

# A posteriori error control for the heat equation

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Lecture II/IV

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# Outline

- 1 Introduction
- 2 Setting
  - Heat equation
  - Equivalence of error and residual
  - Discrete setting
- 3 A posteriori error estimates and their efficiency
  - Potential and flux reconstructions
  - A posteriori error estimates
  - Balancing the spatial and temporal error components
  - Efficiency
- 4 Applications
- 5 Numerical results
- 6 References and bibliography

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# An optimal a posteriori estimate for evolutive problems

## Guaranteed upper bound

- $\|u - u_{h\tau}\|_{?, \Omega \times (0, T)}^2 \leq \sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2$
- no undetermined constant: **error control**

## Local efficiency

- $\eta_K^n(u_{h\tau}) \leq C_{\text{eff}} \|u - u_{h\tau}\|_{\mathcal{T}_K \times (t^{n-1}, t^n)}$
- **optimal space–time mesh refinement**
- **will only be global in space**

## Asymptotic exactness

- $\sum_{n=1}^N \sum_{K \in \mathcal{T}^n} \eta_K^n(u_{h\tau})^2 / \|u - u_{h\tau}\|_{?, \Omega \times (0, T)}^2 \searrow 1$
- **overestimation factor goes to one** with meshes size

## Robustness

- $C_{\text{eff}}$  indep. of data, domain, **final time**, meshes, or solution

## Small evaluation cost

- estimators can be evaluated **locally in space** and **in time**

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# The heat equation

## The heat equation

$$\partial_t u - \Delta u = f \quad \text{a.e. in } Q := \Omega \times (0, T),$$

$$u = 0 \quad \text{a.e. on } \partial\Omega \times (0, T),$$

$$u(\cdot, 0) = u_0 \quad \text{a.e. in } \Omega$$

## Assumptions

- $\Omega \subset \mathbb{R}^d$ ,  $d \geq 2$ , polygonal domain
- $T > 0$ , final simulation time
- $f \in L^2(Q)$ ,  $u_0 \in L^2(\Omega)$

## Spaces

- $X := L^2(0, T; H_0^1(\Omega))$ ,  $X' = L^2(0, T, H^{-1}(\Omega))$
- $Y := \{y \in X; \partial_t y \in X'\}$

## Weak solution

Find  $u \in Y$  with  $u(\cdot, 0) = u_0$  such that, for a.e.  $t \in (0, T)$ ,

$$\langle \partial_t u, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t) + (\nabla u, \nabla v)(t) = (f, v)(t) \quad \forall v \in H_0^1(\Omega).$$

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# Error and residual in the steady case

## Laplace equation

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

## Weak formulation

Find  $u \in H_0^1(\Omega)$  such that

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$$

## Residual of $u_h \in H_0^1(\Omega)$

- $\mathcal{R}(u_h) \in H^{-1}(\Omega)$ , the **misfit** of  $u_h$  in the **weak formulation**:

$$\langle \mathcal{R}(u_h), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} := (f, \varphi) - (\nabla u_h, \nabla \varphi) \quad \varphi \in H_0^1(\Omega)$$

- dual norm of the residual

$$\|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)} := \sup_{\varphi \in H_0^1(\Omega); \|\nabla \varphi\|=1} \langle \mathcal{R}(u_h), \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$$

**Energy error** is the dual norm of the residual

$$\|\nabla(u - u_h)\| = \sup_{\varphi \in H_0^1(\Omega); \|\nabla \varphi\|=1} (\nabla(u - u_h), \nabla \varphi) = \|\mathcal{R}(u_h)\|_{H^{-1}(\Omega)}$$

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# Intrinsic norms for the heat equation

## Norms

- **energy norm:**  $\|v\|_X^2 := \int_0^T \|\nabla v\|^2(t) dt, v \in X$
- **augmented norm:** (Verfürth (2003))  $\|v\|_Y := \|v\|_X + \|\partial_t v\|_{X'}$ ,  
 $\|\partial_t v\|_{X'} = \left\{ \int_0^T \|\partial_t v\|_{H^{-1}(\Omega)}^2(t) dt \right\}^{1/2}, v \in Y$
- **link:**  $\|\partial_t v\|_{X'} = \sup_{\varphi \in X, \|\varphi\|_X=1} \int_0^T \langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t) dt$
- $\|v\|_{X^*} := \sup_{\varphi \in X, \|\varphi\|_X=1} \int_0^T \{ \langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + (\nabla v, \nabla \varphi) \}(t) dt$

