Polynomial-degree-robust a posteriori estimates in a unified setting

Martin Vohralík

INRIA Paris

Prague, March 21, 2016

Outline

Outline

What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE ($-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$).
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|\nabla(u - u_h)\| \leq C(u)h^k$. Dependent on u, not computable. Useful in theoretical assessment of convergence.
- A posteriori error estimate: $\|\nabla(u - u_h)\| \leq C\eta(u_h)$. Only uses u_h, computable. Great in practical calculation.

Usual form

- Element indicators $\eta_K(u_h)$, $K \in \mathcal{T}_h$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE ($-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$).
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|\nabla(u - u_h)\| \leq C(u)h^k$. Dependent on u, not computable. Useful in theoretical assessment of convergence.
- A posteriori error estimate: $\|\nabla(u - u_h)\| \leq C\eta(u_h)$. Only uses u_h, computable. Great in practical calculation.

Usual form

- Element indicators $\eta_K(u_h)$, $K \in \mathcal{T}_h$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE ($-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$).
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|\nabla(u - u_h)\| \leq C(u)h^k$. Dependent on u, not computable. Useful in theoretical assessment of convergence.
- A posteriori error estimate: $\|\nabla(u - u_h)\| \leq C\eta(u_h)$. Only uses u_h, computable. Great in practical calculation.

Usual form

- Element indicators $\eta_K(u_h)$, $K \in \mathcal{T}_h$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE ($-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$).
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|\nabla(u - u_h)\| \leq C(u)h^k$. Dependent on u, not computable. Useful in theoretical assessment of convergence.
- A posteriori error estimate: $\|\nabla(u - u_h)\| \leq C\eta(u_h)$. Only uses u_h, computable. Great in practical calculation.

Usual form

- Element indicators $\eta_K(u_h)$, $K \in \mathcal{T}_h$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

What is an a posteriori error estimate

A posteriori error estimate

- Let u be a weak solution of a PDE ($-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$).
- Let u_h be its approximate numerical solution.
- A priori error estimate: $\|\nabla(u - u_h)\| \leq C(u)h^k$. Dependent on u, not computable. Useful in theoretical assessment of convergence.
- A posteriori error estimate: $\|\nabla(u - u_h)\| \leq C\eta(u_h)$. Only uses u_h, computable. Great in practical calculation.

Usual form

- Element indicators $\eta_K(u_h)$, $K \in \mathcal{T}_h$.
- Can be used to determine mesh elements with large error.
- We can then refine these elements: mesh adaptivity.

What an a posteriori error estimate should fulfill

Optimal estimate for $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

- **guaranteed upper bound**:

$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}$$

- **local efficiency**:

$$\eta_K(u_h) \leq C_{\mathrm{eff}} \|\nabla(u - u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- **asymptotic exactness**:

$$\frac{\left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}}{\|\nabla(u - u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K, polynomial degree of u_h)
- **small evaluation cost** of $\eta_K(u_h)$
- *error components identification*

What an a posteriori error estimate should fulfill

Optimal estimate for $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

- **guaranteed upper bound**:

$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}$$

- **local efficiency**:

$$\eta_K(u_h) \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- **asymptotic exactness**:

$$\frac{\left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}}{\|\nabla(u - u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K, polynomial degree of u_h)
- **small evaluation cost** of $\eta_K(u_h)$
- *error components identification*

What an a posteriori error estimate should fulfill

Optimal estimate for $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

- **guaranteed upper bound**:

$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}$$

- **local efficiency**:

$$\eta_K(u_h) \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- **asymptotic exactness**:

$$\frac{\left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}}{\|\nabla(u - u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K, polynomial degree of u_h)
- **small evaluation cost** of $\eta_K(u_h)$
- *error components identification*

What an a posteriori error estimate should fulfill

Optimal estimate for $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

- **guaranteed upper bound**:

$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}$$

- **local efficiency**:

$$\eta_K(u_h) \leq C_{\mathrm{eff}} \|\nabla(u - u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$

- **asymptotic exactness**:

$$\frac{\left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}}{\|\nabla(u - u_h)\|} \searrow 1$$

- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K, polynomial degree of u_h)
- **small evaluation cost** of $\eta_K(u_h)$
- *error components identification*

What an a posteriori error estimate should fulfill

Optimal estimate for $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

- **guaranteed upper bound**:
$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}$$
- **local efficiency**:
$$\eta_K(u_h) \leq C_{\mathrm{eff}} \|\nabla(u - u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$
- **asymptotic exactness**:
$$\frac{\left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}}{\|\nabla(u - u_h)\|} \searrow 1$$
- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K, polynomial degree of u_h)
- **small evaluation cost** of $\eta_K(u_h)$
- *error components identification*

What an a posteriori error estimate should fulfill

Optimal estimate for $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$, $u = 0$ on $\partial\Omega$

- **guaranteed upper bound**:
$$\|\nabla(u - u_h)\| \leq \left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}$$
- **local efficiency**:
$$\eta_K(u_h) \leq C_{\text{eff}} \|\nabla(u - u_h)\|_{\mathfrak{T}_K} \qquad \forall K \in \mathcal{T}_h$$
- **asymptotic exactness**:
$$\frac{\left\{ \sum_{K \in \mathcal{T}_h} \eta_K(u_h)^2 \right\}^{1/2}}{\|\nabla(u - u_h)\|} \searrow 1$$
- **robustness**: the three previous properties hold independently of the parameters of the problem and of their variation (size of Ω, shape of Ω, regularity of u, local refinement of $\mathcal{T}_h$, sizes h_K, polynomial degree of u_h)
- **small evaluation cost** of $\eta_K(u_h)$
- *error components identification*

Outline

Model problem

Model problem

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

- $u \in H_0^1(\Omega)$ (constraint)
- $\sigma := -\nabla u$ (constitutive relation)
- $\nabla \cdot \sigma = f$ (equilibrium)
- $\sigma \in \mathbf{H}(\mathrm{div}, \Omega)$ (constraint)

Model problem

Model problem

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

- $u \in H_0^1(\Omega)$ (constraint)
- $\sigma := -\nabla u$ (constitutive relation)
- $\nabla{\cdot}\sigma = f$ (equilibrium)
- $\sigma \in \mathbf{H}(\mathrm{div}, \Omega)$ (constraint)

Model problem

Model problem

$$-\Delta u = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ polygon/polyhedron
- $f \in L^2(\Omega)$

Weak formulation

Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Properties of the weak solution

- $u \in H_0^1(\Omega)$ (constraint)
- $\boldsymbol{\sigma} := -\nabla u$ (constitutive relation)
- $\nabla \cdot \boldsymbol{\sigma} = f$ (equilibrium)
- $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div}, \Omega)$ (constraint)

A posteriori error estimate

Theorem (A guaranteed a posteriori error estimate, Prager and Synge (1947), Dari, Durán, Padra, and Vampa (1996), Ainsworth (2005), Kim (2007))

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
- $u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega),\ v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h\}$ *be arbitrary;*
- $s_h \in H_0^1(\Omega)$ *and* $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be such that*

$$(\nabla \cdot \boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \text{ for all } K \in \mathcal{T}_h.$$

Then

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}_h} \Bigg(\underbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \nabla \cdot \boldsymbol{\sigma}_h\|_K}_{\text{equilibrium}} \Bigg)^2$$

$$+ \sum_{K \in \mathcal{T}_h} \underbrace{\|\nabla(u_h - s_h)\|_K^2}_{\text{constraint}}.$$

