A posteriori error estimates for adaptive mesh refinement and error control

Martin Vohralík

Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris 6)

Luminy, February 9, 2009

Outline

Outline

1. Introduction
2. Pure diffusion and conforming methods
   - Optimal abstract framework and a first estimate
   - Optimal a posteriori error estimate
   - Estimates for finite elements
   - Efficiency of the a posteriori error estimate
   - Numerical experiments
3. Convection–reaction–diffusion and nonconforming methods
   - Semi-robust energy norm estimates for DGs
   - Fully robust augmented norm estimates for DGs
   - Numerical experiments (FVs & DGs)
4. Estimates including the algebraic error
   - Problem and estimates
   - Numerical experiments
5. Conclusions and future work

What is an a posteriori error estimate

A posteriori error estimate

- Let p be a weak solution of a PDE.
- Let p_h be its approximate numerical solution.
- A priori error estimate: $\|p - p_h\|_\Omega \leq f(p)h^q$. Dependent on p, not computable. Useful in theory.
- A posteriori error estimate: $\|p - p_h\|_\Omega \lesssim f(p_h)$. Only uses p_h, computable. Great in practice.

Usual form

- $f(p_h)^2 = \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$, where $\eta_K(p_h)$ is an element indicator.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

What is an a posteriori error estimate

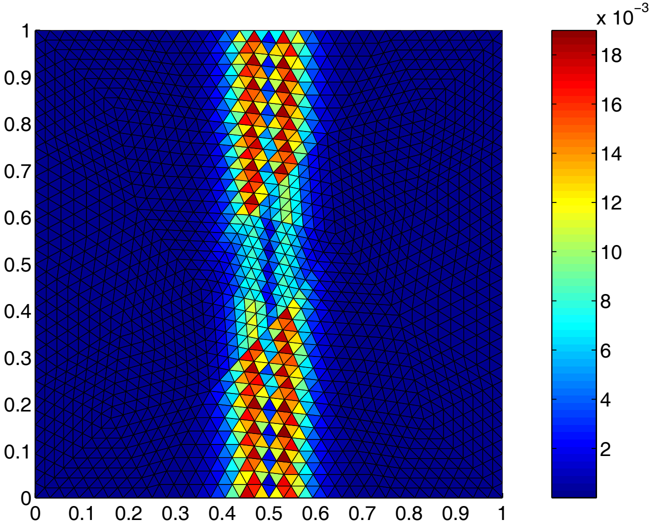
A posteriori error estimate

- Let p be a weak solution of a PDE.
- Let p_h be its approximate numerical solution.
- A priori error estimate: $\|p - p_h\|_\Omega \leq f(p)h^q$. Dependent on p, not computable. Useful in theory.
- A posteriori error estimate: $\|p - p_h\|_\Omega \lesssim f(p_h)$. Only uses p_h, computable. Great in practice.

Usual form

- $f(p_h)^2 = \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$, where $\eta_K(p_h)$ is an element indicator.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

Example of an a posteriori error estimator



Estimated error distribution

What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|p - p_h\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- no undetermined constant: error control
- remark (reliability): $\|p - p_h\|_\Omega^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency (local error lower bound)

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$
- necessary for optimal mesh refinement

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|_\Omega^2 \to 1$
- overestimation factor goes to one with mesh size

Robustness

- $C_{\mathrm{eff},K}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|p - p_h\|_\Omega^2 \leq \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$
- no undetermined constant: error control
- remark (reliability): $\|p - p_h\|_\Omega^2 \leq C \sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency (local error lower bound)

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L\,\text{close to}\,K} \|p - p_h\|_L^2$
- necessary for optimal mesh refinement

Asymptotic exactness

- $\sum_{K\in\mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|_\Omega^2 \to 1$
- overestimation factor goes to one with mesh size

Robustness

- $C_{\mathrm{eff},K}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|p - p_h\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- no undetermined constant: error control
- remark (reliability): $\|p - p_h\|_\Omega^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency (local error lower bound)

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$
- necessary for optimal mesh refinement

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|_\Omega^2 \to 1$
- overestimation factor goes to one with mesh size

Robustness

- $C_{\mathrm{eff},K}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|p - p_h\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- no undetermined constant: error control
- remark (reliability): $\|p - p_h\|_\Omega^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency (local error lower bound)

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L\,\text{close to}\,K} \|p - p_h\|_L^2$
- necessary for optimal mesh refinement

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|_\Omega^2 \to 1$
- overestimation factor goes to one with mesh size

Robustness

- $C_{\mathrm{eff},K}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|p - p_h\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- no undetermined constant: error control
- remark (reliability): $\|p - p_h\|_\Omega^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency (local error lower bound)

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$
- necessary for optimal mesh refinement

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|_\Omega^2 \to 1$
- overestimation factor goes to one with mesh size

Robustness

- $C_{\mathrm{eff},K}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

What an a posteriori error estimate should fulfill

Guaranteed upper bound (global error upper bound)

- $\|p - p_h\|_\Omega^2 \leq \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$
- no undetermined constant: error control
- remark (reliability): $\|p - p_h\|_\Omega^2 \leq C \sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2$

Local efficiency (local error lower bound)

- $\eta_K(p_h)^2 \leq C_{\mathrm{eff},K}^2 \sum_{L \text{ close to } K} \|p - p_h\|_L^2$
- necessary for optimal mesh refinement

Asymptotic exactness

- $\sum_{K \in \mathcal{T}_h} \eta_K(p_h)^2 / \|p - p_h\|_\Omega^2 \to 1$
- overestimation factor goes to one with mesh size

Robustness

- $C_{\mathrm{eff},K}$ does not depend on data, mesh, or solution

Negligible evaluation cost

- estimators can be evaluated locally

Previous results

Continuous finite elements

- Babuška and Rheinboldt (1978), introduction
- Ladevèze and Leguillon (1983), equilibrated fluxes estimates (equality of Prager and Synge (1947))
- Zienkiewicz and Zhu (1987), averaging-based estimates
- Verfürth (1996, book), residual-based estimates
- Repin (1997), functional a posteriori error estimates
- Destuynder and Métivet (1999), equilibrated fluxes estimates
- Ainsworth and Oden (2000, book), equilibrated residual estimates
- Luce and Wohlmuth (2004), equilibrated fluxes estimates
- Braess and Schöberl (2008), equilibrated fluxes estimates

Previous results

Finite volumes

- Ohlberger (2001), non-energy norm estimates

Discontinuous Galerkin finite elements

- Karakashian and Pascal (2003), residual-based estimates
- Ainsworth (2007), reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (2008), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

Problems with discontinuous coefficients

- Bernardi and Verfürth (2000), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

Convection–diffusion problems

- Verfürth (1998, 2005), conforming finite elements
- Sangalli (2008), conforming finite elements

Previous results

Finite volumes

- Ohlberger (2001), non-energy norm estimates

Discontinuous Galerkin finite elements

- Karakashian and Pascal (2003), residual-based estimates
- Ainsworth (2007), reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (2008), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

Problems with discontinuous coefficients

- Bernardi and Verfürth (2000), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

Convection–diffusion problems

- Verfürth (1998, 2005), conforming finite elements
- Sangalli (2008), conforming finite elements

Previous results

Finite volumes

- Ohlberger (2001), non-energy norm estimates

Discontinuous Galerkin finite elements

- Karakashian and Pascal (2003), residual-based estimates
- Ainsworth (2007), reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (2008), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

Problems with discontinuous coefficients

- Bernardi and Verfürth (2000), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

Convection–diffusion problems

- Verfürth (1998, 2005), conforming finite elements
- Sangalli (2008), conforming finite elements

Previous results

Finite volumes

- Ohlberger (2001), non-energy norm estimates

Discontinuous Galerkin finite elements

- Karakashian and Pascal (2003), residual-based estimates
- Ainsworth (2007), reconstruction of side fluxes
- Kim (2007), Cochez-Dhondt and Nicaise (2008), reconstruction of equilibrated $\mathbf{H}(\mathrm{div}, \Omega)$-conforming fluxes

Problems with discontinuous coefficients

- Bernardi and Verfürth (2000), conforming finite elements
- Ainsworth (2005), nonconforming finite elements

Convection–diffusion problems

- Verfürth (1998, 2005), conforming finite elements
- Sangalli (2008), conforming finite elements

Outline

1. Introduction
2. Pure diffusion and conforming methods
   - Optimal abstract framework and a first estimate
   - Optimal a posteriori error estimate
   - Estimates for finite elements
   - Efficiency of the a posteriori error estimate
   - Numerical experiments
3. Convection–reaction–diffusion and nonconforming methods
   - Semi-robust energy norm estimates for DGs
   - Fully robust augmented norm estimates for DGs
   - Numerical experiments (FVs & DGs)
4. Estimates including the algebraic error
   - Problem and estimates
   - Numerical experiments
5. Conclusions and future work

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$
\begin{aligned}
-\nabla \cdot (a \nabla p) &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- a is a piecewise constant scalar, inhomogeneous

Bilinear form $\mathcal{B}$

$$
\mathcal{B}(p, \varphi) := (a \nabla p, \nabla \varphi), \qquad p, \varphi \in H_0^1(\Omega).
$$

Weak solution

Find $p \in H_0^1(\Omega)$ such that $\mathcal{B}(p, \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega)$.

Energy norm

$$
|||\varphi|||^2 := \|a^{\frac{1}{2}} \nabla \varphi\|^2, \qquad \varphi \in H_0^1(\Omega).
$$

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$
\begin{aligned}
-\nabla \cdot (a \nabla p) &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- a is a piecewise constant scalar, inhomogeneous

Bilinear form $\mathcal{B}$

$$\mathcal{B}(p, \varphi) := (a \nabla p, \nabla \varphi), \qquad p, \varphi \in H_0^1(\Omega).$$

Weak solution

Find $p \in H_0^1(\Omega)$ such that $\mathcal{B}(p, \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega)$.

Energy norm

$$|||\varphi|||^2 := \|a^{\frac{1}{2}} \nabla \varphi\|^2, \qquad \varphi \in H_0^1(\Omega).$$

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$\begin{aligned} -\nabla\cdot(a\nabla p) &= f \quad \text{in } \Omega,\\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- a is a piecewise constant scalar, inhomogeneous

Bilinear form $\mathcal{B}$

$$\mathcal{B}(p,\varphi) := (a\nabla p, \nabla\varphi), \qquad p, \varphi \in H^1_0(\Omega).$$

Weak solution

Find $p \in H^1_0(\Omega)$ such that $\mathcal{B}(p,\varphi) = (f,\varphi) \qquad \forall \varphi \in H^1_0(\Omega)$.

Energy norm

$$|||\varphi|||^2 := \|a^{\frac{1}{2}}\nabla\varphi\|^2, \qquad \varphi \in H^1_0(\Omega).$$

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$\begin{aligned} -\nabla \cdot (a\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- a is a piecewise constant scalar, inhomogeneous

Bilinear form $\mathcal{B}$

$$\mathcal{B}(p, \varphi) := (a\nabla p, \nabla\varphi), \qquad p, \varphi \in H_0^1(\Omega).$$

Weak solution

Find $p \in H_0^1(\Omega)$ such that $\mathcal{B}(p, \varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega)$.

Energy norm

$$|||\varphi|||^2 := \|a^{\frac{1}{2}}\nabla\varphi\|^2, \qquad \varphi \in H_0^1(\Omega).$$

A model problem with discontinuous coefficients

Model problem with discontinuous coefficients

$$\begin{aligned} -\nabla\cdot(a\nabla p) &= f \quad \text{in } \Omega,\\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Assumptions

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a polygonal domain
- a is a piecewise constant scalar, inhomogeneous

Bilinear form $\mathcal{B}$

$$\mathcal{B}(p,\varphi) := (a\nabla p, \nabla\varphi), \qquad p,\varphi \in H_0^1(\Omega)\,.$$

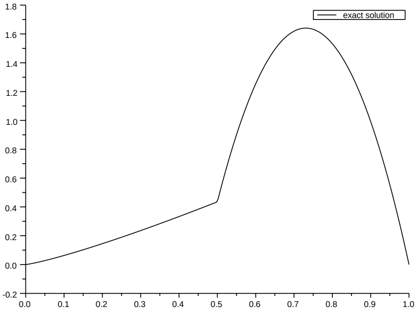
Weak solution

Find $p \in H_0^1(\Omega)$ such that $\mathcal{B}(p,\varphi) = (f,\varphi) \qquad \forall \varphi \in H_0^1(\Omega)$.

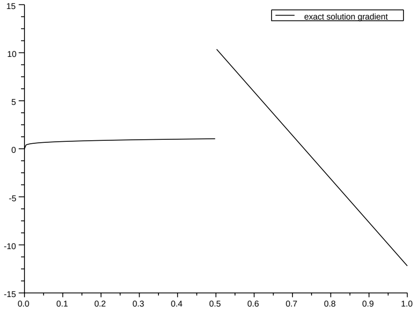
Energy norm

$$|||\varphi|||^2 := \|a^{\frac{1}{2}}\nabla\varphi\|^2, \qquad \varphi \in H_0^1(\Omega).$$

Properties of the weak solution

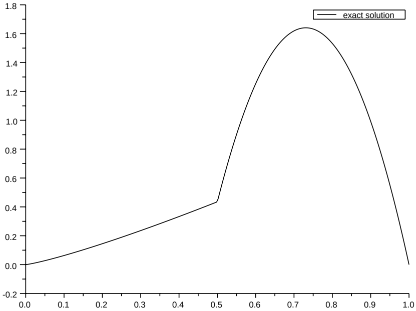


Solution p is in $H_0^1(\Omega)$

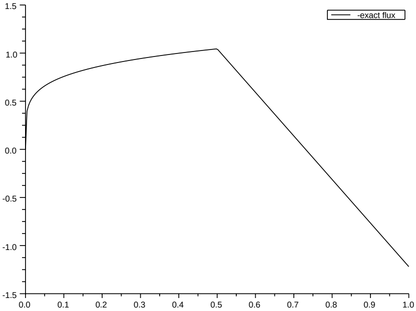


Solution gradient ∇p is not necessarily in $\mathbf{H}(\mathrm{div}, \Omega)$

Properties of the weak solution



Solution p is in $H_0^1(\Omega)$



Flux $-a\nabla p$ is in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

Optimal abstract framework for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal abstract framework, conf. & pure dif. case)

Let $p, p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \mathcal{B}(p - p_h, \varphi) \leq |||p - p_h|||.$$

Proof.

We have

$$\begin{aligned} |||p - p_h||| &= \mathcal{B}\left(p - p_h, \frac{p - p_h}{|||p - p_h|||}\right) \\ &\leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \mathcal{B}(p - p_h, \varphi) \\ &\leq |||p - p_h||| \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} |||\varphi||| \\ &= |||p - p_h|||. \end{aligned}$$

Optimal abstract framework for $-\nabla\cdot(a\nabla p) = f$

Theorem (Optimal abstract framework, conf. & pure dif. case)

Let $p, p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \le \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \mathcal{B}(p - p_h, \varphi) \le |||p - p_h|||.$$

Proof.

We have

$$\begin{aligned} |||p - p_h||| &= \mathcal{B}\left(p - p_h, \frac{p - p_h}{|||p - p_h|||}\right) \\ &\le \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \mathcal{B}(p - p_h, \varphi) \\ &\le |||p - p_h||| \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} |||\varphi||| \\ &= |||p - p_h|||. \end{aligned}$$

Optimal abstract estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal abstract estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \leq \inf_{\mathbf{t}\in\mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot\mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi)\}$$
$$\leq |||p - p_h|||.$$

Proof.

Upper bound: put $\varphi := p - p_h / |||p - p_h|||$ and take $\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)$ arbitrary. Then

$$\begin{aligned}
\mathcal{B}(p - p_h, \varphi) &= (f, \varphi) - (a\nabla p_h, \nabla\varphi) && //\mathcal{B} \text{ lin., weak sol. def.}\\
&= (f, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi) + (\mathbf{t}, \nabla\varphi) && // \pm (\mathbf{t}, \nabla\varphi)\\
&= (f - \nabla\cdot\mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi)\,. && //\text{Green th.}
\end{aligned}$$

Lower bound: put $\mathbf{t} = -a\nabla p$ and use the Schwarz inequality.

Optimal abstract estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal abstract estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \leq \inf_{\mathbf{t}\in\mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot\mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi)\}$$

$$\leq |||p - p_h|||.$$

Proof.

Upper bound: put $\varphi := p - p_h / |||p - p_h|||$ and take $\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)$ arbitrary. Then

$$\begin{aligned}
\mathcal{B}(p - p_h, \varphi) &= (f, \varphi) - (a\nabla p_h, \nabla\varphi) \quad //\mathcal{B} \text{ lin., weak sol. def.}\\
&= (f, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi) + (\mathbf{t}, \nabla\varphi) \quad // \pm (\mathbf{t}, \nabla\varphi)\\
&= (f - \nabla\cdot\mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi)\,. \quad //\text{Green th.}
\end{aligned}$$

Lower bound: put $\mathbf{t} = -a\nabla p$ and use the Schwarz inequality.

Optimal abstract estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal abstract estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \leq \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla \varphi)\}$$
$$\leq |||p - p_h|||.$$

Properties

- Guaranteed upper bound (no undetermined constant).
- Exact and robust.
- Not computable (infimum over an infinite-dimensional space).

Optimal abstract estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal abstract estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$|||p - p_h||| \leq \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla\varphi)\}$$
$$\leq |||p - p_h|||.$$

Properties

- Guaranteed upper bound (no undetermined constant).
- Exact and robust.
- Not computable (infimum over an infinite-dimensional space).

Optimal abstract estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal abstract estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Then

$$\begin{aligned} |||p - p_h||| &\le \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}, \varphi) - (a\nabla p_h + \mathbf{t}, \nabla \varphi)\} \\ &\le |||p - p_h|||. \end{aligned}$$

Properties

- Guaranteed upper bound (no undetermined constant).
- Exact and robust.
- Not computable (infimum over an infinite-dimensional space).

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Proof.

- $|||p - p_h||| \le \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla \varphi)\}$;
- Friedrichs inequality: $\|\varphi\| \le C_{\mathrm{F},\Omega}^{1/2} h_\Omega \|\nabla \varphi\| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||$;
- use this and the Schwarz inequality:

  $$(f - \nabla \cdot \mathbf{t}_h, \varphi) \le \|f - \nabla \cdot \mathbf{t}_h\| \|\varphi\| \le \|f - \nabla \cdot \mathbf{t}_h\| \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||;$$
- use the Schwarz inequality for the second term: $-(a\nabla p_h + \mathbf{t}_h, \nabla \varphi) \le \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\| |||\varphi|||$.

A first computable estimate for $-\nabla\cdot(a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla\cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Proof.

- $|||p - p_h||| \le \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot\mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla\varphi)\};$
- Friedrichs inequality: $\|\varphi\| \le C_{\mathrm{F},\Omega}^{1/2} h_\Omega \|\nabla\varphi\| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||;$
- use this and the Schwarz inequality:

  $$(f - \nabla\cdot\mathbf{t}_h, \varphi) \le \|f - \nabla\cdot\mathbf{t}_h\|\|\varphi\| \le \|f - \nabla\cdot\mathbf{t}_h\| \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||;$$
- use the Schwarz inequality for the second term: $-(a\nabla p_h + \mathbf{t}_h, \nabla\varphi) \le \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\| |||\varphi|||.$

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H^1_0(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Proof.

- $|||p-p_h||| \leq \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla\varphi)\};$
- Friedrichs inequality: $\|\varphi\| \leq C_{\mathrm{F},\Omega}^{1/2} h_\Omega \|\nabla\varphi\| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||;$
- use this and the Schwarz inequality:
  $(f - \nabla \cdot \mathbf{t}_h, \varphi) \leq \|f - \nabla \cdot \mathbf{t}_h\| \|\varphi\| \leq \|f - \nabla \cdot \mathbf{t}_h\| \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||;$
- use the Schwarz inequality for the second term:
  $-(a\nabla p_h + \mathbf{t}_h, \nabla\varphi) \leq \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\| |||\varphi|||.$

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Proof.

- $|||p - p_h||| \le \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla\varphi)\}$;
- Friedrichs inequality: $\|\varphi\| \le C_{\mathrm{F},\Omega}^{1/2} h_\Omega \|\nabla\varphi\| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||$;
- use this and the Schwarz inequality:

  $(f - \nabla \cdot \mathbf{t}_h, \varphi) \le \|f - \nabla \cdot \mathbf{t}_h\| \|\varphi\| \le \|f - \nabla \cdot \mathbf{t}_h\| \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||$;
- use the Schwarz inequality for the second term:

  $-(a\nabla p_h + \mathbf{t}_h, \nabla\varphi) \le \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\| |||\varphi|||$.

A first computable estimate for $-\nabla \cdot (a \nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Proof.

- $|||p - p_h||| \leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}_h, \varphi) - (a \nabla p_h + \mathbf{t}_h, \nabla \varphi)\}$;
- Friedrichs inequality: $\|\varphi\| \leq C_{\mathrm{F},\Omega}^{1/2} h_\Omega \|\nabla \varphi\| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||$;
- use this and the Schwarz inequality:

  $$(f - \nabla \cdot \mathbf{t}_h, \varphi) \leq \|f - \nabla \cdot \mathbf{t}_h\| \|\varphi\| \leq \|f - \nabla \cdot \mathbf{t}_h\| \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} |||\varphi|||;$$
- use the Schwarz inequality for the second term: $-(a \nabla p_h + \mathbf{t}_h, \nabla \varphi) \leq \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\| |||\varphi|||$.

