Full adaptivity for unsteady nonlinear problems

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Lecture IV/IV

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Outline

1. Introduction
2. The Stefan problem
   - Dual norm a posteriori estimate and adaptivity
   - Efficiency
   - Energy error a posteriori estimate
   - Numerical results
3. Two-phase flow
   - A posteriori error estimate
   - Full adaptivity
   - Applications
   - Numerical results
4. References and bibliography

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Full adaptivity for unsteady nonlinear problems

Real (porous media) flows

- systems of PDEs
- nonlinear (degenerate)
- unsteady
- $\Rightarrow$ difficult numerical approximation, large troublesome **systems** of **nonlinear algebraic equations**

Goals

- derive fully computable a posteriori **error upper bounds**
- distinguish different **error components**

Full adaptivity

- time step choice & mesh adaptivity
- **stopping criteria** for **regularization** and **linear** and **nonlinear** solvers

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$$\begin{aligned} \partial_t u - \Delta\beta(u) &= f && \text{in } \Omega \times (0,T), \\ u(\cdot,0) &= u_0 && \text{in } \Omega, \\ \beta(u) &= 0 && \text{on } \partial\Omega \times (0,T) \end{aligned}$$

Nomenclature

- u enthalpy, $\beta(u)$ temperature
- β: L_β-Lipschitz continuous, $\beta(s) = 0$ in $(0,1)$, strictly increasing otherwise
- phase change, degenerate parabolic problem
- $u_0 \in L^2(\Omega)$, $f \in L^2(0,T;L^2(\Omega))$

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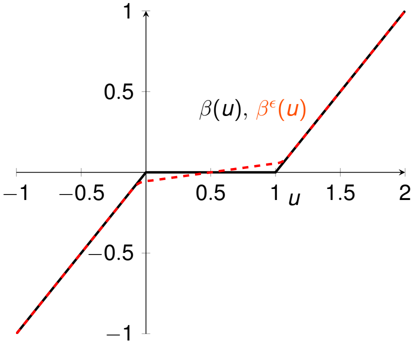
Numerical practice: regularization

Regularization of β, parameter ϵ



- $\beta^\epsilon(\cdot)$ smooth and strictly increasing

Setting

Functional spaces

$$X := L^2(0,T;H_0^1(\Omega)), \qquad Z := H^1(0,T;H^{-1}(\Omega))$$

Weak formulation

$u \in Z$ with $\beta(u) \in X$,

$u(\cdot,0) = u_0$ in Ω,

$$\langle \partial_t u, \varphi\rangle(s) + (\nabla\beta(u), \nabla\varphi)(s) = (f,\varphi)(s) \quad \forall \varphi \in H_0^1(\Omega), s \in (0,T)$$

Approximation (conforming, with linearization & regularization)

$$u_{h\tau}^{\epsilon,k} \in Z, \qquad \partial_t u_{h\tau}^{\epsilon,k} \in L^2(0,T;L^2(\Omega)), \qquad \beta(u_{h\tau}^{\epsilon,k}) \in X,$$

$u_{h\tau}^{\epsilon,k}|_{I_n}$ is affine in time on I_n $\quad \forall 1 \le n \le N$

Residual $\mathcal{R}(u_{h\tau}^{\epsilon,k}) \in X'$ **and its dual norm,** $\varphi \in X$

$$\langle \mathcal{R}(u_{h\tau}^{\epsilon,k}), \varphi\rangle_{X',X} := \int_0^T \Big\{\langle \partial_t(u - u_{h\tau}^{\epsilon,k}), \varphi\rangle + (\nabla(\beta(u) - \beta(u_{h\tau}^{\epsilon,k})), \nabla\varphi)\Big\}(s)\,\mathrm{d}s,$$

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Time-localization of the dual norm of the residual

Time interval I_n

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L^2 in time:

$$\|\mathcal{R}(u_{h\tau}^{\epsilon,k})\|_{X'}^2 = \sum_{1 \le n \le N} \|\mathcal{R}(u_{h\tau}^{\epsilon,k})\|_{X_n'}^2$$

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Outline

Estimate distinguishing different error components

Assumption A (Equilibrated flux reconstruction)

For all $n \geq 1$, $k \geq 1$, and $\epsilon > 0$, there exists $\boldsymbol{\sigma}_h^{n,\epsilon,k} \in \mathbf{H}(\mathrm{div};\Omega)$ s.t.

$$(\nabla\cdot\boldsymbol{\sigma}_h^{n,\epsilon,k}, 1)_K = (f|_{I_n}, 1)_K - (\partial_t u_{h\tau}^{\epsilon,k}|_{I_n}, 1)_K \qquad \forall K \in \mathcal{T}^n.$$

Theorem (An estimate distinguishing the error components)

