A posteriori error estimates via potential and flux reconstructions

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Outline

1. Introduction
2. A posteriori estimates based on potential & flux reconstruction
   - Guaranteed upper bound in a unified framework
   - Potential and flux reconstructions
   - Polynomial-degree-robust local efficiency
   - Applications
   - Numerical illustration
3. Algebraic estimates and stopping criteria for iterative solvers
   - Bounds on the algebraic error
   - Bounds on the total error
   - Stopping criteria
   - Numerical illustration
4. Stokes equation
5. Conclusions and outlook

Optimal a posteriori error estimate

Guaranteed upper bound

- $\|u - u_h\|^2_{?,\Omega} \le \sum_{K\in\mathcal{T}_h} \eta_K(u_h)^2$
- no undetermined constant: **error control**

Local efficiency

- $\eta_K(u_h) \le C_{\text{eff}} \|u - u_h\|_{?,\text{neighbors of } K}$
- **local** error lower bound (optimal space mesh refinement)

Robustness

- C_{eff} independent of data, domain Ω, meshes, solution u, **polynomial degree** of u_h

Asymptotic exactness

- $\sum_{K\in\mathcal{T}_h} \eta_K(u_h)^2 / \|u - u_h\|^2_{?,\Omega} \searrow 1$
- overestimation factor goes to one with meshes size

Small evaluation cost

- estimators can be evaluated cheaply (locally)

Error components identification

- $\eta_K(u_h)$ can distinguish the different error components

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Laplace model problem

Model problem

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

- $u \in H_0^1(\Omega)$ (primal variable constraint)
- $\sigma := -\nabla u$ (constitutive relation)
- $\nabla \cdot \sigma = f$ (equilibrium)
- $\sigma \in \mathbf{H}(\mathrm{div}, \Omega)$ (dual variable constraint)

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Theorem (A guaranteed a posteriori error estimate, Prager and Synge (1947), Dari, Durán, Padra, and Vampa (1996), Ainsworth (2005), Kim (2007), Vohralík (2007), ...**)**

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
- *$u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega),\ v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h\}$ be arbitrary (thus $u_h \notin H_0^1(\Omega)$ and $-\nabla u_h \notin \mathbf{H}(\mathrm{div}, \Omega)$ in gen.);*
- *$s_h \in H_0^1(\Omega)$ and $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ be such that*

$$(\nabla{\cdot}\boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \text{ for all } K \in \mathcal{T}_h.$$

Then

$$\|\nabla(u - u_h)\|^2 \le \sum_{K \in \mathcal{T}_h} \Big(\underbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \nabla{\cdot}\boldsymbol{\sigma}_h\|_K}_{\text{equilibrium}} \Big)^2 + \sum_{K \in \mathcal{T}_h} \underbrace{\|\nabla(u_h - s_h)\|_K^2}_{\text{primal constraint}}.$$

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Proof I

Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\| = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s, definition of u:

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- dual norm characterization, definition of s, definition of u:

$$\|\nabla(u - s)\| = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} \{(f, \varphi) - (\nabla u_h, \nabla \varphi)\}}_{\text{dual norm of the residual}}$$

Proof II

Proof (continuation).

- nonconformity upper bound:

$$\min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\| \le \|\nabla(u_h - s_h)\|$$

- adding and subtracting equilibrated flux, Green theorem:

$$(f,\varphi) - (\nabla u_h, \nabla\varphi) = (f - \nabla\cdot\sigma_h, \varphi) - (\nabla u_h + \sigma_h, \nabla\varphi)$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h + \sigma_h, \nabla\varphi)$$

$$(f - \nabla\cdot\sigma_h, \varphi) = \sum_{K\in\mathcal{T}_h} (f - \nabla\cdot\sigma_h, \varphi \quad)_K$$

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$$\leq \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla \cdot \boldsymbol{\sigma}_h\|_K \|\nabla \varphi\|_K$$

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Global potential and flux reconstructions

Ideally

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h} f} \|\nabla u_h + \mathbf{v}_h\|$$

$$s_h := \arg \min_{v_h \in V_h} \|\nabla(u_h - v_h)\|$$

- $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, $Q_h \subset L^2(\Omega)$, $V_h \subset H_0^1(\Omega)$
- too expensive, **global** minimization problems (the hypercircle method . . .)

Local potential and flux reconstructions

Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

For each $\mathbf{a} \in \mathcal{V}_h$, solve the **local mixed FE problem**

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}.$$

Definition (Construction of s_h, ≈ Carstensen and Merdon (2013), EV (2015))

For each $\mathbf{a} \in \mathcal{V}_h$, solve the **local conforming FE problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}.$$

Key ideas

- local minimizations
- cut-off by hat basis functions $\psi_{\mathbf{a}}$
- $\mathbf{V}_h^{\mathbf{a}}$: homogeneous Neumann BC on $\partial\omega_{\mathbf{a}}$ & $V_h^{\mathbf{a}}$: homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}}$
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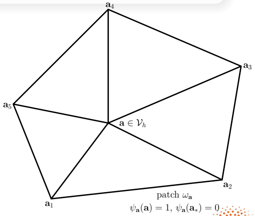

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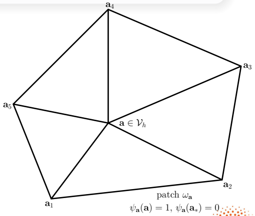

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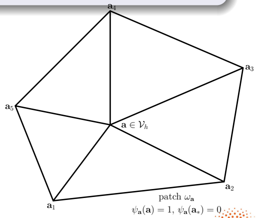

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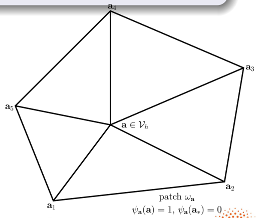
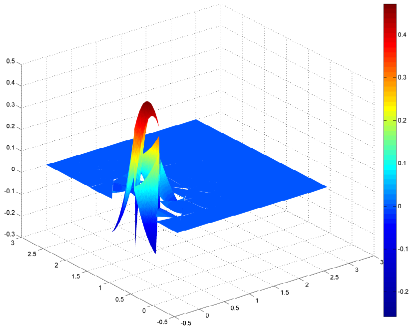
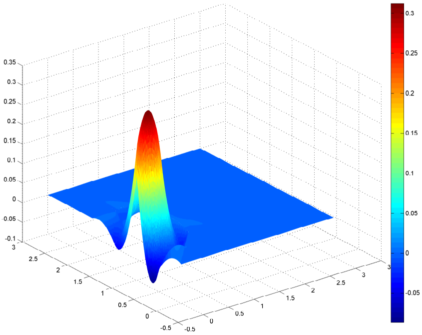


