# A space-time multiscale mortar mixed finite element method for parabolic equations

Manu Jayadharan, Michel Kern, **Martin Vohralík**, and Ivan Yotov

*Inria Paris & Ecole des Ponts*

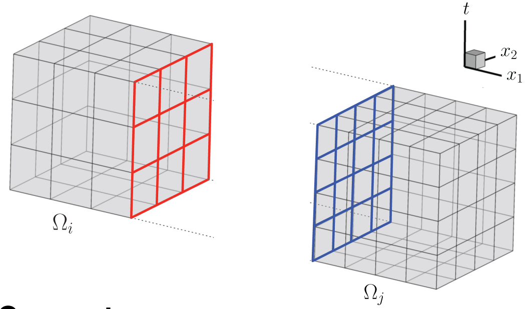
Oberwolfach, February 7, 2022

European Research Council

École des Ponts ParisTech

# Outline

# A space-time multiscale mortar mixed finite element method




## Concepts

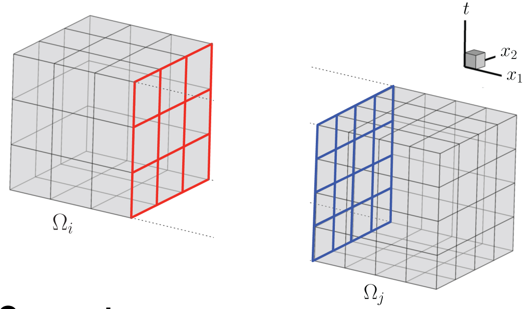
- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega_i}$

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$

- $H$, $\Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars, space-time DD**

# A space-time multiscale mortar mixed finite element method




## Concepts

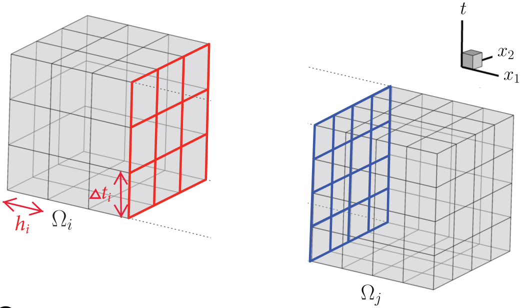
- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega}_i$

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$

- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars, space-time DD**

# A space-time multiscale mortar mixed finite element method




## Concepts

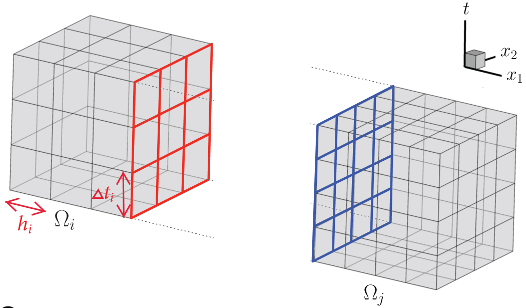
- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega}_i$

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$

- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, **space-time DD**

# A space-time multiscale mortar mixed finite element method



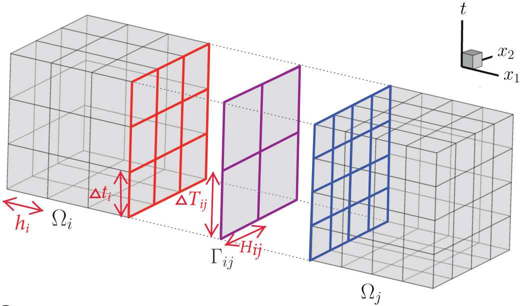

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m, s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

## Concepts

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega}_i$

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, **space-time DD**

# A space-time multiscale mortar mixed finite element method



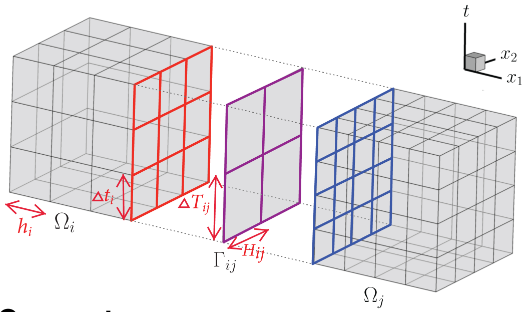

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

## Concepts

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup\overline{\Omega}_i$

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, **space-time DD**

# A space-time multiscale mortar mixed finite element method



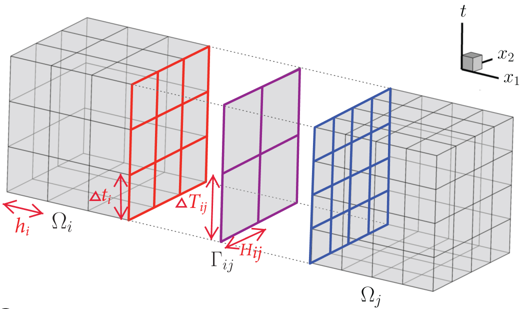

## Concepts

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega}_i$

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, **space-time DD**

# A space-time multiscale mortar mixed finite element method



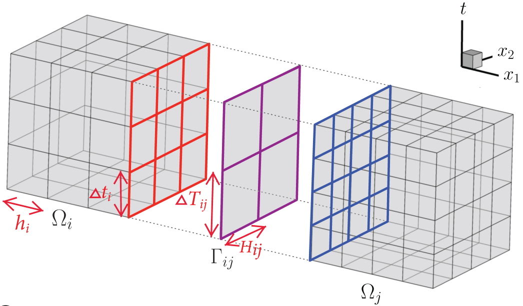

**Concepts**

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega}_i$

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, **space-time DD**

# A space-time multiscale mortar mixed finite element method



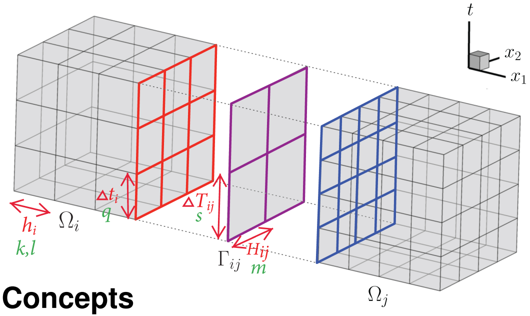

## Concepts

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega_i}$

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, **space-time DD**

# A space-time multiscale mortar mixed finite element method



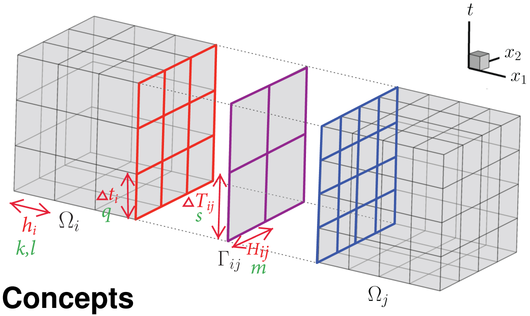

## Concepts

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega_i}$

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, **space-time DD**

# A space-time multiscale mortar mixed finite element method

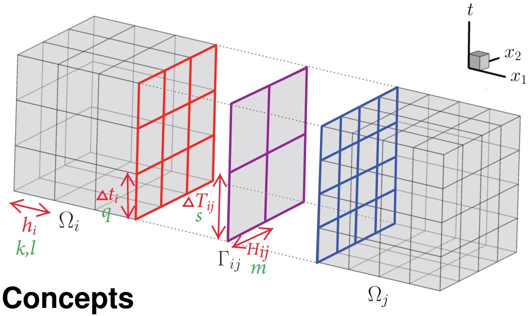



## Concepts

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega}_i$
- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping),
coupling through **multiscale space-time mortars**, **space-time DD**

# A space-time multiscale mortar mixed finite element method

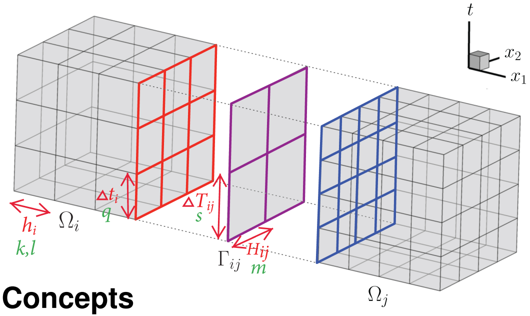



## Concepts

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega}_i$
- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, space-time DD

# A space-time multiscale mortar mixed finite element method




## Concepts

- computational domain $\Omega$ (polytope)
- partition of $\Omega$ into non-overlapping polytopal **subdomains** $\overline{\Omega} = \cup \overline{\Omega_i}$

- $\mathcal{T}_{h,i}$: **individual** mesh of $\Omega_i$ (parallelepipeds or simplices)
- $\mathcal{T}_i^{\Delta t}$: **individual** mesh of $(0, T)$ on $\Omega_i$
- interfaces $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$
- $\mathcal{T}_{H,ij}$ partition of $\Gamma_{ij}$ into $(d-1)$-parallelepipeds or simplices
- $\mathcal{T}_{ij}^{\Delta T}$ partition of $(0, T)$ on $\Gamma_{ij}$
- $H, \Delta T$: **coarser interface grids** wrt the subdomain grids, $m$, $s$ higher polynomial degrees $\Rightarrow$ multiscale approximation

**different space** and **time discretizations** on subdomains (local time stepping), coupling through **multiscale space-time mortars**, **space-time DD**

# Context – steady case

**Hybridized formulation of mixed finite element methods**

- Fraeijs de Veubeke (1960's)
- Arnold and Brezzi (1985)
- Arbogast and Chen (1995)
  - interface mesh given by the neighboring subdomains
  - interface: same mesh and same polynomial degree as in the subdomains
  - hybridized and initial problems equivalent

