Laplace a posteriori error estimates in a unified framework

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Lecture I/IV

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Outline

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What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE ($-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$).
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|\nabla(u - u_h)\| \leq C(u)h^k$. Dependent on u, not computable. Useful in theoretical assessment of convergence.
- A posteriori error estimate: $\|\nabla(u - u_h)\| \leq C\eta(u_h)$. Only uses u_h, computable. Great in practical calculation.

Usual form

- Element indicators $\eta_K(u_h)$, $K \in \mathcal{T}_h$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

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What an a posteriori error estimate should fulfill

Optimal estimate for $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

- **guaranteed upper bound**:

$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}$$

- **local efficiency**:

$$\eta_K(u_h) \leq C_{\mathrm{eff}} \|\nabla(u - u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- **asymptotic exactness**:

$$\frac{\left\{\sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2\right\}^{1/2}}{\|\nabla(u - u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K, polynomial degree of u_h)
- **small evaluation cost** of $\eta_K(u_h)$
- *error components identification*

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Model problem

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$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

- $u \in H_0^1(\Omega)$ (constraint)
- $\sigma := -\nabla u$ (constitutive relation)
- $\nabla{\cdot}\sigma = f$ (equilibrium)
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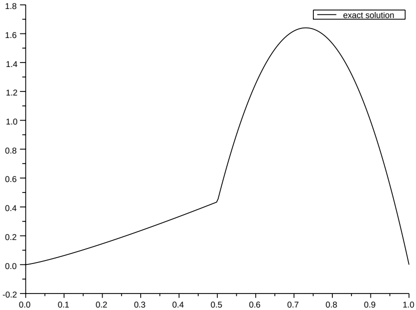
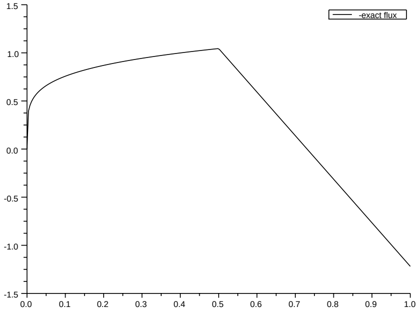
Exact solution and flux

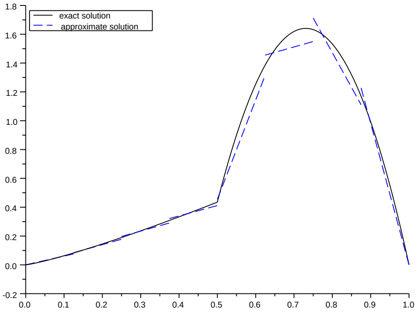


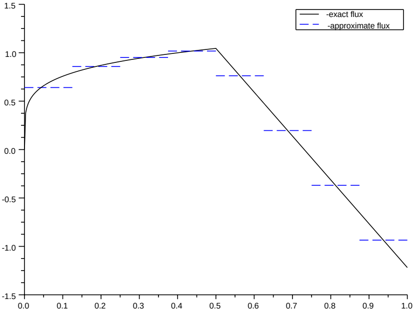
Solution u is continuous



Flux $\boldsymbol{\sigma} := -\underline{\mathbf{K}}\nabla u$ is continuous

Approximate solution and flux

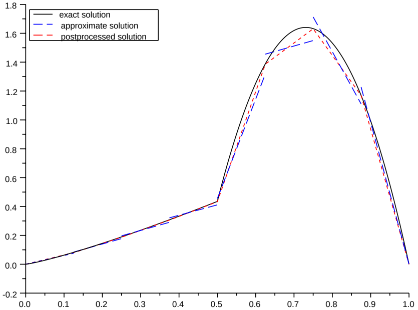




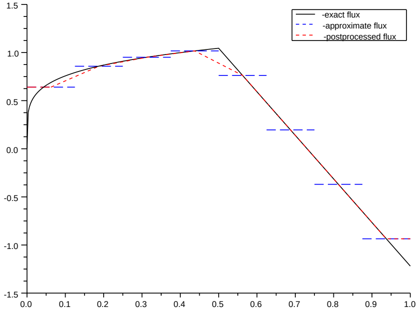
Approximate solution u_h is not necessarily continuous

Approximate flux $-\underline{\mathbf{K}}\nabla u_h$ is not necessarily continuous

