Stable broken H^1 and $\mathbf{H}(\mathrm{div})$ polynomial extensions

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Outline

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Literature

Fundamental results on a reference tetrahedron

- Costabel & McIntosh (2010): bounded right inverse of the divergence operator for polynomial volume data
- Demkowicz, Gopalakrishnan, Schöberl (2009, 2012): polynomial extensions in H^1 and $\mathbf{H}(\mathrm{div})$ for polynomial boundary data

Stable broken $\mathbf{H}(\mathrm{div})$ polynomial extension on a patch

- Braess, Pillwein, & Schöberl (2009), 2D
- volume and boundary data

Stable broken H^1 polynomial extensions on a patch

- Ern & V. (2015), 2D, by rotation from the result of Braess, Pillwein, & Schöberl
- only boundary data (divergence-free vectors are curls)

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Setting

Patches

- $\mathcal{T}_{\boldsymbol{a}} \subset \mathcal{T}_h$: patch of elements sharing $\boldsymbol{a} \in \mathcal{V}_h$, subdomain $\omega_{\boldsymbol{a}}$
- $\mathcal{F}_{\boldsymbol{a}} = \mathcal{F}_{\boldsymbol{a}}^{\mathrm{s}} \cup \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$: faces of the elements in the patch $\mathcal{T}_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{V}_h^{\circ}$

Piecewise H^1 spaces

$$H^1(\mathcal{T}_{\boldsymbol{a}}) := \{v \in L^2(\omega_{\boldsymbol{a}});\ v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_{\boldsymbol{a}}\}$$

Piecewise $\boldsymbol{H}(\mathrm{div})$ spaces

$$\boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\omega_{\boldsymbol{a}});\ \boldsymbol{v}|_K \in \boldsymbol{H}(\mathrm{div}, K) \quad \forall K \in \mathcal{T}_{\boldsymbol{a}}\}$$

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Main result: potentials

Theorem (Broken H^1 polynomial extension; Ern & V. (2015) in 2D)

For $p \geq 1$ and $\mathbf{a} \in \mathcal{V}_h^\circ$, let $r \in \mathbb{P}_p(\mathcal{F}_{\mathbf{a}}^{\mathrm{s}})$. Suppose the compatibility

$$\begin{aligned} r &= 0 && \text{on } \partial\omega_{\mathbf{a}}, \\ \sum_{F \in \mathcal{F}_e} \iota_{F,e}\, r_F|_e &= 0 && \forall e \in \mathcal{E}_{\mathbf{a}}. \end{aligned}$$

Then there exists a constant $C_{\mathrm{st}} > 0$ only depending on the mesh shape-regularity parameter $\kappa_{\mathcal{T}_h}$ such that

$$\min_{\substack{v_h \in \mathbb{P}_p(\mathcal{T}_{\mathbf{a}}) \\ v_h = 0 \text{ on } \partial\omega_{\mathbf{a}}, \\ [\![v_h]\!] = r_F\ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{s}}}} \|\nabla v_h\|_{\omega_{\mathbf{a}}} \leq C_{\mathrm{st}} \min_{\substack{v \in H^1(\mathcal{T}_{\mathbf{a}}) \\ v = 0 \text{ on } \partial\omega_{\mathbf{a}}, \\ [\![v]\!] = r_F\ \forall F \in \mathcal{F}_{\mathbf{a}}^{\mathrm{s}}}} \|\nabla v\|_{\omega_{\mathbf{a}}}.$$

Main result: fluxes

Theorem (Broken $\boldsymbol{H}(\mathrm{div})$ polynomial extension; Braess, Pillwein, & Schöberl (2009) in 2D)

For $p \geq 1$ and $\boldsymbol{a} \in \mathcal{V}_h^{\circ}$, let $r \in \mathbb{P}_p(\mathcal{F}_{\boldsymbol{a}}) \times \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$. Suppose the compatibility

$$\sum_{K \in \mathcal{T}_{\boldsymbol{a}}} (r_K, 1)_K - \sum_{F \in \mathcal{F}_{\boldsymbol{a}}} (r_F, 1)_F = 0.$$

