lambda_h}{\underline{\lambda_2}}\right)^{-2}}_{\text{no if elliptic reg.}} \underbrace{\left(1 - \frac{\alpha_h^2}{4}\right)^{-1}}_{\searrow 1} \|\nabla u_h + \boldsymbol{\sigma}_h\|^2}^{\eta^2},$$

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Theorem (Eigenvector bounds)

Let the assumptions of the eigenvalue theorem be verified. Then

$$\|\nabla(u_1 - u_h)\| \leq \eta.$$

Moreover, under Assumption C,

$$\eta \leq (d+1) C_{\mathrm{cont,PF}} C_{\mathrm{st}} \underbrace{\left(\frac{\|\nabla(u_1 - u_h)\|^2}{\lambda_1} + 1 \right)^{\frac{1}{2}}}_{\searrow 1}$$

$$\underbrace{\left(1 - \frac{\lambda_h}{\lambda_2}\right)^{-1}}_{\searrow \left(1 - \frac{\lambda_1}{\lambda_2}\right)^{-1}} \underbrace{\left(1 - \frac{\alpha_h^2}{4}\right)^{-\frac{1}{2}}}_{\searrow 1} \|\nabla(u_1 - u_h)\|.$$

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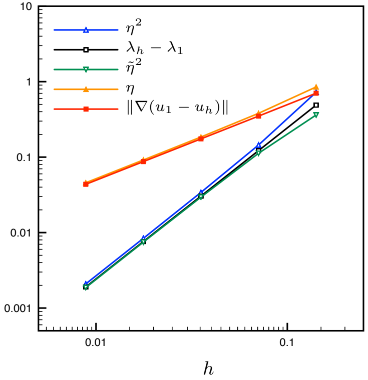
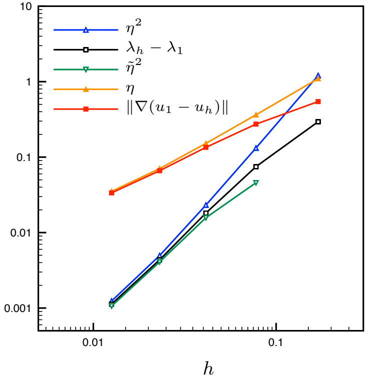
Outline

Errors and estimators, unit square (elliptic reg.)



Structured meshes



Unstructured meshes

Errors and estimators, unit square (elliptic reg.)

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\mathrm{lb}}_{\lambda,\mathrm{eff}}$	$I^{\mathrm{ub}}_{\lambda,\mathrm{eff}}$	$E_{\lambda,\mathrm{rel}}$	$I^{\mathrm{ub}}_{u,\mathrm{eff}}$
10	0.1414	121	19.7392	20.2284	19.5054	19.8667	1.35	1.48	1.84E-02	1.21
20	0.0707	441	19.7392	19.8611	19.7164	19.7486	1.08	1.19	1.63E-03	1.09
40	0.0354	1,681	19.7392	19.7696	19.7356	19.7401	1.03	1.12	2.28E-04	1.06
80	0.0177	6,561	19.7392	19.7468	19.7384	19.7393	1.02	1.10	4.56E-05	1.05
160	0.0088	25,921	19.7392	19.7411	19.7390	19.7392	1.02	1.10	1.01E-05	1.05

Structured meshes

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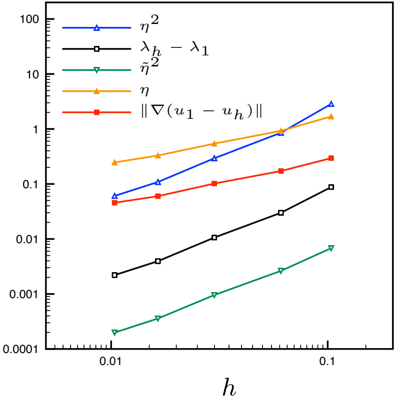
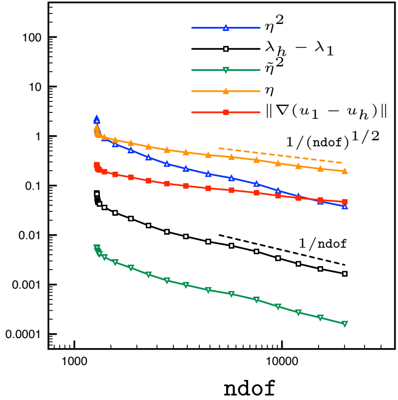
10	0.1698	143	19.7392	20.0336	18.8265	–	–	4.10	–	2.02
20	0.0776	523	19.7392	19.8139	19.6820	19.7682	1.63	1.77	4.37E-03	1.33
40	0.0413	1,975	19.7392	19.7573	19.7342	19.7416	1.15	1.28	3.75E-04	1.13
80	0.0230	7,704	19.7392	19.7436	19.7386	19.7395	1.07	1.14	4.56E-05	1.07
160	0.0126	30,666	19.7392	19.7403	19.7391	19.7393	1.06	1.10	1.01E-05	1.05