Potential reconstruction

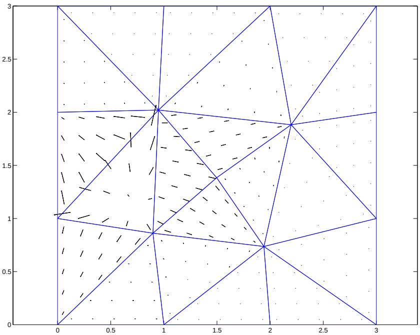


Potential u_h

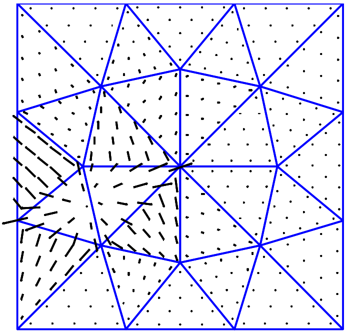


Potential reconstruction s_h

Equilibrated flux reconstruction



Flux $-\nabla u_h$



Flux reconstruction σ_h

Comments

$\mathbf{H}(\mathrm{div}, \Omega)$-conformity

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Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs $(\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0 \Rightarrow$

Assumption A (Galerkin orthogonality wrt hat functions)

There holds

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

Divergence

- Neumann compatibility condition gives

$$\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a} \in \mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a} \in \mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

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Continuous-level patch problems

Definition (Continuous-level flux reconstruction)

For each $\mathbf{a} \in \mathcal{V}_h$, set

$$\boldsymbol{\sigma}^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div},\omega_{\mathbf{a}}),\ \nabla\cdot\mathbf{v} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h) \\ \mathbf{v}\cdot\mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}}\ (\setminus\partial\Omega)}} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}\|_{\omega_{\mathbf{a}}}.$$

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Assumptions for efficiency

Assumption B (Weak continuity)

There holds

$$\langle [\![u_h]\!], 1 \rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$$

Assumption C (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The degrees of the MFE reconstructions σ_h and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.

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Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency via MFE / FE / continuous stability Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015)**)**

Let u be the weak solution and let Assumptions A, B, and C hold. Then there exists constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ only depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that

$$\|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\boldsymbol{\sigma}^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} C_{\rm cont,PF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}};$$

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Remarks

- $C_{\rm st}$ can be bounded by solving the local Neumann problems by conforming FEs
- $\Rightarrow$ maximal overestimation factor guaranteed

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Conforming finite elements

Conforming finite elements

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$
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Discontinuous Galerkin finite elements

Discontinuous Galerkin finite elements

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$$\sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v_h)_K - \sum_{e \in \mathcal{E}_h} \{ \langle \{\!\{ \nabla u_h \}\!\} \cdot \mathbf{n}_e, [\![v_h]\!] \rangle_e + \theta \langle \{\!\{ \nabla v_h \}\!\} \cdot \mathbf{n}_e, [\![u_h]\!] \rangle_e \}$$

$$+ \sum_{e \in \mathcal{E}_h} \langle \alpha h_e^{-1} [\![u_h]\!], [\![v_h]\!] \rangle_e = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$ for $\theta = 0$, otherwise:
  - estimates for the discrete gradient

$$\nabla_{\rm d} u_h := \nabla u_h - \theta \sum_{e \in \mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

  - jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_e)]^2$

$$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle \{\!\{ \mathbf{v}_h \}\!\} \cdot \mathbf{n}_e, [\![u_h]\!] \rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_e)]^2$$

  - $\Rightarrow$ modified Galerkin orthogonality

$$(\nabla_{\rm d} u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\rm int}$$

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$$\sum_{K\in\mathcal{T}_h}(\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h}\{\langle\{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle\{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\}$$

$$+\sum_{e\in\mathcal{E}_h}\langle\alpha h_e^{-1}[\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$ for $\theta = 0$, otherwise:
  - estimates for the discrete gradient

    $$\nabla_{\mathrm{d}} u_h := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

  - jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_e)]^2$

    $$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle\{\!\{\mathbf{v}_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_e)]^2$$

  - $\Rightarrow$ modified Galerkin orthogonality

    $$(\nabla_{\mathrm{d}} u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

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Discontinuous Galerkin finite elements: Assumption B

Nonsymmetric and incomplete versions

- broken Poincaré–Friedrichs inequality with jumps:

$$\|\nabla(\psi_{\mathbf{a}}(\tilde{u} - u_h))\|_{\omega_{\mathbf{a}}} \leq (1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}})\|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}} + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} \left\{ \sum_{e\in\mathcal{E}_h,\, \mathbf{a}\in e} h_e^{-1} \|\Pi_e^0 [\![u_h]\!]\|_e^2 \right\}^{1/2}$$

- include the jump terms in the error and estimators

Symmetric version

- discrete gradient $\mathfrak{G}$ satisfies

$$(\nabla_{\mathrm{d}} u_h, \mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h$$

- modified potential reconstruction: local MFE problems with data $\tau_h^{\mathbf{a}} := \psi_{\mathbf{a}} \mathrm{R}_{\frac{\pi}{2}} \nabla_{\mathrm{d}} u_h$ and $g^{\mathbf{a}} := (\mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}}) \cdot \nabla_{\mathrm{d}} u_h$
- local efficiency

$$\|\mathfrak{G}(u_h - s_h)\|_K \leq C_{\mathrm{st}} C_{\mathrm{cont,P}} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\mathfrak{G}(u - u_h)\|_{\omega_{\mathbf{a}}}$$

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Mixed finite elements

Mixed finite elements

Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} (\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla\cdot\mathbf{v}_h) &= 0 \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla\cdot\boldsymbol{\sigma}_h, q_h) &= (f, q_h) \qquad && \forall q_h \in Q_h. \end{aligned}$$

- postprocessed solution $u_h \in V_h$, $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$; $v_h \in V_h$ satisfy

$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p'}(e), \ \forall e \in \mathcal{E}_h$$

- Assumption A: no need for flux reconstruction, $\boldsymbol{\sigma}_h$ comes from the discretization
- Assumption B satisfied, building requirement for the space V_h

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Numerics: smooth case

Model problem

$$-\Delta u = f \quad \text{in } \Omega := (0,1)^2,$$
$$u = 0 \quad \text{on } \partial\Omega$$