A posteriori error estimate

Theorem (A guaranteed a posteriori error estimate, Prager and Synge (1947), Dari, Durán, Padra, and Vampa (1996), Ainsworth (2005), Kim (2007))

- *Let $u \in H_0^1(\Omega)$ be the weak solution;*
- $u_h \in H^1(\mathcal{T}_h) := \{v \in L^2(\Omega),\ v|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h\}$ *be arbitrary;*
- $s_h \in H_0^1(\Omega)$ *and* $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ *be such that*

$$(\nabla\cdot\boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \text{ for all } K \in \mathcal{T}_h.$$

Then

$$\|\nabla(u - u_h)\|^2 \leq \sum_{K\in\mathcal{T}_h} \Bigg(\underbrace{\|\nabla u_h + \boldsymbol{\sigma}_h\|_K}_{\text{constitutive relation}} + \underbrace{\frac{h_K}{\pi}\|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K}_{\text{equilibrium}} \Bigg)^2 + \sum_{K\in\mathcal{T}_h} \underbrace{\|\nabla(u_h - s_h)\|_K^2}_{\text{constraint}}.$$

A posteriori error estimate

Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s:

$$\|\nabla(u - s)\|^2 = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\ \|\nabla\varphi\|=1} (\nabla(u - \ ), \nabla\varphi)^2}_{\text{dual norm of the residual}}$$

A posteriori error estimate

Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s:

$$\|\nabla(u - s)\|^2 = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\| = 1} (\nabla(u - \ ), \nabla\varphi)^2}_{\text{dual norm of the residual}}$$

A posteriori error estimate

Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s:

$$\|\nabla(u - s)\|^2 = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\nabla(u - \ ), \nabla\varphi)^2}_{\text{dual norm of the residual}}$$

A posteriori error estimate

Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s:

$$\|\nabla(u - s)\|^2 = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\| = 1} (\nabla(u - s), \nabla\varphi)^2}_{\text{dual norm of the residual}}$$

A posteriori error estimate

Proof.

- define $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

- develop (Pythagoras)

$$\|\nabla(u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla(s - u_h)\|^2$$

- projection definition of s:

$$\|\nabla(s - u_h)\|^2 = \underbrace{\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\|^2}_{\text{distance of } u_h \text{ to } H_0^1(\Omega)}$$

- dual norm characterization, definition of s:

$$\|\nabla(u - s)\|^2 = \underbrace{\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\nabla(u - u_h), \nabla\varphi)^2}_{\text{dual norm of the residual}}$$

A posteriori error estimate

Proof (continuation).

- nonconformity upper bound:

$$\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\| \le \|\nabla(u_h - s_h)\|$$

- weak solution definition, equilibrated flux, Green theorem:

$$\begin{aligned}(\nabla(u - u_h), \nabla\varphi) &= (f, \varphi) - (\nabla u_h, \nabla\varphi)\\ &= (f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi)\end{aligned}$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi) \le \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K,$$

$$\begin{aligned}(f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi) &= \sum_{K \in \mathcal{T}_h} (f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi - \varphi_K)_K\\ &\le \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla{\cdot}\boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K\end{aligned}$$

A posteriori error estimate

Proof (continuation).

- nonconformity upper bound:

$$\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\| \le \|\nabla(u_h - s_h)\|$$

- weak solution definition, equilibrated flux, Green theorem:

$$\begin{aligned}(\nabla(u - u_h), \nabla\varphi) &= (f, \varphi) - (\nabla u_h, \nabla\varphi)\\ &= (f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi)\end{aligned}$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi) \le \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K,$$

$$\begin{aligned}(f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi) &= \sum_{K \in \mathcal{T}_h} (f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi - \varphi_K)_K\\ &\le \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K\end{aligned}$$

A posteriori error estimate

Proof (continuation).

- nonconformity upper bound:

$$\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\| \leq \|\nabla(u_h - s_h)\|$$

- weak solution definition, equilibrated flux, Green theorem:

$$\begin{aligned}(\nabla(u - u_h), \nabla\varphi) &= (f, \varphi) - (\nabla u_h, \nabla\varphi) \\ &= (f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi)\end{aligned}$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi) \leq \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K,$$

$$\begin{aligned}(f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi) &= \sum_{K \in \mathcal{T}_h} (f - \nabla\cdot\boldsymbol{\sigma}_h, \varphi - \varphi_K)_K \\ &\leq \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla\cdot\boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K\end{aligned}$$

A posteriori error estimate

Proof (continuation).

- nonconformity upper bound:

$$\min_{v\in H_0^1(\Omega)} \|\nabla(v-u_h)\| \leq \|\nabla(u_h - s_h)\|$$

- weak solution definition, equilibrated flux, Green theorem:

$$\begin{aligned}(\nabla(u-u_h),\nabla\varphi) &= (f,\varphi)-(\nabla u_h,\nabla\varphi)\\ &= (f-\nabla\cdot\boldsymbol{\sigma}_h,\varphi)-(\nabla u_h+\boldsymbol{\sigma}_h,\nabla\varphi)\end{aligned}$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h+\boldsymbol{\sigma}_h,\nabla\varphi) \leq \sum_{K\in\mathcal{T}_h}\|\nabla u_h+\boldsymbol{\sigma}_h\|_K\|\nabla\varphi\|_K,$$

$$\begin{aligned}(f-\nabla\cdot\boldsymbol{\sigma}_h,\varphi) &= \sum_{K\in\mathcal{T}_h}(f-\nabla\cdot\boldsymbol{\sigma}_h,\varphi-\varphi_K)_K\\ &\leq \sum_{K\in\mathcal{T}_h}\frac{h_K}{\pi}\|f-\nabla\cdot\boldsymbol{\sigma}_h\|_K\|\nabla\varphi\|_K\end{aligned}$$

A posteriori error estimate

Proof (continuation).

- nonconformity upper bound:

$$\min_{v \in H_0^1(\Omega)} \|\nabla(v - u_h)\| \leq \|\nabla(u_h - s_h)\|$$

- weak solution definition, equilibrated flux, Green theorem:

$$\begin{aligned}(\nabla(u - u_h), \nabla\varphi) &= (f, \varphi) - (\nabla u_h, \nabla\varphi) \\ &= (f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi)\end{aligned}$$

- Cauchy–Schwarz and Poincaré inequalities, equilibration:

$$-(\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi) \leq \sum_{K \in \mathcal{T}_h} \|\nabla u_h + \boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K,$$

$$\begin{aligned}(f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi) &= \sum_{K \in \mathcal{T}_h} (f - \nabla{\cdot}\boldsymbol{\sigma}_h, \varphi - \varphi_K)_K \\ &\leq \sum_{K \in \mathcal{T}_h} \frac{h_K}{\pi} \|f - \nabla{\cdot}\boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K\end{aligned}$$

Global potential and flux reconstructions

Ideally

$$\boldsymbol{\sigma}_h := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h,\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h} f} \|\nabla u_h + \mathbf{v}_h\|$$

$$s_h := \arg \min_{v_h \in V_h} \|\nabla(u_h - v_h)\|$$

- $\mathbf{V}_h \subset \mathbf{H}(\mathrm{div}, \Omega)$, $Q_h \subset L^2(\Omega)$, $V_h \subset H_0^1(\Omega)$
- too expensive, **global** minimization problems (the hypercircle method)

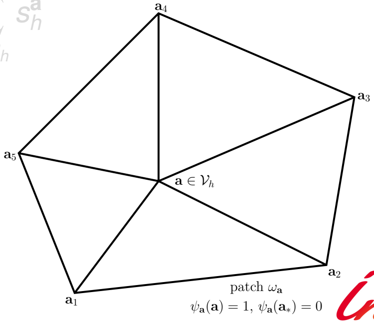
Local potential and flux reconstructions