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega} \leq 1$, Friedrichs constant).
- $\|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|$ penalizes $-a\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $\|f - \nabla \cdot \mathbf{t}_h\|$ is a residual term, evaluated for $\mathbf{t}_h$.
- Advantage: scheme-independent (works for all schemes) (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used), $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{a,\Omega}^{1/2}$ may be too big.

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega} \leq 1$, Friedrichs constant).
- $\|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|$ penalizes $-a\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $\|f - \nabla \cdot \mathbf{t}_h\|$ is a residual term, evaluated for $\mathbf{t}_h$.
- Advantage: scheme-independent (works for all schemes) (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used), $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{a,\Omega}^{1/2}$ may be too big.

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega} \leq 1$, Friedrichs constant).
- $\|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|$ penalizes $-a\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $\|f - \nabla \cdot \mathbf{t}_h\|$ is a residual term, evaluated for $\mathbf{t}_h$.
- Advantage: scheme-independent (works for all schemes) (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used), $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{a,\Omega}^{1/2}$ may be too big.

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega} \le 1$, Friedrichs constant).
- $\|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|$ penalizes $-a\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $\|f - \nabla \cdot \mathbf{t}_h\|$ is a residual term, evaluated for $\mathbf{t}_h$.
- Advantage: scheme-independent (works for all schemes) (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used), $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{a,\Omega}^{1/2}$ may be too big.

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (A first computable estimate, conf. & pure dif. case)

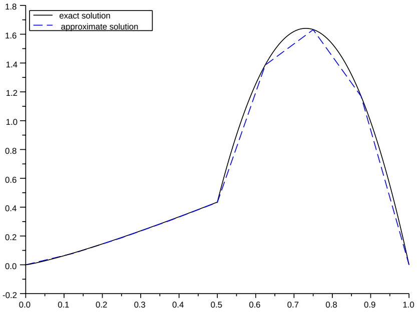
Let p be the weak solution and let $p_h \in H_0^1(\Omega)$ be arbitrary. Take any $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$. Then

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|.$$

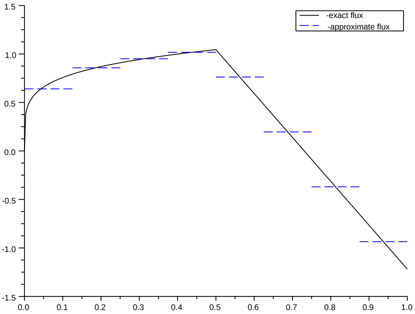
Properties

- Guaranteed upper bound ($C_{\mathrm{F},\Omega} \leq 1$, Friedrichs constant).
- $\|a^{\frac{1}{2}} \nabla p_h + a^{-\frac{1}{2}} \mathbf{t}_h\|$ penalizes $-a\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$.
- $\|f - \nabla \cdot \mathbf{t}_h\|$ is a residual term, evaluated for $\mathbf{t}_h$.
- Advantage: scheme-independent (works for all schemes) (promoted by Repin).
- Disadvantage: scheme-independent (no information from the computation used), $C_{\mathrm{F},\Omega}^{1/2} h_\Omega / c_{a,\Omega}^{1/2}$ may be too big.

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$

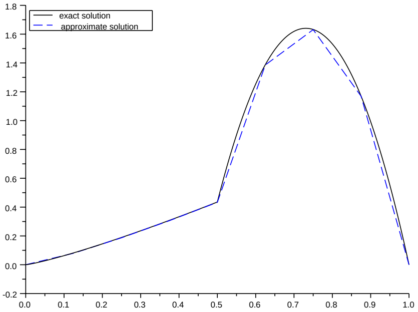


Approximate solution p_h is in $H_0^1(\Omega)$

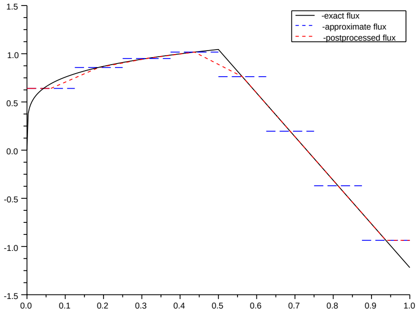


Approximate flux $-a\nabla p_h$ is not in $\mathbf{H}(\mathrm{div}, \Omega)$

A first computable estimate for $-\nabla \cdot (a\nabla p) = f$



Approximate solution p_h is in $H_0^1(\Omega)$



Construct a postprocessed flux $\mathbf{t}_h$ in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

Optimal a posteriori error estimate for $-\nabla \cdot (a\nabla p) = f$

Theorem (Optimal a posteriori error estimate)

Let

- *p be the weak solution,*
- *$p_h \in H_0^1(\Omega)$ be arbitrary,*
- *$\mathcal{D}_h = \mathcal{D}_h^{\rm int} \cup \mathcal{D}_h^{\rm ext}$ be a partition of Ω,*
- *$\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ be arbitrary but such that $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$ for all $D \in \mathcal{D}_h^{\rm int}$.*

Then

$$|||p - p_h||| \leq \left\{ \sum_{D \in \mathcal{D}_h} (\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D})^2 \right\}^{1/2}.$$

Optimal a posteriori error estimate for $-\nabla \cdot (a\nabla p) = f$

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},D} := \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_D$
  - penalizes the fact that $-a\nabla p_h \notin \mathbf{H}(\mathrm{div},\Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},D} := m_{D,a}\|f - \nabla \cdot \mathbf{t}_h\|_D$
  - residue evaluated for $\mathbf{t}_h$
  - $m_{D,a}^2 := C_{\mathrm{P},D} h_D{}^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,a}^2 := C_{\mathrm{F},D} h_D{}^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{a,D}$ is the smallest value of a on D

Comparison with the first computable estimate

Recall that

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|$$

Optimal a posteriori error estimate for $-\nabla \cdot (a\nabla p) = f$

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},D} := \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_D$
  - penalizes the fact that $-a\nabla p_h \notin \mathbf{H}(\mathrm{div},\Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},D} := m_{D,a}\|f - \nabla \cdot \mathbf{t}_h\|_D$
  - residue evaluated for $\mathbf{t}_h$
  - $m_{D,a}^2 := C_{\mathrm{P},D} h_D{}^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,a}^2 := C_{\mathrm{F},D} h_D{}^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{a,D}$ is the smallest value of a on D

Comparison with the first computable estimate

Recall that

$$|||p - p_h||| \leq \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}} \|f - \nabla \cdot \mathbf{t}_h\| + \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|$$

Optimal a posteriori error estimate for $-\nabla \cdot (a\nabla p) = f$

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},D} := \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_D$
  - penalizes the fact that $-a\nabla p_h \notin \mathbf{H}(\mathrm{div},\Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},D} := m_{D,a}\|f - \nabla\cdot\mathbf{t}_h\|_D$
  - residue evaluated for $\mathbf{t}_h$
  - $m_{D,a}^2 := C_{\mathrm{P},D} h_D{}^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,a}^2 := C_{\mathrm{F},D} h_D{}^2 / c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{a,D}$ is the smallest value of a on D

Comparison with the first computable estimate

Recall that

$$|||p - p_h||| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}}\|f - \nabla\cdot\mathbf{t}_h\| + \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|$$

Optimal a posteriori error estimate for $-\nabla \cdot (a\nabla p) = f$

Estimators

- *diffusive flux estimator*
  - $\eta_{\mathrm{DF},D} := \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_D$
  - penalizes the fact that $-a\nabla p_h \notin \mathbf{H}(\mathrm{div},\Omega)$
- *residual estimator*
  - $\eta_{\mathrm{R},D} := m_{D,a}\|f - \nabla\cdot\mathbf{t}_h\|_D$
  - residue evaluated for $\mathbf{t}_h$
  - $m_{D,a}^2 := C_{\mathrm{P},D} h_D{}^2/c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{int}}$, $C_{\mathrm{P},D} = 1/\pi^2$ if D convex
  - $m_{D,a}^2 := C_{\mathrm{F},D} h_D{}^2/c_{a,D}$ for $D \in \mathcal{D}_h^{\mathrm{ext}}$, $C_{\mathrm{F},D} = 1$ in general
  - $c_{a,D}$ is the smallest value of a on D

Comparison with the first computable estimate

Recall that

$$|||p - p_h||| \le \frac{C_{\mathrm{F},\Omega}^{1/2} h_\Omega}{c_{a,\Omega}^{1/2}}\|f - \nabla\cdot\mathbf{t}_h\| + \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|$$

Proof of the optimal estimate for $-\nabla \cdot (a\nabla p) = f$

Proof, part 1.

- Recall that

$$|||p - p_h||| \le \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot \mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla\varphi)\};$$

- Poincaré inequality, $D \in \mathcal{D}_h^{\mathrm{int}}$

$$\|\varphi - \varphi_D\|_D^2 \le C_{\mathrm{P},D} h_D^2 \|\nabla\varphi\|_D^2,$$

where φ_D is the mean value of φ over D;

- Friedrichs inequality, $D \in \mathcal{D}_h^{\mathrm{ext}}$

$$\|\varphi\|_D^2 \le C_{\mathrm{F},D,\partial\Omega} h_D^2 \|\nabla\varphi\|_D^2,$$

where $\varphi = 0$ on $\partial\Omega \cap \partial D \neq \emptyset$;

- energy norm:

$$\|\nabla\varphi\|_D^2 \le \frac{1}{c_{a,D}} |||\varphi|||_D^2;$$

Proof of the optimal estimate for $-\nabla \cdot (a\nabla p) = f$

Proof, part 1.

- Recall that

$$|||p - p_h||| \le \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot\mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla\varphi)\};$$

- Poincaré inequality, $D \in \mathcal{D}_h^{\mathrm{int}}$

$$\|\varphi - \varphi_D\|_D^2 \le C_{\mathrm{P},D} h_D^2 \|\nabla\varphi\|_D^2,$$

  where φ_D is the mean value of φ over D;

- Friedrichs inequality, $D \in \mathcal{D}_h^{\mathrm{ext}}$

$$\|\varphi\|_D^2 \le C_{\mathrm{F},D,\partial\Omega} h_D^2 \|\nabla\varphi\|_D^2,$$

  where $\varphi = 0$ on $\partial\Omega \cap \partial D \neq \emptyset$;

- energy norm:

$$\|\nabla\varphi\|_D^2 \le \frac{1}{c_{a,D}} |||\varphi|||_D^2;$$

Proof of the optimal estimate for $-\nabla \cdot (a\nabla p) = f$

Proof, part 1.

- Recall that

$$|||p - p_h||| \leq \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla \cdot \mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla \varphi)\};$$

- Poincaré inequality, $D \in \mathcal{D}_h^{\mathrm{int}}$

$$\|\varphi - \varphi_D\|_D^2 \leq C_{\mathrm{P},D} h_D^2 \|\nabla \varphi\|_D^2,$$

where φ_D is the mean value of φ over D;

- Friedrichs inequality, $D \in \mathcal{D}_h^{\mathrm{ext}}$

$$\|\varphi\|_D^2 \leq C_{\mathrm{F},D,\partial\Omega} h_D^2 \|\nabla \varphi\|_D^2,$$

where $\varphi = 0$ on $\partial\Omega \cap \partial D \neq \emptyset$;

- energy norm:

$$\|\nabla \varphi\|_D^2 \leq \frac{1}{c_{a,D}} |||\varphi|||_D^2;$$

Proof of the optimal estimate for $-\nabla \cdot (a\nabla p) = f$

Proof, part 1.

- Recall that

$$|||p - p_h||| \le \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{(f - \nabla\cdot\mathbf{t}_h, \varphi) - (a\nabla p_h + \mathbf{t}_h, \nabla\varphi)\};$$

- Poincaré inequality, $D \in \mathcal{D}_h^{\mathrm{int}}$

$$\|\varphi - \varphi_D\|_D^2 \le C_{\mathrm{P},D} h_D^2 \|\nabla\varphi\|_D^2,$$

where φ_D is the mean value of φ over D;

- Friedrichs inequality, $D \in \mathcal{D}_h^{\mathrm{ext}}$

$$\|\varphi\|_D^2 \le C_{\mathrm{F},D,\partial\Omega} h_D^2 \|\nabla\varphi\|_D^2,$$

where $\varphi = 0$ on $\partial\Omega \cap \partial D \neq \emptyset$;

- energy norm:

$$\|\nabla\varphi\|_D^2 \le \frac{1}{c_{a,D}} |||\varphi|||_D^2;$$

Proof of the optimal estimate for $-\nabla \cdot (a\nabla p) = f$

Proof, part 2.

- $D \in \mathcal{D}_h^{\text{int}}$: conservativity of $\mathbf{t}_h$, i.e. $(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1)_D$, Schwarz inequality, and Poincaré inequality:

$$
\begin{aligned}
(f - \nabla\cdot\mathbf{t}_h, \varphi)_D &= (f - \nabla\cdot\mathbf{t}_h, \varphi - \varphi_D)_D \leq \|f - \nabla\cdot\mathbf{t}_h\|_D \|\varphi - \varphi_D\|_D \\
&\leq \|f - \nabla\cdot\mathbf{t}_h\|_D C_{\mathrm{P},D}^{\frac{1}{2}} h_D \|\nabla\varphi\|_D \\
&\leq m_{D,a} \|f - \nabla\cdot\mathbf{t}_h\|_D |||\varphi|||_D;
\end{aligned}
$$

- $D \in \mathcal{D}_h^{\text{ext}}$: Schwarz and Friedrichs inequalities:

$$
\begin{aligned}
(f - \nabla\cdot\mathbf{t}_h, \varphi)_D &\leq \|f - \nabla\cdot\mathbf{t}_h\|_D \|\varphi\|_D \leq \|f - \nabla\cdot\mathbf{t}_h\|_D C_{\mathrm{F},D,\partial\Omega}^{\frac{1}{2}} h_D \|\nabla\varphi\|_D \\
&\leq m_{D,a} \|f - \nabla\cdot\mathbf{t}_h\|_D |||\varphi|||_D;
\end{aligned}
$$

- the Schwarz inequality for the second term:

$$
-(a\nabla p_h + \mathbf{t}_h, \nabla\varphi)_D \leq \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_D |||\varphi|||_D.
$$

Proof of the optimal estimate for $-\nabla \cdot (a\nabla p) = f$

Proof, part 2.

- $D \in \mathcal{D}_h^{\rm int}$: conservativity of $\mathbf{t}_h$, i.e. $(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1)_D$, Schwarz inequality, and Poincaré inequality:

$$
\begin{aligned}
(f - \nabla\cdot\mathbf{t}_h, \varphi)_D &= (f - \nabla\cdot\mathbf{t}_h, \varphi - \varphi_D)_D \le \|f - \nabla\cdot\mathbf{t}_h\|_D \|\varphi - \varphi_D\|_D \\
&\le \|f - \nabla\cdot\mathbf{t}_h\|_D C_{\mathrm{P},D}^{\frac{1}{2}} h_D \|\nabla\varphi\|_D \\
&\le m_{D,a} \|f - \nabla\cdot\mathbf{t}_h\|_D |||\varphi|||_D;
\end{aligned}
$$

- $D \in \mathcal{D}_h^{\rm ext}$: Schwarz and Friedrichs inequalities:

$$
\begin{aligned}
(f - \nabla\cdot\mathbf{t}_h, \varphi)_D &\le \|f - \nabla\cdot\mathbf{t}_h\|_D \|\varphi\|_D \le \|f - \nabla\cdot\mathbf{t}_h\|_D C_{\mathrm{F},D,\partial\Omega}^{\frac{1}{2}} h_D \|\nabla\varphi\|_D \\
&\le m_{D,a} \|f - \nabla\cdot\mathbf{t}_h\|_D |||\varphi|||_D;
\end{aligned}
$$

- the Schwarz inequality for the second term:

$$
-(a\nabla p_h + \mathbf{t}_h, \nabla\varphi)_D \le \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_D |||\varphi|||_D.
$$

Proof of the optimal estimate for $-\nabla\cdot(a\nabla p) = f$

Proof, part 2.

- $D \in \mathcal{D}_h^{\mathrm{int}}$: conservativity of $\mathbf{t}_h$, i.e. $(\nabla\cdot\mathbf{t}_h, 1)_D = (f, 1)_D$, Schwarz inequality, and Poincaré inequality:

$$
\begin{aligned}
(f - \nabla\cdot\mathbf{t}_h, \varphi)_D &= (f - \nabla\cdot\mathbf{t}_h, \varphi - \varphi_D)_D \leq \|f - \nabla\cdot\mathbf{t}_h\|_D \|\varphi - \varphi_D\|_D \\
&\leq \|f - \nabla\cdot\mathbf{t}_h\|_D C_{\mathrm{P},D}^{\frac{1}{2}} h_D \|\nabla\varphi\|_D \\
&\leq m_{D,a} \|f - \nabla\cdot\mathbf{t}_h\|_D |||\varphi|||_D;
\end{aligned}
$$

- $D \in \mathcal{D}_h^{\mathrm{ext}}$: Schwarz and Friedrichs inequalities:

$$
\begin{aligned}
(f - \nabla\cdot\mathbf{t}_h, \varphi)_D &\leq \|f - \nabla\cdot\mathbf{t}_h\|_D \|\varphi\|_D \leq \|f - \nabla\cdot\mathbf{t}_h\|_D C_{\mathrm{F},D,\partial\Omega}^{\frac{1}{2}} h_D \|\nabla\varphi\|_D \\
&\leq m_{D,a} \|f - \nabla\cdot\mathbf{t}_h\|_D |||\varphi|||_D;
\end{aligned}
$$

- the Schwarz inequality for the second term:

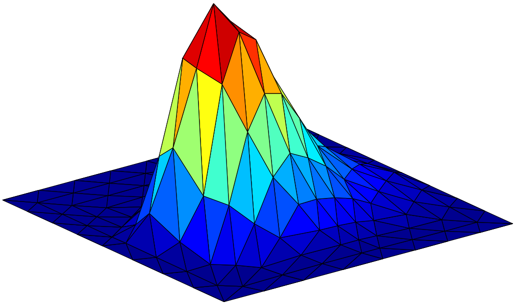
$$
-(a\nabla p_h + \mathbf{t}_h, \nabla\varphi)_D \leq \|a^{\frac{1}{2}}\nabla p_h + a^{-\frac{1}{2}}\mathbf{t}_h\|_D |||\varphi|||_D.
$$

Outline

Finite elements for $-\nabla \cdot (a\nabla p) = f$

Finite element method

- Find $p_h \in V_h$ such that
$$(a\nabla p_h, \nabla\varphi_h) = (f, \varphi_h) \qquad \forall \varphi_h \in V_h.$$
- $p_h \in H_0^1(\Omega)$:



Local conservativity of finite elements

Equivalent form of the FE method

Find $p_h \in V_h$ such that

$$(a\nabla p_h, \nabla\psi_V)_{\mathcal{T}_V} = (f, \psi_V)_{\mathcal{T}_V} \qquad \forall V \in \mathcal{V}_h^{\rm int}.$$

- ψ_V – FE basis function associated with vertex V
- $\mathcal{T}_V$ – simplices of $\mathcal{T}_h$ sharing V

Local conservativity of finite elements

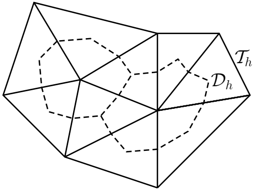
Equivalent form of the FE method

Find $p_h \in V_h$ such that

$$(a\nabla p_h, \nabla\psi_V)_{\mathcal{T}_V} = (f, \psi_V)_{\mathcal{T}_V} \qquad \forall V \in \mathcal{V}_h^{\mathrm{int}}.$$

- ψ_V – FE basis function associated with vertex V
- $\mathcal{T}_V$ – simplices of $\mathcal{T}_h$ sharing V

Construct a dual mesh $\mathcal{D}_h$



Local conservativity of finite elements

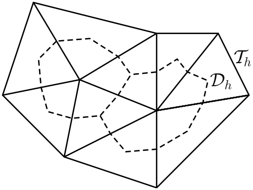
Equivalent form of the FE method

Find $p_h \in V_h$ such that

$$(a\nabla p_h, \nabla\psi_V)_{\mathcal{T}_V} = (f, \psi_V)_{\mathcal{T}_V} \qquad \forall V \in \mathcal{V}_h^{\rm int}.$$

- ψ_V – FE basis function associated with vertex V
- $\mathcal{T}_V$ – simplices of $\mathcal{T}_h$ sharing V

Construct a dual mesh $\mathcal{D}_h$



Equivalences

$$(a\nabla p_h, \nabla\psi_{V_D})_{\mathcal{T}_{V_D}} = -\langle a\nabla p_h \cdot \mathbf{n}, 1\rangle_{\partial D} \qquad \forall D \in \mathcal{D}_h^{\rm int}$$

$$(f, \psi_{V_D})_{\mathcal{T}_{V_D}} = (f, 1)_D \qquad \forall D \in \mathcal{D}_h^{\rm int}$$

for f piecewise constant on $\mathcal{T}_h$

Local conservativity of finite elements

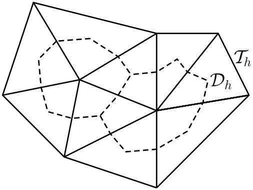
Equivalent form of the FE method

Find $p_h \in V_h$ such that

$$(a\nabla p_h, \nabla\psi_V)_{\mathcal{T}_V} = (f, \psi_V)_{\mathcal{T}_V} \qquad \forall V \in \mathcal{V}_h^{\text{int}}.$$

- ψ_V – FE basis function associated with vertex V
- $\mathcal{T}_V$ – simplices of $\mathcal{T}_h$ sharing V

Construct a dual mesh $\mathcal{D}_h$



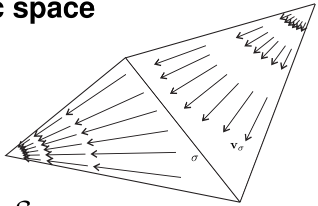
Thus a locally conservative form of the FE method

Find $p_h \in V_h$ such that

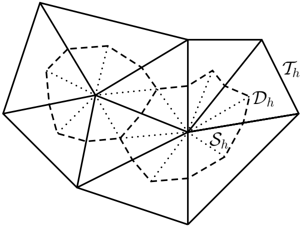
$$-\langle a\nabla p_h \cdot \mathbf{n}, 1\rangle_{\partial D} = (f, 1)_D \qquad \forall D \in \mathcal{D}_h^{\text{int}}.$$

Choice of $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$

Raviart–Thomas–Nédélec space



Construct a ternary mesh $\mathcal{S}_h$

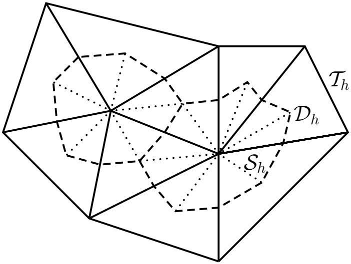


Choice of $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$

- $\mathbf{t}_h \cdot \mathbf{n}_\sigma := -\{\!\{a\nabla p_h \cdot \mathbf{n}_\sigma\}\!\}$ for all sides σ of $\mathcal{S}_h$
- $\langle \mathbf{t}_h \cdot \mathbf{n}, 1\rangle_{\partial D} = (\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D \quad \forall D \in \mathcal{D}_h^{\mathrm{int}}$.