Exact solution

$$u(x,y) = \sin(2\pi x)\sin(2\pi y)$$

Discretization

- symmetric interior penalty discontinuous Galerkin method
- unstructured triangular grids
- uniform h refinement

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Uniform refinement: **asymptotic exactness**

h	p	$\|\nabla_{\mathrm{d}}(u-u_h)\|$	$\|u-u_h\|_{\mathrm{DG}}$	$\|\nabla_{\mathrm{d}} u_h+\boldsymbol{\sigma}_h\|$	η_{osc}	$\|\nabla_{\mathrm{d}}(u_h-s_h)\|$	η	η_{DG}	I^{eff}	$I^{\mathrm{eff}}_{\mathrm{DG}}$
h_0	1	1.07E-00	1.09E-00	1.12E-00	5.55E-02	4.16E-01	1.25E-00	1.26E-00	1.17	1.16
$\approx h_0/2$		5.56E-01	5.61E-01	5.71E-01	7.42E-03	1.82E-01	6.07E-01	6.11E-01	1.09	1.09
$\approx h_0/4$		2.92E-01	2.93E-01	2.96E-01	1.04E-03	8.77E-02	3.10E-01	3.11E-01	1.06	1.06
$\approx h_0/8$		1.39E-01	1.39E-01	1.40E-01	1.10E-04	3.85E-02	1.45E-01	1.45E-01	1.04	1.04
h_0	2	1.54E-01	1.55E-01	1.55E-01	5.10E-03	3.05E-02	1.63E-01	1.64E-01	1.06	1.06
$\approx h_0/2$		4.07E-02	4.09E-02	4.13E-02	3.53E-04	7.55E-03	4.23E-02	4.26E-02	1.04	1.04
$\approx h_0/4$		1.10E-02	1.11E-02	1.12E-02	2.51E-05	1.97E-03	1.14E-02	1.15E-02	1.03	1.03
$\approx h_0/8$		2.50E-03	2.52E-03	2.54E-03	1.30E-06	4.21E-04	2.57E-03	2.59E-03	1.03	1.03
h_0	3	1.37E-02	1.37E-02	1.37E-02	3.58E-04	1.74E-03	1.41E-02	1.41E-02	1.03	1.03
$\approx h_0/2$		1.85E-03	1.85E-03	1.85E-03	1.26E-05	2.10E-04	1.88E-03	1.88E-03	1.01	1.01
$\approx h_0/4$		2.60E-04	2.60E-04	2.60E-04	4.73E-07	2.54E-05	2.62E-04	2.62E-04	1.01	1.01
$\approx h_0/8$		2.75E-05	2.75E-05	2.75E-05	1.15E-08	2.55E-06	2.76E-05	2.76E-05	1.01	1.01
h_0	4	9.87E-04	9.87E-04	9.84E-04	2.12E-05	1.11E-04	1.01E-03	1.01E-03	1.02	1.02
$\approx h_0/2$		6.92E-05	6.93E-05	6.92E-05	3.96E-07	7.44E-06	7.00E-05	7.00E-05	1.01	1.01
$\approx h_0/4$		5.04E-06	5.04E-06	5.04E-06	7.58E-09	4.98E-07	5.07E-06	5.07E-06	1.01	1.01
$\approx h_0/8$		2.58E-07	2.59E-07	2.58E-07	8.96E-11	2.47E-08	2.60E-07	2.60E-07	1.01	1.01
h_0	5	5.64E-05	5.64E-05	5.63E-05	1.06E-06	4.50E-06	5.75E-05	5.75E-05	1.02	1.02
$\approx h_0/2$		2.01E-06	2.01E-06	2.01E-06	9.88E-09	1.46E-07	2.03E-06	2.03E-06	1.01	1.01
$\approx h_0/4$		7.74E-08	7.74E-08	7.73E-08	1.01E-10	4.35E-09	7.76E-08	7.76E-08	1.00	1.00
$\approx h_0/8$		1.86E-09	1.86E-09	1.86E-09	1.70E-12	1.00E-10	1.86E-09	1.86E-09	1.00	1.00
h_0	6	2.85E-06	2.85E-06	2.85E-06	4.70E-08	2.18E-07	2.90E-06	2.90E-06	1.02	1.02
$\approx h_0/2$		5.42E-08	5.42E-08	5.42E-08	2.40E-10	4.02E-09	5.46E-08	5.46E-08	1.01	1.01
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Numerics: singular case

Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := (-1,1)^2 \setminus [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

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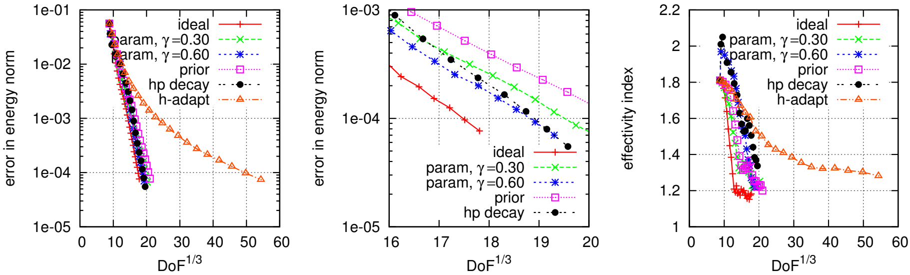
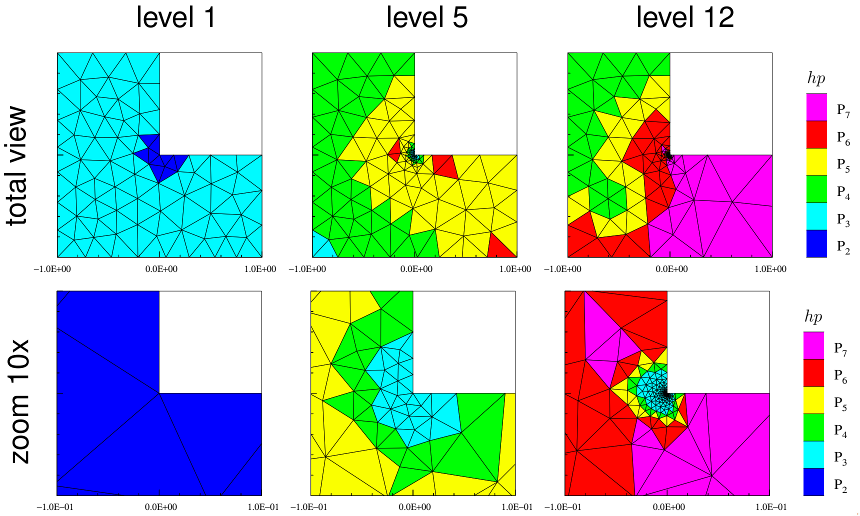
hp-adaptive refinement: **exponential convergence**



hp-refinement grids



Outline

Including iterative algebraic solver (conforming FEs)