Partition of unity localization

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

- cut-off by hat basis functions $\psi_{\mathbf{a}} \in \mathbb{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$
- $\boldsymbol{\sigma}_h := \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}, \quad s_h := \sum_{\mathbf{a}\in\mathcal{V}_h} s_h^{\mathbf{a}}$
- **local** minimizations



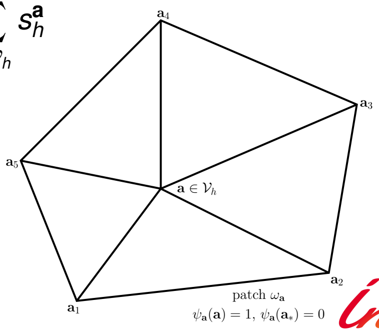
Local potential and flux reconstructions

Partition of unity localization

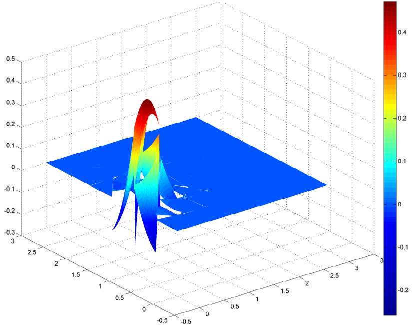
$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = ?} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

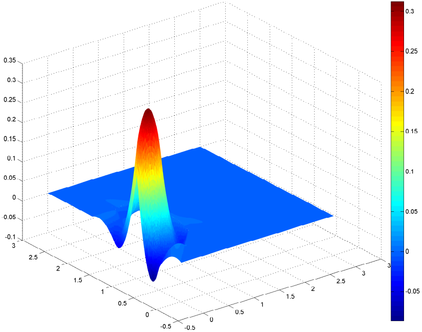
- cut-off by hat basis functions $\psi_{\mathbf{a}} \in \mathbb{P}_1(\mathcal{T}_h) \cap H^1(\Omega)$
- $\boldsymbol{\sigma}_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}}, \quad s_h := \sum_{\mathbf{a} \in \mathcal{V}_h} s_h^{\mathbf{a}}$
- **local** minimizations



Potential reconstruction

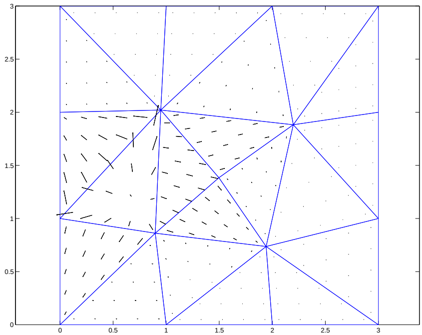


Potential u_h

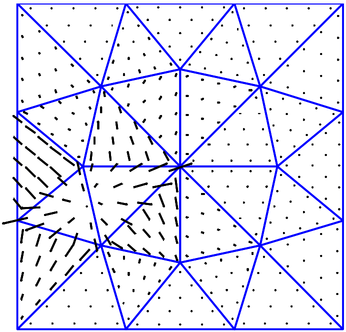


Potential reconstruction s_h

Equilibrated flux reconstruction



Flux $-\nabla u_h$



Flux reconstruction σ_h

Local equilibrated flux reconstruction

Assumption A (Galerkin orthogonality wrt hat functions)

There holds $u_h \in H^1(\mathcal{T}_h)$ *and*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

$\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$: MFE spaces (hom. Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$)

Definition (Constr. of σ_h, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\sigma_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving the **local mixed FE problem**

$$\sigma_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$\Updownarrow$$

$$(\sigma_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},$$

$$(\nabla\cdot\sigma_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, q_h)_{\omega_{\mathbf{a}}} \qquad \forall q_h \in Q_h^{\mathbf{a}}.$$

Local equilibrated flux reconstruction

Assumption A (Galerkin orthogonality wrt hat functions)

There holds $u_h \in H^1(\mathcal{T}_h)$ *and*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

$\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$: MFE spaces (hom. Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$)

Definition (Constr. of σ_h, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\sigma_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving the **local mixed FE problem**

$$\sigma_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$\Updownarrow$$

$$(\sigma_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},$$

$$(\nabla\cdot\sigma_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, q_h)_{\omega_{\mathbf{a}}} \qquad \forall q_h \in Q_h^{\mathbf{a}}.$$

Local equilibrated flux reconstruction

Assumption A (Galerkin orthogonality wrt hat functions)

There holds $u_h \in H^1(\mathcal{T}_h)$ *and*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\text{int}}.$$

$\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$: MFE spaces (hom. Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\text{int}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\text{ext}}$)

Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving the **local mixed FE problem**

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$\Updownarrow$$

$$(\boldsymbol{\sigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} = -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},$$

$$(\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, q_h)_{\omega_{\mathbf{a}}} \qquad \forall q_h \in Q_h^{\mathbf{a}}.$$

Local equilibrated flux reconstruction

Assumption A (Galerkin orthogonality wrt hat functions)

There holds $u_h \in H^1(\mathcal{T}_h)$ *and*

$$(\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}.$$

$\mathbf{V}_h^{\mathbf{a}} \times Q_h^{\mathbf{a}}$: MFE spaces (hom. Neumann BC for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, homogeneous Dirichlet BC on $\partial\omega_{\mathbf{a}} \cap \partial\Omega$ for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{ext}}$)

Definition (Constr. of $\boldsymbol{\sigma}_h$, Destuynder and Métivet (1999) & Braess and Schöberl (2008))

Let Assumption A be satisfied. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{V}_h^{\mathbf{a}}$ and $\bar{r}_h^{\mathbf{a}} \in Q_h^{\mathbf{a}}$ by solving the **local mixed FE problem**

$$\boldsymbol{\sigma}_h^{\mathbf{a}} := \arg \min_{\mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_h = \Pi_{Q_h^{\mathbf{a}}}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}_h\|_{\omega_{\mathbf{a}}}$$

$$\Updownarrow$$

$$\begin{aligned}
(\boldsymbol{\sigma}_h^{\mathbf{a}}, \mathbf{v}_h)_{\omega_{\mathbf{a}}} - (\bar{r}_h^{\mathbf{a}}, \nabla\cdot\mathbf{v}_h)_{\omega_{\mathbf{a}}} &= -(\psi_{\mathbf{a}} \nabla u_h, \mathbf{v}_h)_{\omega_{\mathbf{a}}} && \forall \mathbf{v}_h \in \mathbf{V}_h^{\mathbf{a}},\\
(\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}, q_h)_{\omega_{\mathbf{a}}} &= (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, q_h)_{\omega_{\mathbf{a}}} && \forall q_h \in Q_h^{\mathbf{a}}.
\end{aligned}$$

Comments

$\mathbf{H}(\mathrm{div},\Omega)$-conformity

- $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega) \Rightarrow \boldsymbol{\sigma}_h = \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs

$$(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0$$

- but Assumption A gives

$$0 = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}}$$

Divergence

- Neumann compatibility condition gives

$$\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

$$\nabla\cdot\boldsymbol{\sigma}_h|_K = \Pi_{Q_h} f|_K \qquad \forall K \in \mathcal{T}_h$$

Comments

$\mathbf{H}(\mathrm{div},\Omega)$-conformity

- $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega) \Rightarrow \boldsymbol{\sigma}_h = \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs

$$(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0$$

- but Assumption A gives

$$0 = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}}$$

Divergence

- Neumann compatibility condition gives

$$\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

$$\nabla\cdot\boldsymbol{\sigma}_h|_K = \Pi_{Q_h} f|_K \qquad \forall K \in \mathcal{T}_h$$