Unstructured meshes

Errors and estimators, L-shaped domain (no el. reg.)





Unstructured meshes

Adaptively refined meshes

Errors and estimators, L-shaped domain (no el. reg.)

N	h	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\text{lb}}_{\lambda,\text{eff}}$	$I^{\text{ub}}_{\lambda,\text{eff}}$	$E_{\lambda,\text{rel}}$	$I^{\text{ub}}_{u,\text{eff}}$
30	0.1038	826	9.63972	9.72744	6.88126	9.72064	12.90	32.45	3.42E-01	5.72
60	0.0608	3,154	9.63972	9.66968	8.81618	9.66705	11.39	28.49	9.21E-02	5.38
120	0.0299	12,747	9.63972	9.65032	9.35716	9.64937	11.08	27.65	3.07E-02	5.32
240	0.0164	49,119	9.63972	9.64367	9.53508	9.64331	11.03	27.51	1.13E-02	5.49
360	0.0104	114,806	9.63972	9.64192	9.58128	9.64173	11.08	27.55	6.29E-03	5.40

Unstructured meshes

Level	ndof	λ_1	λ_h	$\lambda_h - \eta^2$	$\lambda_h - \tilde{\eta}^2$	$I^{\text{lb}}_{\lambda,\text{eff}}$	$I^{\text{ub}}_{\lambda,\text{eff}}$	$E_{\lambda,\text{rel}}$	$I^{\text{ub}}_{u,\text{eff}}$
2	1,282	9.63972	9.70858	7.56083	9.70303	12.39	31.19	2.48E-01	5.62
6	1,294	9.63972	9.68971	8.35342	9.68509	10.83	26.73	1.48E-01	5.19
10	1,396	9.63972	9.67581	8.77643	9.67225	10.12	24.92	9.71E-02	4.98
14	2,792	9.63972	9.65137	9.37756	9.65016	9.63	23.51	2.87E-02	4.80
18	7,538	9.63972	9.64438	9.53634	9.64389	9.44	23.19	1.12E-02	4.60
22	20,071	9.63972	9.64137	9.60336	9.64122	10.30	23.01	3.93E-03	4.16

Adaptively refined meshes

Outline

Outline

Localization of the dual residual norm

Weak formulation

Find $u \in V := W_0^{1,p}(\Omega)$ such that

$$(\overline{\sigma}(u, \nabla u), \nabla v) = (f, v) \qquad \forall v \in V$$

Residual and its dual norm ($u_h \in V$, $\mathcal{R}(u_h) \in V'$)

$$\langle \mathcal{R}(u_h), v \rangle := (f, v) - (\overline{\sigma}(u_h, \nabla u_h), \nabla v), \qquad v \in V$$

$$\|\mathcal{R}(u_h)\|_{V'} := \sup_{v \in V;\, \|\nabla v\|_p = 1} \langle \mathcal{R}(u_h), v \rangle$$

Theorem (Localization of $\|\mathcal{R}\|_{V'}$)

Let $\langle \mathcal{R}, \psi_{\mathbf{a}} \rangle = 0$ for all $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$. Then, with $V^{\mathbf{a}} := W_0^{1,p}(\omega_{\mathbf{a}})$,

$$\|\mathcal{R}\|_{V'} \le (d+1)^{\frac{1}{p}} C_{\mathrm{cont,PF}} \left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q \right\}^{\frac{1}{q}},$$

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Localization of the dual residual norm

Upper bound.

