A posteriori error control for transmission problems with sign changing coefficients using localization of dual norms

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Outline

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Sign changing coefficients

Model problem

$$-\nabla\cdot(\underline{\Sigma}\nabla u) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- $\underline{\Sigma}$ **not positive definite** (and symmetric)
- $\Omega = \Omega_+ \cup \Omega_-$, $\sigma_+ > 0$ and $\sigma_- < 0$,

$$\underline{\Sigma}|_{\Omega_+} = \sigma_+ \underline{I}, \quad \underline{\Sigma}|_{\Omega_-} = \sigma_- \underline{I}$$

- specific application: electromagnetism for interfaces between dielectrics and (negative) metamaterials
- weak solution: $u \in H_0^1(\Omega)$ such that

$$(\underline{\Sigma}\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

- conditions for well-posedness follow from the Banach–Nečas–Babuška theorem – Bonnet-Ben Dhia, Chesnel, Ciarlet Jr. (2012): T-coercivity
- numerical discretization: Chesnel and Ciarlet Jr. (2013)

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Assumptions

We suppose

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$, an open polytope
- $\{\mathcal{T}_h\}_h$ is a family of shape-regular simplicial partitions;
- $\underline{\boldsymbol{\Sigma}} \in [L^\infty(\Omega)]^{d \times d}$ is piecewise constant on each given $\mathcal{T}_h$;
- $f \in L^2(\Omega)$;
- there exists a linear bijective operator $\mathrm{T} : H_0^1(\Omega) \to H_0^1(\Omega)$:

$$\|\nabla(\mathrm{T}v)\| \leq \|\mathrm{T}\| \|\nabla v\| \qquad \forall v \in H_0^1(\Omega), \qquad \text{(boundedness)}$$

$$\underline{\alpha} \|\nabla v\|^2 \leq (\underline{\boldsymbol{\Sigma}} \nabla v, \nabla(\mathrm{T}v)) \qquad \forall v \in H_0^1(\Omega) \qquad \text{(T-coercivity)}$$

In which norm to measure the error?

Energy norm

- $\|v\|_{\mathrm{en}}^2 := (\underline{\Sigma}\nabla v, \nabla v)$ for $v \in H_0^1(\Omega)$
- **not well-defined**: $(\underline{\Sigma}\nabla v, \nabla v) < 0$ may happen

Intrinsic problem-dependent norm

- definition

$$\|v\|_{\#} := \sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\underline{\Sigma}\nabla v, \nabla\varphi) \qquad v \in H_0^1(\Omega)$$

- one choice in the sup & the Cauchy–Schwarz inequality:

$$\frac{(\underline{\Sigma}\nabla v, \nabla(\mathrm{T}v))}{\|\nabla(\mathrm{T}v)\|} \le \|v\|_{\#} \le \|\underline{\Sigma}\nabla v\| \qquad \forall v \in H_0^1(\Omega)$$

- T-coercivity & boundedness of $\underline{\Sigma}$ and T:

$$\frac{\alpha}{\|\mathrm{T}\|}\|\nabla v\| \le \|v\|_{\#} \le \|\underline{\Sigma}\|_{\infty}\|\nabla v\| \qquad \forall v \in H_0^1(\Omega)$$

- $\Rightarrow$ **equivalence of** $\|\nabla\cdot\|$ **and** $\|\cdot\|_{\#}$

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Nonconforming discretizations

Broken Sobolev space

$$H^1(\mathcal{T}_h) := \{v \in L^2(\Omega);\ v|_K \in H^1(K) \qquad \forall K \in \mathcal{T}_h\}$$

Equality for $\underline{\Sigma} = \underline{I}$, $u_h \notin H^1_0(\Omega)$

$$\|\nabla(u-u_h)\|^2 = \underbrace{\sup_{\varphi \in H^1_0(\Omega);\, \|\nabla\varphi\|=1} (\nabla(u-u_h), \nabla\varphi)^2}_{\text{dual norm of the residual}} + \underbrace{\inf_{s \in H^1_0(\Omega)} \|\nabla(u_h - s)\|^2}_{\text{distance to } H^1_0(\Omega)}$$