Potential and flux reconstructions



Potential reconstruction



Flux reconstruction

A posteriori error estimate

Theorem (A guaranteed a posteriori error estimate, Prager and Synge (1947), Dari, Durán, Padra, and Vampa (1996), Ainsworth (2005), Kim (2007)**)**

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
- $u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega),\ v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h\}$ *be arbitrary;*
- $s_h \in H_0^1(\Omega)$ *and* $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be such that*

$$(\nabla\cdot\boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \text{ for all } K \in \mathcal{T}_h.$$

Then

$$\|\nabla(u - u_h)\|^2 \le \sum_{K\in\mathcal{T}_h} \Bigg(\underbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K}_{\text{equilibrium}} \Bigg)^2$$

$$+ \sum_{K\in\mathcal{T}_h} \underbrace{\|\nabla(u_h - s_h)\|_K^2}_{\text{constraint}}.$$

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A posteriori error estimate

Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s:

$$\|\nabla(u - s)\|^2 = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\nabla(u - \quad), \nabla\varphi)^2}_{\text{dual norm of the residual}}$$

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- nonconformity upper bound:

$$\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\| \le \|\nabla(u_h - s_h)\|$$

- weak solution definition, equilibrated flux, Green theorem:

$$\begin{aligned}(\nabla(u - u_h), \nabla\varphi) &= (f, \varphi) - (\nabla u_h, \nabla\varphi) \\ &= (f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi)\end{aligned}$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

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Global potential and flux reconstructions

Ideally

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h} f} \|\nabla u_h + \mathbf{v}_h\|$$

$$s_h := \arg \min_{v_h \in V_h} \|\nabla(u_h - v_h)\|$$

- $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, $Q_h \subset L^2(\Omega)$, $V_h \subset H_0^1(\Omega)$
- too expensive, **global** minimization problems (the hypercircle method)

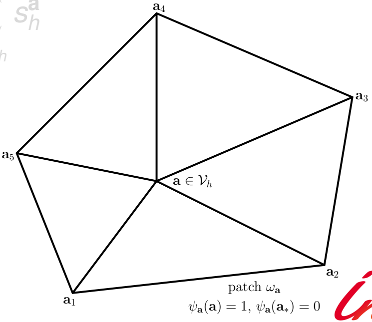
Local potential and flux reconstructions

Partition of unity localization

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

- cut-off by hat basis functions $\psi_{\mathbf{a}} \in \mathbb{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$
- $\boldsymbol{\sigma}_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}, \quad s_h := \sum_{\mathbf{a}\in\mathcal{V}_h} s_h^{\mathbf{a}}$
- local minimizations



Local potential and flux reconstructions

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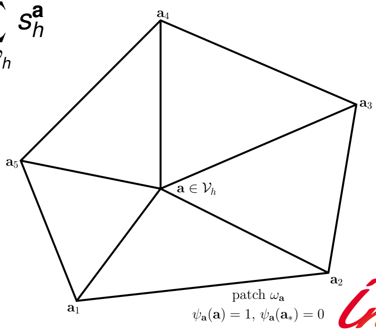
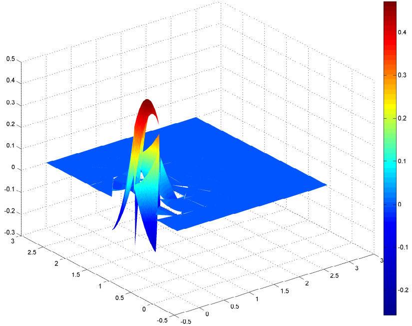
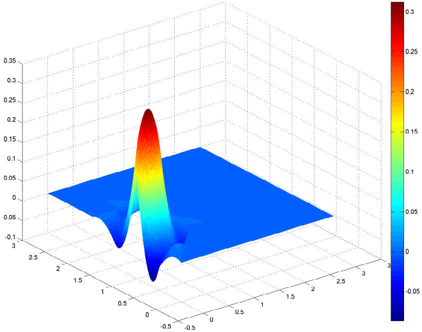