Theorem (Norm equivalence, cf. Verfürth (2003))

Let  $v \in Y$ . Then

$$\|v\|_Y \leq 3\|v\|_{X^*} + 2^{1/2}\|v(\cdot, 0)\|,$$

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# Norm equivalence I

## Proof.

- straightforwardly:

$$\begin{aligned}
 \|v\|_{X^*} &= \sup_{\varphi \in X, \|\varphi\|_X=1} \int_0^T \{ \langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + (\nabla v, \nabla \varphi) \}(t) dt \\
 &\leq \sup_{\varphi \in X, \|\varphi\|_X=1} \int_0^T \langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t) dt \\
 &\quad + \sup_{\varphi \in X, \|\varphi\|_X=1} \int_0^T (\nabla v, \nabla \varphi)(t) dt \\
 &= \|\partial_t v\|_{X'} + \|v\|_X = \|v\|_Y
 \end{aligned}$$

- classically:

$$\frac{1}{2} \|v(\cdot, T)\|^2 = \frac{1}{2} \|v(\cdot, 0)\|^2 + \int_0^T \langle \partial_t v, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}(t) dt$$

- thus  $\|v\|_X^2 \leq \frac{1}{2} \|v(\cdot, T)\|^2 + \|v\|_X^2 =$

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- classically:

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- thus  $\|v\|_X^2 \leq \frac{1}{2} \|v(\cdot, T)\|^2 + \|v\|_X^2 =$

$$\frac{1}{2} \|v(\cdot, 0)\|^2 + \int_0^T \{ \langle \partial_t v, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + (\nabla v, \nabla v) \}(t) dt$$

# Norm equivalence II

## Proof.

- passing to supremum:

$$\|v\|_X^2 \leq \sup_{\varphi \in X, \|\varphi\|_X=1} \int_0^T \{ \langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + (\nabla v, \nabla \varphi) \} (t) dt \|v\|_X + \frac{1}{2} \|v(\cdot, 0)\|^2$$

- $x^2 \leq ax + b^2 \Rightarrow x \leq a + b, a, b \geq 0$ :

$$\|v\|_X \leq \|v\|_{X^*} + 2^{-1/2} \|v(\cdot, 0)\|$$

- for any  $\varphi \in X$ :

$$\int_0^T \langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} (t) dt = \int_0^T \{ \langle \partial_t v, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + (\nabla v, \nabla \varphi) - (\nabla v, \nabla \varphi) \} (t) dt$$

- thus  $\|\partial_t v\|_{X'} \leq \|v\|_{X^*} + \|v\|_X$
- finally  $\|v\|_Y = \|\partial_t v\|_{X'} + \|v\|_X \leq 3\|v\|_{X^*} + 2^{1/2} \|v(\cdot, 0)\|$

# Error and residual for the heat equation

## Residual and its dual norm for the heat equation

For  $u_{h\tau} \in Y$ , the residual  $\mathcal{R}(u_{h\tau}) \in X'$ :

$$\langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X', X} := \int_0^T \{ (f, \varphi) - \langle \partial_t u_{h\tau}, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - (\nabla u_{h\tau}, \nabla \varphi) \} (t) dt \quad \varphi \in X,$$

$$\|\mathcal{R}(u_{h\tau})\|_{X'} := \sup_{\varphi \in X, \|\varphi\|_X=1} \langle \mathcal{R}(u_{h\tau}), \varphi \rangle_{X', X} = \|u - u_{h\tau}\|_{X'}.$$

Corollary (Equivalence of the  $\|\cdot\|_Y$  error and of the residual)

Let  $u$  be the weak solution. Let  $u_{h\tau} \in Y$  be arbitrary. Then

$$\|u - u_{h\tau}\|_Y \leq 3 \|\mathcal{R}(u_{h\tau})\|_{X'} + 2^{1/2} \|(u - u_{h\tau})(\cdot, 0)\|,$$

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  - **Discrete setting**
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# Time-dependent meshes and discrete solutions