Let Assumption A hold. Then, for any $n \geq 1$, $k \geq 1$, and $\epsilon > 0$,

$$\|\mathcal{R}(u_{h\tau}^{\epsilon,k})\|_{X_n'} \leq \eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k} + \eta_{\mathrm{lin}}^{n,\epsilon,k}.$$

$$(\eta_{\mathrm{sp}}^{n,\epsilon,k})^2 := \tau^n \sum_{K\in\mathcal{T}^n} \left(\eta_{\mathrm{E},K}^{n,\epsilon,k} + \|\nabla\beta^{\epsilon,k-1}(u_h^{n,\epsilon,k}) + \boldsymbol{\sigma}_h^{n,\epsilon,k}\|_K\right)^2,$$

$$(\eta_{\mathrm{tm}}^{n,\epsilon,k})^2 := \int_{I_n} \sum_{K\in\mathcal{T}^n} \|\nabla(\beta(u_{h\tau}^{\epsilon,k})(t) - \beta(u_h^{n,\epsilon,k}))\|_K^2 \,\mathrm{d}t,$$

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Outline

Efficiency assumptions

Assumption B (Piecewise polynomials, data, and meshes)

The approximations and the data f and u_0 are piecewise polynomial. The meshes are shape-regular.

Residual estimators

$$\left(\eta_{\mathrm{res},1}^{n,\epsilon,k}\right)^2 := \tau^n \sum_{K\in\mathcal{T}^{n-1,n}} h_K^2 \|f|_{I_n} - \partial_t u_{h\tau}^{\epsilon,k}|_{I_n} + \Delta\beta^{\epsilon,k-1}(u_h^{n,\epsilon,k})\|_K^2,$$

$$\left(\eta_{\mathrm{res},2}^{n,\epsilon,k}\right)^2 := \tau^n \sum_{e\in\mathcal{E}^{\mathrm{int},n-1,n}} h_e \|[\![\nabla\beta^{\epsilon,k-1}(u_h^{n,\epsilon,k})\cdot\mathbf{n}_e]\!]\|_e^2$$

Assumption C (Approximation property)

For all $1 \le n \le N$, there holds

$$\tau^n \sum_{K\in\mathcal{T}^{n-1,n}} \|\nabla\beta^{\epsilon,k-1}(u_h^{n,\epsilon,k}) + \boldsymbol{\sigma}_h^{n,\epsilon,k}\|_K^2 \le C\left(\left(\eta_{\mathrm{res},1}^{n,\epsilon,k}\right)^2 + \left(\eta_{\mathrm{res},2}^{n,\epsilon,k}\right)^2\right).$$

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Efficiency assumptions

Theorem (Efficiency)

Let, for all $1 \leq n \leq N$, the stopping and balancing criteria be satisfied with the parameters small enough. Let Assumptions B and C hold. Then

$$\eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} + \eta_{\mathrm{reg}}^{n,\epsilon,k} + \eta_{\mathrm{lin}}^{n,\epsilon,k} \leq C \|\mathcal{R}(u_h^{n,\epsilon,k})\|_{X_n'}.$$

Outline

Relation residual–energy norm

Energy estimate (by the Gronwall lemma)

$$\frac{L_\beta}{2}\|u-u_{h\tau}\|_{X'}^2+\frac{L_\beta}{2}\|(u-u_{h\tau})(\cdot,T)\|_{H^{-1}(\Omega)}^2+\|\beta(u)-\beta(u_{h\tau})\|_{Q_T}^2$$

$$\leq\frac{L_\beta}{2}(2e^T-1)\left(\|\mathcal{R}(u_{h\tau})\|_{X'}^2+\|(u-u_{h\tau})(\cdot,0)\|_{H^{-1}(\Omega)}^2\right)$$

Theorem (Temperature and enthalpy errors, tight Gronwall)

Let $u_{h\tau}\in Z$ such that $\beta(u_{h\tau})\in X$ be arbitrary. There holds

$$\frac{L_\beta}{2}\|u-u_{h\tau}\|_{X'}^2+\frac{L_\beta}{2}\|(u-u_{h\tau})(\cdot,T)\|_{H^{-1}(\Omega)}^2+\|\beta(u)-\beta(u_{h\tau})\|_{Q_T}^2$$

$$+2\int_0^T\left(\|\beta(u)-\beta(u_{h\tau})\|_{Q_t}^2+\int_0^t\|\beta(u)-\beta(u_{h\tau})\|_{Q_s}^2e^{t-s}\,\mathrm{d}s\right)\mathrm{d}t$$

$$\leq\frac{L_\beta}{2}\Bigg\{(2e^T-1)\|(u-u_{h\tau})(\cdot,0)\|_{H^{-1}(\Omega)}^2+\|\mathcal{R}(u_{h\tau})\|_{X'}^2$$

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$$\leq\frac{L_\beta}{2}\left\{(2e^T-1)\|(u-u_{h\tau})(\cdot,0)\|^2_{H^{-1}(\Omega)}+\|\mathcal{R}(u_{h\tau})\|^2_{X'}\right.$$
$$\left.+2\int_0^T\left(\|\mathcal{R}(u_{h\tau})\|^2_{X'_t}+\int_0^t\|\mathcal{R}(u_{h\tau})\|^2_{X'_s}e^{t-s}\,\mathrm{d}s\right)\mathrm{d}t\right\}.$$