Finite element approximation of the Laplace problem

Find $u_h \in V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$, such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

Linear algebraic system

Find $U_h \in \mathbb{R}^N$ such that

$$\mathbb{A}_h U_h = F_h$$

Algebraic solver (iterative)

On each iteration $i \geq 1$: approximate vector $U_h^i \in \mathbb{R}^N$ such that

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Algebraic error representer

On each iteration $i \geq 1$: approximate solution $u_h^i \in V_h$ such that

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where the algebraic error representer $r_h^i \in L^2(\Omega)$ is such that $(r_h^i, \psi_l) = (R_h^i)_l$, $l = 1, \ldots, N$;

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Algebraic error upper bound

Theorem (Upper bound via algebraic error flux reconstruction)

Let $\boldsymbol{\sigma}^i_{h,\mathrm{alg}} \in \mathbf{H}(\mathrm{div}, \Omega)$ *be such that* $\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}} = r^i_h$. *Then*

$$\underbrace{\|\nabla(u_h - u_h^i)\|}_{\text{algebraic error}} \le \underbrace{\|\boldsymbol{\sigma}^i_{h,\mathrm{alg}}\|}_{\text{upper algebraic est.}} .$$

Proof.

$$\|\nabla(u_h - u_h^i)\| = \sup_{v_h \in V_h, \|\nabla v_h\|=1} (\nabla(u_h - u_h^i), \nabla v_h);$$

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Constructions of $\boldsymbol{\sigma}^i_{h,\mathrm{alg}}$

1. sequential sweep trough $\mathcal{T}_h$, local min. (JSV (2010))
2. approximate by precomputing ν iterations (EV (2013))
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$$\underbrace{\|\nabla(u_h - u^i_h)\|}_{\text{algebraic error}} \leq \underbrace{\|\boldsymbol{\sigma}^i_{h,\mathrm{alg}}\|}_{\text{upper algebraic est.}} .$$

Proof.

$$\|\nabla(u_h - u^i_h)\| = \sup_{v_h\in V_h, \|\nabla v_h\|=1} (\nabla(u_h - u^i_h), \nabla v_h);$$

$$(\nabla(u_h - u^i_h), \nabla v_h) = (r^i_h, v_h) = (\nabla\cdot\boldsymbol{\sigma}^i_{h,\mathrm{alg}}, v_h) = -(\boldsymbol{\sigma}^i_{h,\mathrm{alg}}, \nabla v_h).$$

Constructions of $\boldsymbol{\sigma}^i_{h,\mathrm{alg}}$

1. sequential sweep trough $\mathcal{T}_h$, local min. (JSV (2010))
2. approximate by precomputing ν iterations (EV (2013))
3. **multilevel** flux reconstruction

Algebraic error flux reconstruction, two-level setting

Definition (Coarse grid Riesz representer)

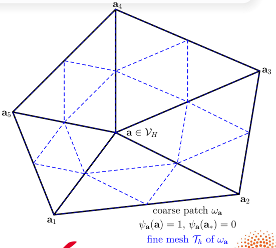
Find $v_H^i \in V_H := \mathbb{P}_1(\mathcal{T}_H) \cap H_0^1(\Omega)$ such that

$$(\nabla v_H^i, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (r_h^i, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \quad \forall \mathbf{a} \in \mathcal{V}_H$$

Definition (Algebraic error flux reconstruction)

$$\boldsymbol{\sigma}_{h,\mathrm{alg}}^{\mathbf{a},i} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(\psi_{\mathbf{a}} r_h^i - \nabla\psi_{\mathbf{a}}\cdot\nabla v_H^i)} \|\mathbf{v}_h\|_{\omega_{\mathbf{a}}}, \quad \boldsymbol{\sigma}_{h,\mathrm{alg}}^i := \sum_{\mathbf{a}\in\mathcal{V}_H} \boldsymbol{\sigma}_{h,\mathrm{alg}}^{\mathbf{a},i}$$

- homogeneous Neumann problems
- mixed FE spaces
- fine meshes of coarse patches $\omega_{\mathbf{a}}$
- Riesz representer (solve on $\mathcal{T}_H$) $\Rightarrow$ hat function orthogonality on $\mathcal{T}_H$
- extends to arbitrary number of levels



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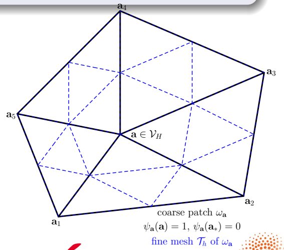

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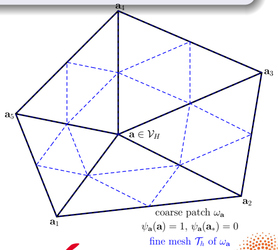

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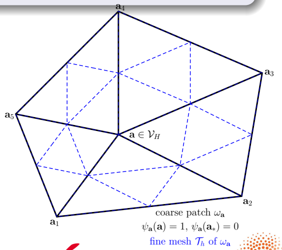

Divergence of the algebraic error flux reconstruction

Lemma (Divergence of $\boldsymbol{\sigma}_{h,\mathrm{alg}}^{\mathbf{a},i}$)

There holds $\nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{alg}}^{i} = r_h^i$.

Proof.