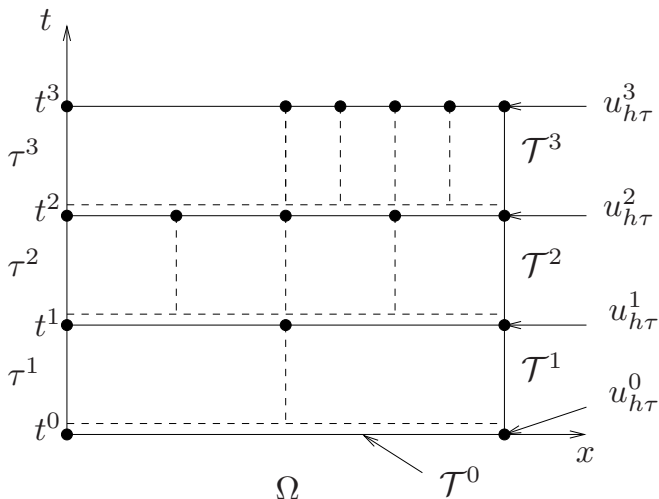
## Approximate solutions

- discrete times  $\{t^n\}_{0 \leq n \leq N}$ ,  $t^0 = 0$  and  $t^N = T$
- $I_n := (t^{n-1}, t^n]$ ,  $\tau^n := t^n - t^{n-1}$ ,  $1 \leq n \leq N$
- a different simplicial mesh  $\mathcal{T}^n$  on all  $0 \leq n \leq N$
- $u_h^n \in V_h^n$ ,  $0 \leq n \leq N$ , piecewise polynomial for simplicity
- $u_h^n$  possibly nonconforming,  $V_h^n \notin H_0^1(\Omega)$
- $u_{h\tau} : Q \rightarrow \mathbb{R}$  continuous and piecewise affine in time

$$u_{h\tau}(\cdot, t) := (1 - \varrho(t))u_h^{n-1} + \varrho(t)u_h^n, \quad \varrho(t) = \frac{1}{\tau^n}(t - t^{n-1})$$

- tailored to the backward Euler discretization

# Time-dependent meshes and discrete solutions



Time-dependent meshes and discrete solutions

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# Potential and flux reconstructions

## General form

- potential reconstruction  $s_{h\tau}$  is **continuous** and **piecewise affine in time** with  $s_h^n \in H_0^1(\Omega)$  for all  $0 \leq n \leq N$
- flux reconstruction  $\sigma_{h\tau}$  is **piecewise constant in time** with  $\sigma_h^n := \sigma_{h\tau}|_{I_n} \in \mathbf{H}(\text{div}, \Omega)$  for all  $1 \leq n \leq N$
- $\mathcal{T}^{n,n+1}$  is a common refinement of  $\mathcal{T}^n$  and  $\mathcal{T}^{n+1}$
- $\tilde{f}^n := \int_{I_n} f(\cdot, t) dt / \tau^n$  is piecewise constant in time

## Assumption A (Potential and flux reconstructions)

Reconstructions  $s_h^n$  **preserve mean values** of  $u_h^n$  on  $\mathcal{T}^{n,n+1}$

$$(s_h^n, 1)_K = (u_h^n, 1)_K \quad \forall K \in \mathcal{T}^{n,n+1}, \forall 1 \leq n \leq N.$$

Reconstructions  $\sigma_h^n$  satisfy a **local conservation property**

$$(\tilde{f}^n - \partial_t u_{h\tau}|_{I_n} - \nabla \cdot \sigma_h^n, 1)_K = 0 \quad \forall K \in \mathcal{T}^n, \forall 1 \leq n \leq N.$$

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# Potential and flux reconstructions

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Reconstructions  $\sigma_h^n$  satisfy a **local conservation property**

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# A posteriori error estimate

## Theorem (A posteriori error estimate)

Let

- $u$  be the weak solution
- $u_{h\tau}$  given by  $u_h^n \in V_h^n$ ,  $0 \leq n \leq N$ , be *arbitrary*
- the potential reconstruction  $s_{h\tau}$  and the flux reconstruction  $\sigma_{h\tau}$  satisfy *Assumption A*.