Outline

Regularization stopping criterion

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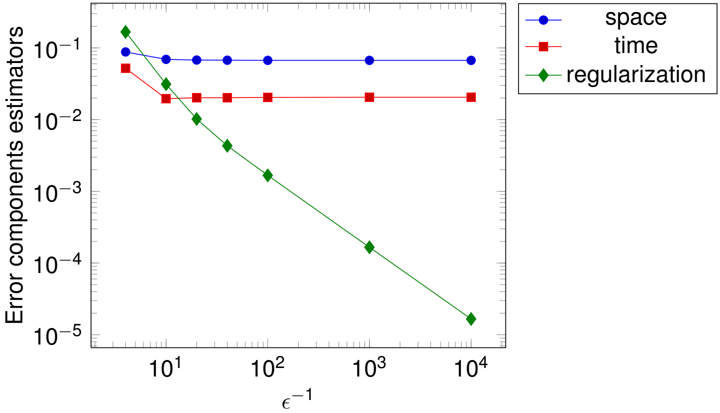
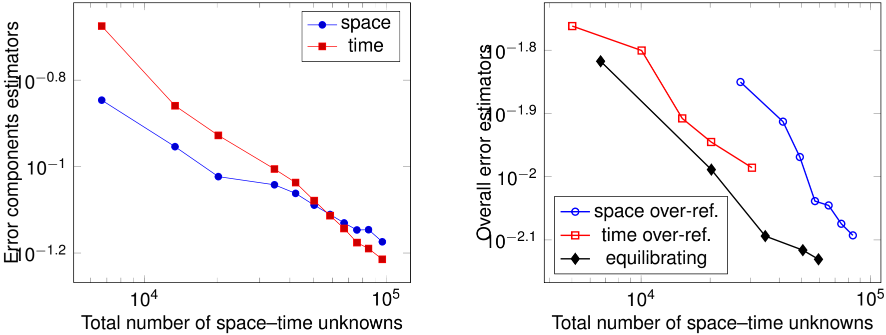
$$\eta_{\mathrm{reg}}^{n,\epsilon,k} \leq \Gamma_{\mathrm{reg}} \left(\eta_{\mathrm{sp}}^{n,\epsilon,k} + \eta_{\mathrm{tm}}^{n,\epsilon,k} \right)$$



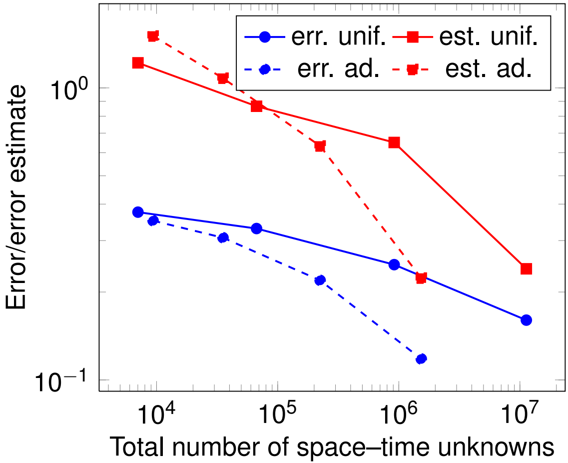
Equilibrating time and space errors

Equilibrating time and space errors

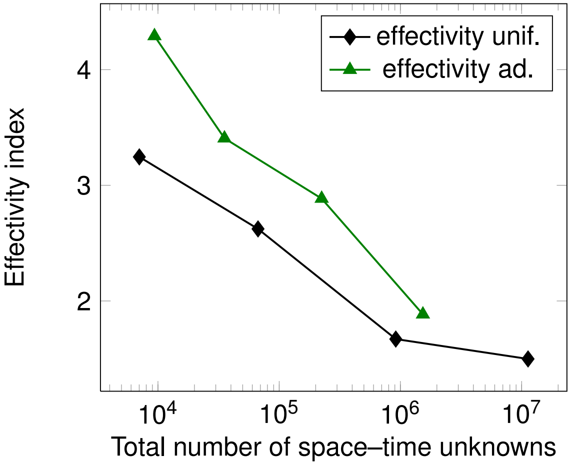
$$\gamma_{\mathrm{tm}}\eta_{\mathrm{sp}}^{n,\epsilon,k} \le \eta_{\mathrm{tm}}^{n,\epsilon,k} \le \Gamma_{\mathrm{tm}}\eta_{\mathrm{sp}}^{n,\epsilon,k}$$



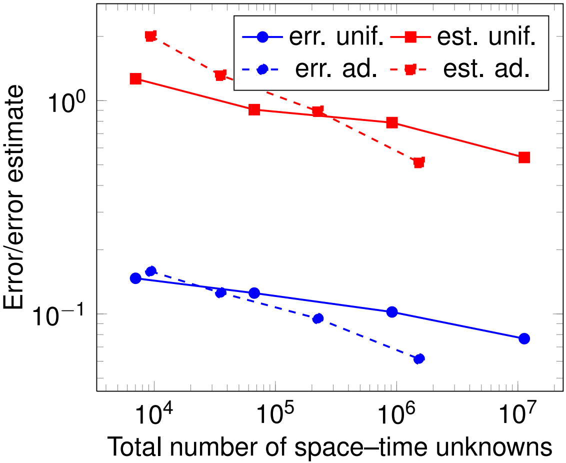
Error and estimate (dual norm of the residual)



