Estimations d'erreur a posteriori et adaptivité pour des écoulements diphasiques

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Outline

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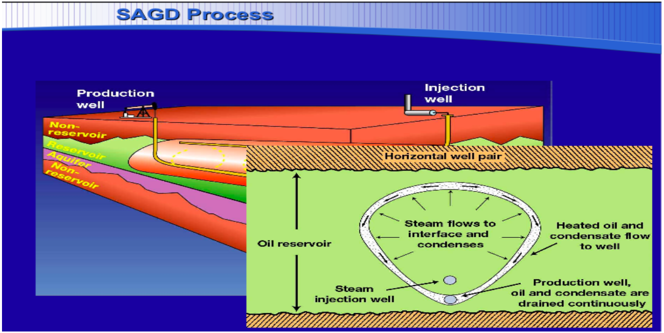
Oil production

Oil production

- oil – one of the major energy supply of today's world
- need for efficient production
- high prices – question of rentability



Oil reservoir & steam-assisted gravity drainage

Numerical simulation difficulties

Numerical simulation difficulties

- highly nonlinear (degenerate) system of partial differential equations
- involves phase transitions
- different time and space scales (orders of magnitude difference)
- highly contrasted, discontinuous coefficients
- complicated 3D geometries
- unstructured and nonmatching grids
- presence of evolving sharp fronts

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Previous results

Model analysis

- Kröner & Luckhaus (1984)
- Chavent & Jaffré (1986)
- Antontsev, Kazhikhov, & Monakhov (1990)
- Chen (2001)

Convergence and a priori estimates

- Chen & Ewing (2001)
- Michel (2003)
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A posteriori indicators

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Two-phase flow

The model

$$\partial_t s_\alpha + \nabla\cdot\mathbf{u}_\alpha = q_\alpha(s_\alpha), \qquad \alpha \in \{\mathrm{n}, \mathrm{w}\},$$
$$\mathbf{u}_\alpha = -\underline{\mathbf{K}}\eta_\alpha(s_\alpha)\nabla p_\alpha, \qquad \alpha \in \{\mathrm{n}, \mathrm{w}\},$$
$$s_\mathrm{n} + s_\mathrm{w} = 1,$$
$$p_\mathrm{n} - p_\mathrm{w} = \pi(s_\mathrm{n})$$

- two immiscible, incompressible fluids
- p_n, p_w: unknown nonwetting and wetting phase pressures
- s_n, s_w: unknown nonwetting and wetting phase saturations
- $\pi(\cdot)$: the nonlinear capillary pressure function
- $\eta_\mathrm{n}(\cdot)$, $\eta_\mathrm{w}(\cdot)$: the nonlinear phase mobilities functions
- $\underline{\mathbf{K}}$ permeability tensor, $q_\mathrm{n}(\cdot)$, $q_\mathrm{w}(\cdot)$ sources

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Outline

Notation and transformations

Notation and transformations

- $s := s_{\mathrm{n}}$
- $$f(s) := \frac{\eta_{\mathrm{n}}(s)}{\eta_{\mathrm{n}}(s) + \eta_{\mathrm{w}}(1-s)}, \quad \lambda(s) := \eta_{\mathrm{w}}(1-s) f(s)$$
- Kirchhoff transform
$$\varphi(s) := \int_0^s \lambda(a) \pi'(a) \, \mathrm{d}a$$
- global pressure
$$P := P(s, p_{\mathrm{n}}) := p_{\mathrm{n}} - \int_0^{\pi(s)} \frac{\eta_{\mathrm{w}}(1-\pi^{-1}(a))}{\eta_{\mathrm{n}}(\pi^{-1}(a)) + \eta_{\mathrm{w}}(1-\pi^{-1}(a))} \, \mathrm{d}a$$
- $M(s) := \eta_{\mathrm{w}}(1-s) + \eta_{\mathrm{n}}(s)$
- $q_{\mathrm{t}}(s) := q_{\mathrm{n}}(s) + q_{\mathrm{w}}(1-s)$
- f, λ, φ, P, M, q_{t} only needed for the theoretical analysis, not in the scheme
- s^0: initial condition
- $\overline{s}$, $\overline{P}$: Dirichlet boundary conditions

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Weak formulation

Functional space for the weak solution

$$\mathcal{E} := \big\{(s,P) \mid s \in \mathcal{C}([0,T];L^2(\Omega)),\ \partial_t s \in L^2((0,T);H^{-1}(\Omega)),$$
$$\varphi(s)-\varphi(\overline{s}) \in L^2((0,T);H^1_0(\Omega)),\ P-\overline{P} \in L^2((0,T);H^1_0(\Omega))\big\}$$