Then

$$\|u - u_{h\tau}\|_Y \leq 3 \left\{ \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}^n} (\eta_{E,K}^n + \eta_{CR,K}^n(t))^2 dt \right\}^{1/2} \\ + \left\{ \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}^n} (\eta_{NC1,K}^n)^2(t) dt \right\}^{1/2} \\ + \left\{ \sum_{n=1}^N \tau^n \sum_{K \in \mathcal{T}^n} (\eta_{NC2,K}^n)^2 \right\}^{1/2} + \eta_{IC} + 3\|f - \tilde{f}\|_{X'}.$$

# Estimators

## Estimators

- *constitutive relation estimator*

- $\eta_{\text{CR},K}^n(t) := \|\nabla \mathbf{s}_{h\tau}(t) + \boldsymbol{\sigma}_h^n\|_K, \quad t \in I_n$
- evaluates that  $-\nabla \mathbf{s}_h^n \notin \mathbf{H}(\text{div}, \Omega)$  & temporal error

- *equilibrium estimator*

- $\eta_{\text{E},K}^n := \frac{h_K}{\pi} \|\tilde{f}^n - \partial_t \mathbf{s}_{h\tau}|_{I_n} - \nabla \cdot \boldsymbol{\sigma}_h^n\|_K$
- equilibrium evaluated for  $\partial_t \mathbf{s}_{h\tau}|_{I_n}$

- *nonconformity (constraint) estimators*

- $\eta_{\text{NC1},K}^n(t) := \|\nabla(\mathbf{s}_{h\tau} - \mathbf{u}_{h\tau})(t)\|_K, \quad t \in I_n$
- $\eta_{\text{NC2},K}^n := \frac{h_K}{\pi} \|\partial_t(\mathbf{s}_{h\tau} - \mathbf{u}_{h\tau})|_{I_n}\|_K$
- evaluate the fact that  $\mathbf{u}_h^n \notin H_0^1(\Omega)$

- *initial condition estimator*

- $\eta_{\text{IC}} := 2^{1/2} \|\mathbf{s}_h^0 - \mathbf{u}^0\|$

- *data oscillation estimator*

- $\|f - \tilde{f}\|_{X'}$

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# Proof

## Proof.

- by the triangle inequality and the **residual equivalence**:

$$\|u - u_{h_T}\|_Y \leq \|s_{h_T} - u_{h_T}\|_Y + 3\|\mathcal{R}(s_{h_T})\|_{X'} + 2^{1/2}\|s_{h_T}(\cdot, 0) - u_0\|$$

- $\|\mathcal{R}(s_{h_T})\|_{X'}$ ;  $(\nabla \cdot \sigma_{h_T}, \varphi) + (\sigma_{h_T}, \nabla \varphi) = 0$  & loc. cons. of  $\sigma_h^n$ :

$$\begin{aligned} & \langle \mathcal{R}(s_{h_T}), \varphi \rangle_{X', X} \\ &= \int_0^T \{ (f - \partial_t s_{h_T} - \nabla \cdot \sigma_{h_T}, \varphi) - (\nabla s_{h_T} + \sigma_{h_T}, \nabla \varphi) \} (t) dt \\ &= \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}^n} \{ (-\partial_t s_{h_T} - \nabla \cdot \sigma_h^n, \varphi)_K - (\nabla s_{h_T} + \sigma_h^n, \nabla \varphi)_K \} (t) dt \\ &\leq \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}^n} (\eta_{E,K}^n + \eta_{CR,K}^n) \|\nabla \varphi\|_K(t) dt + \|f - \tilde{f}\|_{X'} \|\varphi\|_X \end{aligned}$$

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$$\|u - u_{h\tau}\|_Y \leq \|s_{h\tau} - u_{h\tau}\|_Y + 3\|\mathcal{R}(s_{h\tau})\|_{X'} + 2^{1/2}\|s_{h\tau}(\cdot, 0) - u_0\|$$

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$$\begin{aligned} & \langle \mathcal{R}(s_{h\tau}), \varphi \rangle_{X', X} \\ &= \int_0^T \{ (f - \partial_t s_{h\tau} - \nabla \cdot \sigma_{h\tau}, \varphi) - (\nabla s_{h\tau} + \sigma_{h\tau}, \nabla \varphi) \} (t) dt \\ &= \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}^n} \{ (-\partial_t s_{h\tau} - \nabla \cdot \sigma_h^n, \varphi - \varphi_K)_K - (\nabla s_{h\tau} + \sigma_h^n, \nabla \varphi)_K \} (t) dt \\ &\leq \sum_{n=1}^N \int_{I_n} \sum_{K \in \mathcal{T}^n} (\eta_{E,K}^n + \eta_{CR,K}^n) \|\nabla \varphi\|_K(t) dt + \|f - \tilde{f}\|_{X'} \|\varphi\|_X \end{aligned}$$