Then there exists a constant $C_{\mathrm{st}} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}_h}$ such that

$$\min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RTN}_p(\mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}} \\ [\![\boldsymbol{v}_h \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{s}} \\ \nabla_{\mathcal{T}} \cdot \boldsymbol{v}_h|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}_h\|_{\omega_{\boldsymbol{a}}} \leq C_{\mathrm{st}} \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}, \mathcal{T}_{\boldsymbol{a}}) \\ \boldsymbol{v} \cdot \boldsymbol{n}_F = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}} \\ [\![\boldsymbol{v} \cdot \boldsymbol{n}_F]\!] = r_F \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{s}} \\ \nabla_{\mathcal{T}} \cdot \boldsymbol{v}|_K = r_K \ \forall K \in \mathcal{T}_{\boldsymbol{a}}}} \|\boldsymbol{v}\|_{\omega_{\boldsymbol{a}}}.$$

Application to piecewise polynomial approximation

Volume liftings

- $\tau_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$ so that $\tau_h|_F = 0 \ \forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{b}}$, and $[\![\tau_h]\!]_F = r_F$ $\forall F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{s}}$

Corollary (Stability of best piecewise polynomial approximation)

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Corollary (Stability of best piecewise polynomial approximation)

There holds

$$\min_{v_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla(\tau_h - v_h)\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} \min_{v \in H_0^1(\omega_{\boldsymbol{a}})} \|\nabla(\tau_h - v)\|_{\omega_{\boldsymbol{a}}},$$

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Application to a posteriori error analysis

Laplace model problem

For $f \in \mathbb{P}_{p'-1}(\mathcal{T}_h)$, $p' \geq 1$, find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Approximate solution with hat-function orthogonality

$u_h \in \mathbb{P}_{p'}(\mathcal{T}_h)$, $u_h \notin H_0^1(\Omega)$, $-\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega)$

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\circ}$$

Potential case ($p = p' + 1$)

$$r_F := \psi_{\mathbf{a}} [\![u_h]\!] |_F,$$
$$\tau_h := \psi_{\mathbf{a}} u_h$$

Flux case ($p = p'$)

$$r_F := \psi_{\mathbf{a}} [\![\nabla u_h \cdot \mathbf{n}_F]\!] |_F,$$
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Approximate solution with hat-function orthogonality

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Potential reconstruction

Definition (Potential reconstruction)

For each $\boldsymbol{a} \in \mathcal{V}_h$, let $s_h^{\boldsymbol{a}}$ be given by

$$s_h^{\boldsymbol{a}} := \arg \min_{v_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})} \|\nabla(\psi_{\boldsymbol{a}} u_h - v_h)\|_{\omega_{\boldsymbol{a}}}.$$

Then set $s_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} s_h^{\boldsymbol{a}} \in \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$.

Equivalent form

Find $s_h^{\boldsymbol{a}} \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}}) \cap H_0^1(\omega_{\boldsymbol{a}})$ such that

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For each $\boldsymbol{a} \in \mathcal{V}_h$, let $\boldsymbol{\sigma}_h^{\boldsymbol{a}}$ be given by

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$$(\boldsymbol{\sigma}_h^{\boldsymbol{a}}, \boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} - (r_h^{\boldsymbol{a}}, \nabla\cdot\boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} = -(\psi_{\boldsymbol{a}} \nabla u_h, \boldsymbol{v}_h)_{\omega_{\boldsymbol{a}}} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^{\boldsymbol{a}},$$
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Guaranteed reliability and p-robust efficiency

Guaranteed reliability

$$\|\nabla(u-u_h)\|^2 \le \sum_{K\in\mathcal{T}_h} \|\nabla u_h + \boldsymbol{\sigma}_h\|_K^2 + \sum_{K\in\mathcal{T}_h} \|\nabla(u_h - s_h)\|_K^2$$

Potential local p-robust efficiency

$$\|\nabla(\psi_{\boldsymbol{a}} u_h - s_h^{\boldsymbol{a}})\|_{\omega_{\boldsymbol{a}}} \le C_{\mathrm{st}} C_{\mathrm{cont,bPF}} \|\nabla(u-u_h)\|_{\omega_{\boldsymbol{a}}}$$

Flux local p-robust efficiency

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Applications

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- mixed finite elements
- ...