**Multiscale mortar mixed finite element method**

- Arbogast, Pencheva, Wheeler, and Yotov (2007)
  - independent interface mesh
  - typically coarser but one employs polynomials of higher degree
  - (multiscale) weak continuity of the normal flux component over the interfaces between subdomains
  - efficient parallelization via a non-overlapping domain decomposition algorithm reducing to an interface problem
- ...related to numerous other multiscale approaches

# Context – steady case

**Hybridized formulation of mixed finite element methods**

- Fraeijs de Veubeke (1960's)
- Arnold and Brezzi (1985)
- Arbogast and Chen (1995)
  - interface mesh given by the neighboring subdomains
  - interface: same mesh and same polynomial degree as in the subdomains
  - hybridized and initial problems equivalent

**Multiscale mortar mixed finite element method**

- Arbogast, Pencheva, Wheeler, and Yotov (2007)
  - independent interface mesh
  - typically coarser but one employs polynomials of higher degree
  - (multiscale) weak continuity of the normal flux component over the interfaces between subdomains
  - efficient parallelization via a non-overlapping domain decomposition algorithm reducing to an interface problem
- . . . related to numerous other multiscale approaches

# Context – steady case

## Hybridized formulation of mixed finite element methods

- Fraeijs de Veubeke (1960's)
- Arnold and Brezzi (1985)
- Arbogast and Chen (1995)
  - interface mesh given by the neighboring subdomains
  - interface: same mesh and same polynomial degree as in the subdomains
  - hybridized and initial problems equivalent

## Multiscale mortar mixed finite element method

- Arbogast, Pencheva, Wheeler, and Yotov (2007)
  - independent interface mesh
  - typically coarser but one employs polynomials of higher degree
  - (multiscale) weak continuity of the normal flux component over the interfaces between subdomains
  - efficient parallelization via a non-overlapping domain decomposition algorithm reducing to an interface problem
- ...related to numerous other multiscale approaches

# Context – unsteady case

**Local time stepping for parabolic problems**

- Ewing, Lazarov, and Vassilevski (1990), Delpopolo Carciopolo, Cusini, Formaggia, and Hajibeygi (2020), . . .

**Domain decomposition methods with local time stepping**

- Dawson, Du, and Dupont (1991), Yu (2001), Gaiffe, Glowinski, and Masson (2002), Faucher and Combescure (2003), Gander and Halpern (2007), Nakshatrala, Nakshatrala, and Tortorelli (2009), Hager, Hauret, Le Tallec, and Wohlmuth (2012), Kheriji, Masson, and A. Moncorgé (2015), Krause and Krause (2016), Arshad, Park, and Shin (2021), . . .

**Space-time domain decomposition**

- Halpern, Japhet, and Szeftel (2012), Hoang, Jaffré, Japhet, Kern, and Roberts (2013), Gander, Kwok, and Mandal (2016), . . .

**Parareal algorithm & multigrid in time**

- Lions, Maday, and Turinici (2001), Gander and Vandewalle (2007), Falgout, Friedhoff, Kolev, MacLachlan, and Schroder (2014), Gander and Neumüller (2016), . . .

# Context – unsteady case

**Local time stepping for parabolic problems**

- Ewing, Lazarov, and Vassilevski (1990), Delpopolo Carciopolo, Cusini, Formaggia, and Hajibeygi (2020), . . .

**Domain decomposition methods with local time stepping**

- Dawson, Du, and Dupont (1991), Yu (2001), Gaiffe, Glowinski, and Masson (2002), Faucher and Combescure (2003), Gander and Halpern (2007), Nakshatrala, Nakshatrala, and Tortorelli (2009), Hager, Hauret, Le Tallec, and Wohlmuth (2012), Kheriji, Masson, and A. Moncorgé (2015), Krause and Krause (2016), Arshad, Park, and Shin (2021), . . .

**Space-time domain decomposition**

- Halpern, Japhet, and Szeftel (2012), Hoang, Jaffré, Japhet, Kern, and Roberts (2013), Gander, Kwok, and Mandal (2016), . . .

**Parareal algorithm & multigrid in time**

- Lions, Maday, and Turinici (2001), Gander and Vandewalle (2007), Falgout, Friedhoff, Kolev, MacLachlan, and Schroder (2014), Gander and Neumüller (2016), . . .

# Context – unsteady case

**Local time stepping for parabolic problems**

- Ewing, Lazarov, and Vassilevski (1990), Delpopolo Carciopolo, Cusini, Formaggia, and Hajibeygi (2020), ...

**Domain decomposition methods with local time stepping**

- Dawson, Du, and Dupont (1991), Yu (2001), Gaiffe, Glowinski, and Masson (2002), Faucher and Combescure (2003), Gander and Halpern (2007), Nakshatrala, Nakshatrala, and Tortorelli (2009), Hager, Hauret, Le Tallec, and Wohlmuth (2012), Kheriji, Masson, and A. Moncorgé (2015), Krause and Krause (2016), Arshad, Park, and Shin (2021), ...

**Space-time domain decomposition**

- Halpern, Japhet, and Szeftel (2012), Hoang, Jaffré, Japhet, Kern, and Roberts (2013), Gander, Kwok, and Mandal (2016), ...

**Parareal algorithm & multigrid in time**

- Lions, Maday, and Turinici (2001), Gander and Vandewalle (2007), Falgout, Friedhoff, Kolev, MacLachlan, and Schroder (2014), Gander and Neumüller (2016), ...

# Context – unsteady case

**Local time stepping for parabolic problems**

- Ewing, Lazarov, and Vassilevski (1990), Delpopolo Carciopolo, Cusini, Formaggia, and Hajibeygi (2020), . . .

**Domain decomposition methods with local time stepping**

- Dawson, Du, and Dupont (1991), Yu (2001), Gaiffe, Glowinski, and Masson (2002), Faucher and Combescure (2003), Gander and Halpern (2007), Nakshatrala, Nakshatrala, and Tortorelli (2009), Hager, Hauret, Le Tallec, and Wohlmuth (2012), Kheriji, Masson, and A. Moncorgé (2015), Krause and Krause (2016), Arshad, Park, and Shin (2021), . . .

**Space-time domain decomposition**

- Halpern, Japhet, and Szeftel (2012), Hoang, Jaffré, Japhet, Kern, and Roberts (2013), Gander, Kwok, and Mandal (2016), . . .

**Parareal algorithm & multigrid in time**

- Lions, Maday, and Turinici (2001), Gander and Vandewalle (2007), Falgout, Friedhoff, Kolev, MacLachlan, and Schroder (2014), Gander and Neumüller (2016), . . .

# Outline

# Outline

# Outline

# Setting

## The heat equation

Find $p : \Omega \times [0, T] \to \mathbb{R}$ and $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$
\begin{aligned}
\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} &= q && \text{in } \Omega \times (0, T], \\
\mathbf{u} &= -K \nabla p && \text{in } \Omega \times (0, T], \\
p &= 0 && \text{on } \partial\Omega \times (0, T], \\
p &= p_0(x) && \text{on } \Omega.
\end{aligned}
$$

- $q \in L^2(0, T; L^2(\Omega))$, $p_0 \in H_0^1(\Omega)$, $\nabla \cdot K \nabla p_0 \in L^2(\Omega)$
- $K$: time-independent, uniformly bounded, symmetric, and positive definite

## Weak solution

Find $\mathbf{u} \in L^2(0, T; \mathbf{H}(\mathrm{div}; \Omega))$, $p \in H^1(0, T; L^2(\Omega))$ s.t. $p(\cdot, 0) = p_0$ & a.e. in $(0, T)$,

$$
\begin{aligned}
(K^{-1}\mathbf{u}, \mathbf{v})_\Omega - (p, \nabla \cdot \mathbf{v})_\Omega &= 0 \quad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div}; \Omega), \\
(\partial_t p, w)_\Omega + (\nabla \cdot \mathbf{u}, w)_\Omega &= (q, w)_\Omega \quad \forall w \in L^2(\Omega).
\end{aligned}
$$

- actually also $\mathbf{u} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$ and $p \in H^1(0, T; H_0^1(\Omega))$

# Setting

**The heat equation**

Find $p : \Omega \times [0, T] \to \mathbb{R}$ and $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$\begin{aligned} \frac{\partial p}{\partial t} + \nabla \cdot \mathbf{u} &= q && \text{in } \Omega \times (0, T], \\ \mathbf{u} &= -K \nabla p && \text{in } \Omega \times (0, T], \\ p &= 0 && \text{on } \partial\Omega \times (0, T], \\ p &= p_0(x) && \text{on } \Omega. \end{aligned}$$

- $q \in L^2(0, T; L^2(\Omega))$, $p_0 \in H_0^1(\Omega)$, $\nabla \cdot K \nabla p_0 \in L^2(\Omega)$
- $K$: time-independent, uniformly bounded, symmetric, and positive definite

**Weak solution**

Find $\mathbf{u} \in L^2(0, T; \mathbf{H}(\mathrm{div}; \Omega))$, $p \in H^1(0, T; L^2(\Omega))$ s.t. $p(\cdot, 0) = p_0$ & a.e. in $(0, T)$,

$$\begin{aligned} (K^{-1}\mathbf{u}, \mathbf{v})_\Omega - (p, \nabla \cdot \mathbf{v})_\Omega &= 0 \quad \forall \mathbf{v} \in \mathbf{H}(\mathrm{div}; \Omega), \\ (\partial_t p, w)_\Omega + (\nabla \cdot \mathbf{u}, w)_\Omega &= (q, w)_\Omega \quad \forall w \in L^2(\Omega). \end{aligned}$$

- actually also $\mathbf{u} \in L^\infty(0, T; \mathbf{L}^2(\Omega))$ and $p \in H^1(0, T; H_0^1(\Omega))$