- every fine grid element $K \in \mathcal{T}_h$ lies exactly in $(d+1)$ coarse patches $\omega_{\mathbf{a}}$, $\mathbf{a} \in \mathcal{V}_H$
- partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \psi^{\mathbf{a}} = 1|_K$
-

$$\nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{alg}}^{i}|_K = \sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{alg}}^{\mathbf{a},i}|_K$$

$$= \sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \Pi_{Q_h}(\psi_{\mathbf{a}} r_h^i - \nabla\psi_{\mathbf{a}}\cdot\nabla v_H^i)|_K = r_h^i|_K$$

Divergence of the algebraic error flux reconstruction

Lemma (Divergence of $\boldsymbol{\sigma}^{\mathbf{a},i}_{h,\mathrm{alg}}$)

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  $$= \sum_{\mathbf{a}\in\mathcal{V}_H, K\subset\overline{\omega_{\mathbf{a}}}} \Pi_{Q_h}(\psi_{\mathbf{a}} r^i_h - \nabla\psi_{\mathbf{a}}{\cdot}\nabla v^i_H)|_K = r^i_h|_K$$

Algebraic residual lifting

Definition (Algebraic residual lifting, ≈ Babuška and Strouboulis (2001), Repin (2008))

Find $\upsilon_h^{\mathbf{a},i} \in X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\omega_{\mathbf{a}})$ such that

$$(\nabla \upsilon_h^{\mathbf{a},i}, \nabla v_h)_{\omega_{\mathbf{a}}} = (r_h^i, v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in X_h^{\mathbf{a}}.$$

Set

$$\upsilon_h^i := \sum_{\mathbf{a} \in \mathcal{V}_H} \upsilon_h^{\mathbf{a},i}.$$

- homogeneous Dirichlet problems
- conforming FE spaces
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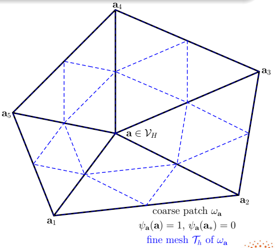

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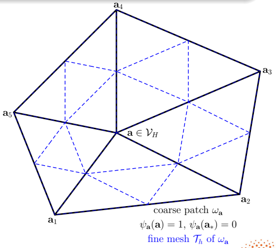

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Find $\rho_h^{\mathbf{a},i} \in X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\omega_{\mathbf{a}})$ such that

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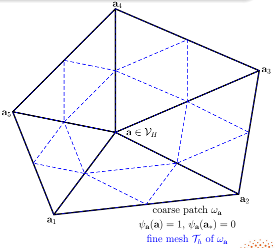

Algebraic error lower bound

Theorem (Lower bound via algebraic residual liftings)

There holds
$$\underbrace{\|\nabla(u_h - u_h^i)\|}_{\text{algebraic error}} \geq \underbrace{\frac{\sum_{\mathbf{a}\in\mathcal{V}_H}\|\nabla \upsilon_h^{\mathbf{a},i}\|^2_{\omega_{\mathbf{a}}}}{\|\nabla \upsilon_h^i\|}}_{\text{lower algebraic est.}}.$$

Proof.

$$\begin{aligned}\|\nabla(u_h - u_h^i)\| &= \sup_{v_h\in V_h, \|\nabla v_h\|=1} (r_h^i, v_h)\\ &\geq \frac{(r_h^i, \upsilon_h^i)}{\|\nabla \upsilon_h^i\|} = \frac{\sum_{\mathbf{a}\in\mathcal{V}_H}(r_h^i, \upsilon_h^{\mathbf{a},i})_{\omega_{\mathbf{a}}}}{\|\nabla \upsilon_h^i\|}\\ &= \frac{\sum_{\mathbf{a}\in\mathcal{V}_H}\|\nabla \upsilon_h^{\mathbf{a},i}\|^2_{\omega_{\mathbf{a}}}}{\|\nabla \upsilon_h^i\|}.\end{aligned}$$

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Proof.

$$\|\nabla(u_h - u_h^i)\| = \sup_{v_h\in V_h, \|\nabla v_h\|=1} (r_h^i, v_h)$$

$$\geq \frac{(r_h^i, v_h^i)}{\|\nabla v_h^i\|} = \frac{\sum_{\mathbf{a}\in\mathcal{V}_H} (r_h^i, v_h^{\mathbf{a},i})_{\omega_{\mathbf{a}}}}{\|\nabla v_h^i\|}$$

$$= \frac{\sum_{\mathbf{a}\in\mathcal{V}_H} \|\nabla v_h^{\mathbf{a},i}\|^2_{\omega_{\mathbf{a}}}}{\|\nabla v_h^i\|}.$$

Outline

Discretization flux reconstruction

Definition (Discretization flux reconstruction)

$$\boldsymbol{\sigma}_{h,\mathrm{dis}}^{\mathbf{a},i} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(f\psi^{\mathbf{a}} - \nabla u_h^i \cdot \nabla \psi_{\mathbf{a}} - r_h^i \psi^{\mathbf{a}})} \|\psi^{\mathbf{a}} \nabla u_h^i + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$\boldsymbol{\sigma}_{h,\mathrm{dis}}^{i} := \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_{h,\mathrm{dis}}^{\mathbf{a},i}$$

Neumann compatibility condition

$$(\nabla u_h^i, \nabla \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} - (r_h^i, \psi^{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

Lemma (Divergence of $\boldsymbol{\sigma}_{h,\mathrm{dis}}^{i}$)

There holds

$$\nabla\cdot\boldsymbol{\sigma}_{h,\mathrm{dis}}^{i} = \Pi_{Q_h} f - r_h^i.$$

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Upper bound on the total error

Theorem (Total error upper bound)

On each iteration $i \geq 1$, there holds

$$\underbrace{\|\nabla(u-u_h^i)\|}_{\text{total error}} \leq \underbrace{\|\nabla u_h^i + \boldsymbol{\sigma}^i_{h,\mathrm{dis}}\|}_{\text{discretization est.}} + \underbrace{\|\boldsymbol{\sigma}^i_{h,\mathrm{alg}}\|}_{\text{algebraic est.}} + \underbrace{\left\{\sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\pi^2}\|f-\Pi_{Q_h} f\|_K^2\right\}^{1/2}}_{\text{data oscillation est.}}.$$

Proof.

$$\|\nabla(u-u_h^i)\| = \sup_{v\in H_0^1(\Omega),\, \|\nabla v\|=1} (\nabla(u-u_h^i), \nabla v)$$

$$(\nabla(u-u_h^i),\nabla v) = (f,v) - (\nabla u_h^i, \nabla v) = (f - \overbrace{\nabla\cdot(\boldsymbol{\sigma}^i_{h,\mathrm{alg}} + \boldsymbol{\sigma}^i_{h,\mathrm{dis}})}, v) - (\boldsymbol{\sigma}^i_{h,\mathrm{alg}} + \boldsymbol{\sigma}^i_{h,\mathrm{dis}} + \nabla u_h^i, \nabla v)$$