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Potential local **p-robust efficiency**

$$\left\|\nabla(\psi_{\boldsymbol{a}} u_h - s_h^{\boldsymbol{a}})\right\|_{\omega_{\boldsymbol{a}}} \le C_{\rm st} C_{\rm cont,bPF} \left(\|\nabla(u-u_h)\|_{\omega_{\boldsymbol{a}}} + \left\{ \sum_{F\in\mathcal{F}_{\boldsymbol{a}}\setminus\mathcal{F}_{\boldsymbol{a}}^{\rm b}} h_F^{-1} \left\| \Pi_F^0 [\![u_h]\!] \right\|_F^2 \right\}^{\frac{1}{2}} \right)$$

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$$\left\|\nabla(\psi_{\boldsymbol{a}} u_h - s_h^{\boldsymbol{a}})\right\|_{\omega_{\boldsymbol{a}}} \leq C_{\rm st} C_{\rm cont,bPF} \, \|\nabla(u-u_h)\|_{\omega_{\boldsymbol{a}}}$$

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Applications

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- mixed finite elements
- ...

Outline

1. Main results & applications
2. Key ingredients
   - Stable polynomial extensions on a tetrahedron
   - 3D patch enumeration
3. Proof sketch (potentials)
4. Numerical illustration in 2D a posteriori estimates
5. Conclusions and future directions

Outline

Potentials (following Demkowicz, Gopalakrishnan, Schöberl (2009))

Lemma (H^1 polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}_h$, $\mathcal{F}_K^{\mathrm{D}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{D}})$ be continuous on $\mathcal{F}_K^{\mathrm{D}}$. Then for $C = C(\kappa_K) > 0$,

$$\min_{\substack{v_h \in \mathbb{P}_p(K) \\ v_h = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v_h\|_K \leq C \min_{\substack{v \in H^1(K) \\ v = r_F \text{ on all } F \in \mathcal{F}_K^{\mathrm{D}}}} \|\nabla v\|_K .$$

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$$\begin{aligned} -\Delta \zeta_K &= 0 && \text{in } K, \\ \zeta_K &= r_F && \text{on all } F \in \mathcal{F}_K^{\mathrm{D}}, \\ -\nabla \zeta_K \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_K \setminus \mathcal{F}_K^{\mathrm{D}}. \end{aligned}$$

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Fluxes (fol. Costabel & McIntosh (2010), Demkowicz, Gopalakrishnan, Schöberl (2012))

Lemma ($\boldsymbol{H}(\mathrm{div})$ polynomial extension on a tetrahedron)

Let $K \in \mathcal{T}_h$, $\mathcal{F}_K^{\mathrm{N}} \subset \mathcal{F}_K$. Let $r \in \mathbb{P}_p(\mathcal{F}_K^{\mathrm{N}}) \times \mathbb{P}_p(K)$, satisfying $\sum_{F \in \mathcal{F}_K} (r_F, 1)_F = (r_K, 1)_K$ if $\mathcal{F}_K^{\mathrm{N}} = \mathcal{F}_K$. Then for $C = C(\kappa_K) > 0$,

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Outline

A graph result for patch enumerations (shellability of polytopes, e.g. Ziegler, Lectures on Polytopes)

Two families of faces

- already visited faces: $\mathcal{F}_i^\sharp := \{F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{s}},\ F = \partial K_i \cap \partial K_j,\ j < i\}$
- yet unvisited faces: $\mathcal{F}_i^\flat := \mathcal{F}_{\boldsymbol{a}}^{\mathrm{s}} \cap \mathcal{F}_K \setminus \mathcal{F}_i^\sharp$
- $|\mathcal{F}_i^\flat| + |\mathcal{F}_i^\sharp| = 3$, $\mathcal{F}_1^\sharp = \emptyset$, and $\mathcal{F}_{|\mathcal{T}_{\boldsymbol{a}}|}^\flat = \emptyset$

Lemma (Interior patch enumeration)

There exists an enumeration of the patch $\mathcal{T}_{\boldsymbol{a}}$ so that

(i) *For all $1 < i < |\mathcal{T}_{\boldsymbol{a}}|$, $|\mathcal{F}_i^\sharp| \in \{1, 2\}$.*

(ii) *If $|\mathcal{F}_i^\sharp| \geq 2$ then $K_j \in \mathcal{T}_{F_i^1 \cap F_i^2} \setminus \{K_i\}$, $\{F_i^1, F_i^2\} \subset \mathcal{F}_i^\sharp$, implies $j < i$.*