Definition (Weak solution)

A weak solution is a pair $(s,P) \in \mathcal{E}$ such that $s(\cdot,0)=s^0$ and for all $\psi \in L^2((0,T);H^1_0(\Omega))$,

$$\int_0^T \langle \partial_t s(\cdot,\theta);\psi(\cdot,\theta)\rangle_{H^{-1},H^1_0}\,\mathrm{d}\theta + \iint_{Q_T} \underline{\mathbf{K}}(\eta_{\mathrm{n}}(s)\nabla P + \nabla\varphi(s))\cdot\nabla\psi\,\mathrm{d}\mathbf{x}\mathrm{d}\theta$$
$$= \iint_{Q_T} q_{\mathrm{n}}(s)\psi\,\mathrm{d}\mathbf{x}\mathrm{d}\theta,$$

$$\iint_{Q_T} \underline{\mathbf{K}}M(s)\nabla P\cdot\nabla\psi\,\mathrm{d}\mathbf{x}\mathrm{d}\theta = \iint_{Q_T} q_{\mathrm{t}}(s)\psi\,\mathrm{d}\mathbf{x}\mathrm{d}\theta.$$

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A posteriori error estimate

Functional space for the approximate solution

$$\mathcal{E}_\tau := \{(s,P) \mid s \in V_\tau,\ \text{pw affine-in-time subspace of } \mathcal{C}([0,T];L^2(\Omega)),\ \varphi(s)-\varphi(\overline{s}) \in L^2((0,T);H_0^1(\Omega)),\ P-\overline{P} \in L^2((0,T);H_0^1(\Omega))\}$$

Theorem (A posteriori error estimate)

Let (s,P) be the weak solution. Let $(s_{h\tau},P_{h\tau}) \in \mathcal{E}_\tau$ be arbitrary. Let there exist equilibrated fluxes reconstructions $\mathbf{u}_{\alpha,h}$ for each phase $\alpha \in \mathrm{n},\mathrm{w}$. Then there exists $C>0$ such that

$$\|s_{h\tau}-s\|^2_{L^2(0,T;H^{-1}(\Omega))}+\|P_{h\tau}-P\|^2_{L^2(0,T;H_0^1(\Omega))}+\|\varphi(s_{h\tau})-\varphi(s)\|^2_{L^2(Q_T)}$$
$$\leq C\|s_{h\tau}(\cdot,0)-s^0\|^2_{H^{-1}(\Omega)} + C$$
$$\sum_{n=1}^{N}\sum_{\alpha\in\{\mathrm{n},\mathrm{t}\}}\int_{I_n}\left(\left\{\sum_{D\in\mathcal{D}_h^n}(\eta^n_{\mathrm{F},\alpha,D}(t)+\eta^n_{\mathrm{R},\alpha,D})^2\right\}^{\frac{1}{2}}+\left\{\sum_{D\in\mathcal{D}_h^n}(\eta^n_{\mathrm{Q},\alpha,D}(t))^2\right\}^{\frac{1}{2}}\right)^2 \mathrm{d}t.$$

A posteriori error estimate

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Equilibrated fluxes reconstructions

Definition (Equilibrated fluxes reconstructions)

Piecewise constant-in-time vector fields $\mathbf{u}_{\mathrm{n},h\tau}$ and $\mathbf{u}_{\mathrm{t},h\tau}$,

$$\mathbf{u}_{\mathrm{n},h}^n := \mathbf{u}_{\mathrm{n},h\tau}|_{I_n},\ \mathbf{u}_{\mathrm{t},h}^n := \mathbf{u}_{\mathrm{t},h\tau}|_{I_n} \in \mathbf{H}(\mathrm{div},\Omega) \qquad \forall n \in \{1,\ldots,N\},$$

$$\int_D \left(\frac{s_h^n - s_h^{n-1}}{\tau^n} + \nabla\cdot\mathbf{u}_{\mathrm{n},h}^n\right)\mathrm{d}\mathbf{x} = \int_D q_{\mathrm{n}}^n(s_h^n)\,\mathrm{d}\mathbf{x} \qquad \forall n,\ \forall D \in \mathcal{D}_h^{\mathrm{int},n},$$

$$\int_D \nabla\cdot\mathbf{u}_{\mathrm{t},h}^n\,\mathrm{d}\mathbf{x} = \int_D q_{\mathrm{t}}^n(s_h^n)\,\mathrm{d}\mathbf{x} \qquad \forall n,\ \forall D \in \mathcal{D}_h^{\mathrm{int},n}.$$