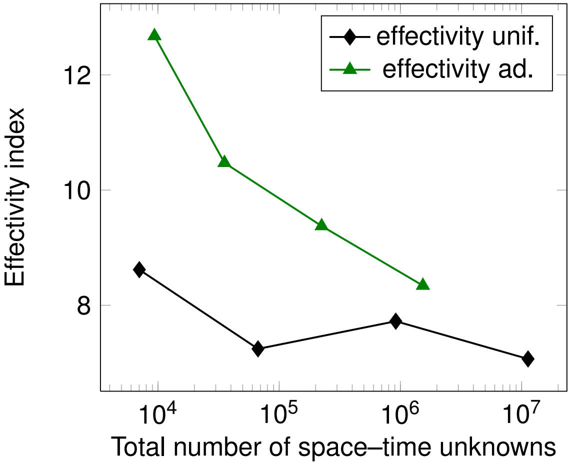
Effectivity indices (dual norm of the residual)



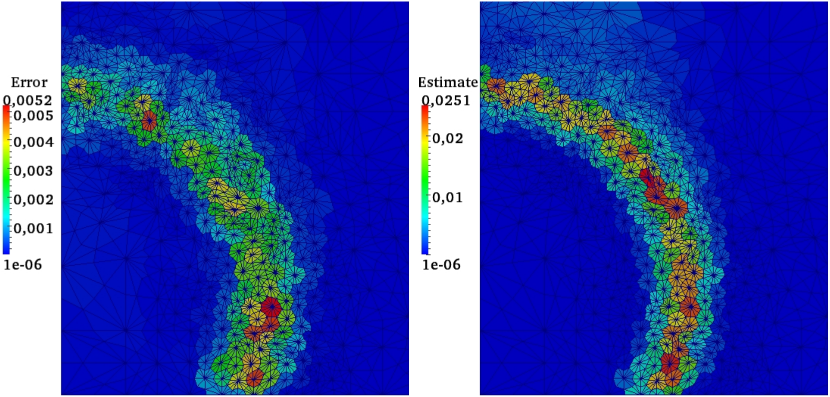
Error and estimate (energy norm)



Effectivity indices (energy norm)



Actual and estimated error distribution



Outline

Two-phase flow in porous media

Two-phase flow in porous media

$$\begin{aligned} \partial_t(\phi s_\alpha) + \nabla\cdot\mathbf{u}_\alpha &= q_\alpha, && \alpha \in \{\mathrm{n}, \mathrm{w}\}, \\ -\lambda_\alpha(s_\mathrm{w})\underline{\mathbf{K}}(\nabla p_\alpha + \rho_\alpha g \nabla z) &= \mathbf{u}_\alpha, && \alpha \in \{\mathrm{n}, \mathrm{w}\}, \\ s_\mathrm{n} + s_\mathrm{w} &= 1, \\ p_\mathrm{n} - p_\mathrm{w} &= p_\mathrm{c}(s_\mathrm{w}) \end{aligned}$$

+ boundary & initial conditions

Mathematical issues

- coupled system
- unsteady, nonlinear
- elliptic–degenerate parabolic type
- dominant advection

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Global and complementary pressures

Global pressure

$$\mathfrak{p}(s_{\rm w}, p_{\rm w}) := p_{\rm w} + \int_0^{s_{\rm w}} \frac{\lambda_{\rm n}(a)}{\lambda_{\rm w}(a) + \lambda_{\rm n}(a)} p_{\rm c}'(a) \mathrm{d}a$$

Complementary pressure

$$\mathfrak{q}(s_{\rm w}) := -\int_0^{s_{\rm w}} \frac{\lambda_{\rm w}(a)\lambda_{\rm n}(a)}{\lambda_{\rm w}(a) + \lambda_{\rm n}(a)} p_{\rm c}'(a) \mathrm{d}a$$

Comments

- necessary for the correct definition of the weak solution
- equivalent Darcy velocities expressions

$$\mathbf{u}_{\rm w}(s_{\rm w}, p_{\rm w}) := -\underline{\mathbf{K}}(\lambda_{\rm w}(s_{\rm w})\nabla \mathfrak{p}(s_{\rm w}, p_{\rm w}) + \nabla \mathfrak{q}(s_{\rm w}) + \lambda_{\rm w}(s_{\rm w})\rho_{\rm w} g \nabla z),$$

$$\mathbf{u}_{\rm n}(s_{\rm w}, p_{\rm w}) := -\underline{\mathbf{K}}(\lambda_{\rm n}(s_{\rm w})\nabla \mathfrak{p}(s_{\rm w}, p_{\rm w}) - \nabla \mathfrak{q}(s_{\rm w}) + \lambda_{\rm n}(s_{\rm w})\rho_{\rm n} g \nabla z)$$

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Weak formulation

Energy space

$$X := L^2((0,T); H^1_{\mathrm{D}}(\Omega))$$

Definition (Weak solution (Arbogast 1992, Chen 2001))