# Proof

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- by the triangle inequality and the **residual equivalence**:

$$\|u - u_{h\tau}\|_Y \leq \|s_{h\tau} - u_{h\tau}\|_Y + 3\|\mathcal{R}(s_{h\tau})\|_{X'} + 2^{1/2}\|s_{h\tau}(\cdot, 0) - u_0\|$$

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- $\|s_{h\tau} - u_{h\tau}\|_X$ : part of the estimator
- $\|\partial_t(s_{h\tau} - u_{h\tau})\|_{X'}$ : same mean values of  $s_h^n$  and  $u_h^n$ :

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# Separating the space and time error components

## Separating the space and time error components

- $\int_{I_n} (\eta_{CR,K}^n(t))^2 dt \leq (\eta_{CR,K,1}^n)^2 + (\eta_{CR,K,2}^n)^2,$

$$(\eta_{CR,K,1}^n)^2 := 2\tau^n \|\nabla s_h^n + \sigma_h^n\|_K^2$$

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- temporal estimator  $\eta_{tm}^n$  uses  $\eta_{CR,K,2}^n$
- spatial estimator  $\eta_{sp}^n$  uses  $\eta_{CR,K,1}^n$ ,  $\eta_{E,K}^n$ ,  $\eta_{NC1,K}^n$ , and  $\eta_{NC2,K}^n$

Corollary (Estimate separating the space and time errors)

$$\|u - u_{h\tau}\|_Y \leq \left\{ \sum_{n=1}^N (\eta_{sp}^n)^2 \right\}^{1/2} + \left\{ \sum_{n=1}^N (\eta_{tm}^n)^2 \right\}^{1/2} + \eta_{IC} + 3\|f - \tilde{f}\|_{X'}$$

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# Assumptions for efficiency

## Assumption B (Piecewise polynomials, data, and meshes)

There holds:

- the *meshes*  $\{\mathcal{T}^n\}_{0 \leq n \leq N}$  are *shape-regular* uniformly in  $n$ ;
- the meshes cannot be *refined* or *coarsened too quickly*;
- for nonconforming methods on time-varying meshes,  $(h^n)^2 \lesssim \tau^n$  (mild inverse parabolic CFL on time step).

Local lower bound for the classical residual estimators:

## Assumption C (Approximation property)

For all  $1 \leq n \leq N$  and all  $K \in \mathcal{T}^n$ , there holds

$$\begin{aligned} \|\nabla u_h^n + \sigma_h^n\|_K^2 &\lesssim \sum_{K' \in \mathfrak{I}_K} h_{K'}^2 \|\tilde{f}^n - \partial_t u_{h\tau}|_{I_n} + \Delta u_h^n\|_{K'}^2 \\ &+ \sum_{e \in \mathfrak{E}_K^{\text{int}}} h_e \|\llbracket \nabla u_h^n \rrbracket \cdot \mathbf{n}_e\|_e^2 + \sum_{e \in \mathfrak{E}_K} h_e^{-1} \|\llbracket u_h^n \rrbracket\|_e^2 \end{aligned}$$

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## Theorem (Efficiency)

Let **Assumptions B** and **C** hold. Then, for all  $1 \leq n \leq N$ ,

$$\eta_{\text{sp}}^n + \eta_{\text{tm}}^n \lesssim \|u - u_{h\tau}\|_{Y(I_n)} + \mathcal{J}^n(u_{h\tau}) + \mathcal{E}_f^n.$$

## Notation

- jump seminorm:

$$\mathcal{J}^n(u_{h\tau})^2 := \tau^n \sum_{e \in \mathcal{E}^{n-1}} h_e^{-1} \|[[u_h^{n-1}]]\|_e^2 + \tau^n \sum_{e \in \mathcal{E}^n} h_e^{-1} \|[[u_h^n]]\|_e^2$$