Comments

$\mathbf{H}(\mathrm{div},\Omega)$-conformity

- $\boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega) \Rightarrow \boldsymbol{\sigma}_h = \sum_{\mathbf{a}\in\mathcal{V}_h} \boldsymbol{\sigma}_h^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

Neumann compatibility condition

- for $\mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$, one needs

$$(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}} = 0$$

- but Assumption A gives

$$0 = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} - (\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h, 1)_{\omega_{\mathbf{a}}}$$

Divergence

- Neumann compatibility condition gives

$$\nabla\cdot\boldsymbol{\sigma}_h^{\mathbf{a}}|_K = \Pi_{Q_h}(\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)|_K \qquad \forall K \in \mathcal{T}_h$$

- the fact that $\boldsymbol{\sigma}_h|_K = \sum_{\mathbf{a}\in\mathcal{V}_K} \boldsymbol{\sigma}_h^{\mathbf{a}}|_K$ and the partition of unity $\sum_{\mathbf{a}\in\mathcal{V}_K} \psi^{\mathbf{a}}|_K = 1|_K$ yield

$$\nabla\cdot\boldsymbol{\sigma}_h|_K = \Pi_{Q_h} f|_K \qquad \forall K \in \mathcal{T}_h$$

Local potential reconstruction

$V_h^{\mathbf{a}}$: FE space (hom. Dirichlet BC on $\partial\omega_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{V}_h$)

Definition (Construction of s_h, ≈ Carstensen and Merdon (2013))

Let $u_h \in H^1(\mathcal{T}_h)$. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ by solving the **local conforming FE problem**

$$s_h^{\mathbf{a}} := \arg\min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

$$\Updownarrow$$

$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla(\psi_{\mathbf{a}} u_h), \nabla v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in V_h^{\mathbf{a}}.$$

Local potential reconstruction

$V_h^{\mathbf{a}}$: FE space (hom. Dirichlet BC on $\partial\omega_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{V}_h$)

Definition (Construction of s_h, ≈ Carstensen and Merdon (2013))

Let $u_h \in H^1(\mathcal{T}_h)$. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ by solving the **local conforming FE problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

$$\Updownarrow$$

$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla(\psi_{\mathbf{a}} u_h), \nabla v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in V_h^{\mathbf{a}}.$$

Local potential reconstruction

$V_h^{\mathbf{a}}$: FE space (hom. Dirichlet BC on $\partial\omega_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{V}_h$)

Definition (Construction of s_h, ≈ Carstensen and Merdon (2013))

Let $u_h \in H^1(\mathcal{T}_h)$. For each $\mathbf{a} \in \mathcal{V}_h$, prescribe $s_h^{\mathbf{a}} \in V_h^{\mathbf{a}}$ by solving the **local conforming FE problem**

$$s_h^{\mathbf{a}} := \arg \min_{v_h \in V_h^{\mathbf{a}}} \|\nabla(\psi_{\mathbf{a}} u_h - v_h)\|_{\omega_{\mathbf{a}}}$$

$$\Updownarrow$$

$$(\nabla s_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = (\nabla(\psi_{\mathbf{a}} u_h), \nabla v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in V_h^{\mathbf{a}}.$$

Outline

Continuous-level patch problems

Definition (Continuous-level flux reconstruction)

For each $\mathbf{a} \in \mathcal{V}_h$, set

$$\boldsymbol{\sigma}^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}}), \\ \nabla \cdot \mathbf{v} = (\psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h) \\ \mathbf{v} \cdot \mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial \omega_{\mathbf{a}} \ (\setminus \partial \Omega)}} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}\|_{\omega_{\mathbf{a}}}.$$

Definition (Continuous-level potential reconstruction)

For each $\mathbf{a} \in \mathcal{V}_h$, set

$$s^{\mathbf{a}} := \arg \min_{v \in H^1_0(\omega_{\mathbf{a}})} \|\nabla(\psi_{\mathbf{a}} u_h - v)\|_{\omega_{\mathbf{a}}}.$$

Continuous-level patch problems

Definition (Continuous-level flux reconstruction)

For each $\mathbf{a} \in \mathcal{V}_h$, set

$$\boldsymbol{\sigma}^{\mathbf{a}} := \arg \min_{\substack{\mathbf{v} \in \mathbf{H}(\mathrm{div}, \omega_{\mathbf{a}}),\\ \nabla\cdot\mathbf{v} = (\psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h)\\ \mathbf{v}\cdot\mathbf{n}_{\omega_{\mathbf{a}}} = 0 \text{ on } \partial\omega_{\mathbf{a}} \ (\setminus\partial\Omega)}} \|\psi_{\mathbf{a}} \nabla u_h + \mathbf{v}\|_{\omega_{\mathbf{a}}}.$$

Definition (Continuous-level potential reconstruction)

For each $\mathbf{a} \in \mathcal{V}_h$, set

$$s^{\mathbf{a}} := \arg \min_{v \in H^1_0(\omega_{\mathbf{a}})} \|\nabla(\psi_{\mathbf{a}} u_h - v)\|_{\omega_{\mathbf{a}}}.$$

Polynomial-degree-robust efficiency

Assumption B (Weak continuity)

There holds $\langle [\![u_h]\!], 1 \rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$

Assumption C (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The **degrees** *of the MFE reconstructions σ_h and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.*

Theorem (MFE / FE / continuous stability Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015)**)**

Let u be the weak solution and let Assumptions A, B, and C hold. Then there exists a constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ **only** *depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that*

$$\|\sigma_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\sigma^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} C_{\rm cont,PF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}};$$

$$\|\nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\nabla(\psi_{\mathbf{a}} u_h - s^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} C_{\rm cont,bPF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Polynomial-degree-robust efficiency

Assumption B (Weak continuity)

There holds
$$\langle [\![u_h]\!], 1 \rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$$

Assumption C (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The **degrees** *of the MFE reconstructions $\boldsymbol{\sigma}_h$ and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.*

Theorem (MFE / FE / continuous stability Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015))

Let u be the weak solution and let Assumptions A, B, and C hold. Then there exists a constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ **only** *depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that*

$$\|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\boldsymbol{\sigma}^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} C_{\rm cont,PF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}};$$

$$\|\nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\nabla(\psi_{\mathbf{a}} u_h - s^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} C_{\rm cont,bPF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Polynomial-degree-robust efficiency

Assumption B (Weak continuity)

There holds $\langle [\![u_h]\!], 1\rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$

Assumption C (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The **degrees** *of the MFE reconstructions $\boldsymbol{\sigma}_h$ and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.*

Theorem (MFE / FE / continuous stability Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015))

Let u be the weak solution and let Assumptions A, B, and C hold. Then there exists a constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ **only** *depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that*

$$\|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\boldsymbol{\sigma}^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} C_{\rm cont,PF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}};$$

$$\|\nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} \|\nabla(\psi_{\mathbf{a}} u_h - s^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \leq C_{\rm st} C_{\rm cont,bPF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Polynomial-degree-robust efficiency

Assumption B (Weak continuity)

There holds $$\langle [\![u_h]\!], 1\rangle_e = 0 \qquad \forall e \in \mathcal{E}_h.$$

Assumption C (Piecewise polynomials, data, and meshes)

The approximation u_h and the datum f are piecewise polynomial. The **degrees** *of the MFE reconstructions $\boldsymbol{\sigma}_h$ and s_h are chosen correspondingly. The meshes $\mathcal{T}_h$ are shape-regular.*