Comments

- $\mathbf{u}_{\mathrm{n},h}^n$: *nonwetting phase flux reconstruction*
- $\mathbf{u}_{\mathrm{t},h}^n$: *total flux reconstruction*
- mimic the basic conservation properties of the model
- $\mathbf{u}_{\mathrm{w},h\tau} := \mathbf{u}_{\mathrm{t},h\tau} - \mathbf{u}_{\mathrm{n},h\tau}$: *wetting phase flux reconstruction*,

$$\int_D \left(-\frac{s_h^n - s_h^{n-1}}{\tau^n} + \nabla\cdot\mathbf{u}_{\mathrm{w},h}^n\right)\mathrm{d}\mathbf{x} = \int_D q_{\mathrm{w}}^n(1-s_h^n)\,\mathrm{d}\mathbf{x} \qquad \forall n,\ \forall D \in \mathcal{D}_h^{\mathrm{int},n},$$

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Outline

Distinguishing different error components

Theorem (Distinguishing different error components)

Consider

- *time step n*
- *linearization step k*
- *iterative algebraic solver step i*

& approximations $(s^{k,i}_{\alpha,h\tau}, P^{k,i}_{\alpha,h\tau})$. Split the flux reconstructions as

$$\mathbf{u}^{n,k,i}_{\alpha,h} := \mathbf{d}^{n,k,i}_{\alpha,h} + \mathbf{l}^{n,k,i}_{\alpha,h} + \mathbf{a}^{n,k,i}_{\alpha,h}, \alpha \in \{\mathrm{n}, \mathrm{w}\}.$$

Then

$$\left(|||\mathcal{R}^n_{\mathrm{n}}(s^{n,k,i}_{h\tau}, P^{n,k,i}_{h\tau})|||^2 + |||\mathcal{R}^n_{\mathrm{t}}(s^{n,k,i}_{h\tau}, P^{n,k,i}_{h\tau})|||^2\right)^{\frac{1}{2}} \leq \eta^{n,k,i}_{\mathrm{sp}} + \eta^{n,k,i}_{\mathrm{tm}} + \eta^{n,k,i}_{\mathrm{lin}} + \eta^{n,k,i}_{\mathrm{alg}}.$$

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Then

$$\begin{aligned}&(|||\mathcal{R}^n_{\mathrm{n}}(s^{n,k,i}_{h\tau}, P^{n,k,i}_{h\tau})|||^2 + |||\mathcal{R}^n_{\mathrm{t}}(s^{n,k,i}_{h\tau}, P^{n,k,i}_{h\tau})|||^2)^{\frac{1}{2}}\\ &\leq \eta^{n,k,i}_{\mathrm{sp}} + \eta^{n,k,i}_{\mathrm{tm}} + \eta^{n,k,i}_{\mathrm{lin}} + \eta^{n,k,i}_{\mathrm{alg}}.\end{aligned}$$

Estimators

- *spatial estimators*

$$\eta_{\mathrm{sp,n},D}^{n,k,i} := \|\mathbf{d}_{\mathrm{n},h}^{n,k,i} - \underline{\mathbf{K}}(\eta(s_{h\tau}^{n,k,i})\nabla P_{h\tau}^{n,k,i} + \nabla\varphi(s_{h\tau}^{n,k,i}))(t^n)\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)},$$

$$\eta_{\mathrm{sp,t},D}^{n,k,i} := \|\mathbf{d}_{\mathrm{t},h}^{n,k,i} - \underline{\mathbf{K}}M(s_{h\tau}^{n,k,i})\nabla P_{h\tau}^{n,k,i}(t^n)\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)}$$

- *temporal estimators*

$$\eta_{\mathrm{tm,n},D}^{n,k,i}(t) := \|\underline{\mathbf{K}}(\eta(s_{h\tau}^{n,k,i})\nabla P_{h\tau}^{n,k,i} + \nabla\varphi(s_{h\tau}^{n,k,i}))(t - t^n)\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)},$$

$$\eta_{\mathrm{tm,t},D}^{n,k,i}(t) := \|\underline{\mathbf{K}}M(s_{h\tau}^{n,k,i})\nabla P_{h\tau}^{n,k,i}(t - t^n)\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)}$$

- *linearization estimators*

$$\eta_{\mathrm{lin,n},D}^{n,k,i} := \|\mathbf{l}_{\mathrm{n},h}^{n,k,i}\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)},$$

$$\eta_{\mathrm{lin,t},D}^{n,k,i} := \|\mathbf{l}_{\mathrm{t},h}^{n,k,i}\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)}$$