Outline

Local efficiency of the estimates for $-\nabla \cdot (a \nabla p) = f$



Theorem (Local efficiency)

Let $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{S}_h)$, $\mathbf{t}_h \cdot \mathbf{n}_\sigma := -\{\!\{ a \nabla p_h \cdot \mathbf{n}_\sigma \}\!\}_\omega$ for all sides σ of $\mathcal{S}_h$. Then

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \le C |||p - p_h|||_{\mathcal{T}_{V_D}},$$

where C depends only on the space dimension d, on the shape regularity parameter $\kappa_\mathcal{T}$, and on the polynomial degree m of f. Moreover, when $a = 1$, one actually has

$$\eta_{\mathrm{R},D} + \eta_{\mathrm{DF},D} \le C |||p - p_h|||_D.$$

Local efficiency of the estimates for $-\nabla \cdot (a\nabla p) = f$

Proof (diffusive flux estimator, case $a = 1$).

- for each $\mathbf{v}_h \in \mathbf{RTN}(K)$, $\|\mathbf{v}_h\|_K^2 \le C h_K \sum_{\sigma \in \mathcal{E}_K} \|\mathbf{v}_h \cdot \mathbf{n}\|_\sigma^2$ (equivalence of norms on finite-dimensional spaces)
- put $\mathbf{v}_h = \nabla p_h + \mathbf{t}_h$; then $\|\nabla p_h + \mathbf{t}_h\|_K^2 = \|\mathbf{v}_h\|_K^2$ $\le C h_K \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\rm int}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma^2 \Rightarrow \eta_{{\rm DF},D}$ is a lower bound for the classical mass balance estimator
- side bubble functions technique of Verfürth: $h_K^{\frac{1}{2}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma \le C \sum_{M \in \{K,L\}} |||p - p_h|||_M$ for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\rm int}$

Proof (residual estimator, case $a = 1$).

- element bubble functions technique of Verfürth: $\|f - \nabla \cdot \mathbf{t}_h\|_K \le C h_K^{-1} \|\nabla p + \mathbf{t}_h\|_K$
- $\|\nabla p + \mathbf{t}_h\|_D \le |||p - p_h|||_D + \|\nabla p_h + \mathbf{t}_h\|_D$
- complete the proof by the previous result

Local efficiency of the estimates for $-\nabla \cdot (a\nabla p) = f$

Proof (diffusive flux estimator, case $a = 1$).

- for each $\mathbf{v}_h \in \mathbf{RTN}(K)$, $\|\mathbf{v}_h\|_K^2 \leq C h_K \sum_{\sigma \in \mathcal{E}_K} \|\mathbf{v}_h \cdot \mathbf{n}\|_\sigma^2$ (equivalence of norms on finite-dimensional spaces)
- put $\mathbf{v}_h = \nabla p_h + \mathbf{t}_h$; then $\|\nabla p_h + \mathbf{t}_h\|_K^2 = \|\mathbf{v}_h\|_K^2$ $\leq C h_K \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{int}}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma^2 \Rightarrow \eta_{\mathrm{DF},D}$ is a lower bound for the classical mass balance estimator
- side bubble functions technique of Verfürth: $h_K^{\frac{1}{2}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma \leq C \sum_{M \in \{K,L\}} |||p - p_h|||_M$ for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{int}}$

Proof (residual estimator, case $a = 1$).

- element bubble functions technique of Verfürth: $\|f - \nabla \cdot \mathbf{t}_h\|_K \leq C h_K^{-1} \|\nabla p + \mathbf{t}_h\|_K$
- $\|\nabla p + \mathbf{t}_h\|_D \leq |||p - p_h|||_D + \|\nabla p_h + \mathbf{t}_h\|_D$
- complete the proof by the previous result

Local efficiency of the estimates for $-\nabla\cdot(a\nabla p) = f$

Proof (diffusive flux estimator, case $a = 1$).

- for each $\mathbf{v}_h \in \mathbf{RTN}(K)$, $\|\mathbf{v}_h\|_K^2 \le C h_K \sum_{\sigma\in\mathcal{E}_K} \|\mathbf{v}_h\cdot\mathbf{n}\|_\sigma^2$ (equivalence of norms on finite-dimensional spaces)
- put $\mathbf{v}_h = \nabla p_h + \mathbf{t}_h$; then $\|\nabla p_h + \mathbf{t}_h\|_K^2 = \|\mathbf{v}_h\|_K^2$ $\le C h_K \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\mathrm{int}}} \|[\![\nabla p_h\cdot\mathbf{n}_\sigma]\!]\|_\sigma^2 \Rightarrow \eta_{\mathrm{DF},D}$ is a lower bound for the classical mass balance estimator
- side bubble functions technique of Verfürth: $h_K^{\frac{1}{2}}\|[\![\nabla p_h\cdot\mathbf{n}_\sigma]\!]\|_\sigma \le C\sum_{M\in\{K,L\}} |||p - p_h|||_M$ for $\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\mathrm{int}}$

Proof (residual estimator, case $a = 1$).

- element bubble functions technique of Verfürth: $\|f - \nabla\cdot\mathbf{t}_h\|_K \le C h_K^{-1}\|\nabla p + \mathbf{t}_h\|_K$
- $\|\nabla p + \mathbf{t}_h\|_D \le |||p - p_h|||_D + \|\nabla p_h + \mathbf{t}_h\|_D$
- complete the proof by the previous result

Local efficiency of the estimates for $-\nabla \cdot (a\nabla p) = f$

Proof (diffusive flux estimator, case $a = 1$).

- for each $\mathbf{v}_h \in \mathbf{RTN}(K)$, $\|\mathbf{v}_h\|_K^2 \leq C h_K \sum_{\sigma \in \mathcal{E}_K} \|\mathbf{v}_h \cdot \mathbf{n}\|_\sigma^2$ (equivalence of norms on finite-dimensional spaces)
- put $\mathbf{v}_h = \nabla p_h + \mathbf{t}_h$; then $\|\nabla p_h + \mathbf{t}_h\|_K^2 = \|\mathbf{v}_h\|_K^2$ $\leq C h_K \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{int}}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma^2 \Rightarrow \eta_{\mathrm{DF},D}$ is a lower bound for the classical mass balance estimator
- side bubble functions technique of Verfürth: $h_K^{\frac{1}{2}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma \leq C \sum_{M \in \{K,L\}} |||p - p_h|||_M$ for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{int}}$

Proof (residual estimator, case $a = 1$).

- element bubble functions technique of Verfürth: $\|f - \nabla \cdot \mathbf{t}_h\|_K \leq C h_K^{-1} \|\nabla p + \mathbf{t}_h\|_K$
- $\|\nabla p + \mathbf{t}_h\|_D \leq |||p - p_h|||_D + \|\nabla p_h + \mathbf{t}_h\|_D$
- complete the proof by the previous result

Local efficiency of the estimates for $-\nabla \cdot (a\nabla p) = f$

Proof (diffusive flux estimator, case $a = 1$).

- for each $\mathbf{v}_h \in \mathbf{RTN}(K)$, $\|\mathbf{v}_h\|_K^2 \leq C h_K \sum_{\sigma \in \mathcal{E}_K} \|\mathbf{v}_h \cdot \mathbf{n}\|_\sigma^2$ (equivalence of norms on finite-dimensional spaces)
- put $\mathbf{v}_h = \nabla p_h + \mathbf{t}_h$; then $\|\nabla p_h + \mathbf{t}_h\|_K^2 = \|\mathbf{v}_h\|_K^2$ $\leq C h_K \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{int}}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma^2 \Rightarrow \eta_{\mathrm{DF},D}$ is a lower bound for the classical mass balance estimator
- side bubble functions technique of Verfürth: $h_K^{\frac{1}{2}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma \leq C \sum_{M \in \{K,L\}} |||p - p_h|||_M$ for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\mathrm{int}}$

Proof (residual estimator, case $a = 1$).

- element bubble functions technique of Verfürth: $\|f - \nabla \cdot \mathbf{t}_h\|_K \leq C h_K^{-1} \|\nabla p + \mathbf{t}_h\|_K$
- $\|\nabla p + \mathbf{t}_h\|_D \leq |||p - p_h|||_D + \|\nabla p_h + \mathbf{t}_h\|_D$
- complete the proof by the previous result

Local efficiency of the estimates for $-\nabla \cdot (a\nabla p) = f$

Proof (diffusive flux estimator, case $a = 1$).

- for each $\mathbf{v}_h \in \mathbf{RTN}(K)$, $\|\mathbf{v}_h\|_K^2 \le C h_K \sum_{\sigma\in\mathcal{E}_K} \|\mathbf{v}_h \cdot \mathbf{n}\|_\sigma^2$ (equivalence of norms on finite-dimensional spaces)
- put $\mathbf{v}_h = \nabla p_h + \mathbf{t}_h$; then $\|\nabla p_h + \mathbf{t}_h\|_K^2 = \|\mathbf{v}_h\|_K^2$ $\le C h_K \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^{\rm int}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma^2 \Rightarrow \eta_{{\rm DF},D}$ is a lower bound for the classical mass balance estimator
- side bubble functions technique of Verfürth: $h_K^{\frac{1}{2}} \|[\![\nabla p_h \cdot \mathbf{n}_\sigma]\!]\|_\sigma \le C \sum_{M\in\{K,L\}} |||p - p_h|||_M$ for $\sigma \in \mathcal{E}_K \cap \mathcal{E}_h^{\rm int}$

Proof (residual estimator, case $a = 1$).

- element bubble functions technique of Verfürth: $\|f - \nabla \cdot \mathbf{t}_h\|_K \le C h_K^{-1} \|\nabla p + \mathbf{t}_h\|_K$
- $\|\nabla p + \mathbf{t}_h\|_D \le |||p - p_h|||_D + \|\nabla p_h + \mathbf{t}_h\|_D$
- complete the proof by the previous result

Local efficiency of the estimates for $-\nabla \cdot (a\nabla p) = f$

Proof (case $a \neq 1$).

- the discontinuities have to be aligned with the dual mesh
- harmonic averaging has to be used in the scheme
- harmonic averaging has to be used in the construction of $\mathbf{t}_h$: $\mathbf{t}_h \cdot \mathbf{n}_\sigma = -\{\!\{\nabla p_h \cdot \mathbf{n}_\sigma\}\!\}_\omega$

Properties

- **guaranteed upper bound**
- local efficiency
- **full robustness**
- negligible evaluation cost
- locally, our estimator is a lower bound for the classical residual one

Local efficiency of the estimates for $-\nabla \cdot (a\nabla p) = f$

Proof (case $a \neq 1$).

- the discontinuities have to be aligned with the dual mesh
- harmonic averaging has to be used in the scheme
- harmonic averaging has to be used in the construction of $\mathbf{t}_h$: $\mathbf{t}_h \cdot \mathbf{n}_\sigma = -\{\!\{\nabla p_h \cdot \mathbf{n}_\sigma\}\!\}_\omega$

Properties

- **guaranteed upper bound**
- local efficiency
- **full robustness**
- negligible evaluation cost
- locally, our estimator is a lower bound for the classical residual one

Local efficiency of the estimates for $-\nabla \cdot (a \nabla p) = f$

Proof (case $a \neq 1$).

- the discontinuities have to be aligned with the dual mesh
- harmonic averaging has to be used in the scheme
- harmonic averaging has to be used in the construction of $\mathbf{t}_h$: $\mathbf{t}_h \cdot \mathbf{n}_\sigma = -\{\!\{\nabla p_h \cdot \mathbf{n}_\sigma\}\!\}_\omega$

Properties

- **guaranteed upper bound**
- local efficiency
- **full robustness**
- negligible evaluation cost
- locally, our estimator is a lower bound for the classical residual one

Outline

The estimate in 1D

Model problem

$$-p'' = \pi^2 sin(\pi x) \quad \text{in }]0,1[,$$
$$p = 0 \quad \text{in } 0, 1$$

Exact solution

$$p(x) = sin(\pi x)$$

Discretization

N given, $h = 1/(N+1)$, $x_k = kh$, $k = 0, \ldots, N+1$ ($x_0 = 0$ and $x_{N+1} = 1$), $x_{k+\frac{1}{2}} = (k + \frac{1}{2})h$, $k = 0, \ldots, N$, $x_{-\frac{1}{2}} = 0$, $x_{N+1+\frac{1}{2}} = 1$

Choice of t_h

$$t_h(x_{k+\frac{1}{2}}) = -p_h'(x_{k+\frac{1}{2}}) \qquad k = 0, \ldots, N,$$
$$t_h(x_k) = -(p_h'|_{]x_{k-1},x_k[} + p_h'|_{]x_k,x_{k+1}[})/2 \qquad k = 1, \ldots, N,$$
$$t_h(x_0) = -p_h'|_{]x_0,x_1[},$$
$$t_h(x_{N+1}) = -p_h'|_{]x_N,x_{N+1}[}$$

The estimate in 1D

Model problem

$$\begin{aligned} -p'' &= \pi^2 sin(\pi x) \quad \text{in }]0,1[, \\ p &= 0 \quad \text{in } 0,1 \end{aligned}$$

Exact solution

$$p(x) = sin(\pi x)$$

Discretization

N given, $h = 1/(N+1)$, $x_k = kh$, $k = 0, \ldots, N+1$ ($x_0 = 0$ and $x_{N+1} = 1$), $x_{k+\frac{1}{2}} = (k + \frac{1}{2})h$, $k = 0, \ldots, N$, $x_{-\frac{1}{2}} = 0$, $x_{N+1+\frac{1}{2}} = 1$

Choice of t_h

$$\begin{aligned} t_h(x_{k+\frac{1}{2}}) &= -p_h'(x_{k+\frac{1}{2}}) \qquad k = 0, \ldots, N, \\ t_h(x_k) &= -(p_h'|_{]x_{k-1},x_k[} + p_h'|_{]x_k,x_{k+1}[})/2 \qquad k = 1, \ldots, N, \\ t_h(x_0) &= -p_h'|_{]x_0,x_1[}, \\ t_h(x_{N+1}) &= -p_h'|_{]x_N,x_{N+1}[} \end{aligned}$$

The estimate in 1D

Model problem

$$-p'' = \pi^2 sin(\pi x) \quad \text{in }]0,1[,$$
$$p = 0 \quad \text{in } 0,1$$

Exact solution

$$p(x) = sin(\pi x)$$

Discretization

N given, $h = 1/(N+1)$, $x_k = kh$, $k = 0, \ldots, N+1$ ($x_0 = 0$ and $x_{N+1} = 1$), $x_{k+\frac{1}{2}} = (k+\frac{1}{2})h$, $k = 0, \ldots, N$, $x_{-\frac{1}{2}} = 0$, $x_{N+1+\frac{1}{2}} = 1$

Choice of t_h

$$t_h(x_{k+\frac{1}{2}}) = -p_h'(x_{k+\frac{1}{2}}) \qquad k = 0, \ldots, N,$$
$$t_h(x_k) = -(p_h'|_{]x_{k-1},x_k[} + p_h'|_{]x_k,x_{k+1}[})/2 \qquad k = 1, \ldots, N,$$
$$t_h(x_0) = -p_h'|_{]x_0,x_1[},$$
$$t_h(x_{N+1}) = -p_h'|_{]x_N,x_{N+1}[}$$

The estimate in 1D

Model problem

$$\begin{aligned} -p'' &= \pi^2 sin(\pi x) \quad \text{in }]0,1[, \\ p &= 0 \quad \text{in } 0,1 \end{aligned}$$

Exact solution

$$p(x) = sin(\pi x)$$

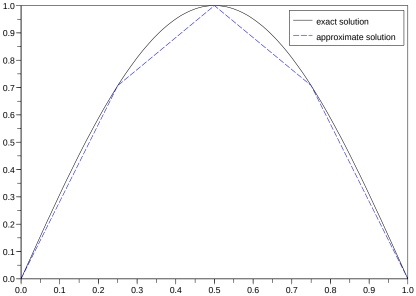
Discretization

N given, $h = 1/(N+1)$, $x_k = kh$, $k = 0, \ldots, N+1$ ($x_0 = 0$ and $x_{N+1} = 1$), $x_{k+\frac{1}{2}} = (k+\frac{1}{2})h$, $k = 0, \ldots, N$, $x_{-\frac{1}{2}} = 0$, $x_{N+1+\frac{1}{2}} = 1$

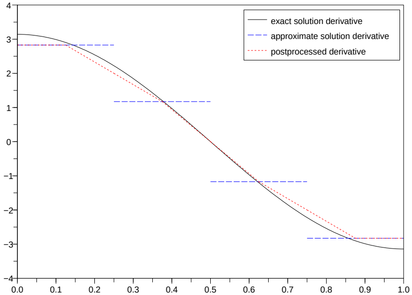
Choice of t_h

$$\begin{aligned} t_h(x_{k+\frac{1}{2}}) &= -p_h'(x_{k+\frac{1}{2}}) \qquad k = 0, \ldots, N, \\ t_h(x_k) &= -(p_h'|_{]x_{k-1},x_k[} + p_h'|_{]x_k,x_{k+1}[})/2 \qquad k = 1, \ldots, N, \\ t_h(x_0) &= -p_h'|_{]x_0,x_1[}, \\ t_h(x_{N+1}) &= -p_h'|_{]x_N,x_{N+1}[} \end{aligned}$$

Plots of p, p_h, and $-t_h$

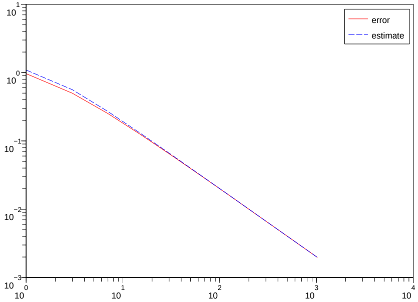


Plot of p and p_h

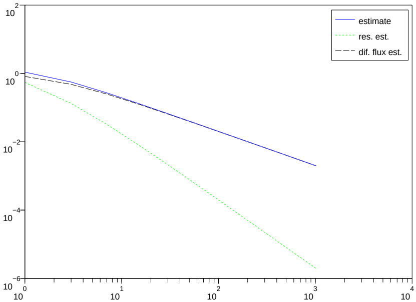


Plot of p', p_h', and $-t_h$

The optimal estimate in 1D

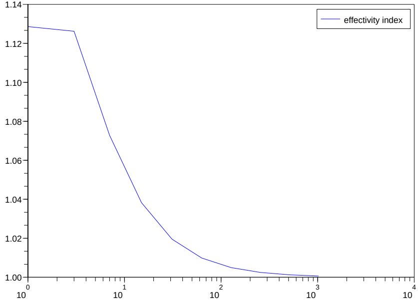


Estimated and actual errors



Estimated error and residual and diffusive flux estimators

The optimal estimate in 1D



Effectivity index

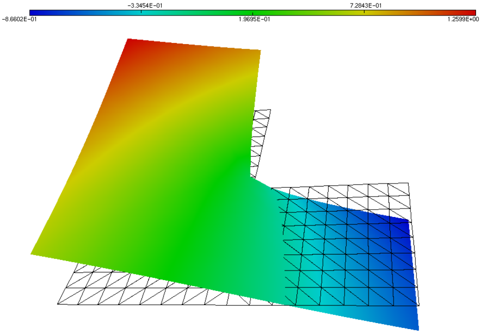
L-shape domain example and finite elements

Problem

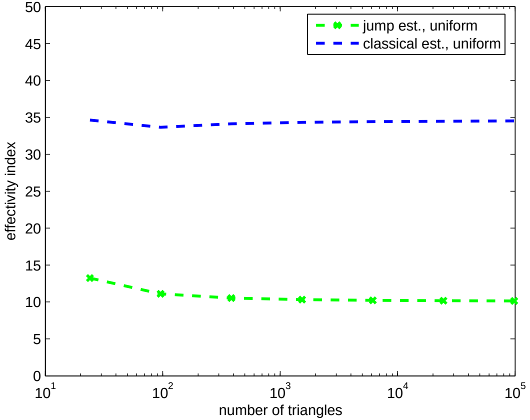
$$
\begin{aligned}
-\triangle p &= 0, &\quad \text{in } \Omega\\
p &= p_0, &\quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution
(polar coordinates)

$$p_0(r,\varphi) = r^{-\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right)$$



Effectivity index – comparison, uniform refinement

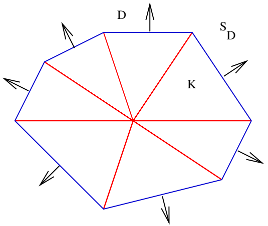


Effectivity indices for the jump and classical estimators

Improvement by local minimization

Observation

- Fluxes of $\mathbf{t}_h$ need to be prescribed on the boundary of dual volumes only to get $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$.
- We can choose them on other edges.