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Outline

Run through the patch following the enumeration: K_1

Construct $\zeta_h \in \mathbb{P}_p(\mathcal{T}_{\boldsymbol{a}})$, $\zeta_h = 0$ on $\partial\omega_{\boldsymbol{a}}$, $[\![\zeta_h]\!] = r_F$ for all $F \in \mathcal{F}_{\boldsymbol{a}}^{\mathrm{s}}$:

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spirit of Braess, Pillwein, & Schöberl (2009), but work with strong norms

On K_i, $1 \le i < |\mathcal{T}_{\boldsymbol{a}}|$, consider the weak form of: find ζ_{K_i} s.t.

$$\begin{aligned} -\Delta \zeta_{K_i} &= 0 && \text{in } K_i, \\ \zeta_{K_i} &= -r_F + \zeta_{h,K_j}|_F && \text{on all } F = \partial K_i \cap \partial K_j \in \mathcal{F}_i^\sharp, \\ \zeta_{K_i} &= 0 && \text{on } \partial K_i \cap \partial\omega_{\boldsymbol{a}}, \\ -\nabla \zeta_{K_i} \cdot \boldsymbol{n}_K &= 0 && \text{on all } F \in \mathcal{F}_i^\flat \end{aligned}$$

1) $i = 1$: trivially $0 = \|\nabla \zeta_{h,K_1}\|_{K_1} \le \|\nabla v^*\|_{\omega_{\boldsymbol{a}}}$

2) only one (Dirichlet) face in the set $\mathcal{F}_i^\sharp$

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3) two faces in $\mathcal{F}_i^\sharp \Leftrightarrow K_i$ last in rotation around e by shellability

- compatibility of the Dirichlet data by assumption
- H^1 extension on a tetrahedron: $\left\|\nabla\zeta_{h,K_i}\right\|_{K_i} \lesssim \left\|\nabla\zeta_{K_i}\right\|_{K_i}$
- binary coloring: affine maps to satisfy the Dirichlet BCs

$$\tilde{\zeta}_{K_i} := v^* - \frac{1}{2} \sum_{\substack{F \in \mathcal{F}_e \setminus \mathcal{F}_i^\sharp \\ F = \partial K_l \cap \partial K_m}} \left\{ (v^*|_{K_l} - \zeta_{h,K_l}) \circ \boldsymbol{T}_{K_l \to K_i}^{-1} - (v^*|_{K_m} - \zeta_{h,K_m}) \circ \boldsymbol{T}_{K_m \to K_i}^{-1} \right\}$$

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$$\lesssim \|\nabla v^*\|_{K_i} + \|\nabla v^*\|_{K_j} + \left\|\zeta_{h,K_j}\right\|_{K_j} \lesssim \|\nabla v^*\|_{\omega_{\boldsymbol{a}}}$$

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- compatibility of the Dirichlet data by assumption
- H^1 extension on a tetrahedron: $\left\|\nabla \zeta_{h,K_i}\right\|_{K_i} \lesssim \left\|\nabla \zeta_{K_i}\right\|_{K_i}$
- binary coloring: affine maps to satisfy the Dirichlet BCs

$$\tilde{\zeta}_{K_i} := v^* - \frac{1}{2} \sum_{\substack{F \in \mathcal{F}_e \setminus \mathcal{F}_i^\sharp \\ F = \partial K_l \cap \partial K_m}} \left\{(v^*|_{K_l} - \zeta_{h,K_l}) \circ \boldsymbol{T}^{-1}_{K_l \to K_i} - (v^*|_{K_m} - \zeta_{h,K_m}) \circ \boldsymbol{T}^{-1}_{K_m \to K_i}\right\}$$

- stability of $\tilde{\zeta}_{K_i}$:

$$\left\|\nabla \zeta_{K_i}\right\|_{K_i} \leq \left\|\nabla \tilde{\zeta}_{K_i}\right\|_{K_i} \lesssim \sum_{K \in \mathcal{T}_e,\, K \neq K_i} \left\{\|\nabla v^*\|_K + \left\|\nabla \zeta_{h,K}\right\|_K\right\} + \|\nabla v^*\|_{K_i} \lesssim \|\nabla v^*\|_{\omega_{\boldsymbol{a}}}$$

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Run through the patch following the enumeration: $K_{|\mathcal{T}_{\boldsymbol{a}}|}$

3) On K_n, $n := |\mathcal{T}_{\boldsymbol{a}}|$, consider the weak form of: find ζ_{K_n} s.t.