# Outline

# Space-time subdomains, forms, and spaces

## Space-time subdomains

$$\Omega^T := \Omega \times (0,T), \Omega_i^T := \Omega_i \times (0,T), \Gamma_i^T := \Gamma_i \times (0,T), \Gamma_{ij}^T := \Gamma_{ij} \times (0,T)$$

## Space-time bilinear forms

$$a^T(\mathbf{u},\mathbf{v}) := \int_0^T \sum_i (K^{-1}\mathbf{u},\mathbf{v})_{\Omega_i}, \quad b^T(\mathbf{v},w) := -\int_0^T \sum_i (\nabla\cdot\mathbf{v},w)_{\Omega_i},$$

$$b_\Gamma^T(\mathbf{v},\mu) := \int_0^T \sum_i \langle \mathbf{v}\cdot\mathbf{n}_i, \mu\rangle_{\Gamma_i}$$

## Tensor product space-time spaces on each space-time subdomain $\Omega_i^T$

$$\mathbf{V}_{h,i}^{\Delta t} := \underbrace{\mathbf{V}_{h,i}}_{\text{MFE spaces}} \otimes \underbrace{W_i^{\Delta t}}_{\text{discontinuous pw polynomials in time}}, \quad W_{h,i}^{\Delta t} := \underbrace{W_{h,i}}_{\text{discontinuous pw polynomials in space}} \otimes W_i^{\Delta t}$$

$$\Lambda_{H,ij}^{\Delta T} := \underbrace{\Lambda_{H,ij} \otimes \Lambda_{ij}^{\Delta T}}_{\text{continuous or discontinuous pw polynomials in space and in time}}$$

# Space-time subdomains, forms, and spaces

## Space-time subdomains

$$\Omega^T := \Omega \times (0,T), \Omega_i^T := \Omega_i \times (0,T), \Gamma_i^T := \Gamma_i \times (0,T), \Gamma_{ij}^T := \Gamma_{ij} \times (0,T)$$

## Space-time bilinear forms

$$a^T(\mathbf{u},\mathbf{v}) := \int_0^T \sum_i (K^{-1}\mathbf{u},\mathbf{v})_{\Omega_i}, \quad b^T(\mathbf{v},w) := -\int_0^T \sum_i (\nabla\cdot\mathbf{v}, w)_{\Omega_i},$$

$$b_\Gamma^T(\mathbf{v},\mu) := \int_0^T \sum_i \langle \mathbf{v}\cdot\mathbf{n}_i, \mu\rangle_{\Gamma_i}$$

## Tensor product space-time spaces on each space-time subdomain $\Omega_i^T$

$$\mathbf{V}_{h,i}^{\Delta t} := \underbrace{\mathbf{V}_{h,i}}_{\text{MFE spaces}} \otimes \underbrace{W_i^{\Delta t}}_{\text{discontinuous pw polynomials in time}}, \quad W_{h,i}^{\Delta t} := \underbrace{W_{h,i}}_{\text{discontinuous pw polynomials in space}} \otimes W_i^{\Delta t}$$

$$\Lambda_{H,ij}^{\Delta T} := \underbrace{\Lambda_{H,ij} \otimes \Lambda_{ij}^{\Delta T}}_{\text{continuous or discontinuous pw polynomials in space and in time}}$$

# Space-time subdomains, forms, and spaces

## Space-time subdomains

$$\Omega^T := \Omega \times (0,T), \Omega_i^T := \Omega_i \times (0,T), \Gamma_i^T := \Gamma_i \times (0,T), \Gamma_{ij}^T := \Gamma_{ij} \times (0,T)$$

## Space-time bilinear forms

$$a^T(\mathbf{u},\mathbf{v}) := \int_0^T \sum_i (K^{-1}\mathbf{u},\mathbf{v})_{\Omega_i}, \quad b^T(\mathbf{v},w) := -\int_0^T \sum_i (\nabla\cdot\mathbf{v},w)_{\Omega_i},$$

$$b_\Gamma^T(\mathbf{v},\mu) := \int_0^T \sum_i \langle \mathbf{v}\cdot\mathbf{n}_i,\mu\rangle_{\Gamma_i}$$

## Tensor product space-time spaces on each space-time subdomain $\Omega_i^T$

$$\mathbf{V}_{h,i}^{\Delta t} := \underbrace{\mathbf{V}_{h,i}}_{\text{MFE spaces}} \otimes \underbrace{W_i^{\Delta t}}_{\text{discontinuous pw polynomials in time}}, \quad W_{h,i}^{\Delta t} := \underbrace{W_{h,i}}_{\text{discontinuous pw polynomials in space}} \otimes W_i^{\Delta t}$$

$$\Lambda_{H,ij}^{\Delta T} := \underbrace{\Lambda_{H,ij} \otimes \Lambda_{ij}^{\Delta T}}_{\text{continuous or discontinuous pw polynomials in space and in time}}$$

# Global spaces and time stepping

## Global space-time finite element spaces

$$\mathbf{V}_h^{\Delta t} := \bigoplus \mathbf{V}_{h,i}^{\Delta t}, \quad W_h^{\Delta t} := \bigoplus W_{h,i}^{\Delta t}, \quad \Lambda_H^{\Delta T} := \bigoplus \Lambda_{H,ij}^{\Delta T}$$

## Space of velocities with space-time weakly continuous normal components

$$\mathbf{V}_{h,0}^{\Delta t} = \left\{ \mathbf{v} \in \mathbf{V}_h^{\Delta t} : b_\Gamma^T(\mathbf{v}, \mu) = 0 \quad \forall \mu \in \Lambda_H^{\Delta T} \right\}$$

## Discontinuous Galerkin time stepping

$$\int_0^T \tilde{\partial}_t \varphi \, \phi = \sum_{k=1}^{N_i} \int_{t_i^{k-1}}^{t_i^k} \partial_t \varphi \, \phi + \sum_{k=1}^{N_i} [\varphi]_{k-1} \, \phi_{k-1}^+,$$

with

$$[\varphi]_k = \varphi_k^+ - \varphi_k^-, \quad \varphi_k^+ = \lim_{t \to t_i^{k,+}} \varphi, \quad \varphi_k^- = \lim_{t \to t_i^{k,-}} \varphi$$

# Global spaces and time stepping

## Global space-time finite element spaces

$$\mathbf{V}_h^{\Delta t} := \bigoplus \mathbf{V}_{h,i}^{\Delta t}, \quad W_h^{\Delta t} := \bigoplus W_{h,i}^{\Delta t}, \quad \Lambda_H^{\Delta T} := \bigoplus \Lambda_{H,ij}^{\Delta T}$$

## Space of velocities with space-time weakly continuous normal components

$$\mathbf{V}_{h,0}^{\Delta t} = \left\{ \mathbf{v} \in \mathbf{V}_h^{\Delta t} : b_\Gamma^T(\mathbf{v}, \mu) = 0 \quad \forall \mu \in \Lambda_H^{\Delta T} \right\}$$

## Discontinuous Galerkin time stepping

$$\int_0^T \tilde{\partial}_t \varphi \, \phi = \sum_{k=1}^{N_i} \int_{t_i^{k-1}}^{t_i^k} \partial_t \varphi \, \phi + \sum_{k=1}^{N_i} [\varphi]_{k-1} \, \phi_{k-1}^+,$$

with

$$[\varphi]_k = \varphi_k^+ - \varphi_k^-, \quad \varphi_k^+ = \lim_{t \to t_i^{k,+}} \varphi, \quad \varphi_k^- = \lim_{t \to t_i^{k,-}} \varphi$$

# Global spaces and time stepping

## Global space-time finite element spaces

$$\mathbf{V}_h^{\Delta t} := \bigoplus \mathbf{V}_{h,i}^{\Delta t}, \quad W_h^{\Delta t} := \bigoplus W_{h,i}^{\Delta t}, \quad \Lambda_H^{\Delta T} := \bigoplus \Lambda_{H,ij}^{\Delta T}$$

## Space of velocities with space-time weakly continuous normal components

$$\mathbf{V}_{h,0}^{\Delta t} = \left\{ \mathbf{v} \in \mathbf{V}_h^{\Delta t} : b_\Gamma^T(\mathbf{v}, \mu) = 0 \quad \forall \mu \in \Lambda_H^{\Delta T} \right\}$$

## Discontinuous Galerkin time stepping

$$\int_0^T \tilde{\partial}_t \varphi \, \phi = \sum_{k=1}^{N_i} \int_{t_i^{k-1}}^{t_i^k} \partial_t \varphi \, \phi + \sum_{k=1}^{N_i} [\varphi]_{k-1} \, \phi_{k-1}^+,$$

with

$$[\varphi]_k = \varphi_k^+ - \varphi_k^-, \quad \varphi_k^+ = \lim_{t \to t_i^{k,+}} \varphi, \quad \varphi_k^- = \lim_{t \to t_i^{k,-}} \varphi$$

# Outline

# Space-time multiscale mortar mixed finite element method

## Definition (Space-time multiscale mortar mixed finite element method)

Find $\mathbf{u}_h^{\Delta t} \in \mathbf{V}_h^{\Delta t}$, $p_h^{\Delta t} \in W_h^{\Delta t}$, and $\lambda_H^{\Delta T} \in \Lambda_H^{\Delta T}$ such that

$$
\begin{aligned}
a^T(\mathbf{u}_h^{\Delta t}, \mathbf{v}) + b^T(\mathbf{v}, p_h^{\Delta t}) + b_\Gamma^T(\mathbf{v}, \lambda_H^{\Delta T}) &= 0 && \forall\, \mathbf{v} \in \mathbf{V}_h^{\Delta t}, \\
(\tilde{\partial}_t p_h^{\Delta t}, w)_{\Omega^T} - b^T(\mathbf{u}_h^{\Delta t}, w) &= (q, w)_{\Omega^T} && \forall\, w \in W_h^{\Delta t}, \\
b_\Gamma^T(\mathbf{u}_h^{\Delta t}, \mu) &= 0 && \forall\, \mu \in \Lambda_H^{\Delta T}.
\end{aligned}
$$