Intrinsic problem-dependent norm, $u_h \in H^1(\mathcal{T}_h)$

$$|||u-u_h|||^2 := \|u-u_h\|^2_{\#} + \inf_{s \in H^1_0(\Omega)} \|\nabla(u_h - s)\|^2$$

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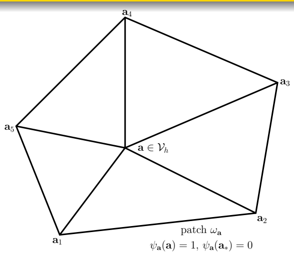
Localizations



Dual norm of the residual

$$\|v\|_{\#,\omega_{\mathbf{a}}} := \sup_{\varphi \in H_0^1(\omega_{\mathbf{a}});\, \|\nabla\varphi\|_{H_0^1(\omega_{\mathbf{a}})}=1} (\underline{\boldsymbol{\Sigma}}\nabla v, \nabla\varphi)_{\omega_{\mathbf{a}}}$$

Distance to $H_0^1(\Omega)$

$$\inf_{s^{\mathbf{a}} \in H_{\#}^1(\omega_{\mathbf{a}})} \|\nabla(u_h - s^{\mathbf{a}})\|_{\omega_{\mathbf{a}}},$$

where

$$H_{\#}^1(\omega_{\mathbf{a}}) := H^1(\omega_{\mathbf{a}}), \qquad \mathbf{a} \in \mathcal{V}_h^{\text{int}},$$

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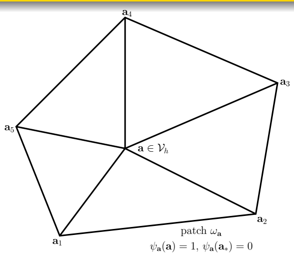

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Outline

Dual norm of the residual

Theorem

Let $\xi \in [L^2(\Omega)]^d$ *such that*

$$(\xi, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\rm int}$$

be arbitrary. Then

$$\sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\xi, \nabla\varphi)^2 \leq (d+1) C_{\rm cont,PF}^2 \sum_{\mathbf{a}\in\mathcal{V}_h} \sup_{\varphi \in H_0^1(\omega_{\mathbf{a}});\, \|\nabla\varphi\|_{\omega_{\mathbf{a}}}=1} (\xi, \nabla\varphi)_{\omega_{\mathbf{a}}}^2,$$

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \sup_{\varphi \in H_0^1(\omega_{\mathbf{a}});\, \|\nabla\varphi\|_{\omega_{\mathbf{a}}}=1} (\xi, \nabla\varphi)_{\omega_{\mathbf{a}}}^2 \leq (d+1) \sup_{\varphi \in H_0^1(\Omega);\, \|\nabla\varphi\|=1} (\xi, \nabla\varphi)^2.$$

Dual norm of the residual

Corollary

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ such that

$$(\underline{\Sigma}\nabla u_h, \nabla \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

be arbitrary. Then

$$\|u - u_h\|_{\#}^2 \le (d+1) C_{\mathrm{cont,PF}}^2 \sum_{\mathbf{a}\in\mathcal{V}_h} \|u - u_h\|_{\#,\omega_{\mathbf{a}}}^2,$$

$$\sum_{\mathbf{a}\in\mathcal{V}_h} \|u - u_h\|_{\#,\omega_{\mathbf{a}}}^2 \le (d+1)\|u - u_h\|_{\#}^2.$$

Distance to $H_0^1(\Omega)$

Theorem

Let $u \in H_0^1(\Omega)$ and $u_h \in H^1(\mathcal{T}_h)$ satisfying

$$\langle [\![u_h]\!], 1 \rangle_e = 0 \qquad \forall e \in \mathcal{E}_h$$

be arbitrary. Then

$$\inf_{s \in H_0^1(\Omega)} \|\nabla(u_h - s)\|^2 \leq (d+1) C_{\mathrm{cont,bPF}}^2 \sum_{\mathbf{a} \in \mathcal{V}_h} \inf_{s^{\mathbf{a}} \in H_\#^1(\omega_{\mathbf{a}})} \|\nabla(u_h - s^{\mathbf{a}})\|_{\omega_{\mathbf{a}}}^2,$$

$$\sum_{\mathbf{a} \in \mathcal{V}_h} \inf_{s^{\mathbf{a}} \in H_\#^1(\omega_{\mathbf{a}})} \|\nabla(u_h - s^{\mathbf{a}})\|_{\omega_{\mathbf{a}}}^2 \leq (d+1) \inf_{s \in H_0^1(\Omega)} \|\nabla(u_h - s)\|^2.$$