Local minimization (for each vertex)

- solve local linear problem (size = number of vertex sides)
- compute the estimators
- the whole estimate still has a linear cost

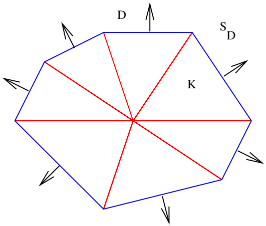
No linear system solution

- choose $\mathbf{t}_h$ such that $(\nabla \cdot \mathbf{t}_h, 1)_K = (f, 1)_K$ for all $K \in \mathcal{S}_h$

Improvement by local minimization

Observation

- Fluxes of $\mathbf{t}_h$ need to be prescribed on the boundary of dual volumes only to get $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$.
- We can choose them on other edges.



Local minimization (for each vertex)

- solve local linear problem (size = number of vertex sides)
- compute the estimators
- the whole estimate still has a linear cost

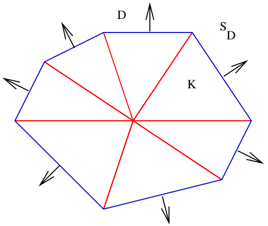
No linear system solution

- choose $\mathbf{t}_h$ such that $(\nabla \cdot \mathbf{t}_h, 1)_K = (f, 1)_K$ for all $K \in \mathcal{S}_h$

Improvement by local minimization

Observation

- Fluxes of $\mathbf{t}_h$ need to be prescribed on the boundary of dual volumes only to get $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$.
- We can choose them on other edges.



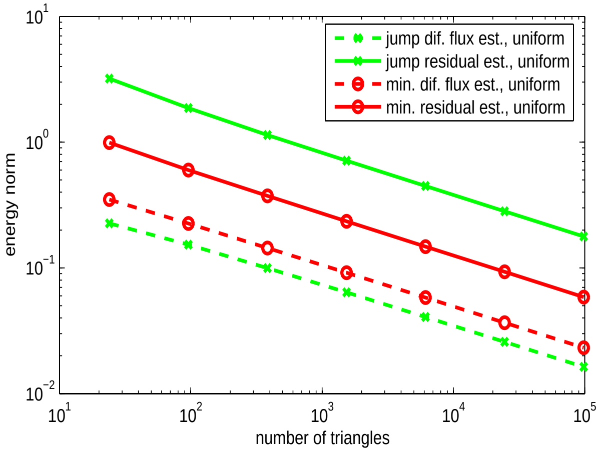
Local minimization (for each vertex)

- solve local linear problem (size = number of vertex sides)
- compute the estimators
- the whole estimate still has a linear cost

No linear system solution

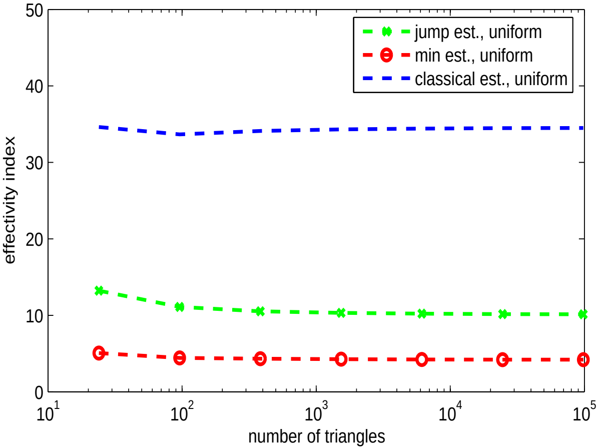
- choose $\mathbf{t}_h$ such that $(\nabla \cdot \mathbf{t}_h, 1)_K = (f, 1)_K$ for all $K \in \mathcal{S}_h$

Residual and diffusive flux estimators, uniform refinement



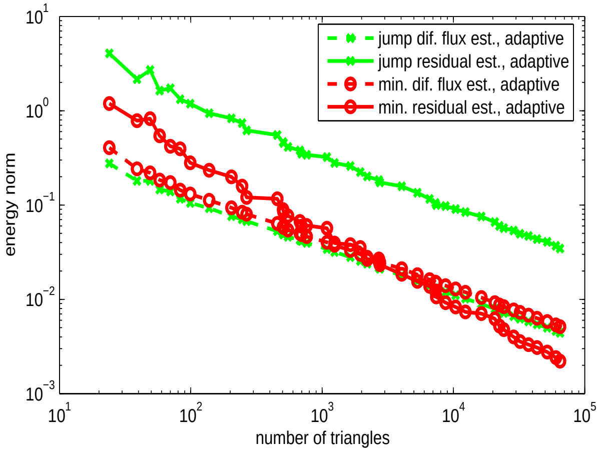
Residual and diffusive flux estimators comparison

Effectivity index – comparison, uniform refinement



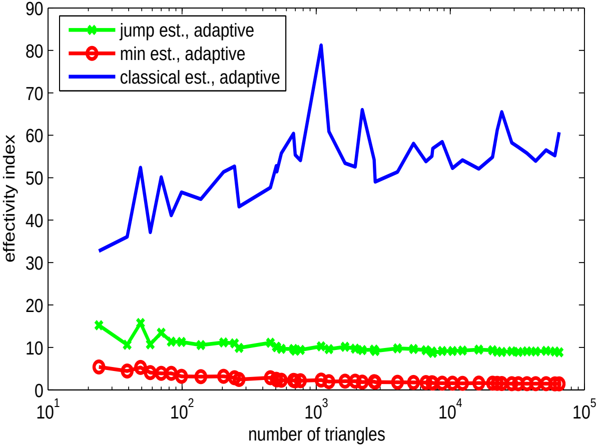
Effectivity indices for the jump, minimization, and classical estimators

Residual and diffusive flux estimators, uniform refinement



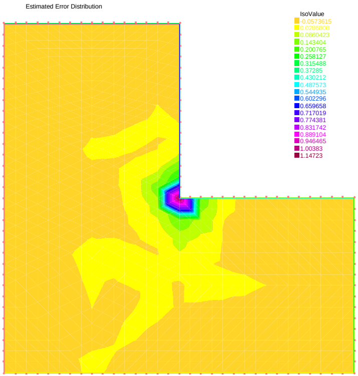
Residual and diffusive flux estimators comparison

Effectivity index – comparison, adaptive refinement

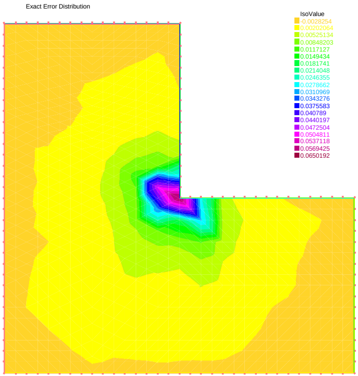


Effectivity indices for the jump, minimization, and classical estimators

Error distribution on a uniformly refined mesh

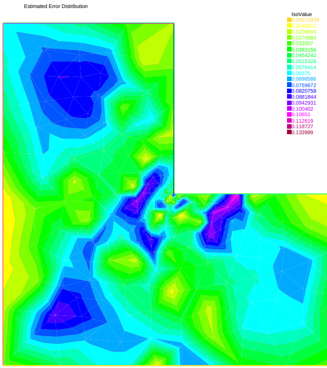


Estimated error distribution

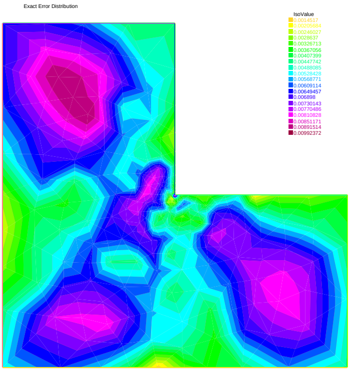


Exact error distribution

Error distribution on an adaptively refined mesh

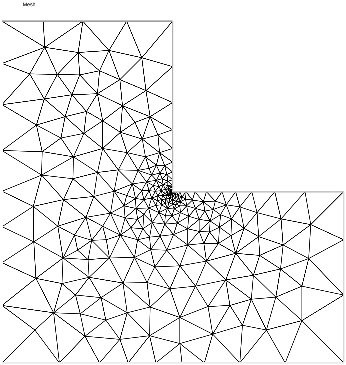


Estimated error distribution



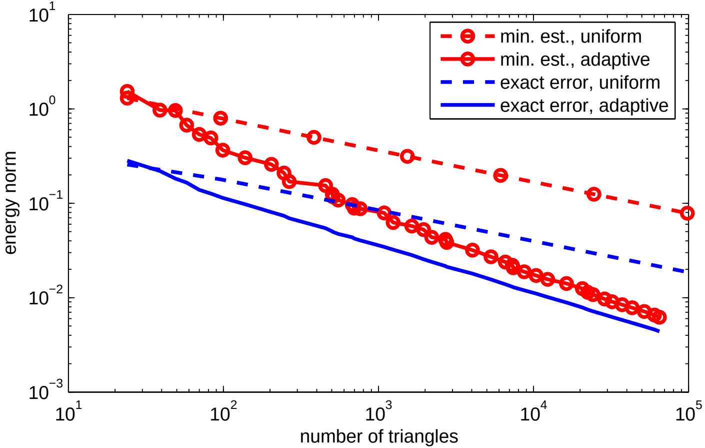
Exact error distribution

Adaptively refined mesh



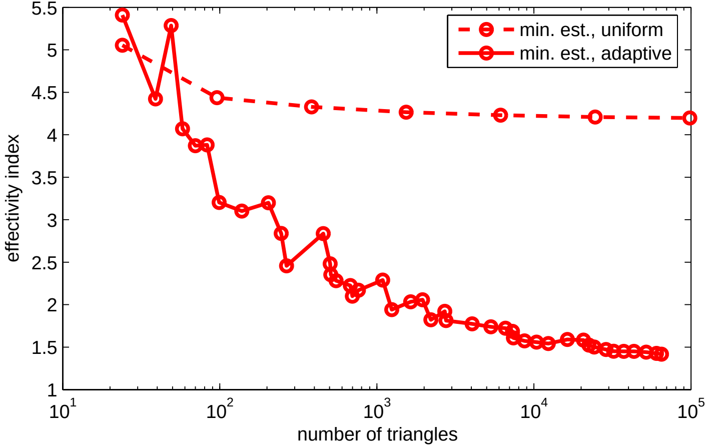
Corresponding adaptively refined mesh

Energy error



Estimated and actual energy error, uniformly/adaptively refined meshes

Effectivity index



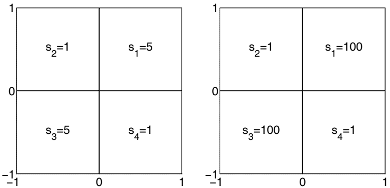
Effectivity index, uniformly/adaptively refined meshes

Discontinuous diffusion tensor and vertex-centered finite volumes

- consider the pure diffusion equation

$$-\nabla \cdot (a\nabla p) = 0 \quad \text{in} \quad \Omega = (-1,1) \times (-1,1)$$

- discontinuous and inhomogeneous a, two cases:

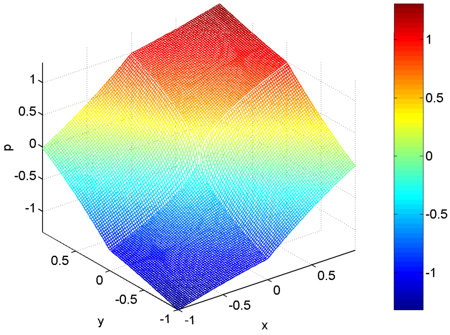


- analytical solution: singularity at the origin

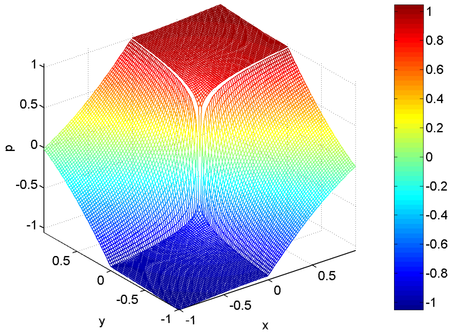
$$p(r,\theta)|_{\Omega_i} = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

Analytical solutions

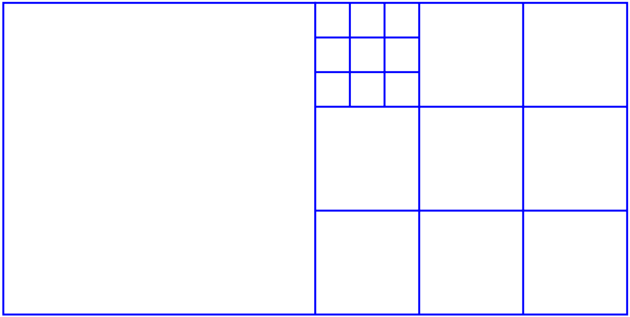


Case 1



Case 2

A vertex-centered FV scheme on nonmatching grids

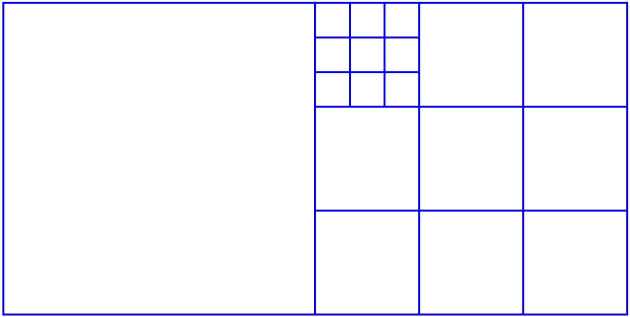


A vertex-centered FV scheme on nonmatching grids

- Suppose that a (nonmatching) grid $\mathcal{D}_h$ is given.
- Construct a conforming simplicial mesh $\mathcal{T}_h$ given by the "centers" of $\mathcal{D}_h$.
- Find $p_h \in V_h$ such that

$$-\langle \{\!\{a\}\!\}_\omega \nabla p_h \cdot \mathbf{n}, 1 \rangle_{\partial D} = (f, 1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

A vertex-centered FV scheme on nonmatching grids

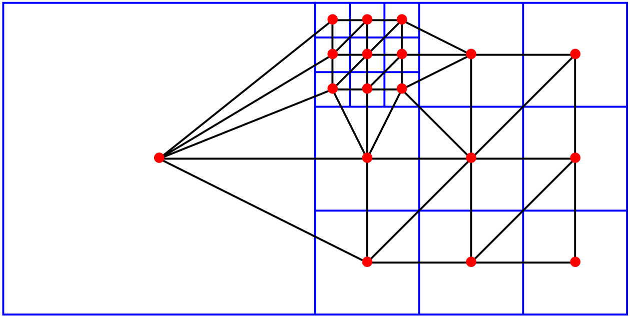


A vertex-centered FV scheme on nonmatching grids

- Suppose that a (nonmatching) grid $\mathcal{D}_h$ is given.
- Construct a conforming simplicial mesh $\mathcal{T}_h$ given by the "centers" of $\mathcal{D}_h$.
- Find $p_h \in V_h$ such that

$$-\langle \{\!\{a\}\!\}_\omega \nabla p_h \cdot \mathbf{n}, 1 \rangle_{\partial D} = (f, 1)_D \qquad \forall D \in \mathcal{D}_h^{\text{int}}.$$

A vertex-centered FV scheme on nonmatching grids

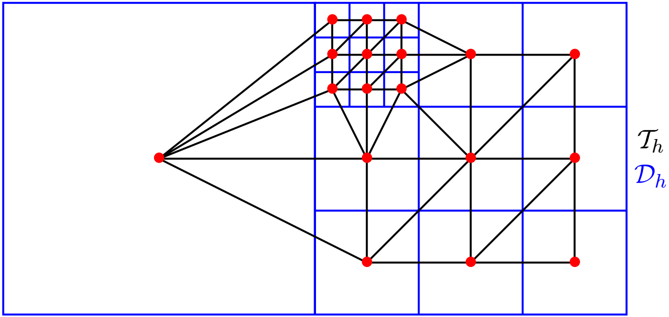


A vertex-centered FV scheme on nonmatching grids

- Suppose that a (nonmatching) grid $\mathcal{D}_h$ is given.
- Construct a conforming simplicial mesh $\mathcal{T}_h$ given by the "centers" of $\mathcal{D}_h$.
- Find $p_h \in V_h$ such that

$$-\langle \{\{a\}\}_\omega \nabla p_h \cdot \mathbf{n}, 1 \rangle_{\partial D} = (f, 1)_D \qquad \forall D \in \mathcal{D}_h^{\text{int}}.$$

A vertex-centered FV scheme on nonmatching grids

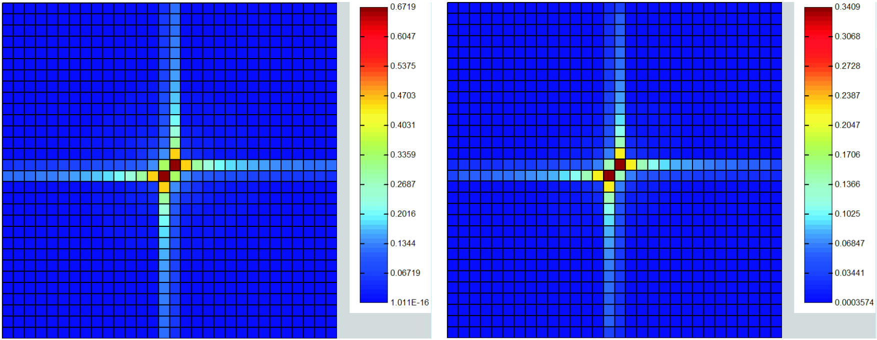


A vertex-centered FV scheme on nonmatching grids

- Suppose that a (nonmatching) grid $\mathcal{D}_h$ is given.
- Construct a conforming simplicial mesh $\mathcal{T}_h$ given by the "centers" of $\mathcal{D}_h$.
- Find $p_h \in V_h$ such that

$$-\langle \{\!\{a\}\!\}_\omega \nabla p_h \cdot \mathbf{n}, 1 \rangle_{\partial D} = (f, 1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int}}.$$

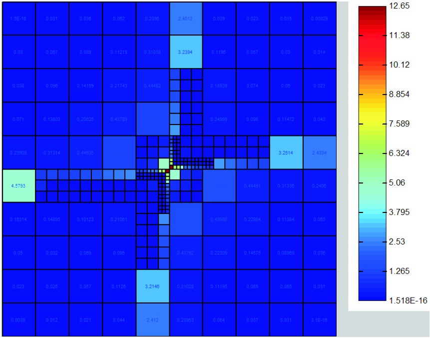
Error distribution on a uniformly refined mesh, case 1