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$$(\nabla(u-u_h^i),\nabla v) =(f,v)-(\nabla u_h^i,\nabla v) = (f - \overbrace{\nabla\cdot(\boldsymbol{\sigma}_{h,\mathrm{alg}}^i+\boldsymbol{\sigma}_{h,\mathrm{dis}}^i)}^{=\Pi_{Q_h}f}, v)$$
$$-(\boldsymbol{\sigma}_{h,\mathrm{alg}}^i+\boldsymbol{\sigma}_{h,\mathrm{dis}}^i+\nabla u_h^i,\nabla v)$$

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Proof.

$$\|\nabla(u-u_h^i)\| = \sup_{v\in H_0^1(\Omega),\ \|\nabla v\|=1} (\nabla(u-u_h^i), \nabla v)$$

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$$- (\boldsymbol{\sigma}_{h,\mathrm{alg}}^i + \boldsymbol{\sigma}_{h,\mathrm{dis}}^i + \nabla u_h^i, \nabla v) \leq \tilde{\eta}_{\mathrm{dis}}^i + \eta_{\mathrm{alg}}^i + \eta_{\mathrm{osc}}$$

Lower bound on the total error

Setting

- vertices $\mathbf{a} \in \mathcal{V}_h$ and patches $\omega_{\mathbf{a}}$
- $X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h)$ with either mean value zero or value on $\partial\Omega$ zero (Lagrange FE space)

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Homogeneous Neumann pbs on patches $\omega_{\mathbf{a}}$

Definition (Total residual lifting)

$$\bar{r}_h^{\mathbf{a},i} \in X_h^{\mathbf{a}} \text{ s.t. } (\nabla \bar{r}_h^{\mathbf{a},i}, \nabla v_h)_{\omega_{\mathbf{a}}} = (f, v_h)_{\omega_{\mathbf{a}}} - (\nabla u_h^i, \nabla v_h)_{\omega_{\mathbf{a}}} \;\; \forall v_h \in X_h^{\mathbf{a}},$$

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Homogeneous Neumann pbs on patches $\omega_\mathbf{a}$

Definition (Total residual lifting)

$$\bar{r}_h^{\mathbf{a},i} \in X_h^\mathbf{a} \text{ s.t. } (\nabla \bar{r}_h^{\mathbf{a},i}, \nabla v_h)_{\omega_\mathbf{a}} = (f, v_h)_{\omega_\mathbf{a}} - (\nabla u_h^i, \nabla v_h)_{\omega_\mathbf{a}} \ \forall v_h \in X_h^\mathbf{a},$$

$$\bar{r}_h^i := \sum_{\mathbf{a} \in \mathcal{V}_h} \psi^\mathbf{a} \bar{r}_h^{\mathbf{a},i}$$

Lower bound on the total error

Setting

- vertices $\mathbf{a} \in \mathcal{V}_h$ and patches $\omega_{\mathbf{a}}$
- $X_h^{\mathbf{a}} := \mathbb{P}_p(\mathcal{T}_h)$ with either mean value zero or value on $\partial\Omega$ zero (Lagrange FE space)

Homogeneous Neumann pbs on patches $\omega_{\mathbf{a}}$

Definition (Total residual lifting)

$$\bar{r}_h^{\mathbf{a},i} \in X_h^{\mathbf{a}} \text{ s.t. } (\nabla \bar{r}_h^{\mathbf{a},i}, \nabla v_h)_{\omega_{\mathbf{a}}} = (f, v_h)_{\omega_{\mathbf{a}}} - (\nabla u_h^i, \nabla v_h)_{\omega_{\mathbf{a}}} \ \forall v_h \in X_h^{\mathbf{a}},$$

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Theorem (Lower bound on the total error)

There holds
$$\underbrace{\|\nabla(u - u_h^i)\|}_{\text{total error}} \geq \underbrace{\frac{\sum_{\mathbf{a} \in \mathcal{V}_h} \|\nabla \bar{r}_h^{\mathbf{a},i}\|_{\omega_{\mathbf{a}}}^2}{\|\nabla \bar{r}_h^i\|}}_{\text{lower total est.}}.$$

Outline

Stopping criteria

Galerkin orthogonality

$$\underbrace{\|\nabla(u-u_h^i)\|^2}_{\text{total error}} = \underbrace{\|\nabla(u-u_h)\|^2}_{\text{discretization error}} + \underbrace{\|\nabla(u_h-u_h^i)\|^2}_{\text{algebraic error}}$$

Discretization error upper and lower bounds

- upper bound on total error & lower bound on algebraic error ⇒ upper bound on the discretization error
- lower bound on total error & upper bound on algebraic error ⇒ lower bound on the discretization error

Safe stopping criterion

upper algebraic est. $\leq \gamma$ lower discretization est.

- local version for local efficiency

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Outline

Numerical illustration

Peak $\Omega = (0,1) \times (0,1)$, $u(x,y) = x(x-1)y(y-1)\exp(-100(x-0.5)^2 - 100(y-117/1000)^2)$

L-shape $(-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $u(r,\theta) = r^{2/3}\sin(2\theta/3)$

Discretization

- conforming finite elements with $p = 1, \ldots, 5$
- unstructured triangular meshes
- 4 uniform refinements

Multigrid setting

- geometric multigrid V-cycle
- 5 pre-smoothing steps of Gauss–Seidel

PCG setting

- incomplete Cholesky with drop-off tolerance 1e-4 preconditioning

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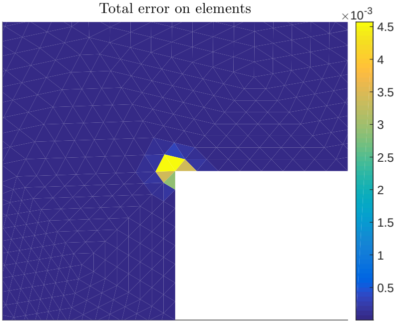
Peak problem, multigrid