- *algebraic estimators*

$$\eta_{\mathrm{alg,n},D}^{n,k,i} := \|\mathbf{a}_{\mathrm{n},h}^{n,k,i}\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)},$$

$$\eta_{\mathrm{alg,t},D}^{n,k,i} := \|\mathbf{a}_{\mathrm{t},h}^{n,k,i}\|_{\underline{\mathbf{K}}^{-\frac{1}{2}};L^2(D)}$$

Outline

Efficiency

Theorem (Efficiency)

Consider the time step n, the linearization step k, and the algebraic solver step i. Let the algebraic, linearization, and temporal estimators do not dominate the overall error estimate. Then there exists $C > 0$ such that

$$\eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}$$
$$\leq C(|||\mathcal{R}_{\mathrm{n}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2 + |||\mathcal{R}_{\mathrm{t}}^{n}(s_{h\tau}^{n,k,i}, P_{h\tau}^{n,k,i})|||^2)^{\frac{1}{2}}.$$

Comments

- algebraic, linearization, and temporal estimators need to be small enough
- local efficiency for the dual norm of the residual

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Outline

Extensions to nonconforming discretizations

Nonconforming discretizations

- $\varphi(s_{h\tau}), P_{h\tau} \notin L^2((0,T); H^1(\Omega))$

Extended dual norm of the residual

- $$\left\{ \inf_{\mathfrak{p} \in L^2((0,T);H^1(\Omega))} \int_0^T \|\underline{\mathbf{K}}(\eta_{\mathrm{w}}(s_{h\tau}) + \eta_{\mathrm{n}}(s_{h\tau}))\nabla(P_{h\tau} - \mathfrak{p})\|^2 \mathrm{d}t \right\}^{\frac{1}{2}}$$
- $$\left\{ \inf_{\mathfrak{q} \in L^2((0,T);H^1(\Omega))} \int_0^T \|\underline{\mathbf{K}}\nabla(\varphi(s_{h\tau}) - \mathfrak{q})\|^2 \mathrm{d}t \right\}^{\frac{1}{2}}$$

Additional nonconformity estimators

- global pressure nonconformity
- Kirchhoff transform nonconformity

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Outline

1. Introduction
2. Mathematical model
   - Global pressure and Kirchhoff transformation
   - Weak solution
3. A posteriori analysis
   - A posteriori error estimate
   - Distinguishing different error components
   - Efficiency
   - Extensions to nonconforming discretizations
4. Applications and numerical experiments
   - Fully implicit cell-centered finite volumes
   - Iteratively coupled implicit pressure–explicit saturation vertex-centered finite volumes
5. Conclusions and future directions

Quarter five spot test problem

Horizontal flow

$$\partial_t(\phi s_\alpha) - \nabla\cdot\left(\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\underline{\mathbf{K}}\nabla p_\alpha\right) = 0,$$

$$s_\mathrm{n} + s_\mathrm{w} = 1,$$

$$p_\mathrm{n} - p_\mathrm{w} = \pi(s_\mathrm{w})$$

Brooks–Corey model

- relative permeabilities

$$k_{\mathrm{r,w}}(s_\mathrm{w}) = s_\mathrm{e}^4, \quad k_{\mathrm{r,n}}(s_\mathrm{w}) = (1-s_\mathrm{e})^2(1-s_\mathrm{e}^2)$$

- capillary pressure

$$\pi(s_\mathrm{w}) = p_\mathrm{d} s_\mathrm{e}^{-\frac{1}{2}}$$

-

$$s_\mathrm{e} := \frac{s_\mathrm{w} - s_\mathrm{rw}}{1 - s_\mathrm{rw} - s_\mathrm{rn}}$$

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Data from Klieber & Rivière (2006)

Data

$$\Omega = (0, 300)\text{m} \times (0, 300)\text{m}, \quad T = 4{\cdot}10^6\text{s},$$

$$\phi = 0.2, \quad \underline{\mathbf{K}} = 10^{-11}\underline{\mathbf{I}}\,\text{m}^2,$$

$$\mu_{\text{w}} = 5{\cdot}10^{-4}\text{kg}\,\text{m}^{-1}\text{s}^{-1}, \quad \mu_{\text{n}} = 2{\cdot}10^{-3}\text{kg}\,\text{m}^{-1}\text{s}^{-1},$$

$$s_{\text{rw}} = s_{\text{rn}} = 0, \quad p_{\text{d}} = 5{\cdot}10^3\text{kg}\,\text{m}^{-1}\text{s}^{-2}$$