Find $(s_{\mathrm{w}}, p_{\mathrm{w}})$ such that, with $s_{\mathrm{n}} := 1 - s_{\mathrm{w}}$,

$$s_{\mathrm{w}} \in C([0,T]; L^2(\Omega)),\ s_{\mathrm{w}}(\cdot,0) = s^0_{\mathrm{w}},$$

$$\partial_t s_{\mathrm{w}} \in L^2((0,T); (H^1_{\mathrm{D}}(\Omega))'),$$

$$\mathfrak{p}(s_{\mathrm{w}}, p_{\mathrm{w}}) \in X,$$

$$\mathfrak{q}(s_{\mathrm{w}}) \in X,$$

$$\int_0^T \{\langle \partial_t(\phi s_\alpha), \varphi\rangle - (\mathbf{u}_\alpha(s_{\mathrm{w}}, p_{\mathrm{w}}), \nabla\varphi) - (q_\alpha, \varphi)\}\mathrm{d}t = 0$$

$$\forall \varphi \in X,\ \alpha \in \{\mathrm{n}, \mathrm{w}\}.$$

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Outline

1. Introduction
2. The Stefan problem
   - Dual norm a posteriori estimate and adaptivity
   - Efficiency
   - Energy error a posteriori estimate
   - Numerical results
3. Two-phase flow
   - A posteriori error estimate
   - Full adaptivity
   - Applications
   - Numerical results
4. References and bibliography

Link energy-type error – dual norm of the residual

Dual norm of the residual on the time interval I_n

$$\mathcal{J}^n_{s_{\mathrm{w}},p_{\mathrm{w}}}(s_{\mathrm{w},h\tau},p_{\mathrm{w},h\tau}) := \left\{ \sum_{\alpha\in\{\mathrm{n,w}\}} \left\{ \sup_{\varphi\in X_n,\,\|\varphi\|_{X_n}=1} \int_{I_n} \{\langle \partial_t(\phi s_\alpha) - \partial_t(\phi s_{\alpha,h\tau}),\varphi\rangle - (\mathbf{u}_\alpha(s_{\mathrm{w}},p_{\mathrm{w}}) - \mathbf{u}_\alpha(s_{\mathrm{w},h\tau},p_{\mathrm{w},h\tau}),\nabla\varphi)\}\mathrm{d}t \right\}^2 \right\}^{\frac{1}{2}}$$

Theorem (Link energy-type error – dual norm of the residual)

Let $(s_{\mathrm{w}},p_{\mathrm{w}})$ be the weak solution. Let $(s_{\mathrm{w},h\tau},p_{\mathrm{w},h\tau})$ be arbitrary such that $\mathfrak{p}(s_{\mathrm{w},h\tau},p_{\mathrm{w},h\tau}) \in X$ and $\mathfrak{q}(s_{\mathrm{w},h\tau}) \in X$ (and satisfying the initial and boundary conditions for simplicity). Then

$$\begin{aligned}&\|s_{\mathrm{w}} - s_{\mathrm{w},h\tau}\|_{L^2((0,T);H^{-1}(\Omega))} + \|\mathfrak{q}(s_{\mathrm{w}}) - \mathfrak{q}(s_{\mathrm{w},h\tau})\|_{L^2(\Omega\times(0,T))}\\ &+ \|\mathfrak{p}(s_{\mathrm{w}},p_{\mathrm{w}}) - \mathfrak{p}(s_{\mathrm{w},h\tau},p_{\mathrm{w},h\tau})\|_{L^2((0,T);H^1_0(\Omega))}\\ &\le C\left\{\sum_{n=1}^N \mathcal{J}^n_{s_{\mathrm{w}},p_{\mathrm{w}}}(s_{\mathrm{w},h\tau},p_{\mathrm{w},h\tau})^2\right\}^{\frac{1}{2}}\end{aligned}$$

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&+ \|\mathfrak{p}(s_{\rm w},p_{\rm w}) - \mathfrak{p}(s_{{\rm w},h\tau},p_{{\rm w},h\tau})\|_{L^2((0,T);H^1_0(\Omega))} \\
&\le C\left\{ \sum_{n=1}^N \mathcal{J}^n_{s_{\rm w},p_{\rm w}}(s_{{\rm w},h\tau},p_{{\rm w},h\tau})^2 \right\}^{\frac{1}{2}}
\end{aligned}$$

Outline

Distinguishing the error components

Theorem (Distinguishing the error components)

Consider a vertex-centered finite volume / backward Euler approximation and Newton linearization. Let

- n *be the* time *step,*
- k *be the* linearization *step,*
- i *be the* algebraic solver *step,*

with the approximations $(s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w},h\tau}^{n,k,i})$*. Then*

$$\mathcal{J}^n_{s_{\mathrm{w}},p_{\mathrm{w}}}(s_{\mathrm{w},h\tau}^{n,k,i}, p_{\mathrm{w},h\tau}^{n,k,i}) \leq \eta_{\mathrm{sp}}^{n,k,i} + \eta_{\mathrm{tm}}^{n,k,i} + \eta_{\mathrm{lin}}^{n,k,i} + \eta_{\mathrm{alg}}^{n,k,i}.$$

Error components

- $\eta_{\mathrm{sp}}^{n,k,i}$: spatial discretization
- $\eta_{\mathrm{tm}}^{n,k,i}$: temporal discretization
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Full adaptivity