Outline

Upper bound on the error

Theorem

Let u be the weak solution and let $u_h \in H^1(\mathcal{T}_h)$ satisfying

$$(\underline{\Sigma}\nabla u_h, \nabla\psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} = (f, \psi_{\mathbf{a}})_{\omega_{\mathbf{a}}} \qquad \forall \mathbf{a} \in \mathcal{V}_h^{\mathrm{int}}$$

be arbitrary. For any equilibrated flux reconstruction and any potential reconstruction,

$$|||u - u_h|||^2 \leq \sum_{K \in \mathcal{T}_h} (\|\underline{\Sigma}\nabla u_h + \boldsymbol{\sigma}_h\|_K + \eta_{\mathrm{osc},K})^2 + \sum_{K \in \mathcal{T}_h} \|\nabla(u_h - s_h)\|_K^2.$$

Lower bound on the error

Theorem

Let u be the weak solution, let u_h be piecewise polynomial, and let $d = 2$. Then, for $\boldsymbol{\sigma}_h$ given by mixed finite element solution of local Neumann problems, for all $K \in \mathcal{T}_h$,

$$\|\underline{\Sigma}\nabla u_h + \boldsymbol{\sigma}_h\|_K \leq C_{\rm st} C_{\rm cont,PF} \sum_{\mathbf{a}\in\mathcal{V}_K} \|u - u_h\|_{\#,\omega_\mathbf{a}} + \eta_{{\rm osc},\omega_K},$$

$$\|\underline{\Sigma}\nabla u_h + \boldsymbol{\sigma}_h\| \leq 3 C_{\rm st} C_{\rm cont,PF} \|u - u_h\|_{\#} + 3 C_{\rm st} \eta_{\rm osc},$$

and, for s_h given by finite element solution of local Dirichlet problems, for all $K \in \mathcal{T}_h$,

$$\|\nabla(u_h - s_h)\|_K \leq C_{\rm st} C_{\rm cont,bPF} \sum_{\mathbf{a}\in\mathcal{V}_K} \inf_{s^\mathbf{a}\in H^1_\#(\omega_\mathbf{a})} \|\nabla(u_h - s^\mathbf{a})\|_{\omega_\mathbf{a}},$$

$$\|\nabla(u_h - s_h)\| \leq 3 C_{\rm st} C_{\rm cont,bPF} \inf_{s\in H^1_0(\Omega)} \|\nabla(u_h - s)\|.$$

Outline

Numerics

Setting

- $\mathbb{P}_1$ conforming finite element discretization
- estimate

$$\eta := \left\{ \sum_{K \in \mathcal{T}_h} \left(\|\underline{\Sigma} \nabla u_h + \boldsymbol{\sigma}_h\|_K + \eta_{\mathrm{osc},K} \right)^2 \right\}^{1/2}$$

- effectivity index

$$\mathrm{Eff} := \frac{\eta}{\|u - u_h\|_\#}$$

- computable effectivity bounds

$$\mathrm{Eff}_+ := \frac{\eta}{\|\underline{\Sigma} \nabla (u - u_h)\|}, \quad \mathrm{Eff}_- := \frac{\eta}{\frac{(\underline{\Sigma}\nabla(u - u_h), \nabla(\mathrm{T}(u - u_h)))}{\|\nabla(\mathrm{T}(u - u_h))\|}}$$