Initial condition ($\widetilde{K}$ 18m × 18m lower left corner block)

$$s_{\text{w}}^0 = 0.2 \text{ on } K \in \mathcal{T}_h,\ K \notin \widetilde{K},$$

$$s_{\text{w}}^0 = 0.95 \text{ on } K \in \mathcal{T}_h,\ K \in \widetilde{K}$$

Boundary conditions ($\widehat{K}$ 18m × 18m upper right corner block)

- no flow Neumann boundary conditions everywhere except of $\partial\widetilde{K} \cap \partial\Omega$ and $\partial\widehat{K} \cap \partial\Omega$
- $\widetilde{K}$ – injection well: $s_{\text{w}} = 0.95$, $p_{\text{w}} = 3.45{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$
- $\widehat{K}$ – production well: $s_{\text{w}} = 0.2$, $p_{\text{w}} = 2.41{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$

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Outline

Cell-centered finite volume scheme

Cell-centered finite volume scheme

For all $1 \leq n \leq N$, look for $s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}$ such that

$$\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{\sigma_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{w},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

$$-\phi \frac{s^n_{\mathrm{w},K} - s^{n-1}_{\mathrm{w},K}}{\tau^n}|K| + \sum_{\sigma_{KL} \in \mathcal{E}^{\mathrm{int}}_K} F_{\mathrm{n},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) = 0,$$

where the fluxes are given by

$$F_{\mathrm{w},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,w}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,w}}(s^n_{\mathrm{w},L})}{2}|\underline{\mathbf{K}}|\frac{\bar{p}^n_{\mathrm{w},L} - \bar{p}^n_{\mathrm{w},K}}{|\mathbf{x}_K - \mathbf{x}_L|}|\sigma_{KL}|,$$

$$F_{\mathrm{n},\sigma_{KL}}(s^n_{\mathrm{w},h}, \bar{p}^n_{\mathrm{w},h}) := -\frac{\eta_{\mathrm{r,n}}(s^n_{\mathrm{w},K}) + \eta_{\mathrm{r,n}}(s^n_{\mathrm{w},L})}{2}|\underline{\mathbf{K}}| \times \frac{\bar{p}^n_{\mathrm{w},L} + \pi(s^n_{\mathrm{w},L}) - (\bar{p}^n_{\mathrm{w},K} + \pi(s^n_{\mathrm{w},K}))}{|\mathbf{x}_K - \mathbf{x}_L|}|\sigma_{KL}|$$

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Linearization and algebraic solution

Linearization step k and algebraic step i

Couple $s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}$ such that

$$\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{\sigma_{KL} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{w},\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{w},K}^{n,k,i},$$

$$-\phi \frac{s_{\mathrm{w},K}^{n,k,i} - s_{\mathrm{w},K}^{n-1}}{\tau^n}|K| + \sum_{\sigma_{KL} \in \mathcal{E}_K^{\mathrm{int}}} F_{\mathrm{n},\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) = -R_{\mathrm{n},K}^{n,k,i},$$

where the linearized fluxes are given by

$$F_{\alpha,\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}) := F_{\alpha,\sigma_{KL}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) + \sum_{M \in \{K,L\}} \frac{\partial F_{\alpha,\sigma_{KL}}}{\partial s_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (s_{\mathrm{w},M}^{n,k,i} - s_{\mathrm{w},M}^{n,k-1}) + \sum_{M \in \{K,L\}} \frac{\partial F_{\alpha,\sigma_{KL}}}{\partial \bar{p}_{\mathrm{w},M}}(s_{\mathrm{w},h}^{n,k-1}, \bar{p}_{\mathrm{w},h}^{n,k-1}) \cdot (\bar{p}_{\mathrm{w},M}^{n,k,i} - \bar{p}_{\mathrm{w},M}^{n,k-1}).$$

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Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

$$(\mathbf{d}_{\alpha,h}^{n,k,i}\cdot\mathbf{n}_K,1)_{\sigma_{KL}} := F_{\alpha,\sigma_{KL}}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),$$
$$((\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K,1)_{\sigma_{KL}} := F_{\alpha,\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i},\bar{p}_{\mathrm{w},h}^{n,k,i}),$$
$$\mathbf{a}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i+\nu}+\mathbf{l}_{\alpha,h}^{n,k,i+\nu}-(\mathbf{d}_{\alpha,h}^{n,k,i}+\mathbf{l}_{\alpha,h}^{n,k,i})$$