- partition of unity, the linearity of $\mathcal{R}$, orthogonality wrt $\psi_{\mathbf{a}}$:

$$\langle \mathcal{R}, v \rangle = \sum_{\mathbf{a} \in \mathcal{V}_h} \langle \mathcal{R}, \psi_{\mathbf{a}} v \rangle = \sum_{\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}} \langle \mathcal{R}, \psi_{\mathbf{a}} (v - \Pi_{0,\omega_{\mathbf{a}}} v) \rangle + \sum_{\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}} \langle \mathcal{R}, \psi_{\mathbf{a}} v \rangle$$

- stability:

$$\|\nabla(\psi_{\mathbf{a}}(v - \Pi_{0,\omega_{\mathbf{a}}} v))\|_{p,\omega_{\mathbf{a}}} \le C_{\mathrm{cont,PF}} \|\nabla v\|_{p,\omega_{\mathbf{a}}}$$

- Hölder inequality:

$$\langle \mathcal{R}, v \rangle \le C_{\mathrm{cont,PF}} \left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \|\mathcal{R}\|^q_{(V^{\mathbf{a}})'} \right\}^{\frac{1}{q}} \left\{ \sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla v\|^p_{p,\omega_{\mathbf{a}}} \right\}^{\frac{1}{p}}$$

- overlapping of the patches:

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla v\|^p_{p,\omega_{\mathbf{a}}} = \sum_{K \in \mathcal{T}_h} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\nabla v\|^p_{p,K} \le (d+1) \overbrace{\sum_{K \in \mathcal{T}_h} \|\nabla v\|^p_{p,K}}^{\|\nabla v\|_p^p}$$

Localization of the dual residual norm

Upper bound.

- partition of unity, the linearity of $\mathcal{R}$, orthogonality wrt $\psi_{\mathbf{a}}$:

$$\langle \mathcal{R}, v\rangle = \sum_{\mathbf{a}\in\mathcal{V}_h} \langle \mathcal{R}, \psi_{\mathbf{a}} v\rangle = \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{int}}} \langle \mathcal{R}, \psi_{\mathbf{a}}(v - \Pi_{0,\omega_{\mathbf{a}}} v)\rangle + \sum_{\mathbf{a}\in\mathcal{V}_h^{\mathrm{ext}}} \langle \mathcal{R}, \psi_{\mathbf{a}} v\rangle$$

- stability:

$$\|\nabla(\psi_{\mathbf{a}}(v - \Pi_{0,\omega_{\mathbf{a}}} v))\|_{p,\omega_{\mathbf{a}}} \le C_{\mathrm{cont,PF}} \|\nabla v\|_{p,\omega_{\mathbf{a}}}$$

- Hölder inequality:

$$\langle \mathcal{R}, v\rangle \le C_{\mathrm{cont,PF}} \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q \right\}^{\frac{1}{q}} \left\{ \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla v\|_{p,\omega_{\mathbf{a}}}^p \right\}^{\frac{1}{p}}$$

- overlapping of the patches:

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla v\|_{p,\omega_{\mathbf{a}}}^p = \sum_{K\in\mathcal{T}_h} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla v\|_{p,K}^p \le (d+1) \overbrace{\sum_{K\in\mathcal{T}_h} \|\nabla v\|_{p,K}^p}^{\|\nabla v\|_p^p}$$

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Localization of the dual residual norm

Lower bound.

- p-Laplacian lifting of the residual on the patch $\omega_{\mathbf{a}}$: $r^{\mathbf{a}} \in V^{\mathbf{a}} = W_0^{1,p}(\omega_{\mathbf{a}})$ such that

$$(|\nabla r^{\mathbf{a}}|^{p-2}\nabla r^{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, v\rangle \qquad \forall v \in V^{\mathbf{a}}$$

- energy equality:

$$\|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p = (|\nabla r^{\mathbf{a}}|^{p-2}\nabla r^{\mathbf{a}}, \nabla r^{\mathbf{a}})_{\omega_{\mathbf{a}}} = \langle \mathcal{R}, r^{\mathbf{a}}\rangle = \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q$$

- setting $r := \sum_{\mathbf{a}\in\mathcal{V}_h} r^{\mathbf{a}} \in V$:

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \|\mathcal{R}\|_{(V^{\mathbf{a}})'}^q = \sum_{\mathbf{a}\in\mathcal{V}_h} \langle \mathcal{R}, r^{\mathbf{a}}\rangle = \langle \mathcal{R}, r\rangle \le \|\mathcal{R}\|_{V'}\|\nabla r\|_p$$