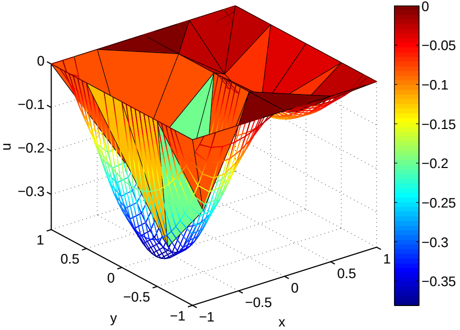
Regular solution

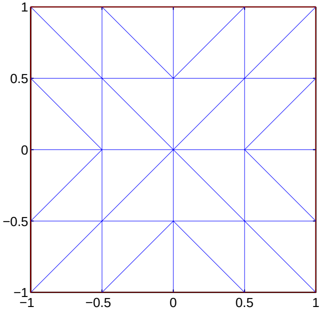
Data

- $\Omega := (-1, 1) \times (-1, 1)$
- $\Omega_+ := (0, 1) \times (-1, 1)$, $\Omega_- := (-1, 0) \times (-1, 1)$
- $\sigma_+ = 1$, $\sigma_- < 0$
- exact solution

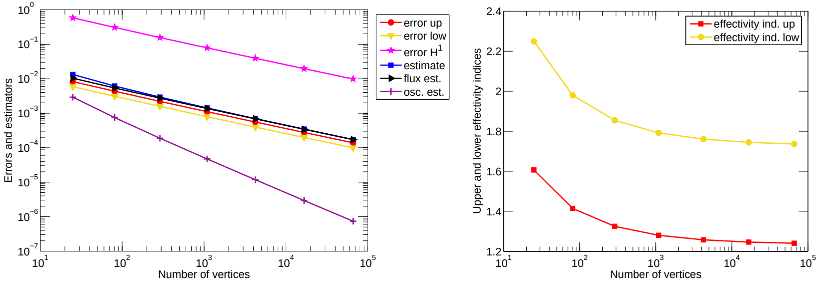
$$u(x, y) = \sigma_- x(x + 1)(x - 1)(y + 1)(y - 1) \text{ for } (x, y) \in \Omega_+,$$
$$u(x, y) = x(x + 1)(x - 1)(y + 1)(y - 1) \text{ for } (x, y) \in \Omega_-$$

Exact solution and mesh

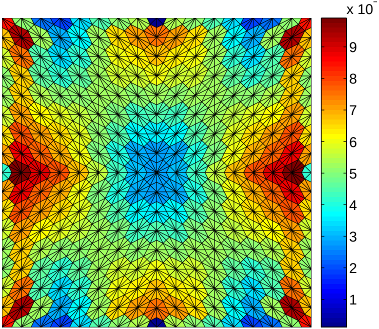


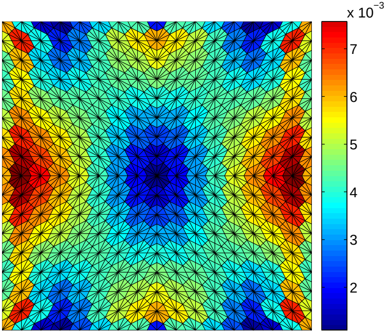


Errors and estimators, $\sigma_- = -0.01$

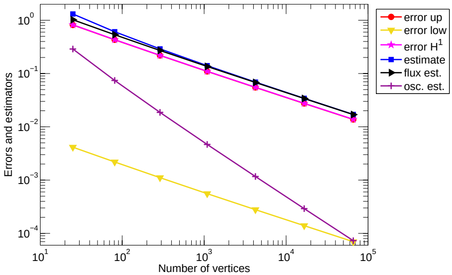


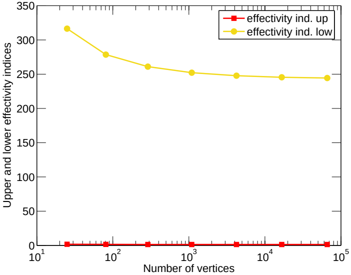
Errors and estimators, $\sigma_- = -1/3$