# Outline

# Assumptions

## Assumption (Mortar grids)

*For C independent of the spatial mesh sizes h and H as well as of the temporal mesh sizes $\Delta t$ and $\Delta T$, there holds*

$$\forall\, \mu \in \Lambda_H,\ \forall\, i,j, \quad \|\mu\|_{\Gamma_{ij}} \leq C(\|\mathcal{Q}_{h,i}\,\mu\|_{\Gamma_{ij}} + \|\mathcal{Q}_{h,j}\,\mu\|_{\Gamma_{ij}}),$$

$$\forall\, i,j, \quad \Lambda_{ij}^{\Delta T} \subset W_i^{\Delta t} \cap W_j^{\Delta t}.$$

### Comments

- $\Lambda_H := \bigoplus \Lambda_{H,ij}$
- $\mathcal{Q}_{h,i} : L^2(\partial\Omega_i) \to \mathbf{V}_{h,i} \cdot \mathbf{n}_i$ is the $L^2$-orthogonal projection
- the spatial mortar assumption as previously: in particular satisfied with $C = \frac{1}{2}$ when $\mathcal{T}_{H,ij}$ is a coarsening of both $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ on $\Gamma_{ij}$ & the space $\Lambda_{H,ij}$ consists of discontinuous pw polynomials contained in $\mathbf{V}_{h,i} \cdot \mathbf{n}_i$ and $\mathbf{V}_{h,j} \cdot \mathbf{n}_j$ on $\Gamma_{ij}$; in general, it requests the mortar space $\Lambda_H$ to be sufficiently coarse
- the temporal mortar assumption: control of the mortar by the subdomain time discretizations; mortar time discretization is a coarsening of each subdomain

# Assumptions

## Assumption (Mortar grids)

*For $C$ independent of the spatial mesh sizes $h$ and $H$ as well as of the temporal mesh sizes $\Delta t$ and $\Delta T$, there holds*

$$\forall\, \mu \in \Lambda_H,\ \forall\, i,j, \quad \|\mu\|_{\Gamma_{ij}} \le C(\|\mathcal{Q}_{h,i}\,\mu\|_{\Gamma_{ij}} + \|\mathcal{Q}_{h,j}\,\mu\|_{\Gamma_{ij}}),$$

$$\forall\, i,j, \quad \Lambda_{ij}^{\Delta T} \subset W_i^{\Delta t} \cap W_j^{\Delta t}.$$

**Comments**

- $\Lambda_H := \bigoplus \Lambda_{H,ij}$
- $\mathcal{Q}_{h,i} : L^2(\partial\Omega_i) \to \mathbf{V}_{h,i} \cdot \mathbf{n}_i$ is the $L^2$-orthogonal projection
- the spatial mortar assumption as previously: in particular satisfied with $C = \frac{1}{2}$ when $\mathcal{T}_{H,ij}$ is a coarsening of both $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ on $\Gamma_{ij}$ & the space $\Lambda_{H,ij}$ consists of discontinuous pw polynomials contained in $\mathbf{V}_{h,i} \cdot \mathbf{n}_i$ and $\mathbf{V}_{h,j} \cdot \mathbf{n}_j$ on $\Gamma_{ij}$; in general, it requests the mortar space $\Lambda_H$ to be sufficiently coarse
- the temporal mortar assumption: control of the mortar by the subdomain time discretizations; mortar time discretization is a coarsening of each subdomain

# Two inf–sup inequalities

**Lemma (Discrete divergence inf–sup condition on $\mathbf{V}_{h,0}^{\Delta t}$)**

*Let the mortar assumptions hold. Then*

$$\forall\, w \in W_h^{\Delta t}, \qquad \sup_{0 \neq \mathbf{v} \in \mathbf{V}_{h,0}^{\Delta t}} \frac{b^T(\mathbf{v}, w)}{\|\mathbf{v}\|_{L^2(0,T;\Pi_i \mathbf{H}(\mathrm{div};\Omega_i))}} \geq \beta \|w\|_{L^2(0,T;L^2(\Omega))}.$$

**Lemma (Discrete mortar inf–sup condition on $\mathbf{V}_h^{\Delta t}$)**

*Let the mortar assumptions hold. Then*

$$\forall\, \mu \in \Lambda_H^{\Delta T}, \qquad \sup_{0 \neq \mathbf{v} \in \mathbf{V}_h^{\Delta t}} \frac{b_\Gamma^T(\mathbf{v}, \mu)}{\|\mathbf{v}\|_{L^2(0,T;\Pi_i \mathbf{H}(\mathrm{div};\Omega_i))}} \geq \beta_\Gamma \|\mu\|_{L^2(0,T;L^2(\Gamma))}.$$

# Two inf–sup inequalities

**Lemma (Discrete divergence inf–sup condition on $\mathbf{V}_{h,0}^{\Delta t}$)**

*Let the mortar assumptions hold. Then*

$$\forall\, w \in W_h^{\Delta t}, \qquad \sup_{0 \neq \mathbf{v} \in \mathbf{V}_{h,0}^{\Delta t}} \frac{b^T(\mathbf{v}, w)}{\|\mathbf{v}\|_{L^2(0,T;\Pi_i \mathbf{H}(\mathrm{div};\Omega_i))}} \geq \beta \|w\|_{L^2(0,T;L^2(\Omega))}.$$

**Lemma (Discrete mortar inf–sup condition on $\mathbf{V}_h^{\Delta t}$)**

*Let the mortar assumptions hold. Then*

$$\forall\, \mu \in \Lambda_H^{\Delta T}, \qquad \sup_{0 \neq \mathbf{v} \in \mathbf{V}_h^{\Delta t}} \frac{b_\Gamma^T(\mathbf{v}, \mu)}{\|\mathbf{v}\|_{L^2(0,T;\Pi_i \mathbf{H}(\mathrm{div};\Omega_i))}} \geq \beta_\Gamma \|\mu\|_{L^2(0,T;L^2(\Gamma))}.$$

# Existence, uniqueness, and stability

**Theorem (Existence and uniqueness of the discrete solution, stability wrt data)**

*Let the mortar assumptions hold. Then the space-time multiscale mortar MFE method has a unique solution. Moreover,*

$$\|p_h^{\Delta t}\|_{\mathrm{DG}} + \|\mathbf{u}_h^{\Delta t}\|_{\Omega^T} + \|p_h^{\Delta t}\|_{\Omega^T} + \|\lambda_H^{\Delta T}\|_{\Gamma^T} \le C(\|q\|_{\Omega^T} + \|\nabla \cdot K\nabla p_0\|_{\Omega}).$$

## Comments

- $\|\varphi\|_{\mathrm{DG}}^2 = \sum_i \left(\|\varphi_{N_i}^-\|_{\Omega_i}^2 + \sum_{k=1}^{N_i} \|[\varphi]_{k-1}\|_{\Omega_i}^2\right)$
- no control of divergence (shown later in a simplified setting)

# Existence, uniqueness, and stability

**Theorem (Existence and uniqueness of the discrete solution, stability wrt data)**

*Let the mortar assumptions hold. Then the space-time multiscale mortar MFE method has a unique solution. Moreover,*

$$\|p_h^{\Delta t}\|_{\mathrm{DG}} + \|\mathbf{u}_h^{\Delta t}\|_{\Omega^T} + \|p_h^{\Delta t}\|_{\Omega^T} + \|\lambda_H^{\Delta T}\|_{\Gamma^T} \leq C(\|q\|_{\Omega^T} + \|\nabla \cdot K \nabla p_0\|_{\Omega}).$$

**Comments**

- $\|\varphi\|_{\mathrm{DG}}^2 = \sum_i \left(\|\varphi_{N_i}^-\|_{\Omega_i}^2 + \sum_{k=1}^{N_i} \|[\varphi]_{k-1}\|_{\Omega_i}^2\right)$
- no control of divergence (shown later in a simplified setting)

# A priori error estimate

**Theorem (A priori error estimate)**

*Let the mortar assumptions hold and let the weak solution be sufficiently smooth. Let the space and time meshes $\mathcal{T}_{h,i}$ and $\mathcal{T}_i^{\Delta t}$ be quasi-uniform and let $h \le Ch_i$ and $\Delta t \le C\Delta t_i$ for all $i$. Then*

$$\|p - p_h^{\Delta t}\|_{\mathrm{DG}} + \|\mathbf{u} - \mathbf{u}_h^{\Delta t}\|_{\Omega^T} + \|p - p_h^{\Delta t}\|_{\Omega^T} + \|\lambda - \lambda_H^{\Delta T}\|_{\Gamma^T}$$
$$\le C\Big(\sum_i \|\mathbf{u}\|_{H^{r_q}(0,T;\mathbf{H}^{r_k}(\Omega_i))}(h^{r_k} + \Delta t^{r_q}) + \|\mathbf{u}\|_{H^{r_q}(0,T;\mathbf{H}^{\tilde r_k + \frac{1}{2}}(\Omega))}(h^{\tilde r_k} H^{\frac{1}{2}} + \Delta t^{r_q})$$
$$+ \sum_i \|p\|_{W^{r_q,\infty}(0,T;H^{r_l}(\Omega_i))} \Delta t^{-\frac{1}{2}}(h^{r_l} + \Delta t^{r_q}) + \sum_{i,j} \|\lambda\|_{H^{r_s}(0,T;H^{r_m}(\Gamma_{ij}))} h^{-\frac{1}{2}}(H^{r_m} + \Delta T^{r_s})\Big),$$

$$\underbrace{0 < r_k \text{ or } \tilde r_k \le k+1}_{\text{MFE space}},\ \underbrace{0 \le r_l \le l+1}_{\text{pw pols space}},\ \underbrace{0 \le r_q \le q+1}_{\text{pw pols time}},\ \underbrace{0 \le r_m \le m+1}_{\text{mortars space}},\ \underbrace{0 \le r_s \le s+1}_{\text{mortars time}}.$$