$$\begin{aligned} -\Delta\zeta_{K_n} &= 0 && \text{in } K_n, \\ \zeta_{K_n} &= -r_F + \zeta_{h,K_j}|_F && \text{on all } F = \partial K_n \cap \partial K_j \in \mathcal{F}_n^\sharp, \\ \zeta_{K_n} &= 0 && \text{on } \partial K_n \cap \partial\omega_{\boldsymbol{a}} \end{aligned}$$

- three faces in $\mathcal{F}_n^\sharp$: pure Dirichlet problem
- compatibility of the Dirichlet data again by assumption
- H^1 extension on a tetrahedron: $\left\|\nabla\zeta_{h,K_n}\right\|_{K_n} \lesssim \|\nabla\zeta_{K_n}\|_{K_n}$
- ternary coloring in a sub-patch: affine maps to construct $\tilde{\zeta}_{K_n}$ satisfying the Dirichlet BCs, thus $\|\nabla\zeta_{K_n}\|_{K_n} \le \left\|\nabla\tilde{\zeta}_{K_n}\right\|_{K_n}$
- stability of $\tilde{\zeta}_{K_n}$: $\left\|\nabla\tilde{\zeta}_{K_n}\right\|_{K_n} \lesssim \|\nabla v^*\|_{\omega_{\boldsymbol{a}}}$

4) $\zeta_h|_{K_i} := \zeta_{h,K_i}$ for all $1 \le i \le n$ meets all the requirements

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\zeta_{K_n} &= 0 && \text{on } \partial K_n \cap \partial\omega_{\boldsymbol{a}}
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Outline

1. Main results & applications
2. Key ingredients
   - Stable polynomial extensions on a tetrahedron
   - 3D patch enumeration
3. Proof sketch (potentials)
4. Numerical illustration in 2D a posteriori estimates
5. Conclusions and future directions

Smooth case

Model problem

$$-\Delta u = f \quad \text{in } \Omega :=]0,1[^2,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Exact solution

$$u(\boldsymbol{x}) = \sin(2\pi \boldsymbol{x}_1)\sin(2\pi \boldsymbol{x}_2)$$

Discretization (with V. Dolejší)

- symmetric, nonsymmetric, and incomplete interior penalty discontinuous Galerkin method
- unstructured triangular grids
- uniform refinement