Full adaptivity

- only a **necessary number** of **algebraic/linearization solver iterations**
- adaptive **regularization**, model adaptation, adaptive choice of the scheme parameters
- **"online decisions"**: algebraic step / linearization step / space mesh refinement / time step modification
- important computational savings
- guaranteed and robust a posteriori error estimates

Not treated for the moment

- convergence and optimality

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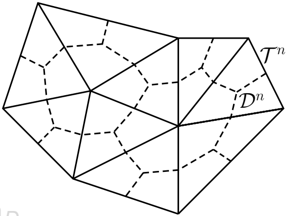
Outline

Iteratively coupled vertex-centered finite volumes

Vertex-centered finite volumes

- simplicial meshes $\mathcal{T}^n$, dual meshes $\mathcal{D}^n$
- saturations & pressures continuous and pw affine on $\mathcal{T}^n$



Implicit pressure equation on step k

$$-\big((\lambda_{\rm w}(s_{{\rm w},h}^{n,k-1}) + \lambda_{\rm n}(s_{{\rm w},h}^{n,k-1}))\underline{\mathbf{K}}\nabla p_{{\rm w},h}^{n,k}\cdot\mathbf{n}_D + \lambda_{\rm n}(s_{{\rm w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\overline{p}_{\rm c}(s_{{\rm w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\big)_{\partial D\setminus\partial\Omega} = 0 \quad \forall D \in \mathcal{D}^{{\rm int},n}$$

Explicit saturation equation on step k

$$s_{{\rm w},D}^{n,k} := \frac{\tau^n}{\phi|D|}(\lambda_{\rm w}(s_{{\rm w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{{\rm w},h}^{n,k}\cdot\mathbf{n}_D, 1)_{\partial D\setminus\partial\Omega} + s_{{\rm w},D}^{n-1} \quad \forall D \in \mathcal{D}^{{\rm int},n}$$

Iteratively coupled vertex-centered finite volumes

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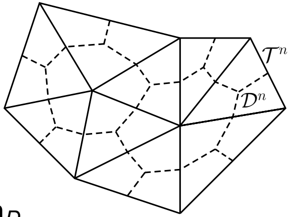

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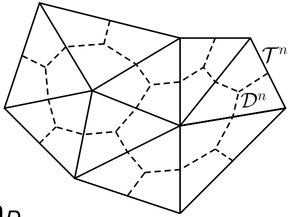

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Linearization and algebraic solution

Iterative coupling step k and algebraic step i

$$-\Big(\big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k-1}) + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla \overline{p}_{\mathrm{c}}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_{\partial D\setminus\partial\Omega} = -R_{\mathrm{t},D}^{n,k,i} \quad \forall D \in \mathcal{D}^{\mathrm{int},n}$$

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Flux reconstructions

Total velocities reconstructions

$$
\begin{aligned}
(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D, 1)_e &:= -\Big(\big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k,i}) + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k,i})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D \\
&\quad + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\bar{p}_{\mathrm{c}}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D, 1\Big)_e, \\
((\mathbf{d}_{\mathrm{t},h}^{n,k,i} + \mathbf{l}_{\mathrm{t},h}^{n,k,i})\cdot\mathbf{n}_D, 1)_e &:= -\Big(\big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k-1}) + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D \\
&\quad + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\bar{p}_{\mathrm{c}}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_e, \\
\mathbf{a}_{\mathrm{t},h}^{n,k,i} &:= \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu} + \mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu} - (\mathbf{d}_{\mathrm{t},h}^{n,k,i} + \mathbf{l}_{\mathrm{t},h}^{n,k,i})
\end{aligned}
$$

Phases velocities reconstructions

$$
\begin{aligned}
(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1)_e &:= -\big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_e, \\
((\mathbf{d}_{\mathrm{w},h}^{n,k,i} + \mathbf{l}_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D, 1)_e &:= -\big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\big)_e, \\
\mathbf{a}_{\mathrm{w},h}^{n,k,i} &:= 0
\end{aligned}
$$

$$
\mathbf{d}_{\mathrm{n},h}^{n,k,i} := \mathbf{d}_{\mathrm{t},h}^{n,k,i} - \mathbf{d}_{\mathrm{w},h}^{n,k,i}, \ \mathbf{l}_{\mathrm{n},h}^{n,k,i} := \mathbf{l}_{\mathrm{t},h}^{n,k,i} - \mathbf{l}_{\mathrm{w},h}^{n,k,i}, \ \mathbf{a}_{\mathrm{n},h}^{n,k,i} := \mathbf{a}_{\mathrm{t},h}^{n,k,i} - \mathbf{a}_{\mathrm{w},h}^{n,k,i}
$$

Equilibrated fluxes

$$
\sigma_{\cdot,h}^{n,k,i} := \mathbf{d}_{\cdot,h}^{n,k,i} + \mathbf{l}_{\cdot,h}^{n,k,i} + \mathbf{a}_{\cdot,h}^{n,k,i}
$$