Phase pressures postprocessing

- Piecewise constant $\bar{p}_{\alpha,h}^{n,k,i}$ postprocessed to piecewise quadratic $p_{\alpha,h}^{n,k,i}$:

$$-\eta_{\mathrm{r,w}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{w},h}^{n,k,i}|_K) = \mathbf{d}_{\mathrm{w},h}^{n,k,i}|_K,$$
$$p_{\mathrm{w},h}^{n,k,i}(\mathbf{x}_K) = \bar{p}_{\mathrm{w},K}^{n,k,i},$$

$$-\eta_{\mathrm{r,n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(p_{\mathrm{n},h}^{n,k,i}|_K) = \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,$$
$$p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) = \pi(s_{\mathrm{w},K}^{n,k,i})+\bar{p}_{\mathrm{w},K}^{n,k,i}$$

Fluxes reconstructions and pressure postprocessing

Fluxes reconstructions

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$$((\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})\cdot\mathbf{n}_K, 1)_{\sigma_{KL}} := F_{\alpha,\sigma_{KL}}^{k-1}(s_{\mathrm{w},h}^{n,k,i}, \bar{p}_{\mathrm{w},h}^{n,k,i}),$$

$$\mathbf{a}_{\alpha,h}^{n,k,i} := \mathbf{d}_{\alpha,h}^{n,k,i+\nu} + \mathbf{l}_{\alpha,h}^{n,k,i+\nu} - (\mathbf{d}_{\alpha,h}^{n,k,i} + \mathbf{l}_{\alpha,h}^{n,k,i})$$

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$$p_{\mathrm{n},h}^{n,k,i}(\mathbf{x}_K) = \pi(s_{\mathrm{w},K}^{n,k,i}) + \bar{p}_{\mathrm{w},K}^{n,k,i}$$

Global pressure and Kirchhoff transform

Global pressure and Kirchhoff transform postprocessing

- Piecewise quadratic global pressure and Kirchhoff transform used in the estimators:

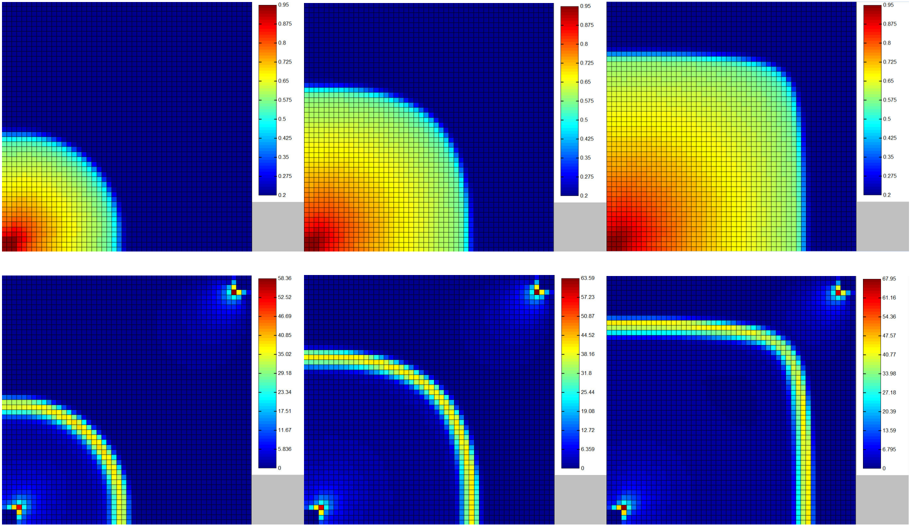
$$-\left(\eta_{\mathrm{w}}(s_{\mathrm{w},K}^{n,k,i}) + \eta_{\mathrm{n}}(s_{\mathrm{w},K}^{n,k,i})\right)\underline{\mathbf{K}}\nabla(\mathfrak{p}_h^{n,k,i}|_K) = (\mathbf{d}_{\mathrm{w},h}^{n,k,i} + \mathbf{d}_{\mathrm{n},h}^{n,k,i})|_K,$$

$$\mathfrak{p}_h^{n,k,i}(\mathbf{x}_K) = P(\bar{p}_{\mathrm{w},K}^{n,k,i}, s_{\mathrm{w},K}^{n,k,i}),$$