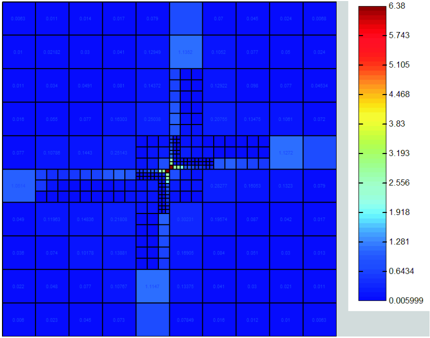
Estimated error distribution

Exact error distribution

Error distribution on an adaptively refined mesh, case 2

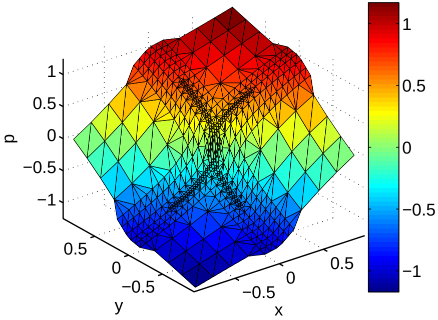


Estimated error distribution

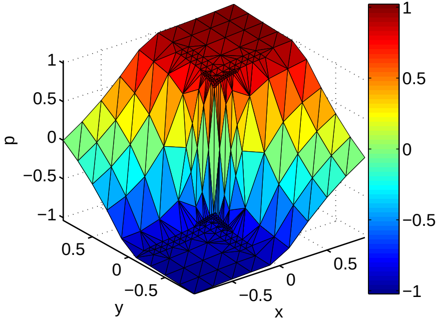


Exact error distribution

Approximate solutions on adaptively refined meshes

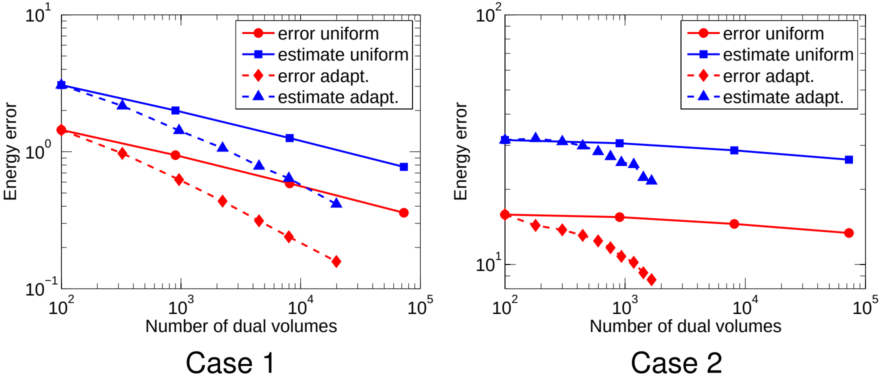


Case 1

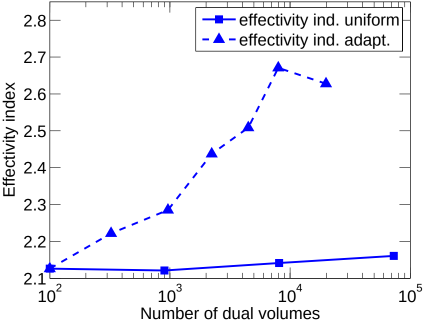


Case 2

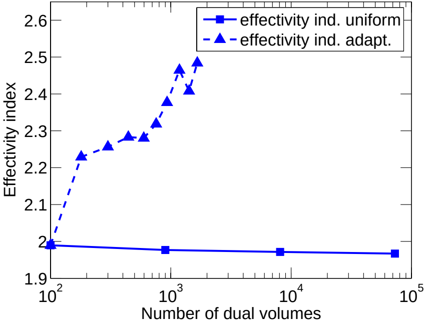
Estimated and actual errors in uniformly/adaptively refined meshes



Original effectivity indices in uniformly/adaptively refined meshes

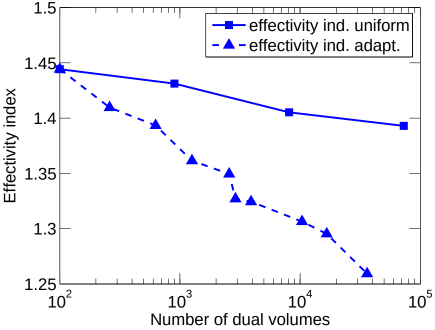


Case 1

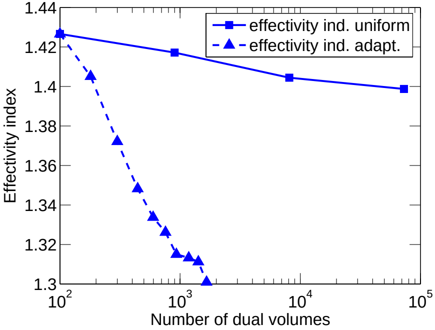


Case 2

Effectivity indices in uniformly/adaptively refined meshes using a simple (no linear system solution) local minimization



Case 1



Case 2

Outline

A model convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Bilinear form

$$
\mathcal{B}(p,\varphi) := (\mathbf{S}\nabla p, \nabla\varphi) + (\mathbf{w}\cdot\nabla p, \varphi) + (rp, \varphi), \qquad p, \varphi \in H^1(\mathcal{T}_h)
$$

Weak solution

Find $p \in H_0^1(\Omega)$ such that $\mathcal{B}(p,\varphi) = (f,\varphi) \quad \forall \varphi \in H_0^1(\Omega)$.

Energy norm

Decompose $\mathcal{B}$ into $\mathcal{B} = \mathcal{B}_{\mathrm{S}} + \mathcal{B}_{\mathrm{A}}$, where

$$
\begin{aligned}
\mathcal{B}_{\mathrm{S}}(p,\varphi) &:= (\mathbf{S}\nabla p, \nabla\varphi) + \left(\left(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\right)p, \varphi\right),\\
\mathcal{B}_{\mathrm{A}}(p,\varphi) &:= \left(\mathbf{w}\cdot\nabla p + \tfrac{1}{2}(\nabla\cdot\mathbf{w})p, \varphi\right).
\end{aligned}
$$

- $\mathcal{B}_{\mathrm{S}}$ is symmetric on $H^1(\mathcal{T}_h)$; put $|||\varphi|||^2 := \mathcal{B}_{\mathrm{S}}(\varphi,\varphi)$
- $\mathcal{B}_{\mathrm{A}}$ is skew-symmetric on $H_0^1(\Omega)$

A model convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$
\begin{aligned}
-\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Bilinear form

$$\mathcal{B}(p,\varphi) := (\mathbf{S}\nabla p, \nabla\varphi) + (\mathbf{w}\cdot\nabla p, \varphi) + (rp, \varphi), \qquad p, \varphi \in H^1(\mathcal{T}_h)$$

Weak solution

Find $p \in H_0^1(\Omega)$ such that $\mathcal{B}(p,\varphi) = (f,\varphi) \quad \forall \varphi \in H_0^1(\Omega)$.

Energy norm

Decompose $\mathcal{B}$ into $\mathcal{B} = \mathcal{B}_\mathrm{S} + \mathcal{B}_\mathrm{A}$, where

$$
\begin{aligned}
\mathcal{B}_\mathrm{S}(p,\varphi) &:= (\mathbf{S}\nabla p, \nabla\varphi) + \left(\left(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\right)p, \varphi\right),\\
\mathcal{B}_\mathrm{A}(p,\varphi) &:= \left(\mathbf{w}\cdot\nabla p + \tfrac{1}{2}(\nabla\cdot\mathbf{w})p, \varphi\right).
\end{aligned}
$$

- $\mathcal{B}_\mathrm{S}$ is symmetric on $H^1(\mathcal{T}_h)$; put $|||\varphi|||^2 := \mathcal{B}_\mathrm{S}(\varphi,\varphi)$
- $\mathcal{B}_\mathrm{A}$ is skew-symmetric on $H_0^1(\Omega)$

A model convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Bilinear form

$$\mathcal{B}(p,\varphi) := (\mathbf{S}\nabla p, \nabla\varphi) + (\mathbf{w}\cdot\nabla p, \varphi) + (rp, \varphi), \qquad p,\varphi \in H^1(\mathcal{T}_h)$$

Weak solution

Find $p \in H_0^1(\Omega)$ such that $\mathcal{B}(p,\varphi) = (f,\varphi) \quad \forall \varphi \in H_0^1(\Omega)$.

Energy norm

Decompose $\mathcal{B}$ into $\mathcal{B} = \mathcal{B}_{\mathrm{S}} + \mathcal{B}_{\mathrm{A}}$, where

$$\begin{aligned} \mathcal{B}_{\mathrm{S}}(p,\varphi) &:= (\mathbf{S}\nabla p, \nabla\varphi) + \left(\left(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\right)p, \varphi\right), \\ \mathcal{B}_{\mathrm{A}}(p,\varphi) &:= \left(\mathbf{w}\cdot\nabla p + \tfrac{1}{2}(\nabla\cdot\mathbf{w})p, \varphi\right). \end{aligned}$$

- $\mathcal{B}_{\mathrm{S}}$ is symmetric on $H^1(\mathcal{T}_h)$; put $|||\varphi|||^2 := \mathcal{B}_{\mathrm{S}}(\varphi,\varphi)$
- $\mathcal{B}_{\mathrm{A}}$ is skew-symmetric on $H_0^1(\Omega)$

A model convection–diffusion–reaction problem

A model convection–diffusion–reaction problem

$$\begin{aligned} -\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Bilinear form

$$\mathcal{B}(p,\varphi) := (\mathbf{S}\nabla p, \nabla\varphi) + (\mathbf{w}\cdot\nabla p, \varphi) + (rp, \varphi), \qquad p, \varphi \in H^1(\mathcal{T}_h)$$

Weak solution

Find $p \in H_0^1(\Omega)$ such that $\mathcal{B}(p,\varphi) = (f,\varphi) \quad \forall \varphi \in H_0^1(\Omega)$.

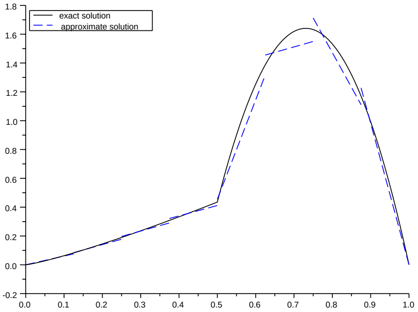
Energy norm

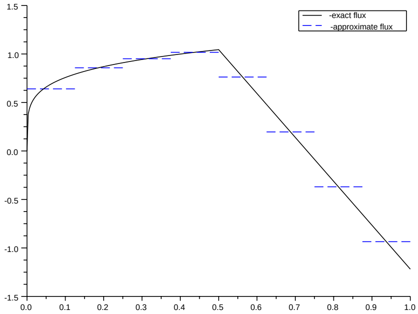
Decompose $\mathcal{B}$ into $\mathcal{B} = \mathcal{B}_\mathrm{S} + \mathcal{B}_\mathrm{A}$, where

$$\begin{aligned} \mathcal{B}_\mathrm{S}(p,\varphi) &:= (\mathbf{S}\nabla p, \nabla\varphi) + \left(\left(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\right)p, \varphi\right), \\ \mathcal{B}_\mathrm{A}(p,\varphi) &:= \left(\mathbf{w}\cdot\nabla p + \tfrac{1}{2}(\nabla\cdot\mathbf{w})p, \varphi\right). \end{aligned}$$

- $\mathcal{B}_\mathrm{S}$ is symmetric on $H^1(\mathcal{T}_h)$; put $|||\varphi|||^2 := \mathcal{B}_\mathrm{S}(\varphi,\varphi)$
- $\mathcal{B}_\mathrm{A}$ is skew-symmetric on $H_0^1(\Omega)$

Approximate solution and approximate flux

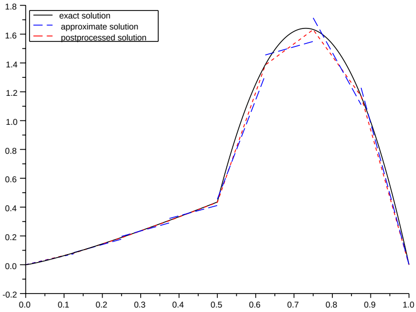




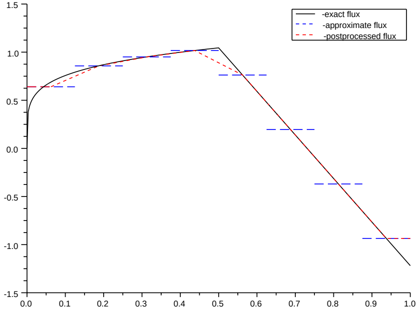
Approximate solution p_h is not in $H_0^1(\Omega)$

Approximate flux $-a\nabla p_h$ is not in $\mathbf{H}(\mathrm{div}, \Omega)$

Approximate solution and approximate flux



Construct a postprocessed approx. solution s_h in $H_0^1(\Omega)$



Construct a postprocessed flux $\mathbf{t}_h$ in $\mathbf{H}(\mathrm{div}, \Omega)$

Outline

Optimal abstract estimate in the energy norm

Theorem (Optimal abstract estimate, energy norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
|||p - p_h||| \; \leq \; & \inf_{s \in H^1_0(\Omega)} \Big\{ |||p_h - s||| \\
& + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H^1_0(\Omega),\, |||\varphi|||=1} \Big| (f - \nabla\cdot\mathbf{t} - \mathbf{w}\cdot\nabla s - rs, \varphi) \\
& - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) + \big((r - \tfrac{1}{2}\nabla\cdot\mathbf{w})(s - p_h), \varphi \big) \Big| \Big\} \\
\leq \; & 2|||p - p_h|||.
\end{aligned}
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of p_h.
- Not computable (infimum over an infinite-dim. space).

Optimal abstract estimate in the energy norm

Theorem (Optimal abstract estimate, energy norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
|||p - p_h||| \leq & \inf_{s \in H_0^1(\Omega)} \Big\{ |||p_h - s||| \\
& + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \Big| (f - \nabla\cdot\mathbf{t} - \mathbf{w}\cdot\nabla s - rs, \varphi) \\
& - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) + \big((r - \tfrac{1}{2}\nabla\cdot\mathbf{w})(s - p_h), \varphi\big) \Big| \Big\} \\
\leq & \; 2|||p - p_h|||.
\end{aligned}
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of p_h.
- Not computable (infimum over an infinite-dim. space).

Optimal abstract estimate in the energy norm

Theorem (Optimal abstract estimate, energy norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
|||p - p_h||| \le & \inf_{s \in H_0^1(\Omega)} \Big\{ |||p_h - s||| \\
& + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \Big| (f - \nabla\cdot\mathbf{t} - \mathbf{w}\cdot\nabla s - rs, \varphi) \\
& - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) + \big((r - \tfrac{1}{2}\nabla\cdot\mathbf{w})(s - p_h), \varphi\big) \Big| \Big\} \\
\le & \ 2|||p - p_h|||.
\end{aligned}
$$

Properties

- Guaranteed upper bound, quasi-exact, and robust.
- Holds uniformly for any mesh (anisotropic) and polynomial degree of p_h.
- Not computable (infimum over an infinite-dim. space).

Discontinuous Galerkin method

Discontinuous Galerkin method

Find $p_h \in \mathbb{P}_k(\mathcal{T}_h)$ such that for all $\varphi_h \in \mathbb{P}_k(\mathcal{T}_h)$

$$(\mathbf{S}\nabla p_h, \nabla \varphi_h) + ((r - \nabla\cdot\mathbf{w})p_h, \varphi_h) - (p_h, \mathbf{w}\cdot\nabla\varphi_h)$$

$$- \sum_{\sigma\in\mathcal{E}_h} \{\langle \mathbf{n}_\sigma \cdot \{\!\{\mathbf{S}\nabla p_h\}\!\}_\omega, [\![\varphi_h]\!]\rangle_\sigma + \theta\langle \mathbf{n}_\sigma \cdot \{\!\{\mathbf{S}\nabla \varphi_h\}\!\}_\omega, [\![p_h]\!]\rangle_\sigma\}$$

$$+\sum_{\sigma\in\mathcal{E}_h}\left\{\langle(\alpha_\sigma\gamma_{\mathbf{S},\sigma}h_\sigma^{-1}+\gamma_{\mathbf{w},\sigma})[\![p_h]\!], [\![\varphi_h]\!]\rangle_\sigma + \langle \mathbf{w}\cdot\mathbf{n}_\sigma\{\!\{p_h\}\!\}, [\![\varphi_h]\!]\rangle_\sigma\right\} = (f, \varphi_h)$$

- jump operator $[\![\varphi]\!]_\sigma = \varphi^- - \varphi^+$
- average operator $\{\!\{\varphi\}\!\} = \frac{1}{2}(\varphi^- + \varphi^+)$
- harmonic-weighted average op. $\{\!\{\varphi\}\!\}_\omega = (\omega^-\varphi^- + \omega^+\varphi^+)$
- diffusivity-dependent penalties $\gamma_{\mathbf{S},\sigma}$ (Ern, Stephansen, and Zunino 08)
- θ: different scheme types (SIPG/NIPG/IIPG/OBB)
- $p_h \notin H_0^1(\Omega)$, $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$

Discontinuous Galerkin method

Discontinuous Galerkin method

Find $p_h \in \mathbb{P}_k(\mathcal{T}_h)$ such that for all $\varphi_h \in \mathbb{P}_k(\mathcal{T}_h)$

$$(\mathbf{S}\nabla p_h, \nabla\varphi_h) + ((r - \nabla\cdot\mathbf{w})p_h, \varphi_h) - (p_h, \mathbf{w}\cdot\nabla\varphi_h)$$

$$- \sum_{\sigma\in\mathcal{E}_h} \{\langle \mathbf{n}_\sigma \cdot \{\!\{\mathbf{S}\nabla p_h\}\!\}_\omega, [\![\varphi_h]\!]\rangle_\sigma + \theta\langle \mathbf{n}_\sigma \cdot \{\!\{\mathbf{S}\nabla\varphi_h\}\!\}_\omega, [\![p_h]\!]\rangle_\sigma\}$$

$$+\sum_{\sigma\in\mathcal{E}_h}\Big\{\langle(\alpha_\sigma\gamma_{\mathbf{S},\sigma}h_\sigma^{-1}+\gamma_{\mathbf{w},\sigma})[\![p_h]\!], [\![\varphi_h]\!]\rangle_\sigma+\langle\mathbf{w}\cdot\mathbf{n}_\sigma\{\!\{p_h\}\!\}, [\![\varphi_h]\!]\rangle_\sigma\Big\}=(f,\varphi_h)$$

- jump operator $[\![\varphi]\!]_\sigma = \varphi^- - \varphi^+$
- average operator $\{\!\{\varphi\}\!\} = \frac{1}{2}(\varphi^- + \varphi^+)$
- harmonic-weighted average op. $\{\!\{\varphi\}\!\}_\omega = (\omega^-\varphi^- + \omega^+\varphi^+)$
- diffusivity-dependent penalties $\gamma_{\mathbf{S},\sigma}$ (Ern, Stephansen, and Zunino 08)
- θ: different scheme types (SIPG/NIPG/IIPG/OBB)
- $p_h \notin H_0^1(\Omega)$, $-\mathbf{S}\nabla p_h \notin \mathbf{H}(\mathrm{div}, \Omega)$

Potential- and flux-conforming reconstructions

Choice of s_h: the Oswald interpolate of p_h

- $\mathcal{I}_{\mathrm{Os}} : \mathbb{P}_k(\mathcal{T}_h) \to \mathbb{P}_k(\mathcal{T}_h) \cap H_0^1(\Omega)$
- prescribed at Lagrange nodes by arithmetic averages

$$\mathcal{I}_{\mathrm{Os}}(\varphi_h)(V) = \frac{1}{\#(\mathcal{T}_V)} \sum_{K \in \mathcal{T}_V} \varphi_h|_K(V)$$

- one can also use diffusivity-weighted averages (Ainsworth '05)

Choice of $\mathbf{t}_h$: a new $\mathbf{H}(\mathrm{div}, \Omega)$ flux reconstruction

- Ern, Nicaise & Vohralík '07 (matching meshes)
- the present work (nonmatching meshes)

Potential- and flux-conforming reconstructions

Choice of s_h: the Oswald interpolate of p_h

- $\mathcal{I}_{\mathrm{Os}} : \mathbb{P}_k(\mathcal{T}_h) \to \mathbb{P}_k(\mathcal{T}_h) \cap H_0^1(\Omega)$
- prescribed at Lagrange nodes by arithmetic averages

$$\mathcal{I}_{\mathrm{Os}}(\varphi_h)(V) = \frac{1}{\#(\mathcal{T}_V)} \sum_{K \in \mathcal{T}_V} \varphi_h|_K(V)$$

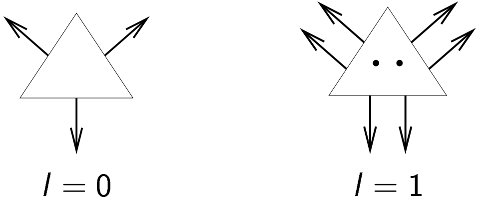
- one can also use diffusivity-weighted averages (Ainsworth '05)

Choice of $\mathbf{t}_h$: a new $\mathbf{H}(\mathrm{div}, \Omega)$ flux reconstruction

- Ern, Nicaise & Vohralík '07 (matching meshes)
- the present work (nonmatching meshes)

Diffusive flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}_h)$: Raviart–Thomas–Nédélec spaces of degree l