Theorem (MFE / FE / continuous stability Braess, Pillwein, and Schöberl (2009); Costabel and McIntosh (2010); Demkowicz, Gopalakrishnan, and Schöberl (2012), EV (2015))

Let u be the weak solution and let Assumptions A, B, and C hold. Then there exists a constants $C_{\rm st}, C_{\rm cont,PF}, C_{\rm cont,bPF} > 0$ **only** *depending on the shape-regularity parameter $\kappa_{\mathcal{T}}$ such that*

$$\|\boldsymbol{\sigma}_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \le C_{\rm st} \|\boldsymbol{\sigma}^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{\omega_{\mathbf{a}}} \le C_{\rm st} C_{\rm cont,PF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}};$$

$$\|\nabla(\psi_{\mathbf{a}} u_h - s_h^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \le C_{\rm st} \|\nabla(\psi_{\mathbf{a}} u_h - s^{\mathbf{a}})\|_{\omega_{\mathbf{a}}} \le C_{\rm st} C_{\rm cont,bPF} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency)

Let u be the weak solution and let Assumptions A, B, and C hold. Then

$$\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \leq C_{\rm st} C_{\rm cont,PF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}},$$

$$\|\nabla(u_h - s_h)\|_K \leq C_{\rm st} C_{\rm cont,bPF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}}.$$

Remarks

- $C_{\rm st}$ can be bounded by solving the local Neumann problems by conforming FEs: find $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}} \subset H^1_*(\omega_{\mathbf{a}})$ s.t.

$$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = -(\tau_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in V_h^{\mathbf{a}};$$

  then $C_{\rm st} \leq \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} / \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$ ($\boldsymbol{\tau}_h^{\mathbf{a}} = \psi^{\mathbf{a}} \nabla u_h$ & $g^{\mathbf{a}} = \psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}}\cdot\nabla u_h$ or $\boldsymbol{\tau}_h^{\mathbf{a}} = \mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h)$ & $g^{\mathbf{a}} = 0$)
- $\Rightarrow$ maximal overestimation factor guaranteed

Polynomial-degree-robust efficiency

Theorem (Polynomial-degree-robust efficiency)

Let u be the weak solution and let Assumptions A, B, and C hold. Then

$$\|\nabla u_h + \boldsymbol{\sigma}_h\|_K \leq C_{\rm st} C_{\rm cont,PF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}},$$

$$\|\nabla(u_h - s_h)\|_K \leq C_{\rm st} C_{\rm cont,bPF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}}.$$

Remarks

- $C_{\rm st}$ can be bounded by solving the local Neumann problems by conforming FEs: find $r_h^{\mathbf{a}} \in V_h^{\mathbf{a}} \subset H_*^1(\omega_{\mathbf{a}})$ s.t.

  $$(\nabla r_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} = -(\boldsymbol{\tau}_h^{\mathbf{a}}, \nabla v_h)_{\omega_{\mathbf{a}}} + (g^{\mathbf{a}}, v_h)_{\omega_{\mathbf{a}}} \qquad \forall v_h \in V_h^{\mathbf{a}};$$

  then $C_{\rm st} \leq \|\boldsymbol{\tau}_h^{\mathbf{a}} + \boldsymbol{\sigma}_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}} / \|\nabla r_h^{\mathbf{a}}\|_{\omega_{\mathbf{a}}}$ ($\boldsymbol{\tau}_h^{\mathbf{a}} = \psi^{\mathbf{a}} \nabla u_h$ & $g^{\mathbf{a}} = \psi_{\mathbf{a}} f - \nabla\psi_{\mathbf{a}} \cdot \nabla u_h$ or $\boldsymbol{\tau}_h^{\mathbf{a}} = \mathrm{R}_{\frac{\pi}{2}} \nabla(\psi_{\mathbf{a}} u_h)$ & $g^{\mathbf{a}} = 0$)
- $\Rightarrow$ maximal overestimation factor guaranteed

Outline

Conforming finite elements

Conforming finite elements

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$
- $V_h \subset H_0^1(\Omega)$: $s_h := u_h$, no need for Assumption B

Conforming finite elements

Conforming finite elements

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$
- $V_h \subset H_0^1(\Omega)$: $s_h := u_h$, no need for Assumption B

Nonconforming finite elements

Nonconforming finite elements

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$, $v_h \in V_h$ satisfy

$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p-1}(e), \forall e \in \mathcal{E}_h$$

- Assumption A: take $v_h = \psi_{\mathbf{a}}$
- Assumption B: building requirement for the space V_h

Nonconforming finite elements

Nonconforming finite elements

Find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$, $v_h \in V_h$ satisfy

$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p-1}(e), \forall e \in \mathcal{E}_h$$

- Assumption A: take $v_h = \psi_{\mathbf{a}}$
- Assumption B: building requirement for the space V_h

Discontinuous Galerkin finite elements

Discontinuous Galerkin finite elements

Find $u_h \in V_h$ such that

$$\sum_{K\in\mathcal{T}_h}(\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h}\{\langle\{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle\{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\}$$

$$+\sum_{e\in\mathcal{E}_h}\langle\alpha h_e^{-1}[\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$ for $\theta = 0$, otherwise:
  - estimates for the discrete gradient

    $$\mathfrak{G}(u_h) := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

  - jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_e)]^2$

    $$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle\{\!\{\mathbf{v}_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_e)]^2$$

  - $\Rightarrow$ modified Galerkin orthogonality

    $$(\mathfrak{G}(u_h), \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

Discontinuous Galerkin finite elements

Discontinuous Galerkin finite elements

Find $u_h \in V_h$ such that

$$\sum_{K\in\mathcal{T}_h}(\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h}\{\langle\{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle\{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\}$$

$$+ \sum_{e\in\mathcal{E}_h}\langle\alpha h_e^{-1}[\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$
- Assumption A: take $v_h = \psi_{\mathbf{a}}$ for $\theta = 0$, otherwise:
  - estimates for the discrete gradient

    $$\mathfrak{G}(u_h) := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h} \mathfrak{l}_e([\![u_h]\!])$$

  - jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_e)]^2$

    $$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle\{\!\{\mathbf{v}_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_e)]^2$$

  - $\Rightarrow$ modified Galerkin orthogonality

    $$(\mathfrak{G}(u_h), \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

Discontinuous Galerkin finite elements

Discontinuous Galerkin finite elements

Find $u_h \in V_h$ such that

$$\sum_{K\in\mathcal{T}_h}(\nabla u_h, \nabla v_h)_K - \sum_{e\in\mathcal{E}_h}\{\langle\{\!\{\nabla u_h\}\!\}\cdot\mathbf{n}_e, [\![v_h]\!]\rangle_e + \theta\langle\{\!\{\nabla v_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e\}$$

$$+ \sum_{e\in\mathcal{E}_h}\langle\alpha h_e^{-1}[\![u_h]\!], [\![v_h]\!]\rangle_e = (f, v_h) \qquad \forall v_h \in V_h.$$

- $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$
- Assumption A: take $v_h = \psi_\mathbf{a}$ for $\theta = 0$, otherwise:
  - estimates for the discrete gradient

    $$\mathfrak{G}(u_h) := \nabla u_h - \theta \sum_{e\in\mathcal{E}_h}\mathfrak{l}_e([\![u_h]\!])$$