Flux reconstructions

Total velocities reconstructions

$$(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D, 1)_e := -\Big(\big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k,i}) + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k,i})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\bar{p}_{\mathrm{c}}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D, 1\Big)_e,$$

$$((\mathbf{d}_{\mathrm{t},h}^{n,k,i} + \mathbf{l}_{\mathrm{t},h}^{n,k,i})\cdot\mathbf{n}_D, 1)_e := -\Big(\big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k-1}) + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k-1})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \lambda_{\mathrm{n}}(s_{\mathrm{w},h}^{n,k-1})\underline{\mathbf{K}}\nabla\bar{p}_{\mathrm{c}}(s_{\mathrm{w},h}^{n,k-1})\cdot\mathbf{n}_D, 1\Big)_e,$$

$$\mathbf{a}_{\mathrm{t},h}^{n,k,i} := \mathbf{d}_{\mathrm{t},h}^{n,k,i+\nu} + \mathbf{l}_{\mathrm{t},h}^{n,k,i+\nu} - (\mathbf{d}_{\mathrm{t},h}^{n,k,i} + \mathbf{l}_{\mathrm{t},h}^{n,k,i})$$

Phases velocities reconstructions

$$(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1)_e := -\Big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D, 1\Big)_e,$$

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$$\mathbf{a}_{\mathrm{w},h}^{n,k,i} := 0$$

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Equilibrated fluxes

$$\sigma_{\cdot,h}^{n,k,i} := \mathbf{d}_{\cdot,h}^{n,k,i} + \mathbf{l}_{\cdot,h}^{n,k,i} + \mathbf{a}_{\cdot,h}^{n,k,i}$$

Flux reconstructions

Total velocities reconstructions

$$(\mathbf{d}_{\mathrm{t},h}^{n,k,i}\cdot\mathbf{n}_D,1)_e := -\Big(\big(\lambda_\mathrm{w}(s_{\mathrm{w},h}^{n,k,i})+\lambda_\mathrm{n}(s_{\mathrm{w},h}^{n,k,i})\big)\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D + \lambda_\mathrm{n}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla\overline{p}_\mathrm{c}(s_{\mathrm{w},h}^{n,k,i})\cdot\mathbf{n}_D,1\Big)_e,$$

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Phases velocities reconstructions

$$(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1)_e := -\Big(\lambda_\mathrm{w}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\Big)_e,$$

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Equilibrated fluxes

$$\sigma_{\cdot,h}^{n,k,i} := \mathbf{d}_{\cdot,h}^{n,k,i}+\mathbf{l}_{\cdot,h}^{n,k,i}+\mathbf{a}_{\cdot,h}^{n,k,i}$$

Flux reconstructions

Total velocities reconstructions

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Phases velocities reconstructions

$$(\mathbf{d}_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1)_e := -\Big(\lambda_{\mathrm{w}}(s_{\mathrm{w},h}^{n,k,i})\underline{\mathbf{K}}\nabla p_{\mathrm{w},h}^{n,k,i}\cdot\mathbf{n}_D,1\Big)_e,$$

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Outline

Model problem

Horizontal flow

$$\partial_t(\phi s_\alpha) - \nabla\cdot\left(\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\underline{\mathbf{K}}\nabla p_\alpha\right) = 0,$$

$$s_\mathrm{n} + s_\mathrm{w} = 1,$$

$$p_\mathrm{n} - p_\mathrm{w} = p_\mathrm{c}(s_\mathrm{w})$$

Brooks–Corey model

- relative permeabilities

$$k_{\mathrm{r,w}}(s_\mathrm{w}) = s_\mathrm{e}^4, \quad k_{\mathrm{r,n}}(s_\mathrm{w}) = (1-s_\mathrm{e})^2(1-s_\mathrm{e}^2)$$

- capillary pressure

$$p_\mathrm{c}(s_\mathrm{w}) = p_\mathrm{d} s_\mathrm{e}^{-\frac{1}{2}}$$

-

$$s_\mathrm{e} := \frac{s_\mathrm{w} - s_\mathrm{rw}}{1 - s_\mathrm{rw} - s_\mathrm{rn}}$$

Model problem

Horizontal flow

$$\partial_t(\phi s_\alpha) - \nabla\cdot\left(\frac{k_{\mathrm{r},\alpha}(s_\mathrm{w})}{\mu_\alpha}\underline{\mathbf{K}}\nabla p_\alpha\right) = 0,$$
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-

$$s_\mathrm{e} := \frac{s_\mathrm{w} - s_\mathrm{rw}}{1 - s_\mathrm{rw} - s_\mathrm{rn}}$$

Data from Klieber & Rivière (2006)

Data

$$\Omega = (0,300)\text{m} \times (0,300)\text{m}, \quad T = 4{\cdot}10^6\text{s},$$
$$\phi = 0.2, \quad \underline{\mathbf{K}} = 10^{-11}\underline{\mathbf{I}}\,\text{m}^2,$$
$$\mu_\text{w} = 5{\cdot}10^{-4}\text{kg}\,\text{m}^{-1}\text{s}^{-1}, \quad \mu_\text{n} = 2{\cdot}10^{-3}\text{kg}\,\text{m}^{-1}\text{s}^{-1},$$
$$s_\text{rw} = s_\text{rn} = 0, \quad p_\text{d} = 5{\cdot}10^3\text{kg}\,\text{m}^{-1}\text{s}^{-2}$$