Construction of $\mathbf{t}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(\sigma)$,
  $$(\mathbf{t}_h{\cdot}\mathbf{n}_\sigma, q_h)_\sigma = \left(-\mathbf{n}_\sigma{\cdot}\{\!\{\mathbf{S}\nabla p_h\}\!\}_\omega + \alpha_\sigma \gamma_{\mathbf{S},\sigma} h_\sigma^{-1}[\![p_h]\!], q_h\right)_\sigma$$
- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}_{l-1}^d(K)$,
  $$(\mathbf{t}_h, \mathbf{r}_h)_K = -(\mathbf{S}\nabla p_h, \mathbf{r}_h)_K + \theta \sum_{\sigma \in \mathcal{E}_K} \omega_{K,\sigma}(\mathbf{n}_\sigma{\cdot}\mathbf{S}\mathbf{r}_h, [\![p_h]\!])_\sigma$$

Crucial property when $\mathbf{w} = r = 0$

$\nabla{\cdot}\mathbf{t}_h = \Pi_l(f)$ (Π_l is the L^2-orthogonal projection onto $\mathbb{P}_k(\mathcal{T}_h)$)

Diffusive flux reconstruction

$\mathbf{RTN}^l(\mathcal{T}_h)$: Raviart–Thomas–Nédélec spaces of degree l

Construction of $\mathbf{t}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(\sigma)$,
$$(\mathbf{t}_h{\cdot}\mathbf{n}_\sigma, q_h)_\sigma = \left(-\mathbf{n}_\sigma{\cdot}\{\!\{\mathbf{S}\nabla p_h\}\!\}_\omega + \alpha_\sigma \gamma_{\mathbf{S},\sigma} h_\sigma^{-1} [\![p_h]\!], q_h\right)_\sigma$$
- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}^d_{l-1}(K)$,
$$(\mathbf{t}_h, \mathbf{r}_h)_K = -(\mathbf{S}\nabla p_h, \mathbf{r}_h)_K + \theta \sum_{\sigma \in \mathcal{E}_K} \omega_{K,\sigma} (\mathbf{n}_\sigma{\cdot}\mathbf{S}\mathbf{r}_h, [\![p_h]\!])_\sigma$$

Crucial property when $\mathbf{w} = r = 0$

$\nabla{\cdot}\mathbf{t}_h = \Pi_l(f)$ (Π_l is the L^2-orthogonal projection onto $\mathbb{P}_k(\mathcal{T}_h)$)

Diffusive flux reconstruction

$\mathbf{RT}\mathbf{N}^l(\mathcal{T}_h)$: Raviart–Thomas–Nédélec spaces of degree l

$l = 0$ $\quad$ $l = 1$

Construction of $\mathbf{t}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(\sigma)$,
$$(\mathbf{t}_h{\cdot}\mathbf{n}_\sigma, q_h)_\sigma = \left(-\mathbf{n}_\sigma{\cdot}\{\!\{\mathbf{S}\nabla p_h\}\!\}_\omega + \alpha_\sigma \gamma_{\mathbf{S},\sigma} h_\sigma^{-1} [\![p_h]\!], q_h\right)_\sigma$$
- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}^d_{l-1}(K)$,
$$(\mathbf{t}_h, \mathbf{r}_h)_K = -(\mathbf{S}\nabla p_h, \mathbf{r}_h)_K + \theta \sum_{\sigma \in \mathcal{E}_K} \omega_{K,\sigma} (\mathbf{n}_\sigma{\cdot}\mathbf{S}\mathbf{r}_h, [\![p_h]\!])_\sigma$$

Crucial property when $\mathbf{w} = r = 0$

$\nabla{\cdot}\mathbf{t}_h = \Pi_l(f)$ (Π_l is the L^2-orthogonal projection onto $\mathbb{P}_k(\mathcal{T}_h)$)

Convective flux reconstruction

Convective flux reconstruction $\mathbf{q}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k-1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(\sigma)$,

$$(\mathbf{q}_h \cdot \mathbf{n}_\sigma, q_h)_\sigma = (\mathbf{w} \cdot \mathbf{n}_\sigma \{\!\{p_h\}\!\} + \gamma_{\mathbf{w},\sigma} [\![p_h]\!], q_h)_\sigma$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}^d_{l-1}(K)$,

$$(\mathbf{q}_h, \mathbf{r}_h)_K = (p_h, \mathbf{w} \cdot \mathbf{r}_h)_K$$

Crucial property

$$(\nabla\cdot \mathbf{t}_h + \nabla\cdot \mathbf{q}_h + (r - \nabla\cdot \mathbf{w}) p_h, \xi_h)_K = (f, \xi_h)_K \qquad \forall K \in \mathcal{T}_h, \forall \xi_h \in \mathbb{P}_l(K)$$

Convective flux reconstruction

Convective flux reconstruction $\mathbf{q}_h \in \mathbf{RTN}^l(\mathcal{T}_h)$, $l = k$ or $l = k - 1$

- normal components on each side: $\forall q_h \in \mathbb{P}_l(\sigma)$,

$$(\mathbf{q}_h{\cdot}\mathbf{n}_\sigma, q_h)_\sigma = (\mathbf{w}{\cdot}\mathbf{n}_\sigma \{\!\{p_h\}\!\} + \gamma_{\mathbf{w},\sigma} [\![p_h]\!], q_h)_\sigma$$

- on each element (only for $l \geq 1$): $\forall \mathbf{r}_h \in \mathbb{P}^d_{l-1}(K)$,

$$(\mathbf{q}_h, \mathbf{r}_h)_K = (p_h, \mathbf{w}{\cdot}\mathbf{r}_h)_K$$

Crucial property

$$(\nabla{\cdot}\mathbf{t}_h + \nabla{\cdot}\mathbf{q}_h + (r - \nabla{\cdot}\mathbf{w})p_h, \xi_h)_K = (f, \xi_h)_K \qquad \forall K \in \mathcal{T}_h,\ \forall \xi_h \in \mathbb{P}_l(K)$$

A post. estimate for $-\nabla\cdot(\mathbf{S}\nabla p)+\mathbf{w}\cdot\nabla p+rp=f$

Theorem (A posteriori error estimate, energy norm)

There holds

$$|||p-p_h|||\leq\eta,$$

$$\eta:=\left\{\sum_{K\in\mathcal{T}_h}\eta_{\mathrm{NC},K}^2\right\}^{\frac{1}{2}}+\left\{\sum_{K\in\mathcal{T}_h}(\eta_{\mathrm{R},K}+\eta_{\mathrm{DF},K}+\eta_{\mathrm{C},1,K}+\eta_{\mathrm{C},2,K}+\eta_{\mathrm{U},K})^2\right\}^{\frac{1}{2}},$$

where

- $\eta_{\mathrm{NC},K}=|||p_h-\mathcal{I}_{\mathrm{Os}}(p_h)|||_K$ (*nonconformity*),
- $\eta_{\mathrm{DF},K}=\min\left\{\eta_{\mathrm{DF},K}^{(1)},\eta_{\mathrm{DF},K}^{(2)}\right\}$ (*diffusive flux*),
- $\eta_{\mathrm{R},K}=m_K\|f-\nabla\cdot\mathbf{t}_h-\nabla\cdot\mathbf{q}_h-(r-\nabla\cdot\mathbf{w})p_h\|_K$ (*residual*),
- $\eta_{\mathrm{C},1,K}=m_K\|(Id-\Pi_0)(\nabla\cdot(\mathbf{q}_h-\mathbf{w}s_h))\|_K$ (*convection*),
- $\eta_{\mathrm{C},2,K}=c_{\mathbf{w},r,K}^{-1/2}\left\|\frac{1}{2}(\nabla\cdot\mathbf{w})(p_h-s_h)\right\|_K$ (*convection*),
- $\eta_{\mathrm{U},K}=\sum_{\sigma\in\mathcal{E}_K}m_\sigma\|\Pi_{0,\sigma}((\mathbf{q}_h-\mathbf{w}s_h)\cdot\mathbf{n}_\sigma)\|_\sigma$ (*upwinding*).

A post. estimate for $-\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp = f$

Theorem (A posteriori error estimate, energy norm)

There holds

$$|||p - p_h||| \le \eta,$$

$$\eta := \left\{\sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2\right\}^{\frac{1}{2}} + \left\{\sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K} + \eta_{\mathrm{C},1,K} + \eta_{\mathrm{C},2,K} + \eta_{\mathrm{U},K})^2\right\}^{\frac{1}{2}},$$

where

- $\eta_{\mathrm{NC},K} = |||p_h - \mathcal{I}_{\mathrm{Os}}(p_h)|||_K$ *(nonconformity),*
- $\eta_{\mathrm{DF},K} = \min\left\{\eta_{\mathrm{DF},K}^{(1)}, \eta_{\mathrm{DF},K}^{(2)}\right\}$ *(diffusive flux),*
- $\eta_{\mathrm{R},K} = m_K\|f - \nabla\cdot\mathbf{t}_h - \nabla\cdot\mathbf{q}_h - (r - \nabla\cdot\mathbf{w})p_h\|_K$ *(residual),*
- $\eta_{\mathrm{C},1,K} = m_K\|(Id - \Pi_0)(\nabla\cdot(\mathbf{q}_h - \mathbf{w}s_h))\|_K$ *(convection),*
- $\eta_{\mathrm{C},2,K} = c_{\mathbf{w},r,K}^{-1/2}\left\|\frac{1}{2}(\nabla\cdot\mathbf{w})(p_h - s_h)\right\|_K$ *(convection),*
- $\eta_{\mathrm{U},K} = \sum_{\sigma\in\mathcal{E}_K} m_\sigma\|\Pi_{0,\sigma}((\mathbf{q}_h - \mathbf{w}s_h)\cdot\mathbf{n}_\sigma)\|_\sigma$ *(upwinding).*

Individual estimators

Diffusive flux estimator $\eta_{\mathrm{DF},K}$

- $\eta^{(1)}_{\mathrm{DF},K} = \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|_K$
- $\eta^{(2)}_{\mathrm{DF},K} = m_K\|(Id - \Pi_0)(\nabla\cdot(\mathbf{S}\nabla p_h + \mathbf{t}_h))\|_K + \widetilde{m}_K^{1/2}\sum_{\sigma\in\mathcal{E}_K} C_{\mathrm{t},K,\sigma}^{1/2}\|(\mathbf{S}\nabla p_h + \mathbf{t}_h)\cdot\mathbf{n}_\sigma\|_\sigma$
- cutoff fcts of local Péclet and Damköhler numbers in $\eta^{(2)}_{\mathrm{DF},K}$:

$$m_K := \min\{C_{\mathrm{P}}^{1/2} h_K c_{\mathbf{S},K}^{-1/2}, c_{\mathbf{w},r,K}^{-1/2}\},$$

$$\widetilde{m}_K := \min\{(C_{\mathrm{P}} + C_{\mathrm{P}}^{1/2}) h_K c_{\mathbf{S},K}^{-1}, h_K^{-1} c_{\mathbf{w},r,K}^{-1} + c_{\mathbf{w},r,K}^{-1/2} c_{\mathbf{S},K}^{-1/2}/2\}$$

Upwinding estimator $\eta_{\mathrm{U},K}$

- $\eta_{\mathrm{U},K} = \sum_{\sigma\in\mathcal{E}_K} m_\sigma\|\Pi_{0,\sigma}((\mathbf{q}_h - \mathbf{w}s_h)\cdot\mathbf{n}_\sigma)\|_\sigma$
- cutoff function of local Péclet and Damköhler numbers:

$$m_\sigma^2 = \min\left\{\max_{K\in\mathcal{T}_\sigma}\left\{C_{\mathrm{F},K,\sigma}\frac{|\sigma|h_K^2}{|K|c_{\mathbf{S},K}}\right\}, \max_{K\in\mathcal{T}_\sigma}\left\{\frac{|\sigma|}{|K|c_{\mathbf{w},r,K}}\right\}\right\}$$

Individual estimators

Diffusive flux estimator $\eta_{\mathrm{DF},K}$

- $\eta_{\mathrm{DF},K}^{(1)} = \|\mathbf{S}^{\frac{1}{2}} \nabla p_h + \mathbf{S}^{-\frac{1}{2}} \mathbf{t}_h\|_K$
- $\eta_{\mathrm{DF},K}^{(2)} = m_K \|(Id - \Pi_0)(\nabla \cdot (\mathbf{S}\nabla p_h + \mathbf{t}_h))\|_K + \widetilde{m}_K^{1/2} \sum_{\sigma \in \mathcal{E}_K} C_{\mathrm{t},K,\sigma}^{1/2} \|(\mathbf{S}\nabla p_h + \mathbf{t}_h) \cdot \mathbf{n}_\sigma\|_\sigma$
- cutoff fcts of local Péclet and Damköhler numbers in $\eta_{\mathrm{DF},K}^{(2)}$:

$$m_K := \min\{C_{\mathrm{P}}^{1/2} h_K c_{\mathbf{S},K}^{-1/2}, c_{\mathbf{w},r,K}^{-1/2}\},$$

$$\widetilde{m}_K := \min\{(C_{\mathrm{P}} + C_{\mathrm{P}}^{1/2}) h_K c_{\mathbf{S},K}^{-1}, h_K^{-1} c_{\mathbf{w},r,K}^{-1} + c_{\mathbf{w},r,K}^{-1/2} c_{\mathbf{S},K}^{-1/2}/2\}$$

Upwinding estimator $\eta_{\mathrm{U},K}$

- $\eta_{\mathrm{U},K} = \sum_{\sigma \in \mathcal{E}_K} m_\sigma \|\Pi_{0,\sigma}((\mathbf{q}_h - \mathbf{w} s_h) \cdot \mathbf{n}_\sigma)\|_\sigma$
- cutoff function of local Péclet and Damköhler numbers:

$$m_\sigma^2 = \min\left\{\max_{K \in \mathcal{T}_\sigma}\left\{C_{\mathrm{F},K,\sigma} \frac{|\sigma| h_K^2}{|K| c_{\mathbf{S},K}}\right\}, \max_{K \in \mathcal{T}_\sigma}\left\{\frac{|\sigma|}{|K| c_{\mathbf{w},r,K}}\right\}\right\}$$

Individual estimators

Diffusive flux estimator $\eta_{\mathrm{DF},K}$

- $\eta^{(1)}_{\mathrm{DF},K} = \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|_K$
- $\eta^{(2)}_{\mathrm{DF},K} = m_K\|(Id - \Pi_0)(\nabla\cdot(\mathbf{S}\nabla p_h + \mathbf{t}_h))\|_K + \widetilde{m}_K^{1/2}\sum_{\sigma\in\mathcal{E}_K} C_{\mathrm{t},K,\sigma}^{1/2}\|(\mathbf{S}\nabla p_h + \mathbf{t}_h)\cdot\mathbf{n}_\sigma\|_\sigma$
- cutoff fcts of local Péclet and Damköhler numbers in $\eta^{(2)}_{\mathrm{DF},K}$:

$$m_K := \min\{C_{\mathrm{P}}^{1/2} h_K c_{\mathbf{S},K}^{-1/2}, c_{\mathbf{w},r,K}^{-1/2}\},$$

$$\widetilde{m}_K := \min\{(C_{\mathrm{P}} + C_{\mathrm{P}}^{1/2}) h_K c_{\mathbf{S},K}^{-1}, h_K^{-1} c_{\mathbf{w},r,K}^{-1} + c_{\mathbf{w},r,K}^{-1/2} c_{\mathbf{S},K}^{-1/2}/2\}$$

Upwinding estimator $\eta_{\mathrm{U},K}$

- $\eta_{\mathrm{U},K} = \sum_{\sigma\in\mathcal{E}_K} m_\sigma\|\Pi_{0,\sigma}((\mathbf{q}_h - \mathbf{w}s_h)\cdot\mathbf{n}_\sigma)\|_\sigma$
- cutoff function of local Péclet and Damköhler numbers:

$$m_\sigma^2 = \min\left\{\max_{K\in\mathcal{T}_\sigma}\left\{C_{\mathrm{F},K,\sigma}\frac{|\sigma| h_K^2}{|K| c_{\mathbf{S},K}}\right\}, \max_{K\in\mathcal{T}_\sigma}\left\{\frac{|\sigma|}{|K| c_{\mathbf{w},r,K}}\right\}\right\}$$

Individual estimators

Diffusive flux estimator $\eta_{\mathrm{DF},K}$

- $\eta^{(1)}_{\mathrm{DF},K} = \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|_K$
- $\eta^{(2)}_{\mathrm{DF},K} = m_K\|(Id - \Pi_0)(\nabla\cdot(\mathbf{S}\nabla p_h + \mathbf{t}_h))\|_K + \widetilde{m}_K^{1/2}\sum_{\sigma\in\mathcal{E}_K} C^{1/2}_{\mathrm{t},K,\sigma}\|(\mathbf{S}\nabla p_h + \mathbf{t}_h)\cdot\mathbf{n}_\sigma\|_\sigma$
- cutoff fcts of local Péclet and Damköhler numbers in $\eta^{(2)}_{\mathrm{DF},K}$:

$$m_K := \min\{C_{\mathrm{P}}^{1/2} h_K c_{\mathbf{S},K}^{-1/2}, c_{\mathbf{w},r,K}^{-1/2}\},$$

$$\widetilde{m}_K := \min\{(C_{\mathrm{P}} + C_{\mathrm{P}}^{1/2}) h_K c_{\mathbf{S},K}^{-1}, h_K^{-1} c_{\mathbf{w},r,K}^{-1} + c_{\mathbf{w},r,K}^{-1/2} c_{\mathbf{S},K}^{-1/2}/2\}$$

Upwinding estimator $\eta_{\mathrm{U},K}$

- $\eta_{\mathrm{U},K} = \sum_{\sigma\in\mathcal{E}_K} m_\sigma\|\Pi_{0,\sigma}((\mathbf{q}_h - \mathbf{w}s_h)\cdot\mathbf{n}_\sigma)\|_\sigma$
- cutoff function of local Péclet and Damköhler numbers:

$$m_\sigma^2 = \min\left\{\max_{K\in\mathcal{T}_\sigma}\left\{C_{\mathrm{F},K,\sigma}\frac{|\sigma|h_K^2}{|K|c_{\mathbf{S},K}}\right\}, \max_{K\in\mathcal{T}_\sigma}\left\{\frac{|\sigma|}{|K|c_{\mathbf{w},r,K}}\right\}\right\}$$

Individual estimators

Diffusive flux estimator $\eta_{\mathrm{DF},K}$

- $\eta^{(1)}_{\mathrm{DF},K} = \|\mathbf{S}^{\frac{1}{2}}\nabla p_h + \mathbf{S}^{-\frac{1}{2}}\mathbf{t}_h\|_K$
- $\eta^{(2)}_{\mathrm{DF},K} = m_K\|(Id - \Pi_0)(\nabla\cdot(\mathbf{S}\nabla p_h + \mathbf{t}_h))\|_K + \widetilde{m}_K^{1/2}\sum_{\sigma\in\mathcal{E}_K} C^{1/2}_{\mathrm{t},K,\sigma}\|(\mathbf{S}\nabla p_h + \mathbf{t}_h)\cdot\mathbf{n}_\sigma\|_\sigma$
- cutoff fcts of local Péclet and Damköhler numbers in $\eta^{(2)}_{\mathrm{DF},K}$:

$$m_K := \min\{C_{\mathrm{P}}^{1/2} h_K c_{\mathbf{S},K}^{-1/2}, c_{\mathbf{w},r,K}^{-1/2}\},$$

$$\widetilde{m}_K := \min\{(C_{\mathrm{P}} + C_{\mathrm{P}}^{1/2}) h_K c_{\mathbf{S},K}^{-1}, h_K^{-1} c_{\mathbf{w},r,K}^{-1} + c_{\mathbf{w},r,K}^{-1/2} c_{\mathbf{S},K}^{-1/2}/2\}$$

Upwinding estimator $\eta_{\mathrm{U},K}$

- $\eta_{\mathrm{U},K} = \sum_{\sigma\in\mathcal{E}_K} m_\sigma\|\Pi_{0,\sigma}((\mathbf{q}_h - \mathbf{w}s_h)\cdot\mathbf{n}_\sigma)\|_\sigma$
- cutoff function of local Péclet and Damköhler numbers:

$$m_\sigma^2 = \min\left\{\max_{K\in\mathcal{T}_\sigma}\left\{C_{\mathrm{F},K,\sigma}\frac{|\sigma| h_K^2}{|K| c_{\mathbf{S},K}}\right\}, \max_{K\in\mathcal{T}_\sigma}\left\{\frac{|\sigma|}{|K| c_{\mathbf{w},r,K}}\right\}\right\}$$

Properties of the estimate

Principal properties

- guaranteed upper bound
- no constants in principal estimators, known constants in the other ones
- cutoff functions of local Péclet ($h_K \|\mathbf{w}\|_{\infty,K} c_{\mathbf{S},K}^{-1}$) and Damköhler ($h_K^2 c_{\mathbf{w},r,K} c_{\mathbf{S},K}^{-1}$) numbers (here $c_{\mathbf{w},r,K}$ is the (essential) minimum of ($r - \frac{1}{2}\nabla\cdot\mathbf{w}$))
- explicit dependence on the mesh and data
- valid for arbitrary polynomial degree and data
- nonmatching meshes
- residual estimator $\eta_{\mathrm{R},K}$ is a higher-order term (data oscillation)

Loc. efficiency for $-\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp = f$

Theorem (Local efficiency, energy norm)