# A priori error estimate

**Comments**

- the term $h^{-\frac{1}{2}}(H^{r_m} + \Delta T^{r_s})$ appears from the discrete trace (inverse) inequality and is suboptimal; can be made comparable to the other error terms by choosing $m$ and $s$ sufficiently large (if the solution is sufficiently smooth)
- the term $\Delta t^{-\frac{1}{2}}(h^{r_l} + \Delta t^{r_q})$ is suboptimal
- both improved if a bound on $\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h^{\Delta t})\|_{\Omega_i^T}$ is available, using the normal trace inequality for $\mathbf{H}(\mathrm{div}; \Omega_i)$

# Improved stability

## Radau reconstruction operator

- for $\varphi(x,\cdot) \in W^{\Delta t}$, $\mathcal{I}\varphi(x,\cdot) \in H^1(0,T)$, $\mathcal{I}\varphi(x,\cdot)|_{(t^{k-1},t^k)} \in P_{q+1}$, such that

$$\int_{t^{k-1}}^{t^k} \partial_t \mathcal{I}\varphi \, \phi = \int_{t^{k-1}}^{t^k} \partial_t \varphi \, \phi + [\varphi]_{k-1} \, \phi_{k-1}^{+} \quad \forall \, \phi(x,\cdot) \in W^{\Delta t}$$

- thus equivalently, $\tilde{\partial}_t p_h^{\Delta t}$ replaced by $\partial_t \mathcal{I} p_h^{\Delta t}$:

$$(\partial_t \mathcal{I} p_h^{\Delta t}, w)_{\Omega^T} - b^T(\mathbf{u}_h^{\Delta t}, w) = (q, w)_{\Omega^T} \quad \forall \, w \in W_h^{\Delta t}$$

**Theorem (Control of divergence)**

*Let $W_i^{\Delta t} = \Lambda_{ij}^{\Delta T} = W_j^{\Delta t}$ (same time discretization everywhere) hold. Then*

$$\|\partial_t \mathcal{I} p_h^{\Delta t}\|_{\Omega^T} + \|\nabla_h \cdot \mathbf{u}_h^{\Delta t}\|_{\Omega^T} + \|\mathbf{u}_h^{\Delta t}\|_{\mathrm{DG}} \le C(\|q\|_{\Omega^T} + \|\nabla \cdot K \nabla p_0\|_{\Omega}).$$

# Improved stability

## Radau reconstruction operator

- for $\varphi(x,\cdot) \in W^{\Delta t}$, $\mathcal{I}\varphi(x,\cdot) \in H^1(0,T)$, $\mathcal{I}\varphi(x,\cdot)|_{(t^{k-1},t^k)} \in P_{q+1}$, such that

$$\int_{t^{k-1}}^{t^k} \partial_t \mathcal{I}\varphi\,\phi = \int_{t^{k-1}}^{t^k} \partial_t \varphi\,\phi + [\varphi]_{k-1}\,\phi_{k-1}^{+} \quad \forall\,\phi(x,\cdot) \in W^{\Delta t}$$

- thus equivalently, $\tilde{\partial}_t p_h^{\Delta t}$ replaced by $\partial_t \mathcal{I} p_h^{\Delta t}$:

$$(\partial_t \mathcal{I} p_h^{\Delta t}, w)_{\Omega^T} - b^T(\mathbf{u}_h^{\Delta t}, w) = (q, w)_{\Omega^T} \quad \forall\,w \in W_h^{\Delta t}$$

**Theorem (Control of divergence)**

*Let $W_i^{\Delta t} = \Lambda_{ij}^{\Delta T} = W_j^{\Delta t}$ (same time discretization everywhere) hold. Then*

$$\|\partial_t \mathcal{I} p_h^{\Delta t}\|_{\Omega^T} + \|\nabla_h \cdot \mathbf{u}_h^{\Delta t}\|_{\Omega^T} + \|\mathbf{u}_h^{\Delta t}\|_{\mathrm{DG}} \leq C(\|q\|_{\Omega^T} + \|\nabla \cdot K\nabla p_0\|_{\Omega}).$$

# Improved a priori error estimate

## Theorem (Improved a priori error estimate)

*Let the mortar space assumption and* $W_i^{\Delta t} = \Lambda_{ij}^{\Delta T} = W_j^{\Delta t}$ *hold and let the weak solution be sufficiently smooth. Then*

$$\begin{aligned}
&\|\mathbf{u}(t^N) - (\mathbf{u}_h^{\Delta t})_N^-\|_\Omega + \|\mathbf{u} - \mathbf{u}_h^{\Delta t}\|_{\Omega^T} + \|\nabla_h \cdot (\mathbf{u} - \mathbf{u}_h^{\Delta t})\|_{\Omega^T} \\
&\qquad + \|p(t^N) - (p_h^{\Delta t})_N^-\|_\Omega + \|p - p_h^{\Delta t}\|_{\Omega^T} + \|\lambda - \lambda_H^{\Delta T}\|_{\Gamma^T} \\
&\leq C\Big( \sum_i \|\mathbf{u}\|_{W^{r_q,\infty}(0,T;\mathbf{H}^{r_k}(\Omega_i))}(h^{r_k} + \Delta t^{r_q}) + \|\mathbf{u}\|_{W^{r_q,\infty}(0,T;\mathbf{H}^{\tilde{r}_k+\frac{1}{2}}(\Omega))}(h^{\tilde{r}_k} H^{\frac{1}{2}} + \Delta t^{r_q}) \\
&\qquad + \sum_i \|p\|_{W^{r_q,\infty}(0,T;H^{r_l}(\Omega_i))}(h^{r_l} + \Delta t^{r_q}) + \sum_{i,j} \|\lambda\|_{H^{r_s}(0,T;H^{r_m}(\Gamma_{ij}))}(H^{r_m-\frac{1}{2}} + \Delta T^{r_s}) \\
&+ \sum_i \|\mathbf{u}\|_{H^1(0,T;\mathbf{H}^{r_k}(\Omega_i))} h^{r_k} + \|\mathbf{u}\|_{H^1(0,T;\mathbf{H}^{\tilde{r}_k+\frac{1}{2}}(\Omega))} h^{\tilde{r}_k} H^{\frac{1}{2}} + \sum_{i,j} \|\lambda\|_{H^1(0,T;H^{r_m}(\Gamma_{ij}))} H^{r_m-\frac{1}{2}}\Big),
\end{aligned}$$

$$\underbrace{0 < r_k \text{ or } \tilde{r}_k \leq k+1}_{\text{MFE space}},\ \underbrace{0 \leq r_l \leq l+1}_{\text{pw pols space}},\ \underbrace{0 \leq r_q \leq q+1}_{\text{pw pols time}},\ \underbrace{\tfrac{1}{2} \leq r_m \leq m+1}_{\text{mortars space}},\ \underbrace{1 \leq r_s \leq s+1}_{\text{mortars time}}.$$

# Outline

# Decomposition of the solution

**Decomposition of the solution**

- $\mathbf{u}_h^{\Delta t} = \mathbf{u}_h^{\Delta t,*}(\lambda_H^{\Delta T}) + \overline{\mathbf{u}}_h^{\Delta t}, \quad p_h^{\Delta t} = p_h^{\Delta t,*}(\lambda_H^{\Delta T}) + \overline{p}_h^{\Delta t}$
- for each $\Omega_i^T$, $\overline{\mathbf{u}}_h^{\Delta t}|_{\Omega_i^T} \in \mathbf{V}_{h,i}^{\Delta t}$, $\overline{p}_h^{\Delta t}|_{\Omega_i^T} \in W_{h,i}^{\Delta t}$ is the solution to (zero Dirichlet data on the space-time interfaces and the prescribed source term $q$, initial data $p_0$, and $0$ boundary data on the external boundary)

$$a_i^T(\overline{\mathbf{u}}_h^{\Delta t}, \mathbf{v}) + b_i^T(\mathbf{v}, \overline{p}_h^{\Delta t}) = 0 \quad \forall\, \mathbf{v} \in \mathbf{V}_{h,i}^{\Delta t}$$

$$(\tilde{\partial}_t \overline{p}_h^{\Delta t}, w)_{\Omega_i^T} - b_i^T(\overline{\mathbf{u}}_h^{\Delta t}, w) = (q, w)_{\Omega_i^T} \quad \forall\, w \in W_{h,i}^{\Delta t}$$

- for a given $\mu \in \Lambda_H^{\Delta T}$, for each $\Omega_i^T$, $\mathbf{u}_h^{\Delta t,*}(\mu)|_{\Omega_i^T} \in \mathbf{V}_{h,i}^{\Delta t}$, $p_h^{\Delta t,*}(\mu)|_{\Omega_i^T} \in W_{h,i}^{\Delta t}$ is the solution to (Dirichlet data $\mu$, zero source term, initial data, and boundary data)

$$a_i^T(\mathbf{u}_h^{\Delta t,*}(\mu), \mathbf{v}) + b_i^T(\mathbf{v}, p_h^{\Delta t,*}(\mu)) = -\langle \mathbf{v} \cdot \mathbf{n}_i, \mu \rangle_{\Gamma_i^T} \quad \forall\, \mathbf{v} \in \mathbf{V}_{h,i}^{\Delta t},$$

$$(\tilde{\partial}_t p_h^{\Delta t,*}(\mu), w)_{\Omega_i^T} - b_i^T(\mathbf{u}_h^{\Delta t,*}(\mu), w) = 0 \quad \forall\, w \in W_{h,i}^{\Delta t}$$