Potential reconstruction

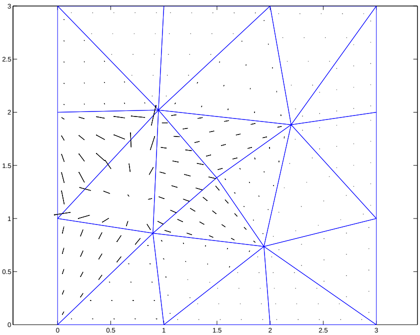


Potential u_h

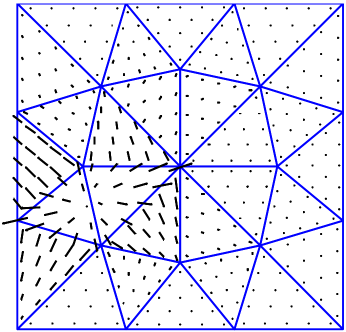


Potential reconstruction s_h

Equilibrated flux reconstruction



Flux $-\nabla u_h$



Flux reconstruction $\boldsymbol{\sigma}_h$

Local equilibrated flux reconstruction

Assumption A (Galerkin orthogonality wrt hat functions)

There holds $u_h \in H^1(\mathcal{T}_h)$ *and*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

$\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$: MFE spaces (hom. Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$)

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$$\Updownarrow$$

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(\boldsymbol{\sigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\\
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Comments

$\mathbf{H}(\mathrm{div}, \Omega)$-conformity

- $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div}, \Omega) \Rightarrow \boldsymbol{\sigma}_h = \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div}, \Omega)$

Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs

$$(\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0$$

- but Assumption A gives

$$0 = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, 1)_{\omega_{\mathbf{a}}}$$

Divergence

- Neumann compatibility condition gives

$$\nabla \cdot \boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a} \in \mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a} \in \mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

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Local potential reconstruction

$V_h^{\mathbf{a}}$: FE space (hom. Dirichlet BC on $\partial\omega_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{V}_h$)

Definition (Construction of s_h, ≈ Carstensen and Merdon (2013))

Let $u_h \in H^1(\mathcal{T}_h)$. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ by solving the **local conforming finite element problem**

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$$\Updownarrow$$

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Remark

- Same problem as for flux, **only** RHS/BC different.

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Outline

Continuous efficiency, flux reconstruction

Theorem (Cont. efficiency Carstensen & Funken (1999), Braess, Pillwein, & Schöberl (2009)**)**

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $r_{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}})$ be such that

$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \nabla v)_{\omega_{\mathbf{a}}} + (\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h, v)_{\omega_{\mathbf{a}}}$$

for all $v \in H^1_*(\omega_{\mathbf{a}})$, where

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$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \le C_{\mathrm{cont,PF}} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Continuous efficiency, flux reconstruction

Theorem (Cont. efficiency Carstensen & Funken (1999), Braess, Pillwein, & Schöberl (2009)**)**

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let $\mathbf{a} \in \mathcal{V}_h$ and let $r_{\mathbf{a}} \in H^1_*(\omega_{\mathbf{a}})$ be such that

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Continuous efficiency, flux reconstruction

Proof.

- dual norm characterization

$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \sup_{v \in H^1_*(\omega_{\mathbf{a}});\, \|\nabla v\|_{\omega_{\mathbf{a}}}=1} (\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}}$$

- weak solution definition, $\psi_{\mathbf{a}} v \in H^1_0(\omega_{\mathbf{a}})$:

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Continuous efficiency, potential reconstruction ($d = 2$)

Assumption B (Weak continuity)

There holds

$$\langle [\![u_h]\!], 1 \rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$$

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Continuous efficiency, potential reconstruction ($d = 2$)

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Proof.

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$$\|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = \sup_{v \in H^1_*(\omega_{\mathbf{a}});\, \|\nabla v\|_{\omega_{\mathbf{a}}}=1} (\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}}$$

- arbitrary $\tilde{u} \in H^1(\omega_{\mathbf{a}})$ with $(\tilde{u}, 1)_{\omega_{\mathbf{a}}} = (u_h, 1)_{\omega_{\mathbf{a}}}$ if $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and $\tilde{u} = 0$ on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ if $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$:

$$(\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} \tilde{u}), \nabla v)_{\omega_{\mathbf{a}}} = 0$$