Initial condition ($\widetilde{K}$ 18m × 18m lower left corner block)

$$s_\text{w}^0 = 0.2 \text{ on } K \in \mathcal{T}_h,\ K \notin \widetilde{K},$$
$$s_\text{w}^0 = 0.95 \text{ on } K \in \mathcal{T}_h,\ K \in \widetilde{K}$$

Boundary conditions ($\widehat{K}$ 18m × 18m upper right corner block)

- no flow Neumann boundary conditions everywhere except of $\partial\widetilde{K} \cap \partial\Omega$ and $\partial\widehat{K} \cap \partial\Omega$
- $\widetilde{K}$ – injection well: $s_\text{w} = 0.95$, $p_\text{w} = 3.45{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$
- $\widehat{K}$ – production well: $s_\text{w} = 0.2$, $p_\text{w} = 2.41{\cdot}10^6\text{kg}\,\text{m}^{-1}\text{s}^{-2}$

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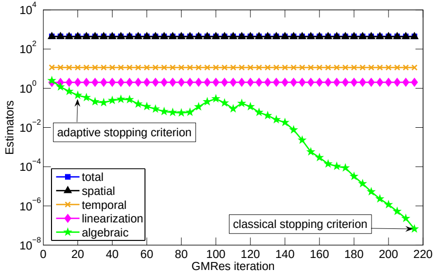
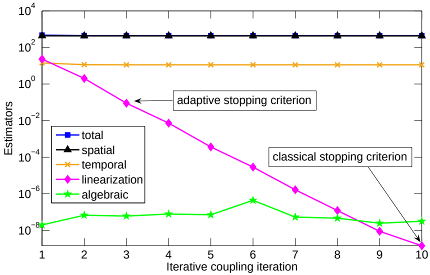
Estimators and stopping criteria

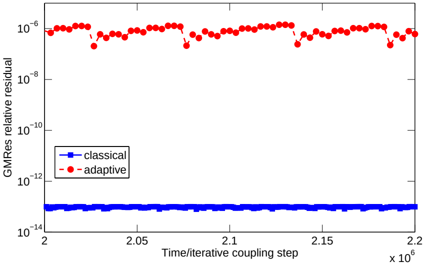


Estimators in function of GMRes iterations

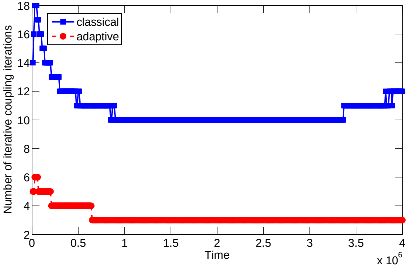


Estimators in function of iterative coupling iterations

GMRes relative residual/iterative coupling iterations

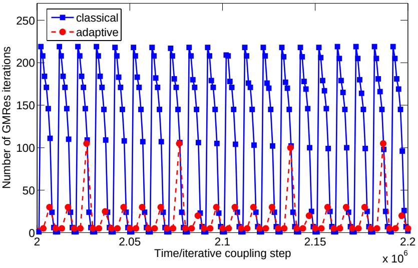


GMRes relative residual

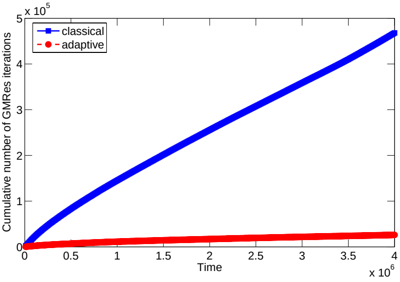


Iterative coupling iterations

GMRes iterations

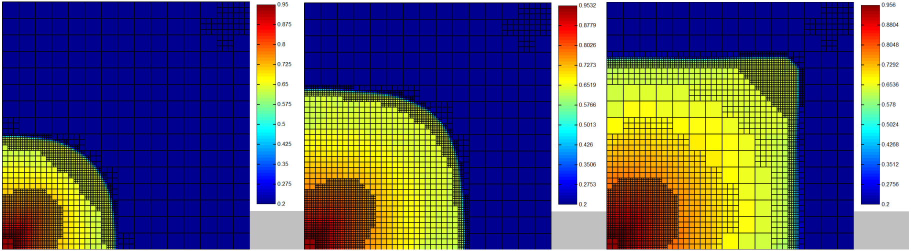


Per time and iterative coupling step



Cumulated

Space/time/nonlinear solver/linear solver adaptivity



Fully adaptive computation

Outline

Previous results

Nonlinear unsteady problems

- Eriksson and Johnson (1995), $L^\infty(0, T; L^2(\Omega))$ estimates exploiting stability of the adjoint problem
- Gallimard, Ladevèze, Pelle (1997), const. rel. estimates
- Verfürth (1998), framework for energy control, efficiency
- Ohlberger (2001), non-energy estimates, hyperbolic limit
- Akrivis, Makridakis, and Nochetto (2006), higher-order temporal discretizations

Degenerate parabolic problems

- Nochetto, Schmidt, Verdi (2000), Stefan problem
- Dolejší, Ern, Vohralík (2013), Richards problem (advection-dominated), robustness in a space–time dual mesh-dependent norm

Two-phase flows

- Chen and Ewing (2003), mesh adaptivity

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Thank you for your attention!