- both above problems are posed in the individual space-time subdomains $\Omega_i^T$ and can thus be solved in parallel (no synchronization on time steps)

# Space-time Steklov–Poincaré operator

**Lemma (Equivalence)**

*The MMMFE method is* **equivalent to**: *find* $\lambda_H^{\Delta T} \in \Lambda_H^{\Delta T}$ *such that*

$$-b_\Gamma^T(\mathbf{u}_h^{\Delta t,*}(\lambda_H^{\Delta T}), \mu) = b_\Gamma^T(\overline{\mathbf{u}}_h^{\Delta t}, \mu) \quad \forall \mu \in \Lambda_H^{\Delta T}.$$

## Space-time Steklov–Poincaré operator

- $S : \Lambda_H^{\Delta T} \to \Lambda_H^{\Delta T}$

$$\langle S\lambda, \mu\rangle_{\Gamma^T} := \sum_i \langle S_i\lambda, \mu\rangle_{\Gamma_i^T}, \quad \langle S_i\lambda, \mu\rangle_{\Gamma_i^T} := -\langle \mathbf{u}_h^{\Delta t,*}(\lambda) \cdot \mathbf{n}_i, \mu\rangle_{\Gamma_i^T} \quad \forall \lambda, \mu \in \Lambda_H^{\Delta T}$$

- $g \in \Lambda_H^{\Delta T}$ is defined as $\langle g, \mu\rangle_{\Gamma^T} := b_\Gamma(\overline{\mathbf{u}}_h^{\Delta t}, \mu) \qquad \forall \mu \in \Lambda_H^{\Delta T}$

**Lemma (Operator form)**

*Equivalent operator form is: find* $\lambda_H^{\Delta T} \in \Lambda_H^{\Delta T}$ *such that*

$$S\,\lambda_H^{\Delta T} = g.$$

# Space-time Steklov–Poincaré operator

## Lemma (Equivalence)

*The MMMFE method is* **equivalent to***: find $\lambda_H^{\Delta T} \in \Lambda_H^{\Delta T}$ such that*

$$-b_\Gamma^T(\mathbf{u}_h^{\Delta t,*}(\lambda_H^{\Delta T}), \mu) = b_\Gamma^T(\overline{\mathbf{u}}_h^{\Delta t}, \mu) \quad \forall\, \mu \in \Lambda_H^{\Delta T}.$$

**Space-time Steklov–Poincaré operator**

- $S : \Lambda_H^{\Delta T} \to \Lambda_H^{\Delta T}$

$$\langle S\lambda, \mu\rangle_{\Gamma^T} := \sum_i \langle S_i\lambda, \mu\rangle_{\Gamma_i^T}, \quad \langle S_i\lambda, \mu\rangle_{\Gamma_i^T} := -\langle \mathbf{u}_h^{\Delta t,*}(\lambda)\cdot \mathbf{n}_i, \mu\rangle_{\Gamma_i^T} \quad \forall\, \lambda, \mu \in \Lambda_H^{\Delta T}$$

- $g \in \Lambda_H^{\Delta T}$ is defined as $\langle g, \mu\rangle_{\Gamma^T} := b_\Gamma(\overline{\mathbf{u}}_h^{\Delta t}, \mu) \qquad \forall\, \mu \in \Lambda_H^{\Delta T}$

## Lemma (Operator form)

*Equivalent operator form is: find $\lambda_H^{\Delta T} \in \Lambda_H^{\Delta T}$ such that*

$$S\,\lambda_H^{\Delta T} = g.$$

# Space-time Steklov–Poincaré operator

**Lemma (Equivalence)**

*The MMMFE method is* **equivalent to**: *find $\lambda_H^{\Delta T} \in \Lambda_H^{\Delta T}$ such that*

$$-b_\Gamma^T(\mathbf{u}_h^{\Delta t,*}(\lambda_H^{\Delta T}), \mu) = b_\Gamma^T(\overline{\mathbf{u}}_h^{\Delta t}, \mu) \quad \forall\, \mu \in \Lambda_H^{\Delta T}.$$

**Space-time Steklov–Poincaré operator**

- $S : \Lambda_H^{\Delta T} \to \Lambda_H^{\Delta T}$

$$\langle S\lambda, \mu\rangle_{\Gamma^T} := \sum_i \langle S_i\lambda, \mu\rangle_{\Gamma_i^T}, \quad \langle S_i\lambda, \mu\rangle_{\Gamma_i^T} := -\langle \mathbf{u}_h^{\Delta t,*}(\lambda)\cdot \mathbf{n}_i, \mu\rangle_{\Gamma_i^T} \quad \forall\, \lambda, \mu \in \Lambda_H^{\Delta T}$$

- $g \in \Lambda_H^{\Delta T}$ is defined as $\langle g, \mu\rangle_{\Gamma^T} := b_\Gamma(\overline{\mathbf{u}}_h^{\Delta t}, \mu) \qquad \forall\, \mu \in \Lambda_H^{\Delta T}$

**Lemma (Operator form)**

*Equivalent operator form is: find $\lambda_H^{\Delta T} \in \Lambda_H^{\Delta T}$ such that*

$$S\,\lambda_H^{\Delta T} = g.$$

# Spectral bound, space-time domain decomposition algorithm

**Theorem (Spectral bound)**

*Let the mortar assumptions hold. Then the operator $S$ is* **positive definite**. *Let moreover $\mathcal{T}_{h,i}$ be quasi-uniform and $h \leq Ch_i$ for all $i$. Then the following* **spectral bound** *holds:*

$$\forall\, \mu \in \Lambda_H^{\Delta T}, \quad C_0 \|\mu\|^2_{\Gamma^T} \leq \langle S\mu, \mu \rangle_{\Gamma^T} \leq C_1 h^{-1} \|\mu\|^2_{\Gamma^T}.$$

## Comments

- well-posed space-time interface problem
- leads to a **space-time domain decomposition** algorithm
- GMRES can be applied; convergence through the field-of-values estimates:

$$\|\mathbf{r}_k\| \leq \left(\sqrt{1-(C_0/C_1)^2 h^2}\right)^k \|\mathbf{r}_0\|$$

- on all iterations: problems posed in the **individual space-time subdomains** $\Omega_i^T$ and **solved in parallel** (no synchronization on time steps)

# Spectral bound, space-time domain decomposition algorithm

**Theorem (Spectral bound)**

*Let the mortar assumptions hold. Then the operator $S$ is* **positive definite**. *Let moreover $\mathcal{T}_{h,i}$ be quasi-uniform and $h \le C h_i$ for all $i$. Then the following* **spectral bound** *holds:*

$$\forall\, \mu \in \Lambda_H^{\Delta T}, \quad C_0 \|\mu\|^2_{\Gamma^T} \le \langle S\mu, \mu\rangle_{\Gamma^T} \le C_1\, h^{-1} \|\mu\|^2_{\Gamma^T}.$$

**Comments**

- well-posed space-time interface problem
- leads to a **space-time domain decomposition** algorithm
- GMRES can be applied; convergence through the field-of-values estimates:
  $\|\mathbf{r}_k\| \le \left(\sqrt{1-(C_0/C_1)^2 h^2}\right)^k \|\mathbf{r}_0\|$
- on all iterations: problems posed in the **individual space-time subdomains** $\Omega_i^T$ and **solved in parallel** (no synchronization on time steps)

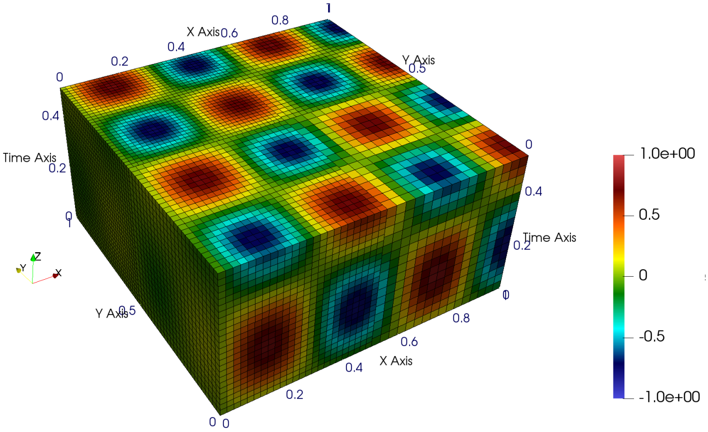
# Outline

# Numerical experiments

## Setting

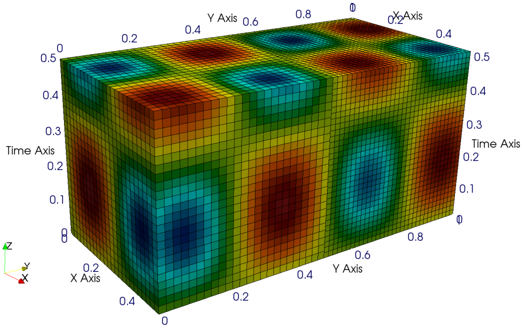
- $d = 2$
- $\mathbf{V}_{h,i} \times W_{h,i}$ on each $\Omega_i$ is the lowest-order Raviart–Thomas pair $RT_0 \times DGQ_0$ ($k = l = 0$)
- backward Euler time discretization in each $\Omega_i^T$ ($q = 0$)
- mortar finite element space $\Lambda_{H,ij}^{\Delta T}$: discontinuous bilinear ($m = s = 1$, $H = 2h$ and $\Delta T = 2\Delta t$) and discontinuous biquadratic ($m = s = 2$, $H = \sqrt{h}$ and $\Delta T = \sqrt{\Delta t}$) mortars
- GMRES without preconditioner for the space-time interface problem
- deal.II package
- $\Delta t^{-\frac{1}{2}}$ loss in convergence rate in the theoretical bound not observed in the numerical results