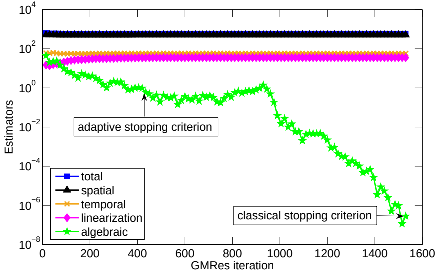
$$\underline{\mathbf{K}}\nabla(\mathfrak{q}_h^{n,k,i}|_K) = \eta_{\mathrm{n}}(s_{\mathrm{w},K}^{n,k,i})\underline{\mathbf{K}}\nabla(\mathfrak{p}_h^{n,k,i}|_K) + \mathbf{d}_{\mathrm{n},h}^{n,k,i}|_K,$$

$$\mathfrak{q}_h^{n,k,i}(\mathbf{x}_K) = \varphi(s_{\mathrm{w},K}^{n,k,i})$$

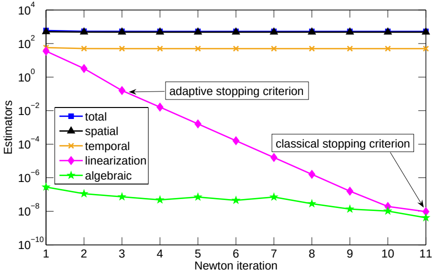
Water saturation/estimators evolution



Estimators and stopping criteria

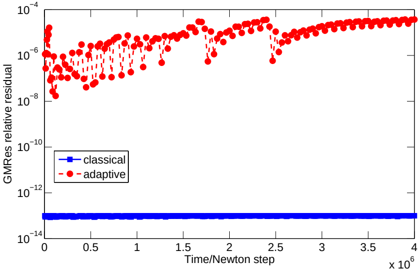


Estimators in function of GMRes iterations

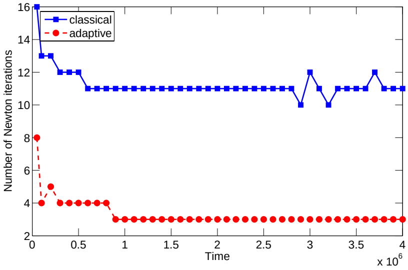


Estimators in function of Newton iterations

GMRes relative residual/Newton iterations

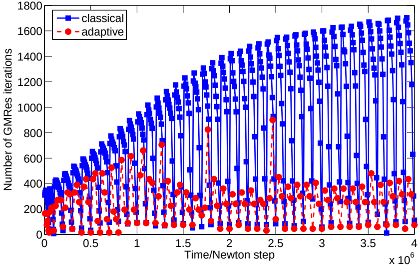


GMRes relative residual

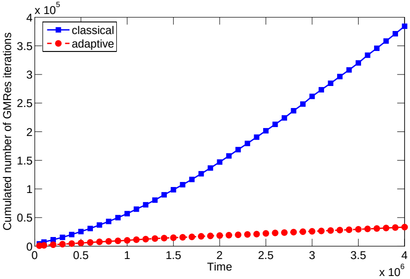


Newton iterations

GMRes iterations



Per time and Newton step



Cumulated

Outline

Vertex-centered finite volumes

Implicit pressure equation on step k

$$-\big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D \\ +\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D,1\big)_{\partial D\setminus\partial\Omega}=0 \quad \forall D\in\mathcal{D}_h^{\mathrm{int},n}$$

Explicit saturation equation on step k

$$s_{\mathrm{w},D}^{n,k}:=\frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D,1\big)_{\partial D\setminus\partial\Omega}+s_{\mathrm{w},D}^{n-1} \quad \forall D\in\mathcal{D}_h^{\mathrm{int},n}$$

Vertex-centered finite volumes

Implicit pressure equation on step k

$$-\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D \\ +\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} = 0 \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

Explicit saturation equation on step k

$$s_{\mathrm{w},D}^{n,k} := \frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k}\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

Linearization and algebraic solution

Iterative coupling step *k* and algebraic step *i*

$$-\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} = -R_{\mathrm{t},D}^{n,k,i} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

$$s_{\mathrm{w},D}^{n,k,i} := \frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1}$$

Linearization and algebraic solution

Iterative coupling step k and algebraic step i

$$-\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1}) + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D \\ +\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} = -R_{\mathrm{t},D}^{n,k,i} \quad \forall D \in \mathcal{D}_h^{\mathrm{int},n}$$

$$s_{\mathrm{w},D}^{n,k,i} := \frac{\tau^n}{\phi|D|}\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} + s_{\mathrm{w},D}^{n-1}$$

Fluxes reconstructions

Total fluxes

$$(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D,1)_\sigma := -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1\Big)_\sigma,$$

$$((\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})\cdot\mathbf{n}_D,1)_\sigma := -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D,1\Big)_\sigma,$$

$$\mathbf{a}_{\mathrm{t},h}^{n,k,i} := \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu}+\mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu}-(\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})$$