- definition of $r_{\mathbf{a}}$, insertion of $\tilde{u}$, Cauchy–Schwarz inequality:

$$(\nabla r_{\mathbf{a}}, \nabla v)_{\omega_{\mathbf{a}}} = (\mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}}(\tilde{u} - u_h)), \nabla v)_{\omega_{\mathbf{a}}} \leq \|\nabla(\psi_{\mathbf{a}}(\tilde{u} - u_h))\|_{\omega_{\mathbf{a}}}$$

- **broken** Poincaré–Friedrichs inequality:

$$\|\nabla(\psi_{\mathbf{a}}(\tilde{u} - u_h))\|_{\omega_{\mathbf{a}}} \leq (1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla \psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}) \|\nabla(\tilde{u} - u_h)\|_{\omega_{\mathbf{a}}}$$

- define $\tilde{u} := u - \mathrm{cnst}$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$ and $\tilde{u} := u$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$

Mixed finite elements stability ($d = 2$)

Assumption C (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The **degrees** *of the MFE reconstructions $\boldsymbol{\sigma}_h$ and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.*

Theorem (MFE stability / continuous right inverse of the divergence operator Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012)**)**

Let u be the weak solution and let u_h, f, and the reconstructions satisfy Assumption C. Then there exists a constant $C_{\rm st} > 0$ ***only depending*** *on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that*

$$\|\boldsymbol{\sigma}_h^{\mathbf{a}} + \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\boldsymbol{\sigma}^{\mathbf{a}} + \boldsymbol{\tau}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} = C_{\rm st} \|\nabla r_{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$$

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Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency)

Let u be the weak solution and let Assumptions A, B, and C hold. Then

$$\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \le C_{\mathrm{st}} C_{\mathrm{cont,PF}} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}},$$

$$\|\nabla(u_h - s_h)\|_K \le C_{\mathrm{st}} C_{\mathrm{cont,bPF}} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Remarks

- C_{st} can be bounded by solving the local Neumann problems by conforming FEs: find $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}} \subset H^1_*(\omega_{\mathbf{a}})$ s.t.

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in V_h^{\mathbf{a}};$$

  then $C_{\mathrm{st}} \le \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} / \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$
- $\Rightarrow$ maximal overestimation factor guaranteed

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Outline

Conforming finite elements

Conforming finite elements

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$
- $V_h \subset H_0^1(\Omega)$: $s_h := u_h$, no need for Assumption B

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Discontinuous Galerkin finite elements

Discontinuous Galerkin finite elements

Find $u_h \in V_h$ such that

$$\sum_{K\in\mathcal{T}_h}(\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h}\{\langle\{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle\{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\}$$

$$+\sum_{e\in\mathcal{E}_h}\langle\alpha h_e^{-1}[\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h) \qquad \forall v_h \in V_h.$$

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- Assumption A: take $v_h = \psi_{\mathbf{a}}$ for $\theta = 0$, otherwise:
  - estimates for the discrete gradient

    $$\mathfrak{G}(u_h) := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

  - jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_e)]^2$

    $$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle\{\!\{\mathbf{v}_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_e)]^2$$

  - $\Rightarrow$ modified Galerkin orthogonality

    $$(\mathfrak{G}(u_h), \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

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Discontinuous Galerkin finite elements: Assumption B

Nonsymmetric and incomplete versions

- broken Poincaré–Friedrichs inequality with jumps:

$$\|\nabla(\psi_{\mathbf{a}}(\tilde{u}-u_h))\|_{\omega_{\mathbf{a}}} \leq (1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}) \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}} + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} \left\{ \sum_{e\in\mathcal{E}_h,\, \mathbf{a}\in e} h_e^{-1} \|\Pi_e^0 [\![u_h]\!]\|_e^2 \right\}^{1/2}$$

- include the jump terms in the error and estimators

Symmetric version

- discrete gradient $\mathfrak{G}$ satisfies

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- modified potential reconstruction: local MFE problems with $\tau_h^{\mathbf{a}} := \psi_{\mathbf{a}} \mathrm{R}_{\frac{\pi}{2}} \mathfrak{G}(u_h)$ and $g^{\mathbf{a}} := (\mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}}) \cdot \mathfrak{G}(u_h)$
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Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} (\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla\cdot\mathbf{v}_h) &= 0 \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla\cdot\boldsymbol{\sigma}_h, q_h) &= (f, q_h) \qquad && \forall q_h \in Q_h. \end{aligned}$$

- postprocessed solution $u_h \in V_h$, $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$; $v_h \in V_h$ satisfy