# Example 1, smooth solution, $\Omega^T$

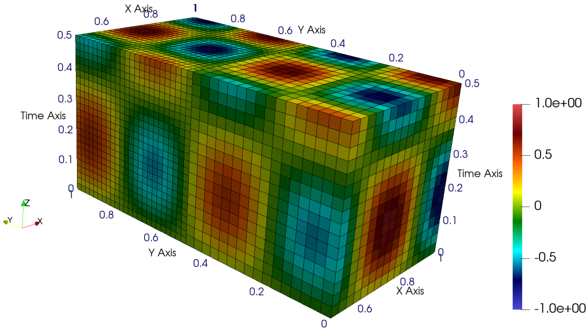



Pressure, bilinear mortars $m = s = 1$, space-time grid at refinement 3, whole $\Omega^T$

# Example 1

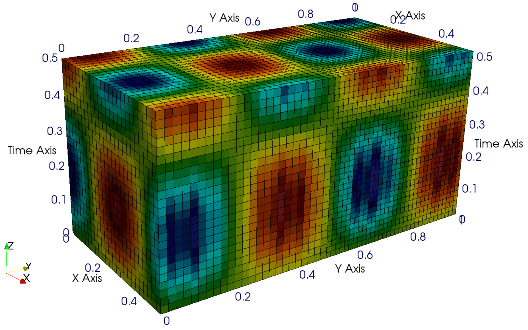



Pressure detail on $\Omega_1^T \cup \Omega_4^T$

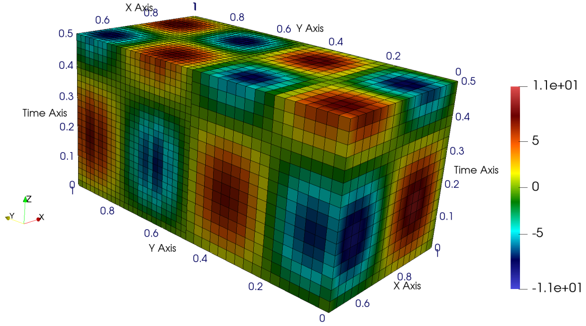



Pressure detail on $\Omega_2^T \cup \Omega_3^T$

# Example 1




$x$-velocity detail on $\Omega_1^T \cup \Omega_4^T$




$x$-velocity detail on $\Omega_2^T \cup \Omega_3^T$

# Example 1

| Ref. No. | $\Omega_1^T$ | | | $\Omega_2^T$ | | | $\Omega_3^T$ | | | $\Omega_4^T$ | | | $\Gamma^T(m=1)$ | | | $\Gamma^T(m=2)$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $n_1$ | $N_1$ | #DoF | $n_2$ | $N_2$ | #DoF | $n_3$ | $N_3$ | #DoF | $n_4$ | $N_4$ | #DoF | $n_\Gamma$ | $N_\Gamma$ | #DoF | $n_\Gamma$ | $N_\Gamma$ | #DoF |
| 0 | 3 | 3 | 33 | 2 | 2 | 16 | 4 | 4 | 56 | 3 | 3 | 33 | 1 | 1 | 16 | 1 | 1 | 36 |
| 1 | 6 | 6 | 120 | 4 | 4 | 56 | 8 | 8 | 208 | 6 | 6 | 120 | 2 | 2 | 64 | | | |
| 2 | 12 | 12 | 456 | 8 | 8 | 208 | 16 | 16 | 800 | 12 | 12 | 456 | 4 | 4 | 256 | 2 | 2 | 144 |
| 3 | 24 | 24 | 1776 | 16 | 16 | 800 | 32 | 32 | 3136 | 24 | 24 | 1776 | 8 | 8 | 1024 | | | |
| 4 | 48 | 48 | 7008 | 32 | 32 | 3136 | 64 | 64 | 12416 | 48 | 48 | 7008 | 16 | 16 | 4096 | 4 | 4 | 576 |

Meshes, polynomial degrees, and number of degrees of freedom

# Example 1: $m = s = 2$ better than $m = s = 1$

| Ref. | # GMRES | | $\|\mathbf{u} - \mathbf{u}_h^{\Delta t}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$ | | $\|p - p_h^{\Delta t}\|_{\mathrm{DG}}$ | | $\|p - p_h^{\Delta t}\|_{L^2(0,T;W)}$ | | $\|\lambda - \lambda_H^{\Delta T}\|_{L^2(0,T;\Lambda_H)}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 11 | Rate | 6.50e-01 | Rate | 1.21e+00 | Rate | 7.91e-01 | Rate | 7.98e-01 | Rate |
| 1 | 23 | -1.06 | 3.63e-01 | 0.84 | 7.21e-01 | 0.75 | 4.76e-01 | 0.73 | 5.11e-01 | 0.64 |
| 2 | 39 | -0.76 | 1.74e-01 | 1.06 | 3.19e-01 | 1.18 | 2.46e-01 | 0.95 | 2.34e-01 | 1.13 |
| 3 | 59 | -0.60 | 8.63e-02 | 1.02 | 1.46e-01 | 1.13 | 1.25e-01 | 0.98 | 1.20e-01 | 0.96 |
| 4 | 86 | -0.54 | 4.29e-02 | 1.01 | 6.93e-02 | 1.08 | 6.25e-02 | 1.00 | 6.11e-02 | 0.97 |

Convergence with bilinear mortars $m = s = 1$

| Ref. | # GMRES | | $\|\mathbf{u} - \mathbf{u}_h^{\Delta t}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$ | | $\|p - p_h^{\Delta t}\|_{\mathrm{DG}}$ | | $\|p - p_h^{\Delta t}\|_{L^2(0,T;W)}$ | | $\|\lambda - \lambda_H^{\Delta T}\|_{L^2(0,T;\Lambda_H)}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 18 | Rate | 6.81e-01 | Rate | 1.35e+00 | Rate | 8.39e-01 | Rate | 2.13e+00 | Rate |
| 2 | 34 | -0.46 | 1.70e-01 | 1.00 | 3.51e-01 | 0.97 | 2.51e-01 | 0.87 | 2.82e-01 | 1.46 |
| 4 | 57 | -0.37 | 4.48e-02 | 0.96 | 8.59e-02 | 1.02 | 6.59e-02 | 0.96 | 9.20e-02 | 0.81 |

Convergence with biquadratic mortars $m = s = 2$

# Example 1: $m = s = 2$ better than $m = s = 1$

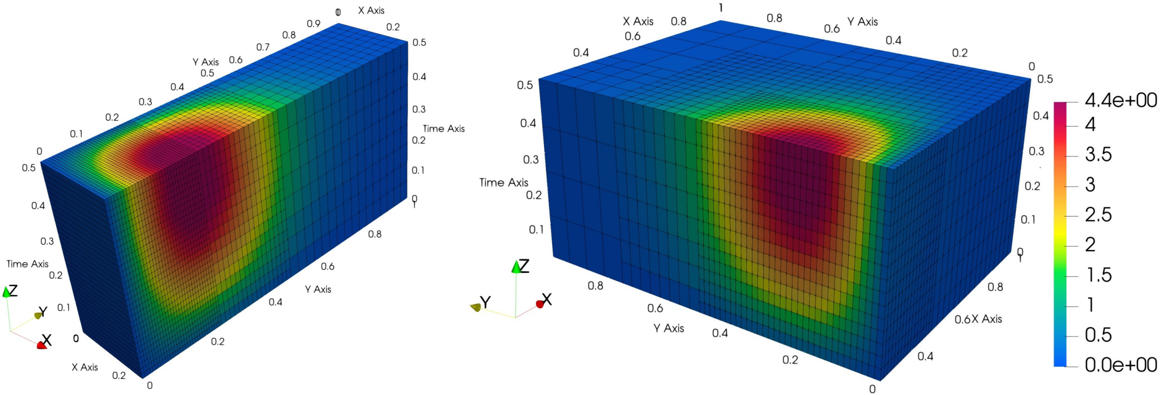
| Ref. | # GMRES | | $\|\mathbf{u} - \mathbf{u}_h^{\Delta t}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$ | | $\|p - p_h^{\Delta t}\|_{\mathrm{DG}}$ | | $\|p - p_h^{\Delta t}\|_{L^2(0,T;W)}$ | | $\|\lambda - \lambda_H^{\Delta T}\|_{L^2(0,T;\Lambda_H)}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 11 | Rate | 6.50e-01 | Rate | 1.21e+00 | Rate | 7.91e-01 | Rate | 7.98e-01 | Rate |
| 1 | 23 | -1.06 | 3.63e-01 | 0.84 | 7.21e-01 | 0.75 | 4.76e-01 | 0.73 | 5.11e-01 | 0.64 |
| 2 | 39 | -0.76 | 1.74e-01 | 1.06 | 3.19e-01 | 1.18 | 2.46e-01 | 0.95 | 2.34e-01 | 1.13 |
| 3 | 59 | -0.60 | 8.63e-02 | 1.02 | 1.46e-01 | 1.13 | 1.25e-01 | 0.98 | 1.20e-01 | 0.96 |
| 4 | 86 | -0.54 | 4.29e-02 | 1.01 | 6.93e-02 | 1.08 | 6.25e-02 | 1.00 | 6.11e-02 | 0.97 |