There holds

$$\eta_{\mathrm{NC},K} + \eta_{\mathrm{DF},K} + \eta_{\mathrm{R},K} + \eta_{\mathrm{C},1,K} + \eta_{\mathrm{C},2,K} + \eta_{\mathrm{U},K} \leq C_{\mathrm{eff},K} |||p - p_h|||_{*,\tilde{\mathcal{E}}_K}.$$

Properties

- the estimates are locally efficient
- only semi-robustness: overestimation is a function of local Péclet and Damköhler numbers

Loc. efficiency for $-\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp = f$

Theorem (Local efficiency, energy norm)

There holds

$$\eta_{\mathrm{NC},K} + \eta_{\mathrm{DF},K} + \eta_{\mathrm{R},K} + \eta_{\mathrm{C},1,K} + \eta_{\mathrm{C},2,K} + \eta_{\mathrm{U},K} \le C_{\mathrm{eff},K} |||p - p_h|||_{*,\widetilde{\mathcal{E}}_K} .$$

Properties

- the estimates are locally efficient
- only semi-robustness: overestimation is a function of local Péclet and Damköhler numbers

Loc. efficiency for $-\nabla\cdot(\mathbf{S}\nabla p) + \mathbf{w}\cdot\nabla p + rp = f$

Theorem (Local efficiency, energy norm)

There holds

$$\eta_{\mathrm{NC},K} + \eta_{\mathrm{DF},K} + \eta_{\mathrm{R},K} + \eta_{\mathrm{C},1,K} + \eta_{\mathrm{C},2,K} + \eta_{\mathrm{U},K} \le C_{\mathrm{eff},K} |||p - p_h|||_{*,\widetilde{\mathcal{E}}_K}.$$

Properties

- the estimates are locally efficient
- only semi-robustness: overestimation is a function of local Péclet and Damköhler numbers

Outline

A dual norm augmented by the convective derivative

- define

$$\mathcal{B}_{\mathrm{D}}(p,\varphi) := -\sum_{\sigma\in\mathcal{E}_h} (\mathbf{w}\cdot\mathbf{n}_\sigma [\![p]\!], \{\!\{\Pi_0\varphi\}\!\})_\sigma$$

- introduce the **augmented norm**

$$|||v|||_{\oplus} := |||v||| + \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}_{\mathrm{A}}(v,\varphi) + \mathcal{B}_{\mathrm{D}}(v,\varphi)\}$$

- when $\|\nabla\cdot\mathbf{w}\|_{\infty,K}$ is controlled by $(r - \frac{1}{2}\nabla\cdot\mathbf{w})$ on K for all K and when $v \in H_0^1(\Omega)$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

A dual norm augmented by the convective derivative

- define

$$\mathcal{B}_{\mathrm{D}}(p,\varphi) := -\sum_{\sigma\in\mathcal{E}_h} (\mathbf{w}\cdot\mathbf{n}_\sigma [\![p]\!], \{\!\{\Pi_0\varphi\}\!\})_\sigma$$

- introduce the **augmented norm**

$$|||v|||_{\oplus} := |||v||| + \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}_{\mathrm{A}}(v,\varphi) + \mathcal{B}_{\mathrm{D}}(v,\varphi)\}$$

- when $\|\nabla\cdot\mathbf{w}\|_{\infty,K}$ is controlled by $(r - \frac{1}{2}\nabla\cdot\mathbf{w})$ on K for all K and when $v \in H_0^1(\Omega)$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

A dual norm augmented by the convective derivative

- define

$$\mathcal{B}_{\mathrm{D}}(p,\varphi) := -\sum_{\sigma\in\mathcal{E}_h}(\mathbf{w}\cdot\mathbf{n}_\sigma[\![p]\!],\{\!\{\Pi_0\varphi\}\!\})_\sigma$$

- introduce the **augmented norm**

$$|||v|||_\oplus := |||v||| + \sup_{\varphi\in H_0^1(\Omega),\,|||\varphi|||=1}\{\mathcal{B}_{\mathrm{A}}(v,\varphi)+\mathcal{B}_{\mathrm{D}}(v,\varphi)\}$$

- when $\|\nabla\cdot\mathbf{w}\|_{\infty,K}$ is controlled by $(r-\frac{1}{2}\nabla\cdot\mathbf{w})$ on K for all K and when $v\in H_0^1(\Omega)$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

A dual norm augmented by the convective derivative

- define

$$\mathcal{B}_{\mathrm{D}}(p,\varphi) := -\sum_{\sigma\in\mathcal{E}_h} (\mathbf{w}\cdot\mathbf{n}_\sigma [\![p]\!], \{\!\{\Pi_0\varphi\}\!\})_\sigma$$

- introduce the **augmented norm**

$$|||v|||_\oplus := |||v||| + \sup_{\varphi\in H_0^1(\Omega),\, |||\varphi|||=1} \{\mathcal{B}_{\mathrm{A}}(v,\varphi) + \mathcal{B}_{\mathrm{D}}(v,\varphi)\}$$

- when $\|\nabla\cdot\mathbf{w}\|_{\infty,K}$ is controlled by $(r - \frac{1}{2}\nabla\cdot\mathbf{w})$ on K for all K and when $v \in H_0^1(\Omega)$, recover the augmented norm introduced by Verfürth '05
- $\mathcal{B}_{\mathrm{D}}$ contribution is new and specific to the nonconforming case

Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
&|||p - p_h|||_\oplus \\
&\leq 2 \inf_{s \in H_0^1(\Omega)} \Big\{ |||p_h - s||| + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{ (f - \nabla\cdot\mathbf{t} - \mathbf{w}\cdot\nabla s - rs, \varphi) \\
&\quad - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) + ((r - \tfrac{1}{2}\nabla\cdot\mathbf{w})(s - p_h), \varphi) \} \Big\} + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \\
&\quad \Big\{ \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \{ (f - \nabla\cdot\mathbf{t} - \mathbf{w}\cdot\nabla p_h - rp_h, \varphi) - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) \} - \mathcal{B}_{\mathrm{D}}(p_h, \varphi) \Big\} \\
&\leq 5 |||p - p_h|||_\oplus .
\end{aligned}
$$

Comments

- only the highlighted terms are new
- their form is similar to the energy estimate
- necessary for robustness in the convection-dominated case

Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
&|||p - p_h|||_{\oplus} \\
&\leq 2 \inf_{s \in H_0^1(\Omega)} \Bigg\{ |||p_h - s||| + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \Big\{ (f - \nabla\cdot\mathbf{t} - \mathbf{w}\cdot\nabla s - rs, \varphi) \\
&\quad - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) + \big(\big(r - \tfrac{1}{2}\nabla\cdot\mathbf{w}\big)(s - p_h), \varphi \big) \Big\} \Bigg\} + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \\
&\quad \Bigg\{ \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \{ (f - \nabla\cdot\mathbf{t} - \mathbf{w}\cdot\nabla p_h - rp_h, \varphi) - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) \} - \mathcal{B}_{\mathrm{D}}(p_h, \varphi) \Bigg\} \\
&\leq 5 |||p - p_h|||_{\oplus}.
\end{aligned}
$$

Comments

- only the highlighted terms are new
- their form is similar to the energy estimate
- necessary for robustness in the convection-dominated case

Optimal abstract estimate in the augmented norm

Theorem (Optimal abstract estimate, augmented norm)

Let p be the weak sol. and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Then

$$
\begin{aligned}
&|||p - p_h|||_\oplus \\
&\leq 2 \inf_{s \in H_0^1(\Omega)} \Big\{ |||p_h - s||| + \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \{ (f - \nabla{\cdot}\mathbf{t} - \mathbf{w}{\cdot}\nabla s - rs, \varphi) \\
&\quad - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) + ((r - \tfrac{1}{2}\nabla{\cdot}\mathbf{w})(s - p_h), \varphi) \} \Big\} + \sup_{\varphi \in H_0^1(\Omega),\, |||\varphi|||=1} \\
&\quad \Big\{ \inf_{\mathbf{t} \in \mathbf{H}(\mathrm{div},\Omega)} \{ (f - \nabla{\cdot}\mathbf{t} - \mathbf{w}{\cdot}\nabla p_h - rp_h, \varphi) - (\mathbf{S}\nabla p_h + \mathbf{t}, \nabla\varphi) \} - \mathcal{B}_{\mathrm{D}}(p_h, \varphi) \Big\} \\
&\leq 5 |||p - p_h|||_\oplus .
\end{aligned}
$$

Comments

- only the highlighted terms are new
- their form is similar to the energy estimate
- necessary for robustness in the convection-dominated case

Augmented norm a posteriori error estimate

Estimator

$$\widetilde{\eta} := 2\eta + \left\{ \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K} + \widetilde{\eta}_{\mathrm{C},1,K} + \widetilde{\eta}_{\mathrm{U},K})^2 \right\}^{1/2}$$

- η defined previously for the energy norm
- $\widetilde{\eta}_{\mathrm{C},1,K}$ and $\widetilde{\eta}_{\mathrm{U},K}$ – slight modifications of $\eta_{\mathrm{C},1,K}$ and $\eta_{\mathrm{U},K}$

Global jump seminorm

- define

$$|||\varphi|||^2_{\#,\mathcal{E}_h} = \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\mathfrak{E}_K} \frac{1}{\#(\mathfrak{T}_\sigma)} \left\{ \frac{c_{\mathbf{S},K}}{c_{\mathbf{S},\mathfrak{T}_K}} \alpha_\sigma \gamma_{\mathbf{S},\sigma} h_\sigma^{-1} \|[\![\varphi]\!]\|^2_\sigma + c_{\mathbf{w},r,K} h_\sigma \|[\![\varphi]\!]\|^2_\sigma + m^2_{\mathfrak{T}_K} \|\mathbf{w}\|^2_{\infty,\mathfrak{T}_K} h_\sigma^{-1} \|[\![\varphi]\!]\|^2_{0,\mathcal{E}_\sigma\cap\mathfrak{E}_K} \right\}$$

- the first two terms are natural for DG methods
- the third term at least contains the cutoff factor $m_{\mathfrak{T}_K}$

Augmented norm a posteriori error estimate

Estimator

$$\widetilde{\eta} := 2\eta + \left\{ \sum_{K\in\mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K} + \widetilde{\eta}_{\mathrm{C},1,K} + \widetilde{\eta}_{\mathrm{U},K})^2 \right\}^{1/2}$$

- η defined previously for the energy norm
- $\widetilde{\eta}_{\mathrm{C},1,K}$ and $\widetilde{\eta}_{\mathrm{U},K}$ – slight modifications of $\eta_{\mathrm{C},1,K}$ and $\eta_{\mathrm{U},K}$

Global jump seminorm

- define

$$|||\varphi|||^2_{\#,\mathcal{E}_h} = \sum_{K\in\mathcal{T}_h} \sum_{\sigma\in\mathfrak{E}_K} \frac{1}{\#(\mathfrak{T}_\sigma)} \left\{ \frac{c_{\mathbf{S},K}}{c_{\mathbf{S},\mathfrak{T}_K}} \alpha_\sigma \gamma_{\mathbf{S},\sigma} h_\sigma^{-1} \|[\![\varphi]\!]\|^2_\sigma \right.$$
$$\left. + c_{\mathbf{w},r,K} h_\sigma \|[\![\varphi]\!]\|^2_\sigma + m^2_{\mathcal{T}_K} \|\mathbf{w}\|^2_{\infty,\mathcal{T}_K} h_\sigma^{-1} \|[\![\varphi]\!]\|^2_{0,\mathcal{E}_\sigma\cap\mathfrak{E}_K} \right\}$$

- the first two terms are natural for DG methods
- the third term at least contains the cutoff factor $m_{\mathcal{T}_K}$

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

There holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \leq \widetilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$$
$$\leq \tilde{C}(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- fully robust with respect to convection or reaction dominance
- sharper than Schötzau & Zhu '08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $|||\cdot|||_{\oplus}$ is a dual norm and cannot be evaluated
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

There holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \le \tilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$$
$$\le \tilde{C}(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- fully robust with respect to convection or reaction dominance
- sharper than Schötzau & Zhu '08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $|||\cdot|||_{\oplus}$ is a dual norm and cannot be evaluated
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

There holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \leq \tilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h} \leq \tilde{C}(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- fully robust with respect to convection or reaction dominance
- sharper than Schötzau & Zhu '08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $|||\cdot|||_{\oplus}$ is a dual norm and cannot be evaluated
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

There holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \leq \widetilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$$
$$\leq \tilde{C}(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- fully robust with respect to convection or reaction dominance
- sharper than Schötzau & Zhu '08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $|||\cdot|||_{\oplus}$ is a dual norm and cannot be evaluated
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

There holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \le \widetilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$$
$$\le \tilde{C}(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- fully robust with respect to convection or reaction dominance
- sharper than Schötzau & Zhu '08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $|||\cdot|||_{\oplus}$ is a dual norm and cannot be evaluated
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)

Augmented norm estimate and its efficiency

Theorem (Fully robust a posteriori estimate)

There holds

$$|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h} \leq \tilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$$
$$\leq \tilde{C}(|||p - p_h|||_{\oplus} + |||p - p_h|||_{\#,\mathcal{E}_h}).$$

- fully robust with respect to convection or reaction dominance
- sharper than Schötzau & Zhu ’08 because of the cutoff factor in the jump seminorm
- only global efficiency
- the norm $|||\cdot|||_{\oplus}$ is a dual norm and cannot be evaluated
- rather theoretical importance, since the estimators for both the energy and the augmented norm are (almost) the same (hence the adaptive strategies are the same)

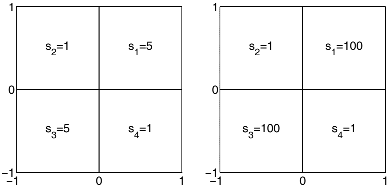
Outline

Discontinuous diffusion tensor and finite volumes

- consider the pure diffusion equation

$$-\nabla \cdot (\mathbf{S}\nabla p) = 0 \quad \text{in} \quad \Omega = (-1,1) \times (-1,1)$$

- discontinuous and inhomogeneous **S**, two cases:

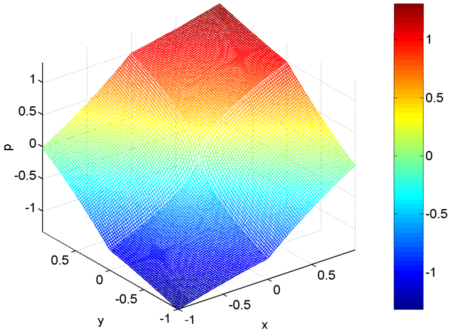


- analytical solution: singularity at the origin

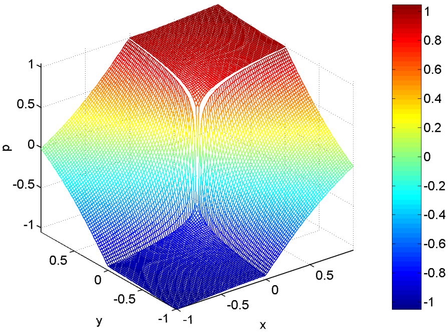
$$p(r,\theta)|_{\Omega_i} = r^{\alpha}(a_i \sin(\alpha\theta) + b_i \cos(\alpha\theta))$$

  - (r,θ) polar coordinates in Ω
  - a_i, b_i constants depending on Ω_i
  - α regularity of the solution

Analytical solutions

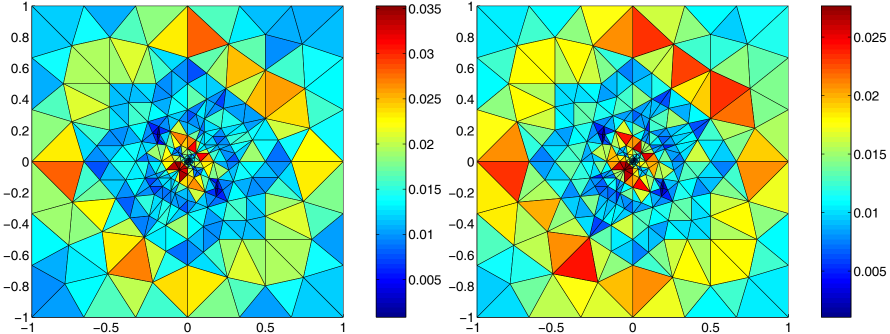


Case 1



Case 2

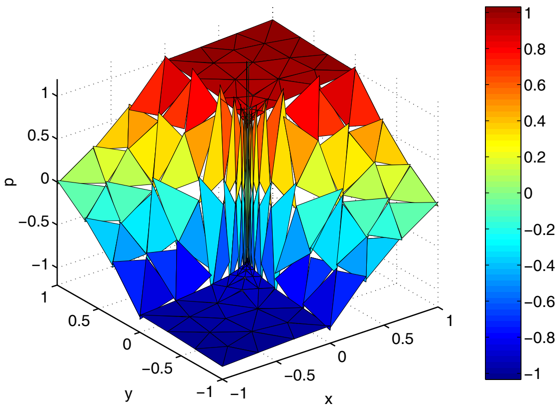
Error distribution on an adaptively refined mesh, case 1

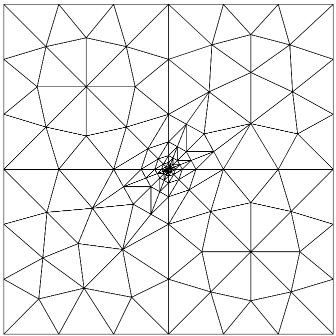


Estimated error distribution

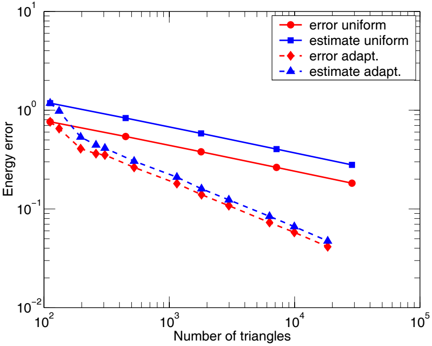
Exact error distribution

Approximate solution and the corresponding adaptively refined mesh, case 2

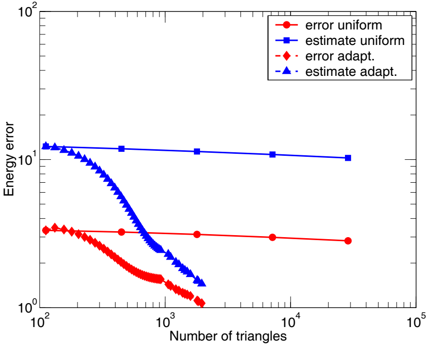




Estimated and actual errors in uniformly/adaptively refined meshes

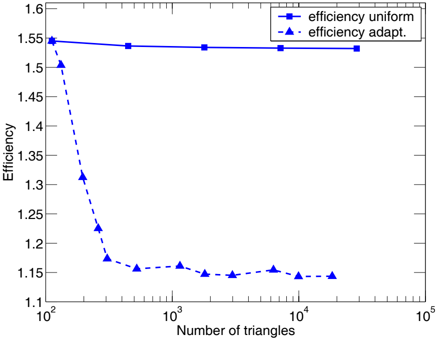


Case 1

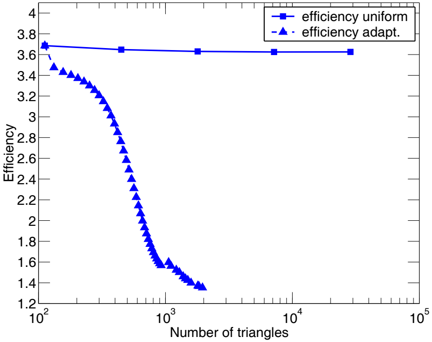


Case 2

Effectivity indices in uniformly/adaptively refined meshes



Case 1



Case 2

Convection-dominated problem, FVs, energy estimates

- consider the convection–diffusion–reaction equation

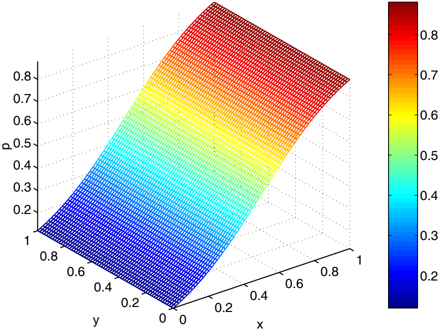
$$-\varepsilon \triangle p + \nabla \cdot (p(0,1)) + p = f \quad \text{in} \quad \Omega = (0,1) \times (0,1)$$

- analytical solution: layer of width a

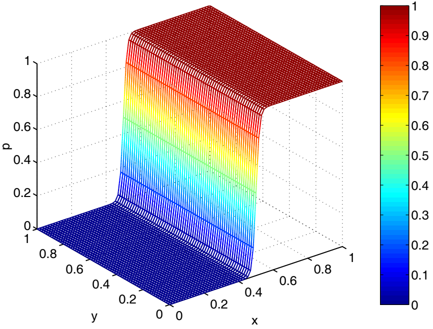
$$p(x,y) = 0.5\left(1 - \tanh\left(\frac{0.5 - x}{a}\right)\right)$$