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Incomplete DG, nested grids

h	p	$\|\nabla(u-u_h)\|$	$\|u-u_h\|_{\mathrm{DG}}$	$\|\nabla u_h+\boldsymbol{\sigma}_h\|$	$\|\nabla(u_h-s_h)\|$	η_{osc}	η	η_{DG}	I^{eff}	$I^{\mathrm{eff}}_{\mathrm{DG}}$
$h_0/1$	1	1.21E+00	1.22E+00	1.24E+00	1.07E-01	5.56E-02	1.30E+00	1.31E+00	1.07	1.07
$h_0/2$		6.18E-01	6.22E-01	6.38E-01	5.09E-02	7.02E-03	6.47E-01	6.50E-01	1.05	1.05
		(0.97)	(0.97)	(0.96)	(1.07)	(2.99)	(1.01)	(1.01)		
$h_0/4$		3.12E-01	3.13E-01	3.22E-01	2.43E-02	8.80E-04	3.24E-01	3.25E-01	1.04	1.04
		(0.99)	(0.99)	(0.99)	(1.07)	(3.00)	(1.00)	(1.00)		
$h_0/8$		1.56E-01	1.57E-01	1.61E-01	1.18E-02	1.10E-04	1.62E-01	1.63E-01	1.04	1.04
		(1.00)	(1.00)	(1.00)	(1.05)	(3.00)	(1.00)	(1.00)		
$h_0/1$	2	1.50E-01	1.53E-01	1.49E-01	2.76E-02	5.10E-03	1.56E-01	1.59E-01	1.04	1.04
$h_0/2$		3.85E-02	3.92E-02	3.83E-02	7.99E-03	3.22E-04	3.94E-02	4.01E-02	1.03	1.02
		(1.96)	(1.96)	(1.96)	(1.79)	(3.98)	(1.98)	(1.98)		
$h_0/4$		9.70E-03	9.88E-03	9.68E-03	2.12E-03	2.02E-05	9.93E-03	1.01E-02	1.02	1.02
		(1.99)	(1.99)	(1.98)	(1.92)	(4.00)	(1.99)	(1.99)		
$h_0/8$		2.43E-03	2.48E-03	2.43E-03	5.42E-04	1.26E-06	2.49E-03	2.54E-03	1.02	1.02
		(1.99)	(1.99)	(1.99)	(1.96)	(4.00)	(1.99)	(1.99)		
$h_0/1$	3	1.32E-02	1.34E-02	1.29E-02	2.52E-03	3.58E-04	1.35E-02	1.37E-02	1.03	1.03
$h_0/2$		1.67E-03	1.69E-03	1.65E-03	3.13E-04	1.13E-05	1.70E-03	1.71E-03	1.01	1.01
		(2.98)	(2.98)	(2.97)	(3.01)	(4.99)	(3.00)	(3.00)		
$h_0/4$		2.11E-04	2.13E-04	2.09E-04	3.83E-05	3.53E-07	2.12E-04	2.15E-04	1.01	1.01
		(2.99)	(2.99)	(2.99)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/8$		2.64E-05	2.67E-05	2.61E-05	4.69E-06	1.10E-08	2.66E-05	2.69E-05	1.01	1.01
		(3.00)	(3.00)	(3.00)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/1$	4	9.36E-04	9.54E-04	9.05E-04	2.41E-04	2.12E-05	9.57E-04	9.74E-04	1.02	1.02
$h_0/2$		5.93E-05	6.05E-05	5.77E-05	1.68E-05	3.36E-07	6.04E-05	6.16E-05	1.02	1.02
		(3.98)	(3.98)	(3.97)	(3.84)	(5.98)	(3.99)	(3.98)		
$h_0/4$		3.72E-06	3.80E-06	3.63E-06	1.10E-06	5.31E-09	3.80E-06	3.87E-06	1.02	1.02
		(3.99)	(3.99)	(3.99)	(3.94)	(5.98)	(3.99)	(3.99)		
$h_0/8$		2.33E-07	2.38E-07	2.27E-07	7.02E-08	8.30E-11	2.38E-07	2.43E-07	1.02	1.02
		(4.00)	(4.00)	(4.00)	(3.97)	(6.00)	(4.00)	(3.99)		
$h_0/1$	5	5.41E-05	5.50E-05	5.22E-05	1.38E-05	1.06E-06	5.50E-05	5.58E-05	1.02	1.02
$h_0/2$		1.70E-06	1.72E-06	1.65E-06	4.39E-07	9.35E-09	1.72E-06	1.74E-06	1.01	1.01
		(4.99)	(5.00)	(4.98)	(4.98)	(6.82)	(5.00)	(5.00)		
$h_0/4$		5.32E-08	5.39E-08	5.19E-08	1.40E-08	7.67E-11	5.38E-08	5.45E-08	1.01	1.01
		(5.00)	(5.00)	(4.99)	(4.97)	(6.93)	(5.00)	(5.00)		
$h_0/8$		1.66E-09	1.69E-09	1.62E-09	4.41E-10	5.99E-13	1.68E-09	1.70E-09	1.01	1.01
		(5.00)	(5.00)	(5.00)	(4.99)	(7.00)	(5.00)	(5.00)		

Symmetric DG, non-nested grids

h	p	$\|\nabla_{\rm d}(u-u_h)\|$	$\|u-u_h\|_{\rm DG}$	$\|\nabla_{\rm d}u_h+\boldsymbol{\sigma}_h\|$	$\eta_{\rm osc}$	$\|\nabla_{\rm d}(u_h-s_h)\|$	η	$\eta_{\rm DG}$	$I^{\rm eff}$	$I^{\rm eff}_{\rm DG}$
h_0	1	1.07E-00	1.09E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.26E-00	1.17	1.16
$\approx h_0/2$		5.56E-01	5.61E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	6.11E-01	1.09	1.09
$\approx h_0/4$		2.92E-01	2.93E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	3.11E-01	1.06	1.06
$\approx h_0/8$		1.39E-01	1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.45E-01	1.04	1.04
h_0	2	1.54E-01	1.55E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.64E-01	1.06	1.06
$\approx h_0/2$		4.07E-02	4.09E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	4.26E-02	1.04	1.04
$\approx h_0/4$		1.10E-02	1.11E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.15E-02	1.03	1.03
$\approx h_0/8$		2.50E-03	2.52E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	2.59E-03	1.03	1.03
h_0	3	1.37E-02	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.41E-02	1.03	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.88E-03	1.01	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	2.62E-04	1.01	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	2.76E-05	1.01	1.01
h_0	4	9.87E-04	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.01E-03	1.02	1.02
$\approx h_0/2$		6.92E-05	6.93E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	7.00E-05	1.01	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	5.07E-06	1.01	1.01
$\approx h_0/8$		2.58E-07	2.59E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	2.60E-07	1.01	1.01
h_0	5	5.64E-05	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	5.75E-05	1.02	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	2.03E-06	1.01	1.01
$\approx h_0/4$		7.74E-08	7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	7.76E-08	1.00	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.86E-09	1.00	1.00
h_0	6	2.85E-06	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	2.90E-06	1.02	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	5.46E-08	1.01	1.01
$\approx h_0/4$		1.07E-09	1.07E-09	1.07E-09	1.03E-11	6.90E-11	1.08E-09	1.08E-09	1.01	1.01