Convergence with bilinear mortars $m = s = 1$

| Ref. | # GMRES | | $\|\mathbf{u} - \mathbf{u}_h^{\Delta t}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$ | | $\|p - p_h^{\Delta t}\|_{\mathrm{DG}}$ | | $\|p - p_h^{\Delta t}\|_{L^2(0,T;W)}$ | | $\|\lambda - \lambda_H^{\Delta T}\|_{L^2(0,T;\Lambda_H)}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 18 | Rate | 6.81e-01 | Rate | 1.35e+00 | Rate | 8.39e-01 | Rate | 2.13e+00 | Rate |
| 2 | 34 | -0.46 | 1.70e-01 | 1.00 | 3.51e-01 | 0.97 | 2.51e-01 | 0.87 | 2.82e-01 | 1.46 |
| 4 | 57 | -0.37 | 4.48e-02 | 0.96 | 8.59e-02 | 1.02 | 6.59e-02 | 0.96 | 9.20e-02 | 0.81 |

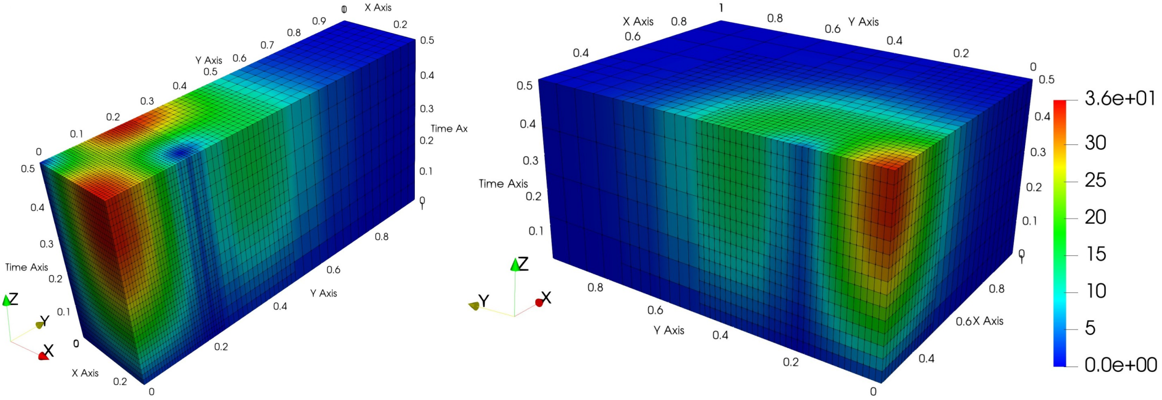
Convergence with biquadratic mortars $m = s = 2$

# Example 2, sharp boundary layer, discontinuous bilinear mortars ($m = s = 1$)



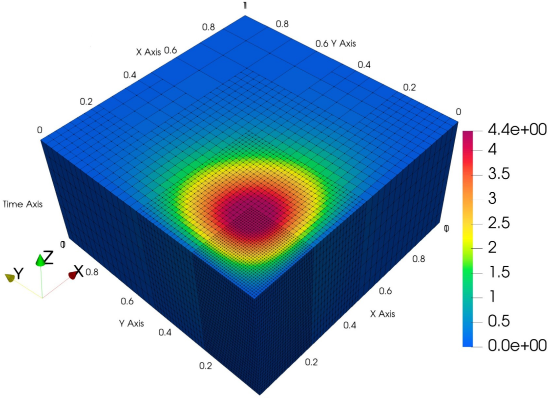

Pressure, cut along the plane $x = 0.25$

# Example 2

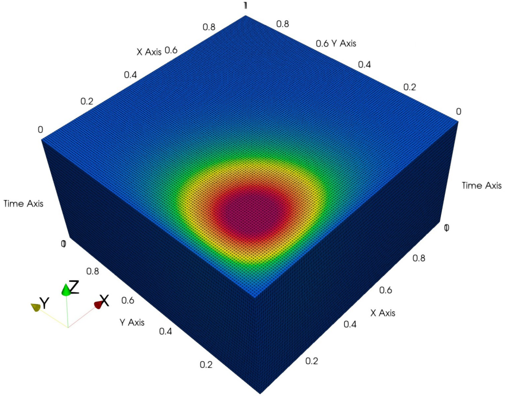



Velocity magnitude, cut along the plane $x = 0.25$

# Example 2, $\Omega^T$

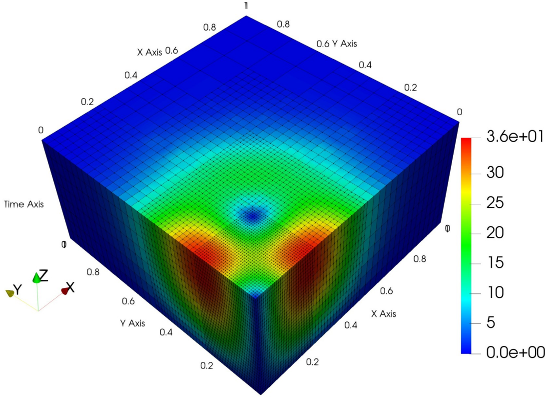



Pressure, mortar multiscale method

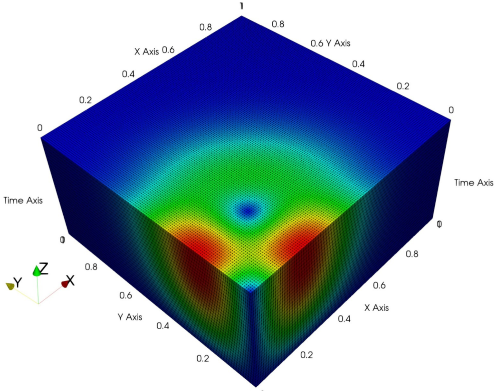



Pressure, fine-scale method

# Example 2, $\Omega^T$




Velocity magnitude, mortar multiscale method




Velocity magnitude, fine-scale method

# Example 2

| Method | # GMRES | $\|\mathbf{u}-\mathbf{u}_h^{\Delta t}\|_{L^2(0,T;\mathbf{L}^2(\Omega))}$ | $\|p-p_h^{\Delta t}\|_{\mathrm{DG}}$ | $\|p-p_h^{\Delta t}\|_{L^2(0,T;W)}$ | $\|\lambda-\lambda_H^{\Delta T}\|_{L^2(0,T;\Lambda_H)}$ |
|---|---|---|---|---|---|
| multiscale | 102 | 5.657e-02 | 8.425e-02 | 6.319e-02 | 5.796e-02 |
| fine-scale | 140 | 1.524e-02 | 2.234e-02 | 2.154e-02 | 3.016e-02 |

Errors and GMRES iterations for the multiscale and fine-scale methods

# Outline

# Conclusions and future directions

## Conclusions

- standard building blocks: **DG time stepping** on **individual subdomains**
- mortar coupling: **space-time interface problem**
- mortars: coarse mesh / high polynomial degree: **multiscale** approximation
- leads to a **space-time domain decomposition** algorithm

## Future directions

- developing a preconditioner for the space-time interface iterative solver
- dedicated a posteriori error analysis

## References


- M. Jayadharan, M. Kern, M. Vohralík, I. Yotov, A space-time multiscale mortar mixed finite element method for parabolic equations, HAL Preprint 03355088, 2021.
- S. Ali Hassan, C. Japhet, M. Vohralík, A posteriori stopping criteria for space-time domain decomposition for the heat equation in mixed formulations, *Electron. Trans. Numer. Anal.* **49** (2018), 151–181.


**Thank you for your attention!**

# Conclusions and future directions

## Conclusions

- standard building blocks: **DG time stepping** on **individual subdomains**
- mortar coupling: **space-time interface problem**
- mortars: coarse mesh / high polynomial degree: **multiscale** approximation
- leads to a **space-time domain decomposition** algorithm

## Future directions

- developing a preconditioner for the space-time interface iterative solver
- dedicated a posteriori error analysis

## References


- M. Jayadharan, M. Kern, M. Vohralík, I. Yotov, A space-time multiscale mortar mixed finite element method for parabolic equations, HAL Preprint 03355088, 2021.
- S. Ali Hassan, C. Japhet, M. Vohralík, A posteriori stopping criteria for space-time domain decomposition for the heat equation in mixed formulations, *Electron. Trans. Numer. Anal.* **49** (2018), 151–181.


**Thank you for your attention!**

# Conclusions and future directions

## Conclusions

- standard building blocks: **DG time stepping** on **individual subdomains**
- mortar coupling: **space-time interface problem**
- mortars: coarse mesh / high polynomial degree: **multiscale** approximation
- leads to a **space-time domain decomposition** algorithm

## Future directions

- developing a preconditioner for the space-time interface iterative solver
- dedicated a posteriori error analysis

## References


- M. Jayadharan, M. Kern, M. Vohralík, I. Yotov, A space-time multiscale mortar mixed finite element method for parabolic equations, HAL Preprint 03355088, 2021.
- S. Ali Hassan, C. Japhet, M. Vohralík, A posteriori stopping criteria for space-time domain decomposition for the heat equation in mixed formulations, *Electron. Trans. Numer. Anal.* **49** (2018), 151–181.


**Thank you for your attention!**

# Conclusions and future directions

**Conclusions**

- standard building blocks: **DG time stepping** on **individual subdomains**
- mortar coupling: **space-time interface problem**
- mortars: coarse mesh / high polynomial degree: **multiscale** approximation
- leads to a **space-time domain decomposition** algorithm

**Future directions**

- developing a preconditioner for the space-time interface iterative solver
- dedicated a posteriori error analysis

**References**


- M. Jayadharan, M. Kern, M. Vohralík, I. Yotov, A space-time multiscale mortar mixed finite element method for parabolic equations, HAL Preprint 03355088, 2021.
- S. Ali Hassan, C. Japhet, M. Vohralík, A posteriori stopping criteria for space-time domain decomposition for the heat equation in mixed formulations, *Electron. Trans. Numer. Anal.* **49** (2018), 151–181.


**Thank you for your attention!**