- overlapping of the patches:

$$\|\nabla r\|_p^p \le (d+1)^{p-1} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^p$$

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  $$\|\nabla r\|_{p}^{p} \leq (d+1)^{p-1} \sum_{\mathbf{a}\in\mathcal{V}_h} \|\nabla r^{\mathbf{a}}\|_{p,\omega_{\mathbf{a}}}^{p}$$

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Outline

Abstract assumptions

Numerical approximation

- simplicial mesh $\mathcal{T}_h$, linearization step k, algebraic step i
- $u_h^{k,i} \in V(\mathcal{T}_h) := \{v \in L^p(\Omega),\ v|_K \in W^{1,p}(K) \quad \forall K \in \mathcal{T}_h\} \not\subset V$

Assumption A (Total flux reconstruction)

There exists $\boldsymbol{\sigma}_h^{k,i} \in \mathbf{H}^q(\mathrm{div},\Omega)$ *and* $\rho_h^{k,i} \in L^q(\Omega)$ *such that*

$$\nabla\cdot\boldsymbol{\sigma}_h^{k,i} = f_h - \underbrace{\rho_h^{k,i}}_{\text{algebraic remainder}}.$$

Assumption B (Discretization, linearization, and alg. fluxes)

There exist fluxes $\mathbf{d}_h^{k,i}, \mathbf{l}_h^{k,i}, \mathbf{a}_h^{k,i} \in [L^q(\Omega)]^d$ *such that*

(i) $\boldsymbol{\sigma}_h^{k,i} = \mathbf{d}_h^{k,i} + \mathbf{l}_h^{k,i} + \mathbf{a}_h^{k,i}$;

(ii) *as the linear solver converges,* $\|\mathbf{a}_h^{k,i}\|_q \to 0$;

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Estimate distinguishing error components

Theorem (Estimate distinguishing different error components)

Let

- $u \in V$ *be the weak solution,*
- $u_h^{k,i} \in V(\mathcal{T}_h)$ *be arbitrary,*
- *Assumptions A and B hold.*

Then there holds

$$\underbrace{\mathcal{J}_u(u_h^{k,i})}_{\text{dual norm of the residual + NC}} \leq \eta_{\text{disc}}^{k,i} + \underbrace{\eta_{\text{lin}}^{k,i}}_{\|\mathbf{l}_h^{k,i}\|_q} + \underbrace{\eta_{\text{alg}}^{k,i}}_{\|\mathbf{a}_h^{k,i}\|_q} + \underbrace{\eta_{\text{rem}}^{k,i}}_{h_\Omega \|\rho_h^{k,i}\|_{q,K}} + \eta_{\text{quad}}^{k,i} + \eta_{\text{osc}}.$$

Outline

Stopping criteria and efficiency

Global stopping criteria (≈ Becker, Johnson, and Rannacher (1995), Arioli (2000's))

$$\eta_{\mathrm{rem}}^{k,i} \le \gamma_{\mathrm{rem}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}, \eta_{\mathrm{alg}}^{k,i}\},$$
$$\eta_{\mathrm{alg}}^{k,i} \le \gamma_{\mathrm{alg}} \max\{\eta_{\mathrm{disc}}^{k,i}, \eta_{\mathrm{lin}}^{k,i}\}, \qquad \gamma_{\mathrm{rem}}, \gamma_{\mathrm{alg}}, \gamma_{\mathrm{lin}} \approx 0.1$$
$$\eta_{\mathrm{lin}}^{k,i} \le \gamma_{\mathrm{lin}} \eta_{\mathrm{disc}}^{k,i}$$

Theorem (Global efficiency)

Under the global stopping criteria and the usual assumptions,

$$\eta_{\mathrm{disc}}^{k,i} + \eta_{\mathrm{lin}}^{k,i} + \eta_{\mathrm{alg}}^{k,i} + \eta_{\mathrm{rem}}^{k,i} \le C(\mathcal{J}_u(u_h^{k,i}) + \eta_{\mathrm{quad}}^{k,i} + \eta_{\mathrm{osc}}),$$

where C is independent of $\overline{\sigma}$ and q.