Nonsymmetric DG, non-nested grids

h	p	$\lVert\nabla_{\mathrm{d}}(u-u_h)\rVert$	$\lVert u-u_h\rVert_{\mathrm{DG}}$	$\lVert\nabla_{\mathrm{d}}u_h+\boldsymbol{\sigma}_h\rVert$	η_{osc}	$\lVert\nabla_{\mathrm{d}}(u_h-s_h)\rVert$	η	η_{DG}	I^{eff}	$I^{\mathrm{eff}}_{\mathrm{DG}}$
h_0	1	1.08E-00	1.09E-00	8.05E-01	5.55E-02	7.98E-01	1.17E-00	1.18E-00	1.09	1.09
$\approx h_0/2$		5.50E-01	5.55E-01	4.18E-01	7.42E-03	3.75E-01	5.66E-01	5.71E-01	1.03	1.03
$\approx h_0/4$		2.84E-01	2.86E-01	2.18E-01	1.04E-03	1.86E-01	2.87E-01	2.89E-01	1.01	1.01
$\approx h_0/8$		1.34E-01	1.35E-01	1.04E-01	1.10E-04	8.64E-02	1.36E-01	1.36E-01	1.01	1.01
h_0	2	1.65E-01	1.72E-01	1.41E-01	5.10E-03	1.71E-01	2.24E-01	2.30E-01	1.36	1.33
$\approx h_0/2$		4.28E-02	4.46E-02	3.67E-02	3.53E-04	4.74E-02	6.01E-02	6.14E-02	1.41	1.38
$\approx h_0/4$		1.14E-02	1.19E-02	9.86E-03	2.51E-05	1.29E-02	1.63E-02	1.66E-02	1.43	1.40
$\approx h_0/8$		2.58E-03	2.70E-03	2.24E-03	1.30E-06	2.99E-03	3.74E-03	3.82E-03	1.45	1.42
h_0	3	1.53E-02	1.54E-02	1.34E-02	3.58E-04	9.19E-03	1.65E-02	1.66E-02	1.08	1.08
$\approx h_0/2$		2.07E-03	2.07E-03	1.79E-03	1.26E-05	1.22E-03	2.18E-03	2.18E-03	1.05	1.05
$\approx h_0/4$		2.99E-04	2.99E-04	2.64E-04	4.73E-07	1.59E-04	3.08E-04	3.09E-04	1.03	1.03
$\approx h_0/8$		3.16E-05	3.17E-05	2.82E-05	1.15E-08	1.60E-05	3.24E-05	3.25E-05	1.02	1.02
h_0	4	1.11E-03	1.12E-03	9.80E-04	2.12E-05	7.21E-04	1.23E-03	1.24E-03	1.11	1.11
$\approx h_0/2$		7.71E-05	7.75E-05	6.89E-05	3.96E-07	5.08E-05	8.59E-05	8.63E-05	1.11	1.11
$\approx h_0/4$		5.66E-06	5.69E-06	5.05E-06	7.58E-09	3.76E-06	6.30E-06	6.33E-06	1.11	1.11
$\approx h_0/8$		2.89E-07	2.91E-07	2.58E-07	8.96E-11	1.96E-07	3.24E-07	3.26E-07	1.12	1.12
h_0	5	6.23E-05	6.24E-05	5.62E-05	1.06E-06	3.23E-05	6.57E-05	6.58E-05	1.05	1.05
$\approx h_0/2$		2.26E-06	2.27E-06	2.04E-06	9.88E-09	1.17E-06	2.36E-06	2.36E-06	1.04	1.04
$\approx h_0/4$		8.86E-08	8.87E-08	8.17E-08	1.01E-10	3.90E-08	9.06E-08	9.06E-08	1.02	1.02
$\approx h_0/8$		2.11E-09	2.12E-09	1.96E-09	1.70E-12	9.02E-10	2.16E-09	2.16E-09	1.02	1.02
h_0	6	3.18E-06	3.18E-06	2.91E-06	4.70E-08	1.66E-06	3.39E-06	3.39E-06	1.07	1.07
$\approx h_0/2$		6.00E-08	6.01E-08	5.57E-08	2.40E-10	3.07E-08	6.38E-08	6.39E-08	1.06	1.06
$\approx h_0/4$		1.20E-09	1.20E-09	1.12E-09	1.03E-11	6.01E-10	1.28E-09	1.28E-09	1.07	1.07