  - jumps lifting operator $\mathfrak{l}_e : L^2(e) \to [\mathbb{P}_0(\mathcal{T}_e)]^2$

    $$(\mathfrak{l}_e([\![u_h]\!]), \mathbf{v}_h) = \langle\{\!\{\mathbf{v}_h\}\!\}\cdot\mathbf{n}_e, [\![u_h]\!]\rangle_e \qquad \forall \mathbf{v}_h \in [\mathbb{P}_0(\mathcal{T}_e)]^2$$

  - $\Rightarrow$ modified Galerkin orthogonality

    $$(\mathfrak{G}(u_h), \nabla\psi_\mathbf{a})_{\omega_\mathbf{a}} = (f, \psi_\mathbf{a})_{\omega_\mathbf{a}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

Discontinuous Galerkin finite elements: Assumption B

Nonsymmetric and incomplete versions

- broken Poincaré–Friedrichs inequality with jumps:

$$\|\nabla(\psi_{\mathbf{a}}(\tilde{u}-u_h))\|_{\omega_{\mathbf{a}}} \leq (1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}) \|\nabla(u-u_h)\|_{\omega_{\mathbf{a}}} + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} \left\{ \sum_{e\in\mathcal{E}_h,\, \mathbf{a}\in e} h_e^{-1} \|\Pi_e^0 [\![u_h]\!]\|_e^2 \right\}^{1/2}$$

- include the jump terms in the error and estimators

Symmetric version

- discrete gradient $\mathfrak{G}$ satisfies

$$(\mathfrak{G}(u_h), \mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h$$

- modified potential reconstruction: local MFE problems with $\tau_h^{\mathbf{a}} := \psi_{\mathbf{a}} \mathrm{R}_{\frac{\pi}{2}} \mathfrak{G}(u_h)$ and $g^{\mathbf{a}} := (\mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}}) \cdot \mathfrak{G}(u_h)$
- local efficiency

$$\|\mathfrak{G}(u_h - s_h)\|_K \leq C_{\mathrm{st}} C_{\mathrm{cont,P}} \sum_{\mathbf{a}\in\mathcal{V}_K} \|\mathfrak{G}(u-u_h)\|_{\omega_{\mathbf{a}}}$$

Discontinuous Galerkin finite elements: Assumption B

Nonsymmetric and incomplete versions

- broken Poincaré–Friedrichs inequality with jumps:

$$\|\nabla(\psi_{\mathbf{a}}(\tilde{u} - u_h))\|_{\omega_{\mathbf{a}}} \leq (1 + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}}) \|\nabla(u - u_h)\|_{\omega_{\mathbf{a}}} + C_{\mathrm{bPF},\omega_{\mathbf{a}}} h_{\omega_{\mathbf{a}}} \|\nabla\psi_{\mathbf{a}}\|_{\infty,\omega_{\mathbf{a}}} \left\{ \sum_{e \in \mathcal{E}_h,\, \mathbf{a} \in e} h_e^{-1} \|\Pi_e^0 [\![u_h]\!]\|_e^2 \right\}^{1/2}$$

- include the jump terms in the error and estimators

Symmetric version

- discrete gradient $\mathfrak{G}$ satisfies

$$(\mathfrak{G}(u_h), \mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h$$

- modified potential reconstruction: local MFE problems with $\boldsymbol{\tau}_h^{\mathbf{a}} := \psi_{\mathbf{a}} \mathrm{R}_{\frac{\pi}{2}} \mathfrak{G}(u_h)$ and $g^{\mathbf{a}} := (\mathrm{R}_{\frac{\pi}{2}} \nabla\psi_{\mathbf{a}}) \cdot \mathfrak{G}(u_h)$
- local efficiency

$$\|\mathfrak{G}(u_h - s_h)\|_K \leq C_{\mathrm{st}} C_{\mathrm{cont,P}} \sum_{\mathbf{a} \in \mathcal{V}_K} \|\mathfrak{G}(u - u_h)\|_{\omega_{\mathbf{a}}}$$

Mixed finite elements

Mixed finite elements

Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} (\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla\cdot\mathbf{v}_h) &= 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ (\nabla\cdot\boldsymbol{\sigma}_h, q_h) &= (f, q_h) \qquad \forall q_h \in Q_h. \end{aligned}$$

- postprocessed solution $u_h \in V_h$, $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$; $v_h \in V_h$ satisfy

$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p'}(e), \forall e \in \mathcal{E}_h$$

- Assumption A: no need for flux reconstruction, $\boldsymbol{\sigma}_h$ comes from the discretization
- Assumption B satisfied, building requirement for the space V_h

Mixed finite elements

Mixed finite elements

Find a couple $(\boldsymbol{\sigma}_h, \bar{u}_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{aligned}(\boldsymbol{\sigma}_h, \mathbf{v}_h) - (\bar{u}_h, \nabla{\cdot}\mathbf{v}_h) &= 0 \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h,\\ (\nabla{\cdot}\boldsymbol{\sigma}_h, q_h) &= (f, q_h) \qquad && \forall q_h \in Q_h.\end{aligned}$$

- postprocessed solution $u_h \in V_h$, $V_h := \mathbb{P}_p(\mathcal{T}_h)$, $p \geq 1$; $v_h \in V_h$ satisfy

$$\langle [\![v_h]\!], q_h \rangle_e = 0 \qquad \forall q_h \in \mathbb{P}_{p'}(e), \forall e \in \mathcal{E}_h$$

- Assumption A: no need for flux reconstruction, $\boldsymbol{\sigma}_h$ comes from the discretization
- Assumption B satisfied, building requirement for the space V_h

Outline

Numerics: smooth test case

Model problem

$$-\Delta u = f \quad \text{in } \Omega :=]0,1[^2$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured nested triangular grids
- uniform refinement

Numerics: smooth test case

Model problem

$$-\Delta u = f \quad \text{in } \Omega :=]0,1[^2$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured nested triangular grids
- uniform refinement

Numerics: smooth test case

Model problem

$$-\Delta u = f \quad \text{in } \Omega :=]0,1[^2$$
$$u = u_{\mathrm{D}} \quad \text{on } \partial\Omega$$

Exact solution

$$u(\mathbf{x}) = (c_1 + c_2(1 - x_1) + e^{-\alpha x_1})(c_1 + c_2(1 - x_2) + e^{-\alpha x_2})$$
$$c_1 = -e^{-\alpha}, \quad c_2 = -1 - c_1, \quad \alpha = 10$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured nested triangular grids
- uniform refinement