- **local** (elementwise) stopping criteria ⇒ **local** efficiency
- **robustness** with respect to the **nonlinearity** thanks to the choice of $\mathcal{J}_u$ as error measure

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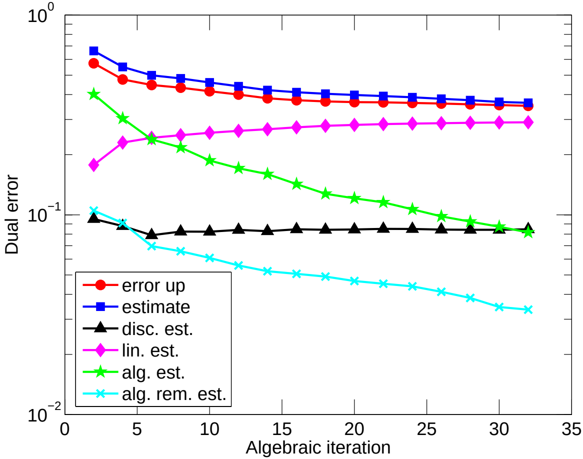
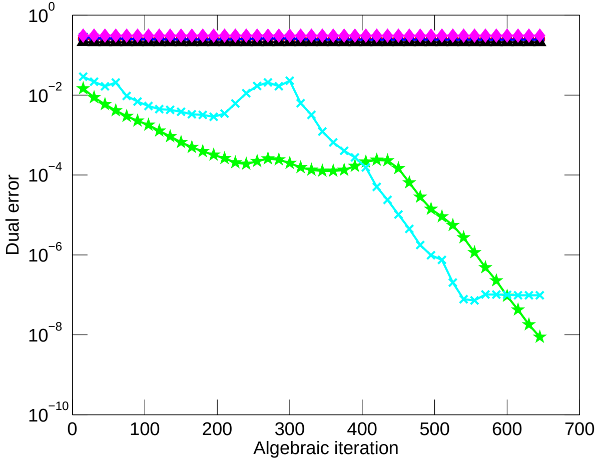
Outline

Error and estimators as a function of CG iterations, regular 10-Laplacian, 6th level mesh, 6th Newton step.



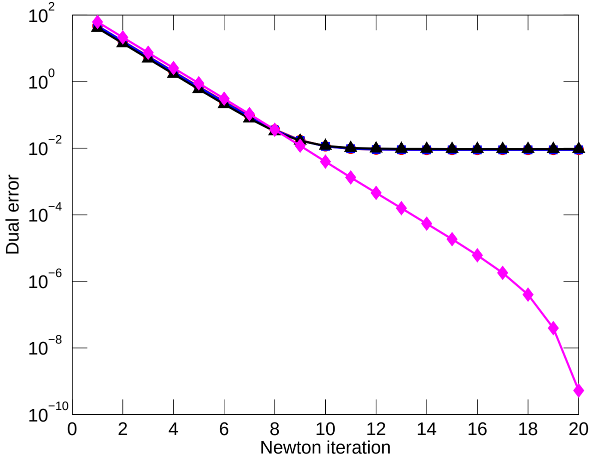
Adaptive stopping criteria

Error and estimators as a function of CG iterations, regular 10-Laplacian, 6th level mesh, 6th Newton step.



"Full" Newton

Error and estimators as a function of Newton iterations, regular 10-Laplacian, 6th level mesh