Errors and estimators, $\sigma_- = -0.99$





Singular solution

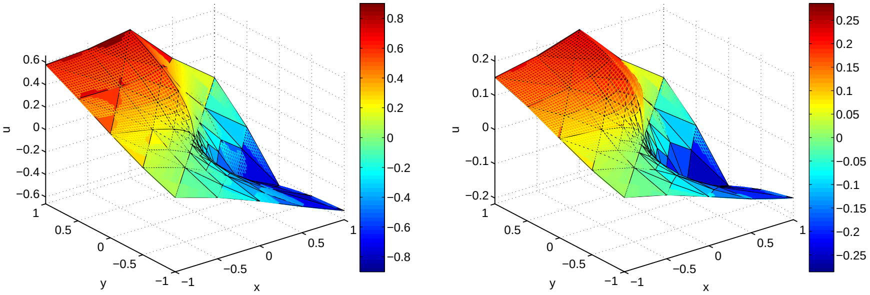
Data

- $\Omega := (-1,1) \times (-1,1)$
- $\Omega_+ := (0,1) \times (0,1)$, $\Omega_- := \Omega \setminus \overline{\Omega_+}$
- $\sigma_+ = 1$, $\sigma_- < 0$
- exact solution

$$u(x,y) = r^\lambda (c_1 \sin(\lambda\theta) + c_2 \sin(\lambda(\pi/2 - \theta))) \text{ for } (x,y) \in \Omega_+,$$
$$u(x,y) = r^\lambda (d_1 \sin(\lambda(\theta - \pi/2)) + d_2 \sin(\lambda(2\pi - \theta)) \text{ for } (x,y) \in \Omega_-$$

- $u \in H^{1+\lambda}(\Omega)$
- $\sigma_- = -5$: $\lambda \approx 0.4601069123$
- $\sigma_- = -3.1$: $\lambda \approx 0.1391989493$

Exact solutions



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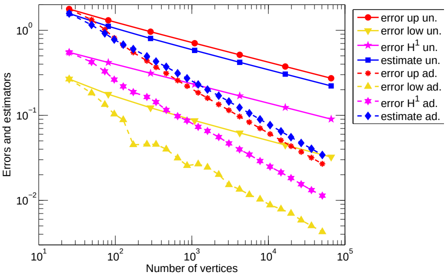
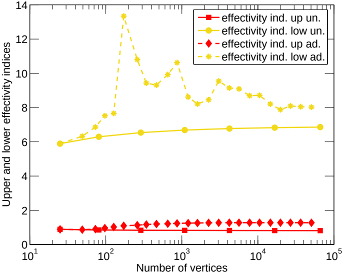
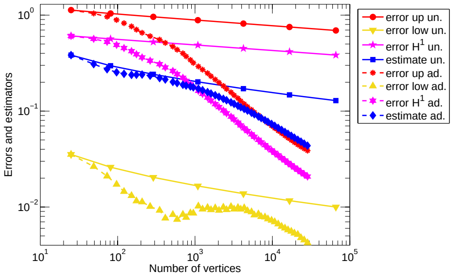
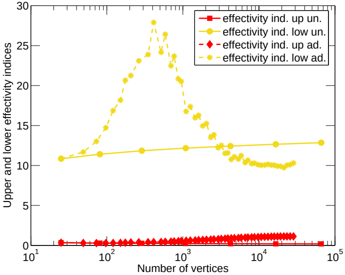




Errors and estimators, $\sigma_- = 3.1$





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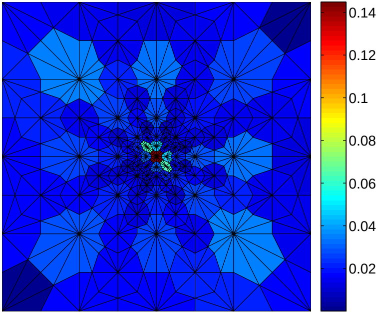

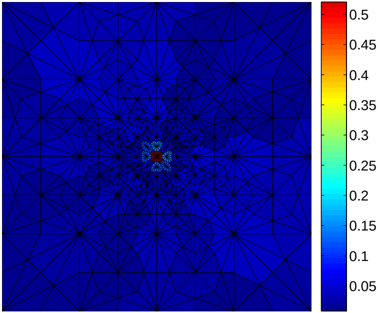

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- robust equivalence between the intrinsic dual norm and computable a posteriori estimators
- unified framework covering all classical numerical methods
- easily computable upper and lower bounds for the dual norm $\Rightarrow$ loss of robustness
- cure: localization of the dual norms

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Bibliography

Bibliography

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Thank you for your attention!