Estimates, errors, and effectivity indices

h	p	$\|\nabla(u-u_h)\|$	$\|u-u_h\|_{DG}$	$\|\nabla u_h+\sigma_h\|$	$\|\nabla(u_h-s_h)\|$	η_{osc}	η	η_{DG}	I^{eff}	I^{eff}_{DG}
$h_0/1$	1	1.21E+00	1.22E+00	1.24E+00	1.07E-01	5.56E-02	1.30E+00	1.31E+00	1.07	1.07
$h_0/2$		6.18E-01	6.22E-01	6.38E-01	5.09E-02	7.02E-03	6.47E-01	6.50E-01	1.05	1.05
		(0.97)	(0.97)	(0.96)	(1.07)	(2.99)	(1.01)	(1.01)		
$h_0/4$		3.12E-01	3.13E-01	3.22E-01	2.43E-02	8.80E-04	3.24E-01	3.25E-01	1.04	1.04
		(0.99)	(0.99)	(0.99)	(1.07)	(3.00)	(1.00)	(1.00)		
$h_0/8$		1.56E-01	1.57E-01	1.61E-01	1.18E-02	1.10E-04	1.62E-01	1.63E-01	1.04	1.04
		(1.00)	(1.00)	(1.00)	(1.05)	(3.00)	(1.00)	(1.00)		
$h_0/1$	2	1.50E-01	1.53E-01	1.49E-01	2.76E-02	5.10E-03	1.56E-01	1.59E-01	1.04	1.04
$h_0/2$		3.85E-02	3.92E-02	3.83E-02	7.99E-03	3.22E-04	3.94E-02	4.01E-02	1.03	1.02
		(1.96)	(1.96)	(1.96)	(1.79)	(3.98)	(1.98)	(1.98)		
$h_0/4$		9.70E-03	9.88E-03	9.68E-03	2.12E-03	2.02E-05	9.93E-03	1.01E-02	1.02	1.02
		(1.99)	(1.99)	(1.98)	(1.92)	(4.00)	(1.99)	(1.99)		
$h_0/8$		2.43E-03	2.48E-03	2.43E-03	5.42E-04	1.26E-06	2.49E-03	2.54E-03	1.02	1.02
		(1.99)	(1.99)	(1.99)	(1.96)	(4.00)	(1.99)	(1.99)		
$h_0/1$	3	1.32E-02	1.34E-02	1.29E-02	2.52E-03	3.58E-04	1.35E-02	1.37E-02	1.03	1.03
$h_0/2$		1.67E-03	1.69E-03	1.65E-03	3.13E-04	1.13E-05	1.70E-03	1.71E-03	1.01	1.01
		(2.98)	(2.98)	(2.97)	(3.01)	(4.99)	(3.00)	(3.00)		
$h_0/4$		2.11E-04	2.13E-04	2.09E-04	3.83E-05	3.53E-07	2.12E-04	2.15E-04	1.01	1.01
		(2.99)	(2.99)	(2.99)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/8$		2.64E-05	2.67E-05	2.61E-05	4.69E-06	1.10E-08	2.66E-05	2.69E-05	1.01	1.01
		(3.00)	(3.00)	(3.00)	(3.03)	(5.00)	(3.00)	(3.00)		
$h_0/1$	4	9.36E-04	9.54E-04	9.05E-04	2.41E-04	2.12E-05	9.57E-04	9.74E-04	1.02	1.02
$h_0/2$		5.93E-05	6.05E-05	5.77E-05	1.68E-05	3.36E-07	6.04E-05	6.16E-05	1.02	1.02
		(3.98)	(3.98)	(3.97)	(3.84)	(5.98)	(3.99)	(3.98)		
$h_0/4$		3.72E-06	3.80E-06	3.63E-06	1.10E-06	5.31E-09	3.80E-06	3.87E-06	1.02	1.02
		(3.99)	(3.99)	(3.99)	(3.94)	(5.98)	(3.99)	(3.99)		
$h_0/8$		2.33E-07	2.38E-07	2.27E-07	7.02E-08	8.30E-11	2.38E-07	2.43E-07	1.02	1.02
		(4.00)	(4.00)	(4.00)	(3.97)	(6.00)	(4.00)	(3.99)		
$h_0/1$	5	5.41E-05	5.50E-05	5.22E-05	1.38E-05	1.06E-06	5.50E-05	5.58E-05	1.02	1.02
$h_0/2$		1.70E-06	1.72E-06	1.65E-06	4.39E-07	9.35E-09	1.72E-06	1.74E-06	1.01	1.01
		(4.99)	(5.00)	(4.98)	(4.98)	(6.82)	(5.00)	(5.00)		
$h_0/4$		5.32E-08	5.39E-08	5.19E-08	1.40E-08	7.67E-11	5.38E-08	5.45E-08	1.01	1.01
		(5.00)	(5.00)	(4.99)	(4.97)	(6.93)	(5.00)	(5.00)		
$h_0/8$		1.66E-09	1.69E-09	1.62E-09	4.41E-10	5.99E-13	1.68E-09	1.70E-09	1.01	1.01
		(5.00)	(5.00)	(5.00)	(4.99)	(7.00)	(5.00)	(5.00)		

Numerics: singular test case & *hp*-adaptivity

Model problem

$$
\begin{aligned}
-\Delta u &= 0 \quad \text{in } \Omega := \Omega :=]-1,1[^2 \backslash [0,1]^2, \\
u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega
\end{aligned}
$$

Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

Numerics: singular test case & *hp*-adaptivity

Model problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega := \Omega :=]-1,1[^2 \backslash [0,1]^2, \\ u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega \end{aligned}$$

Exact solution

$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

Numerics: singular test case & *hp*-adaptivity

Model problem

$$
\begin{aligned}
-\Delta u &= 0 \quad \text{in } \Omega := \Omega :=]-1,1[^2 \backslash [0,1]^2, \\
u &= u_{\mathrm{D}} \quad \text{on } \partial\Omega
\end{aligned}
$$

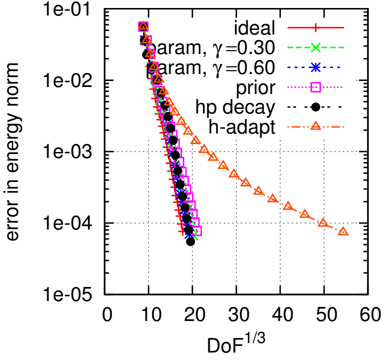
Exact solution

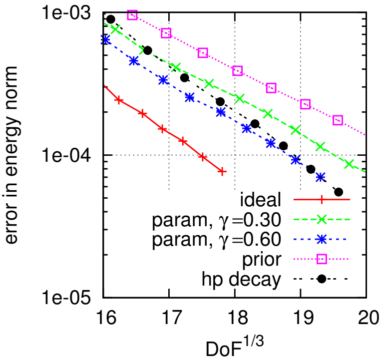
$$u(r,\phi) = r^{2/3} \sin(2\phi/3)$$

Discretization

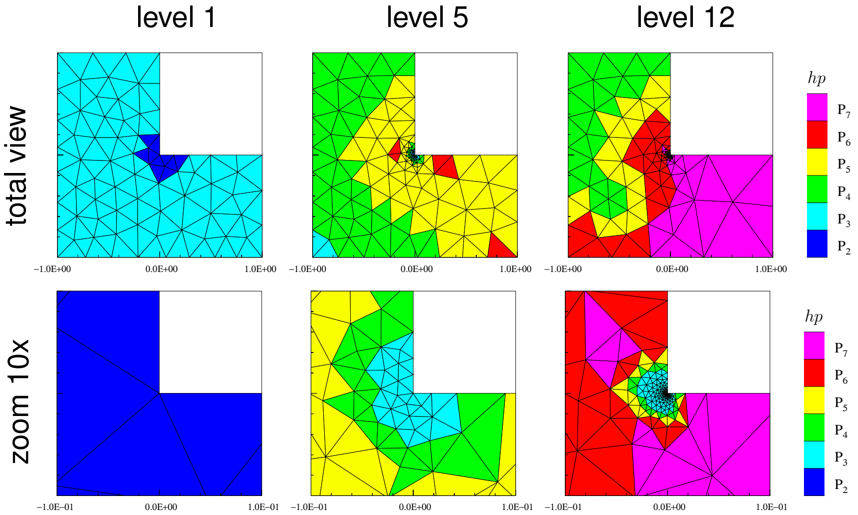
- incomplete interior penalty discontinuous Galerkin method
- unstructured non-nested triangular grids
- *hp*-adaptive refinement