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Singular case & *hp*-adaptivity

Model problem

$$
\begin{aligned}
-\Delta u &= 0 \quad \text{in } \Omega :=]-1,1[^2 \setminus [0,1]^2, \\
u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

$$u(r,\phi) = r^{2/3}\sin(2\phi/3)$$

Discretization (with V. Dolejší)

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

Singular case & *hp*-adaptivity

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Singular case & *hp*-adaptivity

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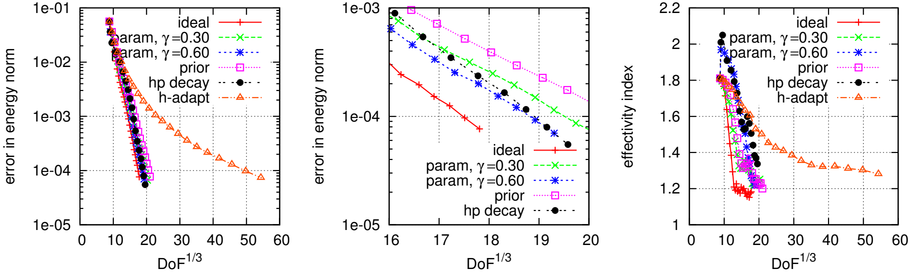
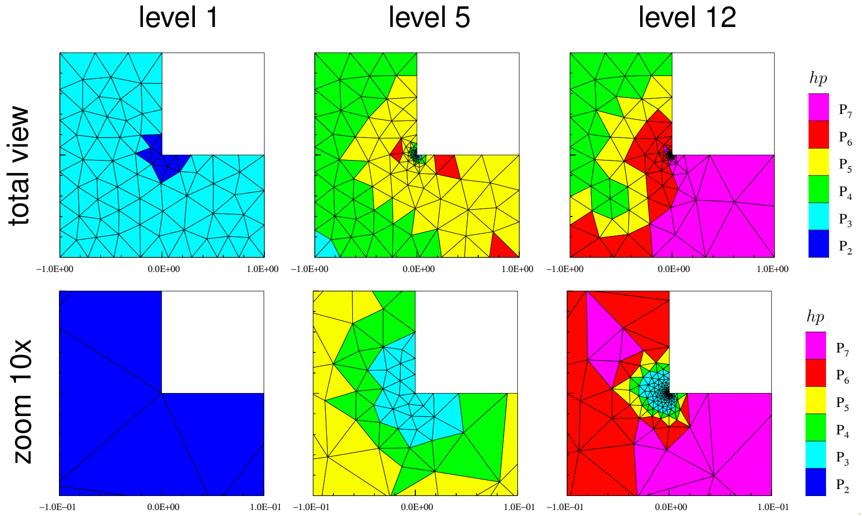
- *hp*-adaptive refinement

hp-adaptive refinement: **exponential convergence**



hp-refinement grids



Outline

Conclusions and future directions

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- stability of the best piecewise polynomial approximation
- polynomial-degree-robust local efficiency of a posteriori error estimates
- a framework covering all standard numerical methods (conforming FEs, nonconforming FEs, discontinuous Galerkin, mixed FEs . . .)

Ongoing generalizations

- transmission problems with sing changing coefficients
- singularly-perturbed reaction-diffusion problems
- Stokes equation
- eigenvalue problems
- heat equation

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Thank you for your attention!