- $(\mathcal{E}_f^n)$  space-time data oscillation term

## Comments

- $\mathcal{J}^n(u_{h\tau}) \lesssim \|u - u_{h\tau}\|_{X(I_n)}$  if the jumps in  $u_{h\tau}$  have zero mean values
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# Applications

## Space discretization

- conforming finite elements
- nonconforming finite elements
- discontinuous Galerkin
- various finite volumes
- mixed finite elements

## Time discretization

- backward Euler

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# Outline

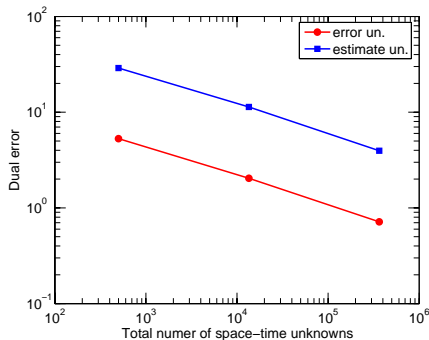
- 1 Introduction
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# Numerical experiment

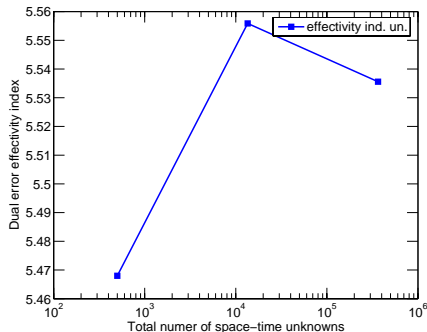
## Numerical experiment

- exact solution  $u = e^{x+y+t-3}$  on square domain  $\Omega = ]0, 3[{}^2$ ,  
 $T = 1.5$  or  $T = 3$
- square meshes:  $10 \times 10$ ,  $30 \times 30$ ,  $90 \times 90$
- time steps: 0.3, 0.1, 0.3333
- vertex-centered finite volumes

# Error, estimate, and effectivity index, $T = 1.5$

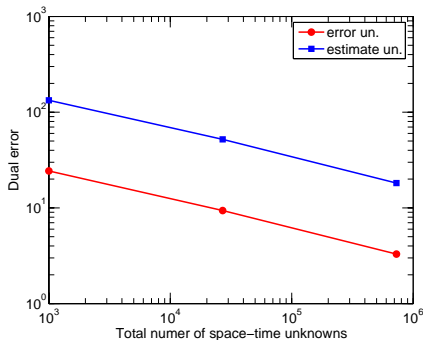


Dual error and estimator

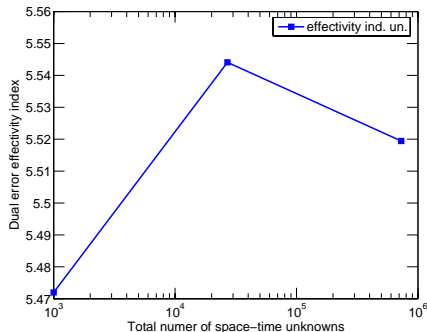


Effectivity index

# Error, estimate, and effectivity index, $T = 3$



Dual error and estimator



Effectivity index

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# Previous results

## Continuous finite elements

- Bieterman and Babuška (1982), introduction
- Eriksson and Johnson (1991), duality techniques
- Picasso (1998), no derefinement allowed
- Babuška, Feistauer, and Šolín (2001), continuous-in-time discretization
- Strouboulis, Babuška, and Datta (2003), guaranteed estimates
- Verfürth (2003), efficiency, **robustness with respect to the final time**
- Makridakis and Nochetto (2003), elliptic reconstruction
- Bergam, Bernardi, and Mghazli (2005), efficiency
- Lakkis and Makridakis (2006), elliptic reconstruction

# Previous results

## Finite volumes

- Ohlberger (2001), non-energy-norm estimates
- Amara, Nadau, and Trujillo (2004), energy-norm estimates

## Discontinuous Galerkin finite elements

- Sun and Wheeler (2005, 2006), non-energy-norm estimates
- Georgoulis and Lakkis (2009)

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# Bibliography

## Bibliography

- ERN A., VOHRALÍK M., A posteriori error estimation based on potential and flux reconstruction for the heat equation, *SIAM J. Numer. Anal.* **48** (2010), 198–223.

**Thank you for your attention!**