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Outline

Numerics: smooth test case

Model problem

$$-\Delta u = f \quad \text{in } \Omega :=]0,1[^2$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured nested triangular grids
- uniform refinement

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Estimates, errors, and effectivity indices

h	p	$\lVert\nabla(u-u_h)\rVert$	$\lVert u-u_h\rVert_{\mathrm{DG}}$	$\lVert\nabla u_h+\boldsymbol{\sigma}_h\rVert$	$\lVert\nabla(u_h-s_h)\rVert$	η_{osc}	η	η_{DG}	I^{eff}	$I^{\mathrm{eff}}_{\mathrm{DG}}$
$h_0/1$	1	1.21E+00	1.22E+00	1.24E+00	1.07E-01	5.56E-02	1.30E+00	1.31E+00	1.07	1.07
$h_0/2$		6.18E-01	6.22E-01	6.38E-01	5.09E-02	7.02E-03	6.47E-01	6.50E-01	1.05	1.05
		(0.97)	(0.97)	(0.96)	(1.07)	(2.99)	(1.01)	(1.01)		
$h_0/4$		3.12E-01	3.13E-01	3.22E-01	2.43E-02	8.80E-04	3.24E-01	3.25E-01	1.04	1.04
		(0.99)	(0.99)	(0.99)	(1.07)	(3.00)	(1.00)	(1.00)		
$h_0/8$		1.56E-01	1.57E-01	1.61E-01	1.18E-02	1.10E-04	1.62E-01	1.63E-01	1.04	1.04
		(1.00)	(1.00)	(1.00)	(1.05)	(3.00)	(1.00)	(1.00)		
$h_0/1$	2	1.50E-01	1.53E-01	1.49E-01	2.76E-02	5.10E-03	1.56E-01	1.59E-01	1.04	1.04
$h_0/2$		3.85E-02	3.92E-02	3.83E-02	7.99E-03	3.22E-04	3.94E-02	4.01E-02	1.03	1.02
		(1.96)	(1.96)	(1.96)	(1.79)	(3.98)	(1.98)	(1.98)		
$h_0/4$		9.70E-03	9.88E-03	9.68E-03	2.12E-03	2.02E-05	9.93E-03	1.01E-02	1.02	1.02
		(1.99)	(1.99)	(1.98)	(1.92)	(4.00)	(1.99)	(1.99)		
$h_0/8$		2.43E-03	2.48E-03	2.43E-03	5.42E-04	1.26E-06	2.49E-03	2.54E-03	1.02	1.02
		(1.99)	(1.99)	(1.99)	(1.96)	(4.00)	(1.99)	(1.99)		
$h_0/1$	3	1.32E-02	1.34E-02	1.29E-02	2.52E-03	3.58E-04	1.35E-02	1.37E-02	1.03	1.03
$h_0/2$		1.67E-03	1.69E-03	1.65E-03	3.13E-04	1.13E-05	1.70E-03	1.71E-03	1.01	1.01
		(2.98)	(2.98)	(2.97)	(3.01)	(4.99)	(3.00)	(3.00)		
$h_0/4$		2.11E-04	2.13E-04	2.09E-04	3.83E-05	3.53E-07	2.12E-04	2.15E-04	1.01	1.01
		(2.99)	(2.99)	(2.99)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/8$		2.64E-05	2.67E-05	2.61E-05	4.69E-06	1.10E-08	2.66E-05	2.69E-05	1.01	1.01
		(3.00)	(3.00)	(3.00)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/1$	4	9.36E-04	9.54E-04	9.05E-04	2.41E-04	2.12E-05	9.57E-04	9.74E-04	1.02	1.02
$h_0/2$		5.93E-05	6.05E-05	5.77E-05	1.68E-05	3.36E-07	6.04E-05	6.16E-05	1.02	1.02
		(3.98)	(3.98)	(3.97)	(3.84)	(5.98)	(3.99)	(3.98)		
$h_0/4$		3.72E-06	3.80E-06	3.63E-06	1.10E-06	5.31E-09	3.80E-06	3.87E-06	1.02	1.02
		(3.99)	(3.99)	(3.99)	(3.94)	(5.98)	(3.99)	(3.99)		
$h_0/8$		2.33E-07	2.38E-07	2.27E-07	7.02E-08	8.30E-11	2.38E-07	2.43E-07	1.02	1.02
		(4.00)	(4.00)	(4.00)	(3.97)	(6.00)	(4.00)	(3.99)		
$h_0/1$	5	5.41E-05	5.50E-05	5.22E-05	1.38E-05	1.06E-06	5.50E-05	5.58E-05	1.02	1.02
$h_0/2$		1.70E-06	1.72E-06	1.65E-06	4.39E-07	9.35E-09	1.72E-06	1.74E-06	1.01	1.01
		(4.99)	(5.00)	(4.98)	(4.98)	(6.82)	(5.00)	(5.00)		
$h_0/4$		5.32E-08	5.39E-08	5.19E-08	1.40E-08	7.67E-11	5.38E-08	5.45E-08	1.01	1.01
		(5.00)	(5.00)	(4.99)	(4.97)	(6.93)	(5.00)	(5.00)		
$h_0/8$		1.66E-09	1.69E-09	1.62E-09	4.41E-10	5.99E-13	1.68E-09	1.70E-09	1.01	1.01
		(5.00)	(5.00)	(5.00)	(4.99)	(7.00)	(5.00)	(5.00)		