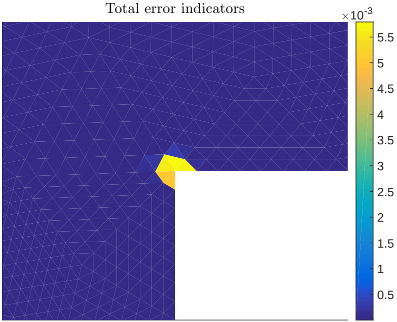
p	MG iter	algebraic error	algebraic eff. UB	algebraic eff. LB	total error	total eff. UB	total eff. LB	discretization error	discretization eff. UB	discretization eff. LB
1 (2.55×10^3)	1	8.1×10^{-3}	1.14	1.10^{-1}	1.0×10^{-2}	1.63	1.19^{-1}	6.1×10^{-3}	2.42	—
	2	4.3×10^{-4}	1.13	1.12^{-1}	6.1×10^{-3}	1.13	1.05^{-1}		1.13	1.06^{-1}
2 (1.03×10^4)	1	8.8×10^{-3}	1.17	1.08^{-1}	8.8×10^{-3}	1.72	1.18^{-1}	3.9×10^{-4}	3.28×10^1	—
	2	6.1×10^{-4}	1.19	1.03^{-1}	7.2×10^{-4}	1.75	1.12^{-1}		2.89	—
	3	2.0×10^{-5}	1.19	1.03^{-1}	3.9×10^{-4}	1.08	1.04^{-1}		1.08	1.04^{-1}
3 (2.34×10^4)	1	4.9×10^{-3}	1.14	1.06^{-1}	4.9×10^{-3}	1.59	1.26^{-1}	1.9×10^{-5}	3.33×10^2	—
	3	2.7×10^{-5}	1.17	1.04^{-1}	3.3×10^{-5}	1.69	1.17^{-1}		2.60	—
	5	1.6×10^{-7}	1.15	1.04^{-1}	1.9×10^{-5}	1.02	1.09^{-1}		1.02	1.09^{-1}
4 (4.17×10^4)	1	5.8×10^{-3}	1.22	1.05^{-1}	5.8×10^{-3}	1.83	1.17^{-1}	8.1×10^{-7}	1.12×10^4	—
	3	1.0×10^{-4}	1.16	1.03^{-1}	1.0×10^{-4}	1.71	1.08^{-1}		1.76×10^2	—
	5	2.4×10^{-6}	1.14	1.03^{-1}	2.5×10^{-6}	1.62	1.10^{-1}		4.12	—
	7	6.7×10^{-8}	1.13	1.03^{-1}	8.2×10^{-7}	1.10	1.16^{-1}		1.10	1.16^{-1}
5 (6.52×10^4)	1	4.8×10^{-3}	1.19	1.04^{-1}	4.8×10^{-3}	1.74	1.19^{-1}	3.1×10^{-8}	2.21×10^5	—
	3	2.1×10^{-4}	1.14	1.03^{-1}	2.1×10^{-4}	1.63	1.09^{-1}		8.78×10^3	—
	5	1.5×10^{-5}	1.11	1.02^{-1}	1.5×10^{-5}	1.55	1.07^{-1}		5.57×10^2	—
	7	1.4×10^{-6}	1.12	1.02^{-1}	1.4×10^{-6}	1.57	1.05^{-1}		5.34×10^1	—
	9	1.4×10^{-7}	1.14	1.01^{-1}	1.4×10^{-7}	1.65	1.06^{-1}		6.06	—
	11	1.3×10^{-8}	1.16	1.01^{-1}	3.4×10^{-8}	1.41	1.38^{-1}		1.47	1.62^{-1}
	13	1.2×10^{-9}	1.16	1.01^{-1}	3.1×10^{-8}	1.05	1.21^{-1}		1.05	1.21^{-1}

L-shape problem, PCG

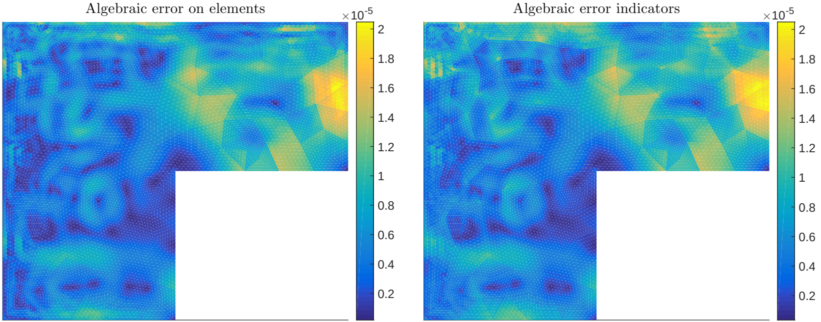
p	PCG iter	algebraic error	algebraic eff. UB	algebraic eff. LB	total error	total eff. UB	total eff. LB	discretization error	discretization eff. UB	discretization eff. LB
1 (7.97×10^3)	2	2.9×10^{-1}	1.25	4.08^{-1}	2.9×10^{-1}	1.38	6.15^{-1}	3.6×10^{-2}	1.11×10^1	—
	4	1.2×10^{-3}	1.24	4.17^{-1}	3.6×10^{-2}	1.24	1.12^{-1}		1.24	1.12^{-1}
2 (3.22×10^4)	3	2.1×10^{-1}	1.14	3.62^{-1}	2.1×10^{-1}	1.26	6.03^{-1}	1.4×10^{-2}	1.76×10^1	—
	6	2.5×10^{-3}	1.18	3.17^{-1}	1.5×10^{-2}	1.47	1.32^{-1}		1.49	1.35^{-1}
	9	9.2×10^{-6}	1.17	3.53^{-1}	1.4×10^{-2}	1.29	1.30^{-1}		1.29	1.30^{-1}
3 (7.27×10^4)	4	1.3	1.06	4.53^{-1}	1.3	1.10	$1.08 \times 10^{1}{}^{-1}$	8.6×10^{-3}	1.58×10^2	—
	8	9.9×10^{-2}	1.10	3.55^{-1}	10.0×10^{-2}	1.24	6.02^{-1}		1.41×10^1	—
	12	1.2×10^{-2}	1.10	3.58^{-1}	1.5×10^{-2}	1.71	2.67^{-1}		2.99	—
	16	8.2×10^{-4}	1.10	3.55^{-1}	8.6×10^{-3}	1.51	1.42^{-1}		1.52	1.43^{-1}
4 (1.29×10^5)	5	1.7×10^{-1}	1.24	2.34^{-1}	1.7×10^{-1}	1.42	3.35^{-1}	6.2×10^{-3}	3.66×10^1	—
	10	2.4×10^{-3}	1.22	2.79^{-1}	6.6×10^{-3}	1.78	1.83^{-1}		1.90	2.93^{-1}
	15	2.3×10^{-5}	1.27	2.33^{-1}	6.2×10^{-3}	1.44	1.62^{-1}		1.44	1.62^{-1}
5 (2.02×10^5)	6	1.1	1.09	4.14^{-1}	1.1	1.16	7.42^{-1}	4.7×10^{-3}	2.71×10^2	—
	12	8.5×10^{-2}	1.11	3.75^{-1}	8.5×10^{-2}	1.23	5.77^{-1}		2.19×10^1	—
	18	7.5×10^{-3}	1.15	3.12^{-1}	8.9×10^{-3}	1.76	3.43^{-1}		3.31	—
	24	3.9×10^{-4}	1.15	3.17^{-1}	4.7×10^{-3}	1.56	1.80^{-1}		1.57	1.82^{-1}