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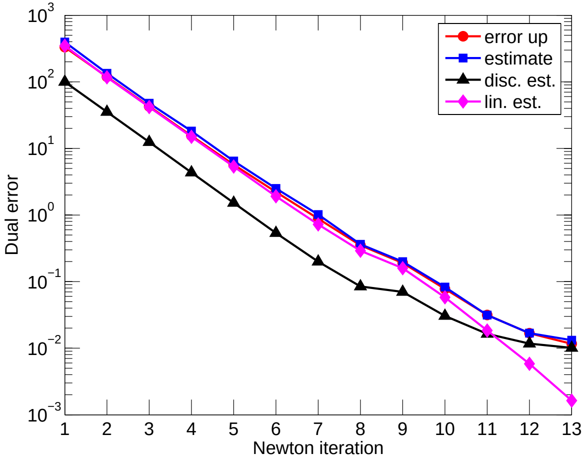
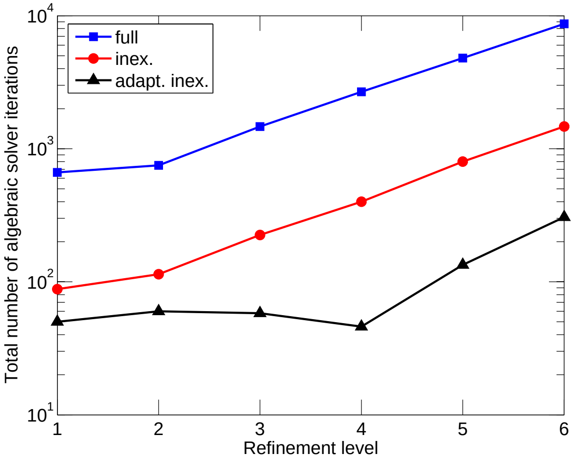
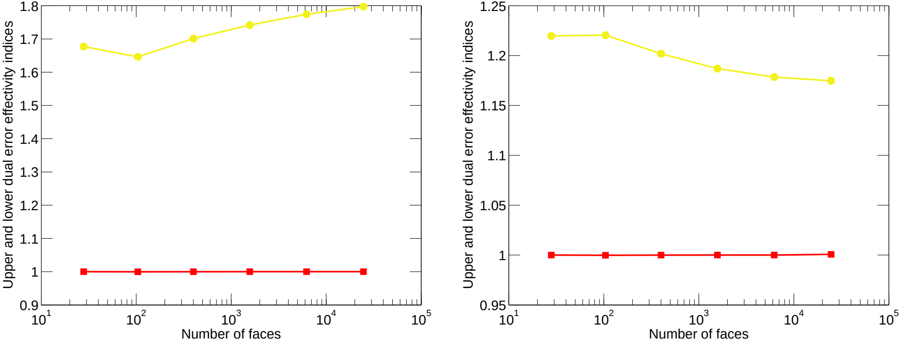


Adaptive stopping criteria

Overall algebraic solver iterations, regular 10-Laplacian

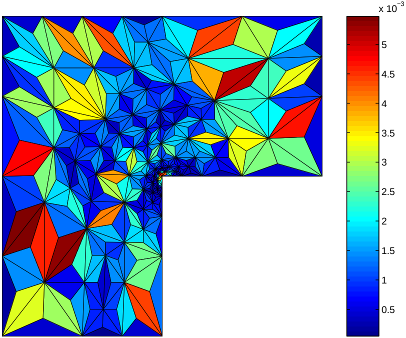


Effectivity indices (**robustness**)

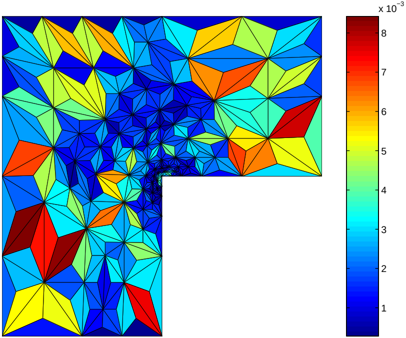


$p = 10$ $\qquad$ $p = 1.5$

Full adaptivity, singular 4-Laplacian



Estimated error distribution



Exact error distribution

Outline

Conclusions and future directions

Conclusions

- guaranteed energy error and first eigenvalue bounds
- robustness (polynomial degree, nonlinearity)
- full adaptivity (linear solver, nonlinear solver, mesh)
- unified framework for all classical numerical schemes

Ongoing work

- multigrid as a linear solver
- convergence and optimality
- higher-order time discretizations

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Bibliography

Bibliography

- ERN A., VOHRALÍK M., Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, *SIAM J. Numer. Anal.*, **53** (2015), 1058–1081.
- DOLEJŠÍ V., ERN A., VOHRALÍK M., *hp*-adaptation driven by polynomial-degree-robust a posteriori error estimates for elliptic problems, HAL Preprint 01165187, submitted for publication.
- CANCÈS E., DUSSON G., MADAY Y., STAMM B., VOHRALÍK M., Guaranteed and robust a posteriori bounds for Laplace eigenvalues and eigenvectors: conforming approximations, HAL Preprint 01194364.
- BLECHTA J., MÁLEK J., VOHRALÍK M., Localization of $W^{-1,q}$ norms for robust local a posteriori efficiency, to be submitted.
- ERN A., VOHRALÍK M., Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs, *SIAM J. Sci. Comput.* **35** (2013), A1761–A1791.

Merci de votre attention !