Numerics: singular test case & *hp*-adaptivity

Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := \Omega :=]-1,1[^2 \backslash [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

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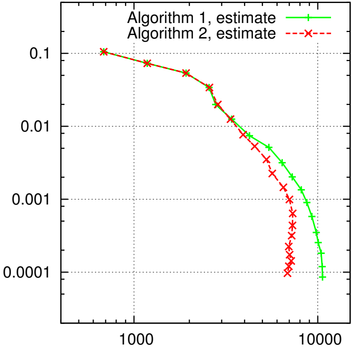
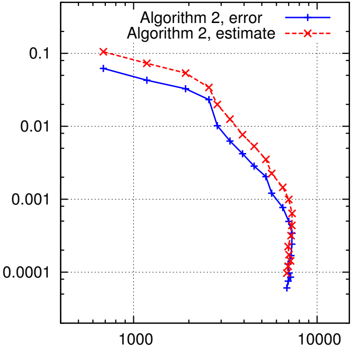
hp-adaptive refinement algorithms

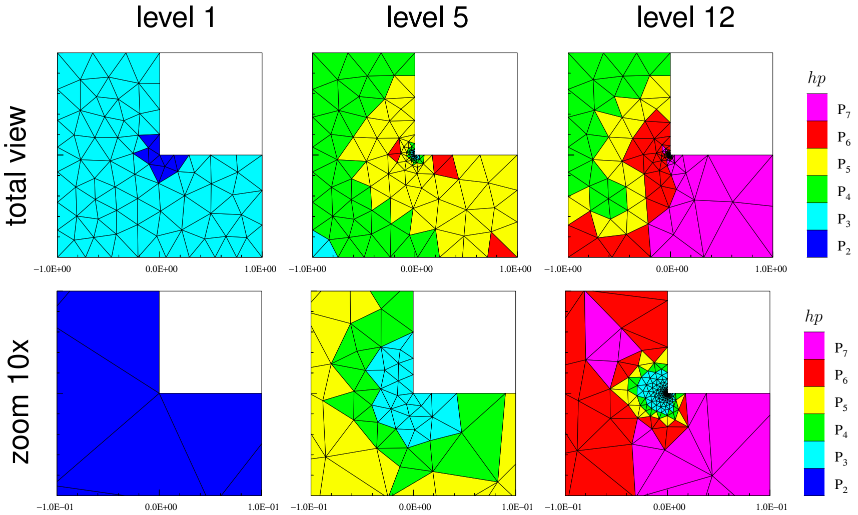


Algorithm 1 (only refinement) and Algorithm 2 (refinement & derefinement) wrt DoF