L-shape problem, $p = 3$, total error, 16th PCG iteration





L-shape problem, $p = 3$, alg. error, 16th PCG iteration



Outline

1. Introduction
2. A posteriori estimates based on potential & flux reconstruction
   - Guaranteed upper bound in a unified framework
   - Potential and flux reconstructions
   - Polynomial-degree-robust local efficiency
   - Applications
   - Numerical illustration
3. Algebraic estimates and stopping criteria for iterative solvers
   - Bounds on the algebraic error
   - Bounds on the total error
   - Stopping criteria
   - Numerical illustration
4. Stokes equation
5. Conclusions and outlook

Stokes problem

Stokes problem

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p = \mathbf{f} \qquad & \text{in } \Omega, \\
\nabla\cdot\mathbf{u} = 0 \qquad & \text{in } \Omega, \\
\mathbf{u} = \mathbf{0} \qquad & \text{on } \partial\Omega
\end{aligned}
$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $\mathbf{f} \in [L^2(\Omega)]^d$
- $\mathbf{V} := [H_0^1(\Omega)]^d$, $Q := L_0^2(\Omega) := \{q \in L^2(\Omega); (q, 1) = 0\}$

Weak formulation

Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$
\begin{aligned}
(\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla\cdot\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \qquad & \forall \mathbf{v} \in \mathbf{V}, \\
(\nabla\cdot\mathbf{u}, q) = 0 \qquad & \forall q \in Q.
\end{aligned}
$$

Inf–sup condition

$$
\inf_{q \in Q} \sup_{\mathbf{v} \in \mathbf{V}} \frac{(q, \nabla\cdot\mathbf{v})}{\|\nabla\mathbf{v}\| \, \|q\|} = \beta > 0
$$

Stokes problem

Stokes problem

$$
\begin{aligned}
-\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega,\\
\nabla{\cdot}\mathbf{u} &= 0 && \text{in } \Omega,\\
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\end{aligned}
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\end{aligned}
$$

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$$

Exact and approximate solutions

Properties of the weak solution

- $\mathbf{u} \in \mathbf{V}$
- $\underline{\boldsymbol{\sigma}} := \nabla \mathbf{u} - p\underline{\boldsymbol{I}}$
- $\nabla \cdot \underline{\boldsymbol{\sigma}} = -\mathbf{f}$
- $\underline{\boldsymbol{\sigma}} \in [\mathbf{H}(\mathrm{div}, \Omega)]^d$

Approximate solution

- $\mathbf{u}_h \in [H^1(\mathcal{T}_h)]^d \not\subset \mathbf{V}$
- $p_h \in Q$
- $\nabla \mathbf{u}_h - p_h\underline{\boldsymbol{I}} \not\in \mathbf{V}$
- $\nabla \cdot (\nabla \mathbf{u}_h - p_h\underline{\boldsymbol{I}}) \neq -\mathbf{f}$

Exact and approximate solutions

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- $\mathbf{u} \in \mathbf{V}$
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- $p_h \in Q$
- $\nabla \mathbf{u}_h - p_h\underline{\boldsymbol{I}} \notin \mathbf{V}$
- $\nabla \cdot (\nabla \mathbf{u}_h - p_h\underline{\boldsymbol{I}}) \neq -\mathbf{f}$

Velocity and equilibrated stress reconstructions

Velocity reconstruction

- $\mathbf{s}_h \in \mathbf{V}$
- $\mathbf{s}_h$ constructed from $\mathbf{u}_h$

Equilibrated stress reconstruction

- $\underline{\sigma}_h \in [\mathbf{H}(\mathrm{div}, \Omega)]^d$
- $-(\nabla \cdot \underline{\sigma}_h, \mathbf{e}_i)_K = (\mathbf{f}, \mathbf{e}_i)_K \qquad i = 1, \ldots, d, \quad \forall K \in \mathcal{T}_h$
- $\underline{\sigma}_h$ constructed from $\mathbf{u}_h$

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- $\underline{\boldsymbol{\sigma}}_h$ constructed from $\mathbf{u}_h$

A guaranteed a posteriori error estimate

Theorem (A guaranteed a posteriori error estimate)

Let $(\mathbf{u}, p) \in \mathbf{V} \times Q$ be the weak solution & $(\mathbf{u}_h, p_h) \in [H^1(\mathcal{T}_h)]^d \times Q$ be arbitrary. Let $\mathbf{s}_h$ be a velocity reconstruction and $\underline{\boldsymbol{\sigma}}_h$ an equilibrated stress reconstruction. For any $K \in \mathcal{T}_h$, define

$$\eta_{\mathrm{R},K} := C_{\mathrm{P},K} h_K \|\nabla \cdot \underline{\boldsymbol{\sigma}}_h + \mathbf{f}\|_K \qquad \text{residual est.,}$$

$$\eta_{\mathrm{F},K} := \|\nabla \mathbf{u}_h - p_h \underline{\boldsymbol{I}} - \underline{\boldsymbol{\sigma}}_h\|_K \qquad \text{flux est.,}$$

$$\eta_{\mathrm{NC},K} := \|\nabla(\mathbf{u}_h - \mathbf{s}_h)\|_K \qquad \text{nonconformity est.,}$$

$$\eta_{\mathrm{D},K} := \frac{\|\nabla \cdot \mathbf{s}_h\|_K}{\beta} \qquad \text{divergence est.}$$

Then

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\|^2 \le \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{F},K})^2 + \left\{ \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{D},K}^2 \right\}^{1/2} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{1/2} \right\}^2,$$

$$\|p - p_h\| \le \frac{1}{\beta} \left(\left\{ \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{F},K})^2 \right\}^{1/2} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{D},K}^2 \right\}^{1/2} + \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \right\}^{1/2} \right).$$

Outline

Conclusions and outlook

Conclusions

- guaranteed energy error estimates
- robustness (polynomial degree)
- unified framework for all classical numerical schemes

Ongoing work

- convergence and optimality
- nonlinear problems

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Ongoing work

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Thank you for your attention!