hp-adaptive refinement: **exponential convergence** wrt DoF





hp-refinement grids



Estimates, errors, and effectivity indices

lev	$\lvert\mathcal{T}_h\rvert$	DoF	$\lVert\nabla(u-u_h)\rVert$	$\lVert\nabla u_h+\boldsymbol{\sigma}_h\rVert$	$\eta_{\rm osc}$	$\lVert\nabla(u_h-s_h)\rVert$	$\eta_{\rm BC}$	η	$I^{\rm eff}$
0	114	684	6.22E-02	6.63E-02	1.89E-15	4.48E-02	3.81E-02	1.05E-01	1.69
1	122	1180	4.28E-02	4.27E-02	1.18E-14	3.08E-02	2.92E-02	7.29E-02	1.70
2	139	1919	3.28E-02	3.37E-02	8.21E-14	2.09E-02	2.12E-02	5.36E-02	1.64
3	165	2573	2.32E-02	2.30E-02	3.88E-13	1.50E-02	1.03E-02	3.41E-02	1.47
4	174	2858	1.02E-02	1.01E-02	4.48E-13	8.22E-03	9.19E-03	1.99E-02	1.96
5	199	3351	6.27E-03	6.21E-03	1.12E-12	4.81E-03	6.18E-03	1.25E-02	2.00
6	237	3926	4.21E-03	4.23E-03	1.98E-12	3.15E-03	3.29E-03	7.66E-03	1.82
7	285	4537	2.84E-03	2.91E-03	7.47E-12	2.13E-03	2.42E-03	5.33E-03	1.88
8	338	5257	2.04E-03	2.19E-03	4.63E-11	1.45E-03	1.32E-03	3.51E-03	1.72
9	372	5658	1.21E-03	1.23E-03	1.11E-11	9.07E-04	9.99E-04	2.26E-03	1.87
10	426	6500	7.70E-04	7.69E-04	5.69E-11	5.55E-04	6.95E-04	1.46E-03	1.89
11	453	7010	4.95E-04	5.04E-04	9.77E-11	3.97E-04	4.74E-04	9.91E-04	2.00
12	469	7308	3.41E-04	3.47E-04	1.13E-10	2.55E-04	2.88E-04	6.40E-04	1.88
13	463	7286	2.42E-04	2.42E-04	1.39E-10	1.73E-04	1.94E-04	4.37E-04	1.81
14	458	7215	1.69E-04	1.69E-04	1.23E-10	1.19E-04	1.53E-04	3.17E-04	1.88
15	440	6955	1.29E-04	1.31E-04	1.45E-10	9.21E-05	9.10E-05	2.24E-04	1.73
16	435	7035	9.71E-05	9.91E-05	1.39E-10	6.89E-05	7.63E-05	1.74E-04	1.79
17	434	7167	8.52E-05	8.97E-05	1.41E-10	5.76E-05	5.47E-05	1.42E-04	1.67
18	419	6960	7.51E-05	7.97E-05	1.44E-10	5.00E-05	4.15E-05	1.21E-04	1.60
19	410	6838	6.06E-05	6.35E-05	1.47E-10	3.87E-05	3.65E-05	9.69E-05	1.60

Outline

Previous results

Global flux reconstructions

- Prager and Synge (1947):

$$\|\nabla u + \boldsymbol{\sigma}_h\|^2 + \|\nabla(u - u_h)\|^2 = \|\nabla u_h + \boldsymbol{\sigma}_h\|^2$$

for **any** $u_h \in H_0^1(\Omega)$ and **any** $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ s.t. $\nabla \cdot \boldsymbol{\sigma}_h = f$

- Hlaváček, Haslinger, Nečas, and Lovíšek (1979), Repin (1997), ...: global construction of $\boldsymbol{\sigma}_h$: unprecise/costly

Local flux reconstructions

- Ladevèze and Leguillon (1983), equilibrated face fluxes
- Destuynder and Métivet (1999), discrete flux $\boldsymbol{\sigma}_h$
- Luce and Wohlmuth (2004), local efficiency proof
- Vejchodský (2006), equilibration–hypercircle approach
- Kim (2007) & Ern, Nicaise, and Vohralík (2007), discontinuous Galerkin method elementwise prescription
- Braess and Schöberl (2008), Vohralík (2008), Ern and Vohralík (2009), **local Neumann MFE problems**

Previous results

Global flux reconstructions

- Prager and Synge (1947):

$$\|\nabla u + \boldsymbol{\sigma}_h\|^2 + \|\nabla(u - u_h)\|^2 = \|\nabla u_h + \boldsymbol{\sigma}_h\|^2$$

for **any** $u_h \in H_0^1(\Omega)$ and **any** $\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega)$ s.t. $\nabla\cdot\boldsymbol{\sigma}_h = f$

- Hlaváček, Haslinger, Nečas, and Lovíšek (1979), Repin (1997), . . . : global construction of $\boldsymbol{\sigma}_h$: unprecise/costly

Local flux reconstructions

- Ladevèze and Leguillon (1983), equilibrated face fluxes
- Destuynder and Métivet (1999), discrete flux $\boldsymbol{\sigma}_h$
- Luce and Wohlmuth (2004), local efficiency proof
- Vejchodský (2006), equilibration–hypercircle approach
- Kim (2007) & Ern, Nicaise, and Vohralík (2007), discontinuous Galerkin method elementwise prescription
- Braess and Schöberl (2008), Vohralík (2008), Ern and Vohralík (2009), **local Neumann MFE problems**

Previous results

Local potential reconstructions ($u_h \notin H_0^1(\Omega)$)

- Achdou, Bernardi, and Coquel (2003) & Karakashian and Pascal (2003), by prescription
- Carstensen and Merdon (2013), **local Neumann MFE problems**

Unified frameworks

- Ainsworth and Oden (1993)
- Carstensen (2005)
- Ainsworth (2010)
- Ern and Vohralík (heat equation 2010, Stokes equation 2012, nonlinear Laplace equation 2013)

Polynomial-degree-robust estimates

- Braess, Pillwein, and Schöberl (2009), conforming finite elements

Previous results

Local potential reconstructions ($u_h \notin H_0^1(\Omega)$)

- Achdou, Bernardi, and Coquel (2003) & Karakashian and Pascal (2003), by prescription
- Carstensen and Merdon (2013), **local Neumann MFE problems**

Unified frameworks

- Ainsworth and Oden (1993)
- Carstensen (2005)
- Ainsworth (2010)
- Ern and Vohralík (heat equation 2010, Stokes equation 2012, nonlinear Laplace equation 2013)

Polynomial-degree-robust estimates

- Braess, Pillwein, and Schöberl (2009), conforming finite elements

Previous results

Local potential reconstructions ($u_h \notin H_0^1(\Omega)$)

- Achdou, Bernardi, and Coquel (2003) & Karakashian and Pascal (2003), by prescription
- Carstensen and Merdon (2013), **local Neumann MFE problems**

Unified frameworks

- Ainsworth and Oden (1993)
- Carstensen (2005)
- Ainsworth (2010)
- Ern and Vohralík (heat equation 2010, Stokes equation 2012, nonlinear Laplace equation 2013)

Polynomial-degree-robust estimates

- Braess, Pillwein, and Schöberl (2009), conforming finite elements

Bibliography

Bibliography

- ERN A., VOHRALÍK M., Polynomial-degree-robust a posteriori estimates in a unified setting for conforming, nonconforming, discontinuous Galerkin, and mixed discretizations, *SIAM J. Numer. Anal.*, **53** (2015), 1058–1081.
- DOLEJŠÍ V., ERN A., VOHRALÍK M., *hp*-adaptation driven by polynomial-degree-robust a posteriori error estimates for elliptic problems, HAL Preprint 01165187, submitted for publication.

Thank you for your attention!