Wetting fluxes

$$(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1)_\sigma := -\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\big)_\sigma,$$

$$((\mathbf{d}_{\mathrm{w},h}^{n,k,i}+\mathbf{l}_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1)_\sigma := -\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\big)_\sigma,$$

$$\mathbf{a}_{\mathrm{w},h}^{n,k,i} := 0$$

Fluxes reconstructions

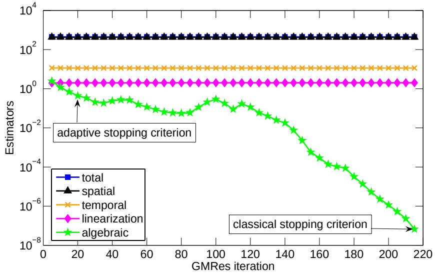
Total fluxes

$$
\begin{aligned}
(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D,1)_\sigma &:= -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D \\
&\qquad + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1\Big)_\sigma, \\
((\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})\cdot\mathbf{n}_D,1)_\sigma &:= -\Big(\big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})+\eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D \\
&\qquad + \eta_{\mathrm{r,n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{\pi}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D,1\Big)_\sigma, \\
\mathbf{a}_{\mathrm{t},h}^{n,k,i} &:= \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu}+\mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu}-(\mathbf{d}_{\mathrm{t},h}^{n,k,i}+\mathbf{l}_{\mathrm{t},h}^{n,k,i})
\end{aligned}
$$

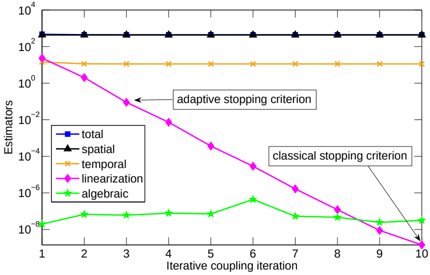
Wetting fluxes

$$
\begin{aligned}
(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1)_\sigma &:= -\Big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\Big)_\sigma, \\
((\mathbf{d}_{\mathrm{w},h}^{n,k,i}+\mathbf{l}_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1)_\sigma &:= -\Big(\eta_{\mathrm{r,w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\Big)_\sigma, \\
\mathbf{a}_{\mathrm{w},h}^{n,k,i} &:= 0
\end{aligned}
$$

Estimators and stopping criteria

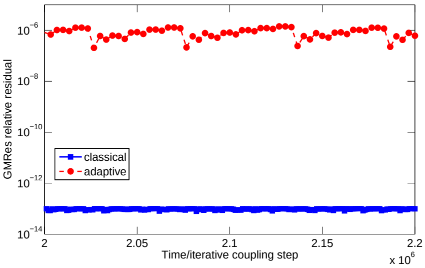


Estimators in function of GMRes iterations

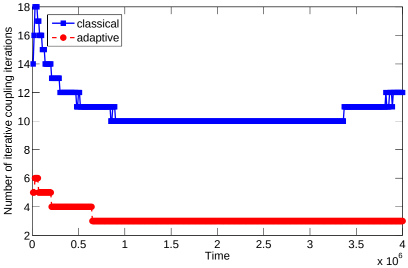


Estimators in function of iterative coupling iterations

GMRes relative residual/iterative coupling iterations

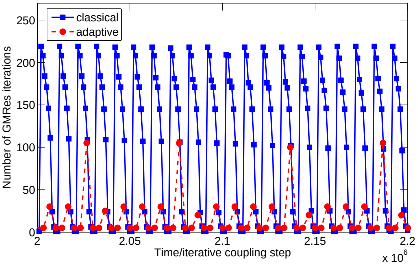


GMRes relative residual

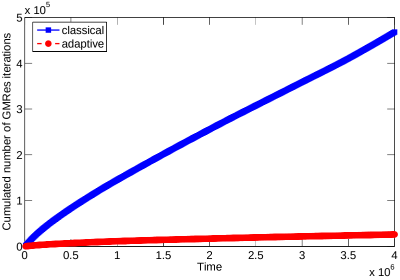


Iterative coupling iterations

GMRes iterations



Per time and iterative coupling step



Cumulated

Outline

Conclusions

Complete adaptivity

- only a **necessary number** of **algebraic solver iterations** on each linearization step
- only a **necessary number** of **linearization iterations**
- **space-time** mesh **adaptivity**
- **“smart online decisions”**: algebraic step / linearization step / time step refinement / space mesh refinement
- important **computational savings**
- error upper bound via **a posteriori error estimates**

Future directions

- other complex problems
- convergence and optimality

Conclusions

Complete adaptivity

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Bibliography

Bibliography

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Merci de votre attention !