- consider
  - $\varepsilon = 1$, $a = 0.5$
  - $\varepsilon = 10^{-2}$, $a = 0.05$
  - $\varepsilon = 10^{-4}$, $a = 0.02$
- unstructured grid of 46 elements given, uniformly/adaptively refined

Analytical solutions

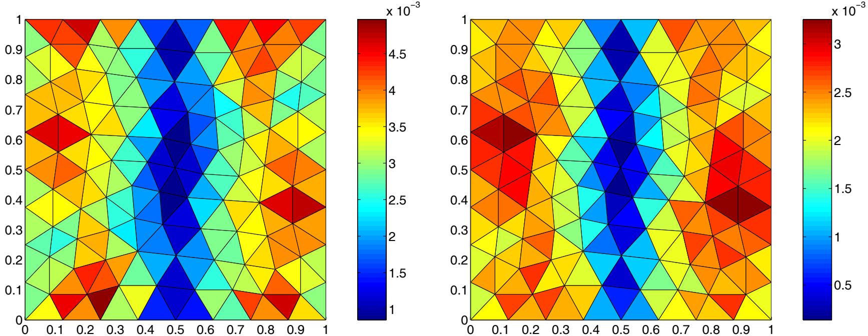


Case $\varepsilon = 1$, $a = 0.5$



Case $\varepsilon = 10^{-4}$, $a = 0.02$

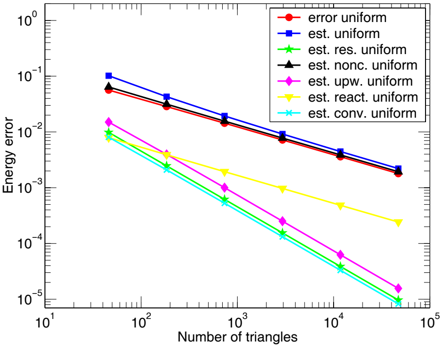
Error distribution on a uniformly refined mesh, $\varepsilon = 1$, $a = 0.5$



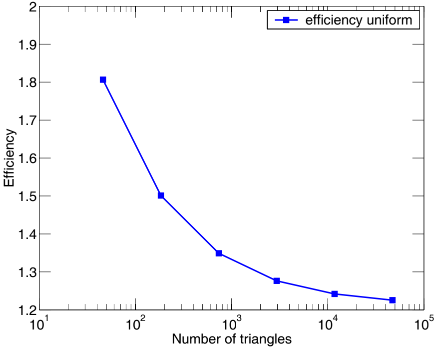
Estimated error distribution

Exact error distribution

Estimated and actual errors and the effectivity index, $\varepsilon = 1$, $a = 0.5$

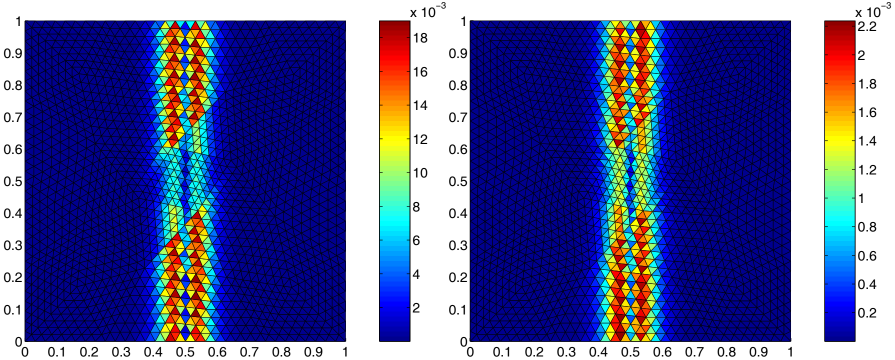


The different estimators



Effectivity index

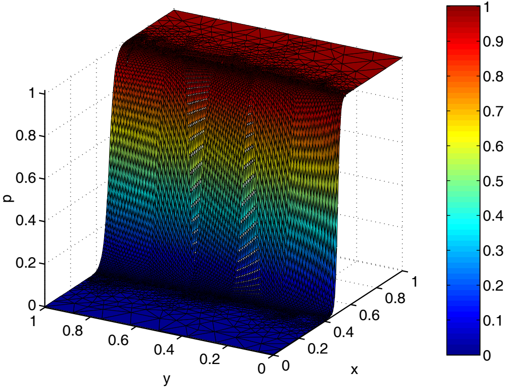
Error distribution on a uniformly refined mesh, $\varepsilon = 10^{-2}$, $a = 0.05$



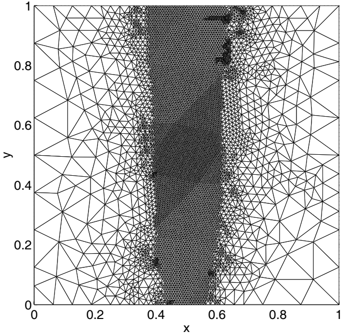
Estimated error distribution

Exact error distribution

Approximate solution and the corresponding adaptively refined mesh, $\varepsilon = 10^{-4}$, $a = 0.02$

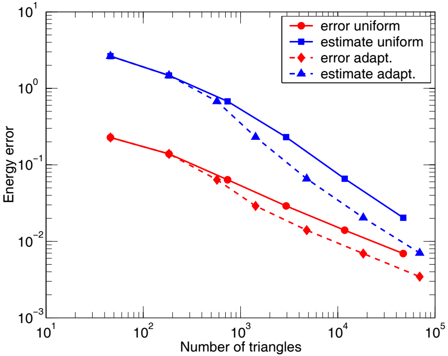


Approximate solution

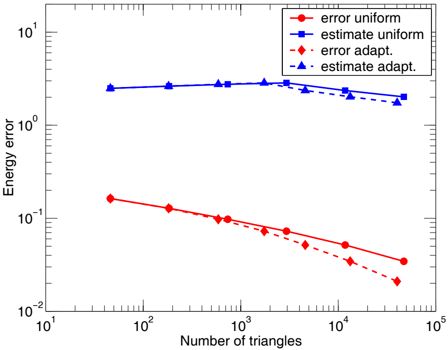


Adaptively refined mesh

Estimated and actual errors in uniformly/adaptively refined meshes

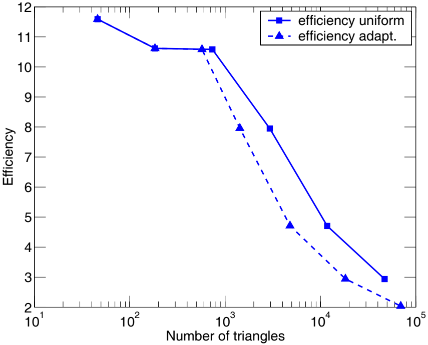


Case $\varepsilon = 10^{-2}$, $a = 0.05$

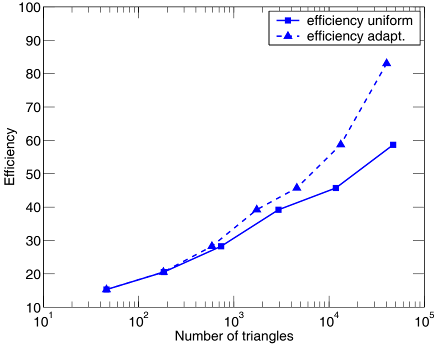


Case $\varepsilon = 10^{-4}$, $a = 0.02$

Effectivity indices in uniformly/adaptively refined meshes



Case $\varepsilon = 10^{-2}$, $a = 0.05$



Case $\varepsilon = 10^{-4}$, $a = 0.02$

Convection-dominated problem, DGs, energy and augmented estimates, $\epsilon = 10^{-2}$

	energy norm			augmented norm			
N	err.	est.	eff.	err.	est.	eff.	$\|\|\|p_h\|\|\|_{\#,\mathcal{E}_h}$
128	7.74e-3	1.10e-1	14	1.40e-1	3.28e-1	2.3	3.40e-2
512	4.03e-3	4.35e-2	11	3.97e-2	1.29e-1	3.3	1.16e-2
2048	1.88e-3	1.43e-2	7.6	9.77e-3	4.14e-2	4.2	2.72e-3
8192	9.30e-4	3.58e-3	3.8	2.98e-3	1.02e-2	3.4	8.25e-4
order	1.0	2.0	-	1.7	2.0	-	1.7

Errors ($|||p - p_h|||$ and $|||p - p_h|||_{\oplus'} + |||p - p_h|||_{\#,\mathcal{E}_h}$), estimates ($\eta$ and $\tilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$), and effectivity indices for the energy and augmented norms; $\epsilon = 10^{-2}$

Convection-dominated problem, DGs, energy and augmented estimates, $\epsilon = 10^{-4}$

	energy norm			augmented norm			
N	err.	est.	eff.	err.	est.	eff.	$\vert\vert\vert p_h \vert\vert\vert_{\#,\mathcal{E}_h}$
128	1.70e-3	1.34e-1	79	3.67e-1	4.05e-1	1.10	4.02e-2
512	5.65e-4	7.01e-2	124	1.44e-1	2.11e-1	1.47	2.11e-2
2048	2.14e-4	3.09e-2	144	5.35e-2	9.36e-2	1.75	9.99e-3
8192	1.00e-4	1.25e-2	125	2.14e-2	3.89e-2	1.82	4.96e-3
order	1.1	1.3	-	1.3	1.3	-	1.0

Errors ($|||p - p_h|||$ and $|||p - p_h|||_{\oplus'} + |||p - p_h|||_{\#,\mathcal{E}_h}$), estimates ($\eta$ and $\tilde{\eta} + |||p_h|||_{\#,\mathcal{E}_h}$), and effectivity indices for the energy and augmented norms; $\epsilon = 10^{-4}$

Outline

Outline

A model pure diffusion problem

A model pure diffusion problem

$$
\begin{aligned}
-\nabla \cdot (\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

Goals

- take into account the algebraic error
- efficiently stop the iterative solver
- **certified error bound** and **huge computational savings**

A model pure diffusion problem

A model pure diffusion problem

$$
\begin{aligned}
-\nabla \cdot (\mathbf{S}\nabla p) &= f \quad \text{in } \Omega,\\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

Goals

- take into account the algebraic error
- efficiently stop the iterative solver
- **certified error bound** and **huge computational savings**

A model pure diffusion problem

A model pure diffusion problem

$$\begin{aligned} -\nabla \cdot (\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

Goals

- take into account the algebraic error
- efficiently stop the iterative solver
- **certified error bound** and **huge computational savings**

A model pure diffusion problem

A model pure diffusion problem

$$
\begin{aligned}
-\nabla \cdot (\mathbf{S} \nabla p) &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

Goals

- take into account the algebraic error
- efficiently stop the iterative solver
- **certified error bound** and **huge computational savings**

A model pure diffusion problem

A model pure diffusion problem

$$\begin{aligned} -\nabla \cdot (\mathbf{S} \nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

Goals

- take into account the algebraic error
- efficiently stop the iterative solver
- **certified error bound** and **huge computational savings**

A model pure diffusion problem

A model pure diffusion problem

$$\begin{aligned} -\nabla \cdot (\mathbf{S} \nabla p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

Goals

- take into account the algebraic error
- efficiently stop the iterative solver
- **certified error bound** and **huge computational savings**

A model pure diffusion problem

A model pure diffusion problem

$$
\begin{aligned}
-\nabla \cdot (\mathbf{S}\nabla p) &= f \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

Algebraic problem

- at some point, we shall solve $\mathbb{A}X = B$
- we only solve it inexactly, $\mathbb{A}X^* \approx B$
- we know the algebraic residual, $R := B - \mathbb{A}X^*$

Goals

- take into account the algebraic error
- efficiently stop the iterative solver
- **certified error bound** and **huge computational savings**

Estimate including inexact linear systems error

Theorem (A posteriori error estimate including inexact linear systems solution error, cell-centered FVs or MFEs)

There holds

$$|||p - \tilde{p}_h^*||| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^* - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^*)|||_K$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := m_K \|f + \nabla \cdot (\mathbf{S}_K \nabla \tilde{p}_h^*)\|_K$
  - $m_K^2 := C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}}$
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \|\mathbf{S}^{-\frac{1}{2}} \mathbf{t}_h\|_K$
  - $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{T}_h)$ is such that $\nabla \cdot \mathbf{t}_h|_K = \frac{R_K}{|K|}$
  - R is the residual vector

Estimate including inexact linear systems error

Theorem (A posteriori error estimate including inexact linear systems solution error, cell-centered FVs or MFEs)

There holds

$$|||p - \tilde{p}_h^*||| \leq \left\{ \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^* - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^*)|||_K$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := m_K \|f + \nabla\cdot(\mathbf{S}_K \nabla \tilde{p}_h^*)\|_K$
  - $m_K^2 := C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}}$
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \|\mathbf{S}^{-\frac{1}{2}} \mathbf{t}_h\|_K$
  - $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{T}_h)$ is such that $\nabla\cdot\mathbf{t}_h|_K = \frac{R_K}{|K|}$
  - R is the residual vector

Estimate including inexact linear systems error

Theorem (A posteriori error estimate including inexact linear systems solution error, cell-centered FVs or MFEs)

There holds

$$|||p - \tilde{p}_h^*||| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^* - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^*)|||_K$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := m_K \|f + \nabla \cdot (\mathbf{S}_K \nabla \tilde{p}_h^*)\|_K$
  - $m_K^2 := C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}}$
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \|\mathbf{S}^{-\frac{1}{2}} \mathbf{t}_h\|_K$
  - $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{T}_h)$ is such that $\nabla \cdot \mathbf{t}_h|_K = \frac{R_K}{|K|}$
  - R is the residual vector

Estimate including inexact linear systems error

Theorem (A posteriori error estimate including inexact linear systems solution error, cell-centered FVs or MFEs)

There holds

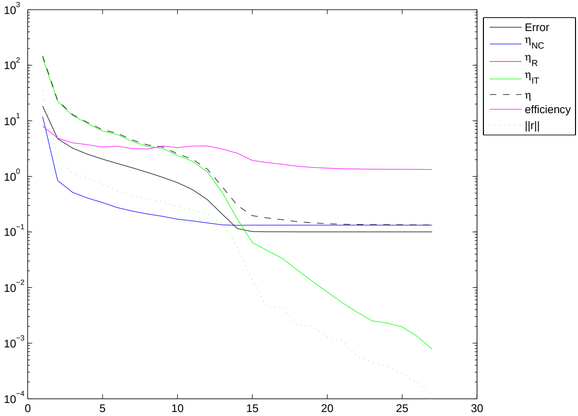
$$|||p - \tilde{p}_h^*||| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{R},K}^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{AE},K}^2 \right\}^{\frac{1}{2}}.$$

- **nonconformity estimator**
  - $\eta_{\mathrm{NC},K} := |||\tilde{p}_h^* - \mathcal{I}_{\mathrm{Os}}(\tilde{p}_h^*)|||_K$
- **residual estimator**
  - $\eta_{\mathrm{R},K} := m_K \|f + \nabla \cdot (\mathbf{S}_K \nabla \tilde{p}_h^*)\|_K$
  - $m_K^2 := C_{\mathrm{P}} \frac{h_K^2}{c_{\mathbf{S},K}}$
- **algebraic error estimator**
  - $\eta_{\mathrm{AE},K} := \|\mathbf{S}^{-\frac{1}{2}} \mathbf{t}_h\|_K$
  - $\mathbf{t}_h \in \mathbf{RTN}(\mathcal{T}_h)$ is such that $\nabla \cdot \mathbf{t}_h|_K = \frac{R_K}{|K|}$
  - R is the residual vector

Outline

1. Introduction
2. Pure diffusion and conforming methods
   - Optimal abstract framework and a first estimate
   - Optimal a posteriori error estimate
   - Estimates for finite elements
   - Efficiency of the a posteriori error estimate
   - Numerical experiments
3. Convection–reaction–diffusion and nonconforming methods
   - Semi-robust energy norm estimates for DGs
   - Fully robust augmented norm estimates for DGs
   - Numerical experiments (FVs & DGs)
4. Estimates including the algebraic error
   - Problem and estimates
   - Numerical experiments
5. Conclusions and future work

Finite volume estimates including inexact linear systems solution



Different estimators, error, and effectivity index as a function of the number of CG iterations

Outline

Comments on the estimates and their efficiency

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no saturation assumption
- no Helmholtz decomposition
- polynomial degree-independent upper bound
- the only important tools: Cauchy–Schwarz and optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases
- based on local conservativity (no global Galerkin orthogonality used)

Comments on the estimates and their efficiency

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no saturation assumption
- no Helmholtz decomposition
- polynomial degree-independent upper bound
- the only important tools: Cauchy–Schwarz and optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases
- based on local conservativity (no global Galerkin orthogonality used)

Comments on the estimates and their efficiency

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no saturation assumption
- no Helmholtz decomposition
- polynomial degree-independent upper bound
- the only important tools: Cauchy–Schwarz and optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases
- based on local conservativity (no global Galerkin orthogonality used)

Comments on the estimates and their efficiency

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no saturation assumption
- no Helmholtz decomposition
- polynomial degree-independent upper bound
- the only important tools: Cauchy–Schwarz and optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases
- based on local conservativity (no global Galerkin orthogonality used)

Comments on the estimates and their efficiency

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no saturation assumption
- no Helmholtz decomposition
- polynomial degree-independent upper bound
- the only important tools: Cauchy–Schwarz and optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases
- based on local conservativity (no global Galerkin orthogonality used)

Comments on the estimates and their efficiency

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no saturation assumption
- no Helmholtz decomposition
- polynomial degree-independent upper bound
- the only important tools: Cauchy–Schwarz and optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases
- based on local conservativity (no global Galerkin orthogonality used)

Comments on the estimates and their efficiency

General comments

- $p \in H^1(\Omega)$, no additional regularity
- no saturation assumption
- no Helmholtz decomposition
- polynomial degree-independent upper bound
- the only important tools: Cauchy–Schwarz and optimal Poincaré–Friedrichs and trace inequalities
- holds from diffusion to convection–diffusion–reaction cases
- based on local conservativity (no global Galerkin orthogonality used)

Essentials of the estimates

Essentials of the estimates

- nonconformity estimate: compare the approximate solution p_h to a $H^1(\Omega)$-conforming potential s_h
- diffusive flux estimate: compare the flux of the approximate solution $-\mathbf{S}\nabla p_h$ to a $\mathbf{H}(\mathrm{div}, \Omega)$-conforming flux $\mathbf{t}_h$
- evaluate the residue for $\mathbf{t}_h$
- in conforming methods ($p_h \in H^1(\Omega)$), there is no nonconformity estimate
- in flux-conforming methods ($-\mathbf{S}\nabla p_h \in \mathbf{H}(\mathrm{div}, \Omega)$), there is no diffusive flux estimate

Conclusions and future work

Conclusions

- guaranteed, locally efficient, and robust a posteriori error estimates
- directly and locally computable
- almost asymptotically exact
- optimal framework (exact and robust)
- works for all major numerical schemes
- based on local conservativity

Future work

- asymptotic exactness
- nonlinear (degenerate) cases
- extensions to other types of problems (Stokes, Maxwell)
- multi-scale, multi-numerics, multi-physics, mortars

Conclusions and future work

Conclusions

- guaranteed, locally efficient, and robust a posteriori error estimates
- directly and locally computable
- almost asymptotically exact
- optimal framework (exact and robust)
- works for all major numerical schemes
- based on local conservativity

Future work

- asymptotic exactness
- nonlinear (degenerate) cases
- extensions to other types of problems (Stokes, Maxwell)
- multi-scale, multi-numerics, multi-physics, mortars

Bibliography 1

Papers and collaborators

- VOHRALÍK M., Guaranteed and fully robust a posteriori error estimates for conforming discretizations of diffusion problems with discontinuous coefficients, submitted to *Math. Comp*.
- VOHRALÍK M., A posteriori error estimation in the conforming finite element method based on its local conservativity and using local minimization, *C. R. Math. Acad. Sci. Paris.* **346** (2008), 687–690.
- CHEDDADI I., FUČÍK R., PRIETO M. I., VOHRALÍK M., Computable a posteriori error estimates in the finite element method based on its local conservativity: improvements using local minimization, *ESAIM: Proc.* **24** (2008), 77–96.
- CHEDDADI I., FUČÍK R., PRIETO M. I., VOHRALÍK M., Guaranteed and robust a posteriori error estimates for singularly perturbed reaction–diffusion problems, to appear in *M2AN Math. Model. Numer. Anal*.

Bibliography 2

- ERN A., STEPHANSEN, A. F., VOHRALÍK M., Guaranteed and robust discontinuous Galerkin a posteriori error estimates for convection–diffusion–reaction problems, submitted to *M3AS*.
- VOHRALÍK M., A posteriori error estimates for lowest-order mixed finite element discretizations of convection–diffusion–reaction equations, *SIAM J. Numer. Anal.* **45** (2007), 1570–1599.
- VOHRALÍK M., Residual flux-based a posteriori error estimates for finite volume and related locally conservative methods, *Numer. Math.* **111** (2008), 121–158.
- JIRÁNEK P., STRAKOŠ Z., VOHRALÍK M., A posteriori error estimates including algebraic error: computable upper bounds and stopping criteria for iterative solvers, submitted to *SIAM J. Sci. Comput.*

Thank you for your attention!