Exponential convergence of Algorithm 2 wrt DoF

hp-refinement grids



Estimates, errors, and effectivity indices

lev	$\|\mathcal{T}_h\|$	DoF	$\|\nabla(u-u_h)\|$	$\|\nabla u_h+\boldsymbol{\sigma}_h\|$	η_{osc}	$\|\nabla(u_h-s_h)\|$	η_{BC}	η	I^{eff}
0	114	684	6.22E-02	6.63E-02	1.89E-15	4.48E-02	3.81E-02	1.05E-01	1.69
1	122	1180	4.28E-02	4.27E-02	1.18E-14	3.08E-02	2.92E-02	7.29E-02	1.70
2	139	1919	3.28E-02	3.37E-02	8.21E-14	2.09E-02	2.12E-02	5.36E-02	1.64
3	165	2573	2.32E-02	2.30E-02	3.88E-13	1.50E-02	1.03E-02	3.41E-02	1.47
4	174	2858	1.02E-02	1.01E-02	4.48E-13	8.22E-03	9.19E-03	1.99E-02	1.96
5	199	3351	6.27E-03	6.21E-03	1.12E-12	4.81E-03	6.18E-03	1.25E-02	2.00
6	237	3926	4.21E-03	4.23E-03	1.98E-12	3.15E-03	3.29E-03	7.66E-03	1.82
7	285	4537	2.84E-03	2.91E-03	7.47E-12	2.13E-03	2.42E-03	5.33E-03	1.88
8	338	5257	2.04E-03	2.19E-03	4.63E-11	1.45E-03	1.32E-03	3.51E-03	1.72
9	372	5658	1.21E-03	1.23E-03	1.11E-11	9.07E-04	9.99E-04	2.26E-03	1.87
10	426	6500	7.70E-04	7.69E-04	5.69E-11	5.55E-04	6.95E-04	1.46E-03	1.89
11	453	7010	4.95E-04	5.04E-04	9.77E-11	3.97E-04	4.74E-04	9.91E-04	2.00
12	469	7308	3.41E-04	3.47E-04	1.13E-10	2.55E-04	2.88E-04	6.40E-04	1.88
13	463	7286	2.42E-04	2.42E-04	1.39E-10	1.73E-04	1.94E-04	4.37E-04	1.81
14	458	7215	1.69E-04	1.69E-04	1.23E-10	1.19E-04	1.53E-04	3.17E-04	1.88
15	440	6955	1.29E-04	1.31E-04	1.45E-10	9.21E-05	9.10E-05	2.24E-04	1.73
16	435	7035	9.71E-05	9.91E-05	1.39E-10	6.89E-05	7.63E-05	1.74E-04	1.79
17	434	7167	8.52E-05	8.97E-05	1.41E-10	5.76E-05	5.47E-05	1.42E-04	1.67
18	419	6960	7.51E-05	7.97E-05	1.44E-10	5.00E-05	4.15E-05	1.21E-04	1.60
19	410	6838	6.06E-05	6.35E-05	1.47E-10	3.87E-05	3.65E-05	9.69E-05	1.60

Outline

Previous results

Global flux reconstructions

- Prager and Synge (1947):

$$\|\nabla u + \boldsymbol{\sigma}_h\|^2 + \|\nabla(u - u_h)\|^2 = \|\nabla u_h + \boldsymbol{\sigma}_h\|^2$$

  for **any** $u_h \in H_0^1(\Omega)$ and **any** $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ s.t. $\nabla \cdot \boldsymbol{\sigma}_h = f$
- Hlaváček, Haslinger, Nečas, and Lovíšek (1979), Repin (1997), ...: global construction of $\boldsymbol{\sigma}_h$: unprecise/costly

Local flux reconstructions

- Ladevèze and Leguillon (1983), equilibrated face fluxes
- Destuynder and Métivet (1999), discrete flux σ_h
- Luce and Wohlmuth (2004), local efficiency proof
- Vejchodský (2006), equilibration–hypercircle approach
- Kim (2007) & Ern, Nicaise, and Vohralík (2007), discontinuous Galerkin method elementwise prescription
- Braess and Schöberl (2008), Vohralík (2008), Ern and Vohralík (2009), **local Neumann MFE problems**

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- Achdou, Bernardi, and Coquel (2003) & Karakashian and Pascal (2003), by prescription
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Unified frameworks

- Ainsworth and Oden (1993)
- Carstensen (2005)
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Polynomial-degree-robust estimates

- Braess, Pillwein, and Schöberl (2009), conforming finite elements

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- DOLEJŠÍ V., ERN A., VOHRALÍK M., *hp*-adaptation driven by polynomial-degree-robust a posteriori error estimates for elliptic problems, HAL Preprint 01165187, submitted for publication.

Thank you for your attention!